Algorithmic reduction of Normal Forms

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Abstract We present an algorithmic procedure to further reduce (Poincaré) Normal Forms; we give both general results and methods, and a concrete application.

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1. In the Poincaré theory of Normal Forms[1], one considers systems of C^∞ ODEs in R^n, having a fixed point – say, the origin – and aims to a classification of perturbative expansions (formal power series) for the systems around the fixed point modulo formal C^∞ changes of coordinates. One obtains then the *Poincaré-Dulac theorem* according to which any system can be formally put into Poincaré Normal Form up to any given order n by means of a formal series of Poincaré transformations.

Concretely, let us denote by V_k the set of homogeneous vector polynomials of degree $k+1$; if we write the system as

$$\dot{y} \ = \ \sum_{m=0}^{\infty} f_m(y) \ , \tag{1}$$

where $f_m \in V_m$, we then consider near-identity changes of coordinates of the form

$$y \ = \ x + h_k(x) \tag{2}$$

with $h_k \in V_k$; as it can be readily checked by direct computation, (2) changes (1) into a system $\dot{x} = \sum_m \widetilde{f}_m(x)$, where $\widetilde{f}_m(x) = f_m(x)$ for $m < k$,

$$\widetilde{f}_k = f_k - \mathcal{L}_0(h_k) \ , \tag{3}$$

and terms with $m > k$ are changed in a more complicate way; in (3) the $\mathcal{L}_0$ denotes the linear *homological operator*: if we define

$$\{f, g\} := (f \cdot \nabla) g - (g \cdot \nabla) f \ , \tag{4}$$

then the $\mathcal{L}_0 : V_k \to V_k$ associated to f is simply given by

$$\mathcal{L}_0(h) = \{f_0, h\} \ . \tag{5}$$

It is clear that, denoting by π the projection on the range of $\mathcal{L}_0$, by choosing h_k as the solution of the *homological equation*

$$\pi f_k = \mathcal{L}_0(h_k) \tag{6}$$

we can eliminate all terms in $V_k \cap \mathrm{Ran}(\mathcal{L}_0)$; we call *resonant* the terms in $[\mathrm{Ran}(\mathcal{L}_0)]^c$ (by introducing a scalar product in each of the V_k, this is $\mathrm{Ker}(\mathcal{L}_0^+)$; the appropriate scalar product is the Bargman one[2]), and *non-resonant* those in $\mathrm{Ran}(\mathcal{L}_0)$. We can thus consider transformations like (2) successively for $k = 1, 2, ..., n$, and eliminate all the non-resonant terms up to order n. When this is done, i.e. when in (1)

$$f_k \in [\mathrm{Ran}(\mathcal{L}_0)]^c \quad \forall k \geq 1 \ , \tag{7}$$

we say the system is in *Poincaré Normal Form.*

2. Some points should be noted in the above construction: (i) the solution to (6) is not unique, being only unique up to a $\delta h_k \in \mathrm{Ker}(\mathcal{L}_0)$; (ii) the h_k and $\widetilde{h}_k = h_k + \delta h_k$ produce the same transformation on the term f_k, but (in general) different ones on terms f_m with $m > k$; (iii) the transformation (2) can, and in general will, produce resonant terms in V_m for $m > k$.

These simple considerations suggest that one could try to use the freedom in choosing h_k in order to produce resonant higher order terms, and thus eliminate from the Normal Form some of the resonant terms (not eliminable by the standard Poincaré algorithm described above).

This is indeed possible; moreover – what is even more relevant – it can be done in a completely algorithmical way, as for the Poincaré procedure. We do now discuss the algorithm and the final result, together with a relevant example; a more complete discussion is given elsewhere[3].

3. Let us consider a formal power series $f(x) = \sum_k f_k(x)$, $f_k \in V_k$ (we denote the linear space of these by V); to this we associate a series of linear *homological operators* $\mathcal{L}_k : V_m \to V_{m+k}$ by

$$\mathcal{L}_k(h) = \{f_k, h\} \ ; \tag{8}$$

also, we denote by π_k the projection on the range of $\mathcal{L}_k$.

We consider now a first chain of subspaces $H^{(p)} \subseteq V$, with $H^{(0)} = V$, and defined for $p > 0$ by

$$H^{(p+1)} = H^{(p)} \cap \mathrm{Ker}(\mathcal{L}_p) \ . \tag{9}$$

We can then consider $\mathcal{M}_p$, the restriction of $\mathcal{L}_p$ to $H^{(p)}$; we define then a second chain of subspaces $F^{(p)} \subseteq V$, with $F^{(0)} = V$, and defined for $p > 0$ by

$$F^{(p+1)} = F^{(p)} \cap [\mathrm{Ran}(\mathcal{M}_p)]^c \tag{10}$$

(having introduced the scalar products in V_k, and thus in V, we would have $F^{(p+1)} = F^{(p)} \cap \mathrm{Ker}(\mathcal{M}_p^+)$).

Our main result is then: *Given any system (1), it is possible to formally reduce it, by a formal double series of Poincaré transformations, to a system*

$$\dot{x} = f^*(x) = \sum_{m=0}^{\infty} f_m^*(x)$$

where $f_k^* \in F^{(k)}$ for all $k = 1, ..., n$, for arbitrary given n. We say then that the system is in Poincaré *renormalized* form (PRF for short).

This result can be proven constructively, giving the concrete algorithm of reduction to renormalized form.

We notice that it would be possible to first reduce the system to the usual Poincaré normal form, than take it into a "second normalized form" using the effect of h_k ($k = 1, 2, ...$) on V_{k+1}, then into a third one using the effect of h_k ($k = 1, 2, ...$) on V_{k+2}, and so on up to an "n-th normalized" which is our renormalized form up to order n; this is the reason for the name we adopt.

However, for computational purposes it appears preferable to proceed in a different way, i.e. order by order, taking f_k into f_k^* before passing to operate on f_{k+1}; we illustrate the algorithm in this approach.

4. We will use Lie-Poincaré transformations, i.e. we will write the change of coordinates as

$$y \;=\; \left[e^{-\lambda H_k} x \right]_{\lambda=1} \;, \tag{11}$$

where H_k is a vector field with components h_k, i.e. $H_k = h_k(x) \cdot \nabla$. The expression of $X = f(x) \cdot \nabla$ in the new coordinates is given by $\widetilde{X} = e^{H_k} X e^{-H_k}$, which can be explicitely computed by means of the Baker-Campbell-Haussdorf formula[4]; in this way the f_m with $m < k$ are unchanged, and f_k follows (3), as in the usual Poincaré transformations. Referring to Ref.3 for details, we introduce the operator $\mathcal{H} = \{h_k, .\}$ and just mention that with (11) the f_m are transformed into

$$\widetilde{f}_m \;=\; \sum_{s=0}^{[m/k]} (1/k!)\, \mathcal{H}^s (f_{m-sk}) \;, \tag{12}$$

where the square bracket denotes integer part, and $\mathcal{H}^0$ is the identity operator.

Let us now suppose the series $f = \sum_m f_m$ is in PRF up to f_{k-1}, and let us try to transform $f_k \equiv f_k^{(0)}$ into f_k^* by means of transformations like (11). In doing this, we should make sure we do not modify the terms $f_1, ..., f_{k-1}$ which are already in PRF. If we use $h_k^{(0)} \in V_k$, this is no restriction, and in this way we can eliminate all terms in $[\mathrm{Ran}(\mathcal{L}_0)] \cap V_k$, as from (12) we get now $f_k^{(0)} \to f_k^{(1)} = f_k^{(0)} - \mathcal{L}_0(h_k^{(0)})$. Let us now choose a $h_{k-1}^{(1)} \in V_{k-1}$; in order to leave f_{k-1} unchanged, we require that $h_{k-1}^{(1)} \in \mathrm{Ker}(\mathcal{L}_0) = H^{(1)}$; from (12) we have $f_k^{(1)} \to f_k^{(2)} = f_k^{(1)} - \mathcal{L}_1(h_{k-1}^{(1)})$ and thus in this way we can eliminate all terms in the image of $H^{(1)}$ under $\mathcal{L}_1$, i.e. in $\mathrm{Ran}(\mathcal{M}_1)$. We can proceed like this for $h_{k-s}^{(s)} \in V_{k-s}$ for all $s = 2, 3, ...k-1$; each time we have to ask, in order to leave the f_m with $m < k$ unchanged, that $h_{k-s}^{(s)} \in H^{(s)}$, and with this requirement (12) just gives

$$f_k^{(s-1)} \to f_k^{(s)} = f_k^{(s-1)} - (1/s!) \mathcal{L}_{k-s} \left(h_{k-s}^{(s)} \right) \;; \tag{13}$$

thus, in this step we can eliminate terms in $\mathrm{Ran}(\mathcal{M}_s)$. In this way, after performing sequentially these transformations for $s = 0, 1, ..., k-1$ we arrive at a $f_k^{(k)} = f_k^* \in F^{(k)}$, as claimed above.

With the notation introduced above – and with the adjoint being meant with respect to the Bargman scalar product[2] – the $h_{k-s}^{(s)}$ to be used at each step are obtained as solution to the higher homological equations

$$h_{k-s}^{(s)} \ = \ \mu_s \circ \mathcal{M}_s^+ \circ P_s \left(f_k^{(s)} \right) \ . \tag{14}$$

Clearly, once we introduce a given orthogonal basis in each of the V_k, (14) amounts to a linear algebra problem, and the whole procedure can be implemented by solving at each stage *finite dimensional linear algebraic equations*, as in the standard (Poincaré) Normal Form reduction/unfolding procedure.

5. In order to illustrate the reduction of Normal Forms obtained here by a significant example, we consider the unfolding of Normal Forms for two dimensional systems with linear part $f_0(x) = Ax$, with A the rotation matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad . \tag{15}$$

It is well known that, with $r^2 = (x_1^2 + x_2^2)$, the Poincaré Normal Form corresponding to this is

$$f(x) \ = \ Ax \ + \ \sum_{k=1}^\infty \ r^2 \left[a_k I + b_k A \right] x \ , \tag{16}$$

with a_k, b_k arbitrary real constants. By the procedure proposed here, we can reduce the system to PRF, whose general form is

$$f^*(x) \ = \ Ax \ + \ \left[\left(r^{2\mu} a_\mu + r^{4\mu} \alpha \right) I + r^{2\nu} \beta A \right] x \ , \tag{17}$$

where μ, ν are respectively the smallest integers such that $a_\mu \neq 0$, $b_\nu \neq 0$, and α, β are real constants (with $\beta = 0$ if $\nu > \mu$); the computation of (17) is immediate[3].

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