Diffusion of a Particle in Presence of N Moving Point Sources

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Abstract

We analyze the diffusion of a particle in R^3 subject to N point interactions moving on preassigned non intersecting paths.
For any regular motion of the sources, we give an existence and uniqueness result for the corresponding parabolic evolution equation, together with a rather explicit representation of the solution.
We exemplify our approach for $N = 1$ in the case of uniform motion on a straight line, exhibiting the complete solution of the problem.

1. Introduction and result

We consider the diffusion of a particle in presence of N point interactions of fixed strengths placed at points in R^3 moving on given smooth paths. The problem is then described by a parabolic evolution equation with generator depending explicitely on time.
This model is of interest in the study of diffusion in fluids. It can also be viewed as a first step in the study of non linear models, in which the paths depend on the solution itself.
¿From the mathematical point of view, the problem is non trivial since the strong time-dependence of the generator prevents the application of known general results in the theory of non autonomous parabolic equations (see e.g. [1],[2]).
We shall prove an existence and uniqueness theorem, exploiting the detailed knowledge of the domain and of the action of the generator.
Let us describe the model.
Let $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_j \in R$, $j = 1, \ldots, N$, and let $Y(t) = (y_1(t), \ldots, y_N(t))$, $t \in [0,T]$, $T > 0$, $y_i(t) \neq y_j(t)\ \ \forall t$, be a collection of N given curves in R^3, of class C^1, which do not intersect at equal times.
For any $t \in [0,T]$, let $-\Delta_{\alpha,Y(t)}$ be the Schrödinger operator in $L^2(R^3)$ with point interactions of strengths α_j placed at $Y(t)$.
For the construction and the main properties of $-\Delta_{\alpha,Y(t)}$ we refer to the monograph [3]. The operator $-\Delta_{\alpha,Y(t)}$ is self-adjoint and bounded from below, with domain and action given by

$$\begin{aligned} D(-\Delta_{\alpha,Y(t)}) = \{u(t) \in L^2(R^3) \mid u(t) = \phi_\lambda(t) + \sum_{j=1}^{N} q_j(t) G_\lambda(\cdot - y_j(t)), \\ \phi_\lambda(t) \in H^2(R^3), \lim_{|x - y_j(t)| \to 0} (u(x,t) - q_j(t) G_0(x - y_j(t))) = \alpha_j q_j(t)\} \end{aligned} \tag{1.1}$$

$$(-\Delta_{\alpha,Y(t)} + \lambda) u(t) = (-\Delta + \lambda)\phi_\lambda(t). \tag{1.2}$$

We have introduced the following notation

$$G_\lambda(x - x') = (-\Delta + \lambda)^{-1}(x - x') = \frac{e^{-\sqrt{\lambda}|x - x'|}}{4\pi|x - x'|},\ \lambda \geq 0. \tag{1.3}$$

$H^n(R^3)$ is the standard Sobolev space of order n. It is clear from (1.1) that the operator domain consists of functions with a regular part $\phi_\lambda(t)$ plus the "potential" of "point charges" $q_j(t)$. The limit in (1.1) is regarded as a boundary condition satisfied by u at $Y(t)$.
We shall study the evolution problem

$$\frac{\partial u(t)}{\partial t} = \Delta_{\alpha,Y(t)} u(t) \quad t \in [0,T] \tag{1.4}$$

$$u(0) = u^0 \in D(\Delta_{\alpha,Y(0)}). \tag{1.5}$$

In previous papers ([4],[5],[6]) problems of this type were considered, where the location of the point sources is fixed and their strengths depend on time. These problems are less singular than the one treated here, since the generator has the same form domain at all times. Our main result is

Theorem 1.
If $Y(t) = (y_1(t), \ldots, y_N(t))$ are N curves in R^3 of class C^1such that

$$\sup_{0<t<T} \sup_{i \neq j} |y_i(t) - y_j(t)| > c > 0$$

than there exists a unique $u \in C^1([0,T], L^2(R^3)) \cap C([0,T], D(\Delta_{\alpha,Y(\cdot)}))$ which solves problem (1.4),(1.5). Moreover $u(t)$ is given by

$$u(t) = P_t u^0 + \sum_{j=1}^{N} \int_0^t ds P_{t-s}(\cdot - y_j(s)) q_j(s) \ , \tag{1.6}$$

where P_t is the integral operator defined by the heat kernel

$$P_t(x - x') = e^{\Delta t}(x - x') = \frac{e^{-\frac{|x-x'|^2}{4t}}}{(4\pi t)^{3/2}} \tag{1.7}$$

and the charges $q_j(t)$ satisfy the system of Volterra integral equations

$$\frac{q_j(t)}{4\sqrt{\pi}} + \alpha_j \int_0^t ds \frac{q_j(s)}{\sqrt{t-s}} + \int_0^t ds q_j(s) C_j(t,s) - \sum_{l=1, l\neq j}^{N} \int_0^t ds q_l(s) Q_{lj}(t,s) = f_j(t), \tag{1.8}$$

$$f_j(t) = \int_0^t ds \frac{P_s u^0(y_j(s))}{\sqrt{t-s}}, \tag{1.9}$$

$$C_j(t,s) = \int_0^1 dz \frac{1}{\sqrt{(1-z)z}} \Big(A_j(s+(t-s)z,s) + \dot{B}_j(s+(t-s)z,s)$$
$$+ \frac{B_j(s+(t-s)z,s) - (4\pi^{3/2})^{-1}}{2(t-s)z} \Big), \tag{1.10}$$

$$A_j(t,s) = \frac{(y_j(t)-y_j(s)) \cdot \dot{y}_j(s)}{8\pi^{3/2}(t-s)} \left(\frac{\sqrt{t-s}}{|y_j(t)-y_j(s)|} \right)^3 \int_0^{\frac{|y_j(t)-y_j(s)|}{\sqrt{t-s}}} dz\, z^2 e^{-\frac{z^2}{4}}, \tag{1.11}$$

$$B_j(t,s) = \frac{1}{4\pi^{3/2}} \frac{\sqrt{t-s}}{|y_j(t)-y_j(s)|} \int_0^{\frac{|y_j(t)-y_j(s)|}{\sqrt{t-s}}} dz\, e^{-\frac{z^2}{4}}, \tag{1.12}$$

$$Q_{lj}(t,s) = \int_s^t d\sigma \frac{P_{\sigma-s}(y_j(\sigma)-y_l(s))}{\sqrt{t-\sigma}}, \tag{1.13}$$

where $\dot{B}_j(t,s)$ denotes derivative with respect to the second argument .

The result described in theorem 1 can be easily generalized to the case in which also the strengths depend on time. Following the line of [6], one can further consider the limit $N \to \infty$ and its relation with diffusion in presence of a time-dependent potential.
A slight modification of the proof should also give the corresponding existence and uniqueness result for the Schrödinger or the wave equation. This will be of interest in the study of quantum mechanical scattering from moving particles and as an approach to the problem of classical electrodynamics in presence of charged point particles ([7]).
The paper is organized as follows.
In Section 2 we briefly analyze equation (1.8) and prove that its solution has some useful regularity properties.
Using this result, in Section 3 we prove that (1.6) is in fact the unique solution of the evolution problem (1.4),(1.5).

As an example, in Section 4 we consider the case $N = 1$ with $y(t) = vt$, $v \in R^3$, and we give the explicit solution of the evolution problem.

2. The integral equation

In this Section we consider the system of Volterra integral equations (1.8). We prove

Lemma 1.
For any $T > 0$, (1.8) has a unique solution $q_j(t)$, $j = 1, \dots, N$, satisfying

$$q_j \in C^0([0,T]) \cap C^1((0,T]) \ , \qquad \sup_{t\in[0,T]} |t^{1/4}\dot{q}_j(t)| < \infty. \tag{2.1}$$

Proof. As a consequence of the regularity assumption on the curves $y_j(t)$ one can easily check that, in (1.10), the integral kernel $C_j(t,s)$ is a continuous function and $Q_{lj}(t,s)$ is of class C^1. Denoting by $\tilde{\eta}$ the Fourier transform of η, we can decompose u^0 as

$$\tilde{u}^0(k) = \tilde{\phi}_0^0(k) + \sum_{j=1}^N q_j(0) \frac{e^{-ik\cdot y_j(0)}}{(2\pi)^{3/2}k^2}, \tag{2.2}$$

where $\phi_0^0 \in L^2_{loc}(R^3)$ and $|\nabla\phi_0^0|, \Delta\phi_0^0 \in L^2(R^3)$. Then an explicit computation gives

$$\begin{aligned} P_s u^0(y_j(s)) &= P_s\phi_0^0(y_j(s)) + \sum_{l=1}^N \frac{q_l(0)}{(2\pi)^3} \int_{R^3} dk \frac{1}{k^2} e^{-k^2 s + ik\cdot(y_j(s)-y_l(0))} \\ &= P_s\phi_0^0(y_j(s)) + q_j(0)\frac{B_j(s,0)}{\sqrt{s}} \\ &+ \sum_{l=1, l\neq j}^N \frac{q_l(0)}{4\pi^{3/2}|y_j(s)-y_l(0)|} \int_0^{\frac{|y_j(s)-y_l(0)|}{\sqrt{s}}} dz\, e^{-\frac{z^2}{4}}, \end{aligned} \tag{2.3}$$

(see e.g. [8]). From (1.9),(2.3) we conclude that $f \in C^0([0,T])$ and there exists a unique solution $q_j \in C^0([0,T])$ of (1.8) (see e.g. [9]).

To establish the required regularity properties of the solution it is convenient to write the corresponding equation for $w_j(t) = q_j(t) - q_j(0)$. A simple computation shows the w_j satisfies the same equation (1.8) but with a different datum $g_j(t)$ given by

$$
\begin{aligned}
g_j(t) &= f_j(t) - \frac{q_j(0)}{4\sqrt{\pi}} - \alpha_j q_j(0) \int_0^t ds \frac{1}{\sqrt{t-s}} \\
&-q_j(0) \int_0^t ds C_j(t,s) + \sum_{l=1, l\neq j}^{N} q_l(0) \int_0^t ds Q_{lj}(t,s) \\
&= \int_0^t ds \frac{P_s \phi_0^0(y_j(s)) - \phi_0^0(y_j(0))}{\sqrt{t-s}} + q_j(0) \int_0^t ds \frac{B_j(s,0) - (4\pi^{3/2})^{-1}}{\sqrt{s(t-s)}} \\
&-q_j(0) \int_0^t ds C_j(t,s) + \sum_{l=1, l\neq j}^{N} q_l(0) \int_0^t ds Q_{lj}(t,s) \\
&+ \sum_{l=1, l\neq j}^{N} q_l(0) \int_0^t ds \frac{1}{\sqrt{t-s}} \left[\frac{1}{4\pi^{3/2}|y_j(s) - y_l(0)|} \int_0^{\frac{|y_j(s)-y_l(0)|}{\sqrt{s}}} dz\, e^{-\frac{z^2}{4}} \right. \\
&\left. - \frac{1}{4\pi |y_j(0) - y_l(0)|} \right], \qquad (2.4)
\end{aligned}
$$

where we have used the boundary condition at $y_j(0)$. The last four terms in the r.h.s. of (2.4) belong to $C^1([0,T])$. The derivative of the first term in (2.4) can only be singular in $t = 0$.
An integration by parts ([9]), the regularity of ϕ_0^0 and Schwartz's inequality give

$$
\begin{aligned}
&\left| \frac{d}{dt} \int_0^t ds \frac{P_s \phi_0^0(y_j(s)) - \phi_0^0(y_j(0))}{\sqrt{t-s}} \right| \\
&= \left| \int_0^t ds \frac{1}{\sqrt{t-s}} \frac{d}{ds} P_s \phi_0^0(y_j(s)) \right| \\
&= \left| \int_0^t ds \frac{1}{(2\pi)^3 \sqrt{t-s}} \int_{R^3} dk\, \tilde{\phi}_0^0(k) (ik \cdot \dot{y}_j(s) - k^2) e^{ik\cdot y_j(s) - k^2 s} \right| \\
&\leq \frac{c}{t^{1/4}} \left[\|\Delta \phi_0^0\|_{L^2(R^3)} + \left(\sup_{t\in[0,T]} |\dot{y}_j(t)| \right) \||\nabla \phi_0^0|\|_{L^2(R^3)} \right], \qquad (2.5)
\end{aligned}
$$

where c denotes a generic positive constant.
Then the datum g_j satisfies the regularity assumptions (2.1). Iterating the integral equation (1.8) and using the smoothing properties of the Abel operator ([9]) one easily sees that the solution q_j also satisfies (2.1), concluding the proof of lemma 1. $\square$

3. Existence and uniqueness

Using the result of lemma 1, we give here the proof of theorem 1.

Proof of theorem 1. Let q_j be the solution of (1.8), satisfying (2.1), and define

$$u(t) = P_t u^0 + \sum_{l=1}^{N} \int_0^t ds P_{t-s}(\cdot - y_l(s)) q_l(s) \; . \tag{3.1}$$

It is obviously true that (3.1) satisfies the initial condition (1.5).
We show next that $u(t) \in \Delta_{\alpha,Y(t)}$. The Fourier transform of $u(t)$ reads

$$\tilde{u}(k,t) = e^{-k^2 t}\tilde{u}^0(k) + \frac{1}{(2\pi)^{3/2}} \sum_{l=1}^{N} \int_0^t ds q_l(s) e^{-k^2(t-s) - ik\cdot y_l(s)}. \tag{3.2}$$

Observe that for any $\lambda > 0$ an integration by parts yields

$$\begin{aligned}
&\int_0^t ds q_l(s) e^{-k^2(t-s)-ik\cdot y_l(s)} \\
&= \frac{1}{k^2+\lambda}\int_0^t ds q_l(s) e^{-ik\cdot y_l(s)} \frac{\partial}{\partial s}\left(e^{-k^2(t-s)}\right) \\
&+ \frac{\lambda}{k^2+\lambda}\int_0^t ds q_l(s) e^{-k^2(t-s)-ik\cdot y_l(s)} \\
&= -\frac{1}{k^2+\lambda}\int_0^t ds \frac{\partial}{\partial s}\left(q_l(s) e^{-ik\cdot y_l(s)}\right) e^{-k^2(t-s)} + q_l(t)\frac{e^{-ik\cdot y_l(t)}}{k^2+\lambda} \\
&- q_l(0)\frac{e^{-ik\cdot y_l(0) - k^2 t}}{k^2+\lambda} + \frac{\lambda}{k^2+\lambda}\int_0^t ds q_l(s) e^{-k^2(t-s)-ik\cdot y_l(s)}.
\end{aligned} \tag{3.3}$$

Since

$$\tilde{u}^0(k) = \tilde{\phi}^0_\lambda(k) + \sum_{l=1}^{N} \frac{q_l(0)}{(2\pi)^{3/2}} \frac{e^{-ik\cdot y_l(0)}}{k^2+\lambda}, \quad \phi^0_\lambda \in H^2(R^3), \tag{3.4}$$

we conclude

$$\begin{aligned}
\tilde{u}(k,t) &= e^{-k^2 t}\tilde{\phi}^0_\lambda(k) + \frac{\lambda(2\pi)^{-3/2}}{k^2+\lambda}\sum_{l=1}^{N}\int_0^t ds q_l(s) e^{-k^2(t-s)-ik\cdot y_l(s)} \\
&-\frac{(2\pi)^{-3/2}}{k^2+\lambda}\sum_{l=1}^{N}\int_0^t ds \dot{q}_l(s) e^{-ik\cdot y_l(s)-k^2(t-s)} \\
&+\frac{(2\pi)^{-3/2}}{k^2+\lambda}\sum_{l=1}^{N}\int_0^t ds q_l(s) ik\cdot\dot{y}_l(s) e^{-ik\cdot y_l(s)-k^2(t-s)} \\
&+\sum_{l=1}^{N}\frac{q_l(t)}{(2\pi)^{3/2}}\frac{e^{-ik\cdot y_l(t)}}{k^2+\lambda} \equiv \tilde{\phi}_\lambda(k,t) + \sum_{l=1}^{N}\frac{q_l(t)}{(2\pi)^{3/2}}\frac{e^{-ik\cdot y_l(t)}}{k^2+\lambda}.
\end{aligned} \tag{3.5}$$

The first three terms in (3.5) are easily seen to be in $H^2(R^3)$. For the fourth term in (3.5) an explicit computation gives

$$\begin{aligned}
&\int_{R^3} dk \left(\frac{k^2}{k^2+\lambda}\right)^2 \left|\int_0^t ds q_l(s) ik\cdot\dot{y}_l(s) e^{-ik\cdot y_l(s)-k^2(t-s)}\right|^2 \\
&= 2\pi i \int_0^t ds q_l(s) \int_0^t ds' q_l(s') |\dot{y}_l(s)-\dot{y}_l(s')| \cos\hat{\theta} \\
&\times \int_0^\infty dr r^3 \left(\frac{r^2}{r^2+\lambda}\right)^2 e^{-r^2(2t-s-s')} \int_0^\pi d\theta \sin\theta\cos\theta e^{-ir|y_l(s)-y_l(s')|\cos\theta} \\
&\le c\left(\sup_{t\in[0,T]} |\dot{y}_l(t)|\right)\left(\sup_{t\in[0,T]} |q_l(t)|\right)^2 \int_0^t ds \int_0^t ds' |y_l(s)-y_l(s')| \\
&\times \int_0^\infty dr r^4 e^{-r^2(2t-s-s')} h(r|y_l(s)-y_l(s')|),
\end{aligned} \tag{3.6}$$

where

$$h(x) \equiv \frac{1}{x^2}\left(\cos x - \frac{\sin x}{x}\right). \tag{3.7}$$

In (3.6) the computation of the integral has been done using a system of spherical coordinates such that $k=(r,\theta,\phi)$, $y_l(s)-y_l(s')=(|y_l(s)-y_l(s')|,0,0)$, $\dot y_l(s)-\dot y_l(s')=(|\dot y_l(s)-\dot y_l(s')|,\hat\theta,0)$ and the formula

$$\xi_1\cdot\xi_2=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2), \tag{3.8}$$

where $\xi_i=(1,\theta_i,\phi_i)$, $i=1,2$. Using the boundedness of the function h and the regularity of the curves, one easily sees that (3.6) is bounded for any $t\in[0,T]$. It is thus proved that $\phi(t)_\lambda\in H^2(R^3)$ for any $t\in[0,T]$.
It remains to verify the boundary condition at $y_j(t)$. Proceeding as in (3.5), one has

$$\begin{aligned}
&\lim_{|x-y_j(t)|\to 0}(u(x,t)-q_j(t)G_0(x-y_j(t)))\\
&=\frac{1}{(2\pi)^{3/2}}\int_{R^3}dke^{ik\cdot y_j(t)}\left(\tilde u(k,t)-\frac{q_j(t)e^{-ik\cdot y_j(t)}}{(2\pi)^{3/2}k^2}\right)\\
&=P_tu^0(y_j(t))+\sum_{l=1,l\neq j}^N\int_0^t dsq_l(s)P_{t-s}(y_j(t)-y_l(s))\\
&+\frac{1}{(2\pi)^3}\int_{R^3}dke^{ik\cdot y_j(t)}\left(\int_0^t dsq_j(s)e^{-k^2(t-s)-ik\cdot y_j(s)}-\frac{q_j(t)e^{-ik\cdot y_j(t)}}{k^2}\right)\\
&=P_tu^0(y_j(t))+\sum_{l=1,l\neq j}^N\int_0^t dsq_l(s)P_{t-s}(y_j(t)-y_l(s))\\
&-\frac{q_j(0)}{(2\pi)^3}\int_{R^3}dk\frac{1}{k^2}e^{-k^2t+ik\cdot(y_j(t)-y_j(0))}\\
&-\frac{1}{(2\pi)^3}\int_0^t ds\dot q_j(s)\int_{R^3}dk\frac{1}{k^2}e^{-k^2(t-s)+ik\cdot(y_j(t)-y_j(s))}\\
&+\frac{i}{(2\pi)^3}\int_0^t dsq_j(s)\int_{R^3}dk\frac{k\cdot\dot y_j(s)}{k^2}e^{-k^2(t-s)+ik\cdot(y_j(t)-y_j(s))}\\
&=P_tu^0(y_j(t))+\sum_{l=1,l\neq j}^N\int_0^t dsq_l(s)P_{t-s}(y_j(t)-y_l(s))\\
&-q_j(0)\frac{B_j(t,0)}{\sqrt t}-\int_0^t ds\dot q_j(s)\frac{B_j(t,s)}{\sqrt{t-s}}-\int_0^t dsq_j(s)\frac{A_j(t,s)}{\sqrt{t-s}},
\end{aligned}\tag{3.9}$$

where the integral in the fourth line of (3.9) can be computed as in (3.6) and $A_j(t,s)$, $B_j(t,s)$ are given by (1.11), (1.12).

Note that $B_j(t,\cdot)\in C^1([0,t])$, $B_j(t,t)=(4\pi^{3/2})^{-1}$ and $A_j(t,s)$ is a continuous function.
The integral containing $\dot{q}_j$ in (3.9) can be transformed. An integration by parts, Abel's inversion formula ([9]) and a change of the order of integration give

$$
\begin{aligned}
&\int_0^\tau dt\frac{1}{\sqrt{\tau-t}}\int_0^t ds\dot{q}_j(s)\frac{B_j(t,s)}{\sqrt{t-s}}\\
&=\int_0^\tau dt\frac{1}{\sqrt{\tau-t}}\frac{1}{4\pi^{3/2}}\frac{d}{dt}\int_0^t ds\frac{q_j(s)}{\sqrt{t-s}}-q_j(0)\int_0^\tau dt\frac{B_j(t,0)}{\sqrt{(\tau-t)t}}\\
&-\int_0^\tau dt\frac{1}{\sqrt{\tau-t}}\int_0^t dsq_j(s)\frac{\partial}{\partial s}\left(\frac{B_j(t,s)-(4\pi^{3/2})^{-1}}{\sqrt{t-s}}\right)\\
&=\frac{q_j(\tau)}{4\sqrt{\pi}}-q_j(0)\int_0^\tau dt\frac{B_j(t,0)}{\sqrt{(\tau-t)t}}\\
&-\int_0^\tau dsq_j(s)\int_s^\tau dt\frac{1}{\sqrt{(\tau-t)(t-s)}}\left(\dot{B}_j(t,s)+\frac{B_j(t,s)-(4\pi^{3/2})^{-1}}{2(t-s)}\right).
\end{aligned}
\tag{3.10}
$$

¿From (3.9),(3.10) and the equation satisfied by q_j it is seen that the boundary condition at $y_j(t)$ is also satisfied. We have thus shown that $u(t)\in D(-\Delta_{\alpha,Y(t)})$.
¿From (1.2) and the expression (3.5) for the Fourier transform of u we have

$$
(\Delta_{\alpha,Y(t)}u)^{\sim}(k,t)=-k^2\tilde{\phi}_\lambda(k,t)+\lambda\sum_{l=1}^N q_l(t)\frac{e^{ik\cdot y_l(t)}}{k^2+\lambda}. \tag{3.11}
$$

On the other hand the time derivative of (3.5), by a straightforward computation, is easily seen to be equal to the r.h.s. of (3.11).
This means that $u(t)$ given by (3.1) solves our evolution problem.
The proof of uniqueness proceeds exactly as in the case of α varying on time ([6]).
Suppose that u_1 and u_2 are two solutions with the same initial datum, so that $v=u_1-u_2$ is a solution with initial datum zero. Since $v(t)\in D(\Delta_{\alpha,Y(t)})$, we know that $v(t)$ has a regular part $\bar{\phi}_\lambda(t)\in H^2(R^3)$ and continuous charges

$\bar{q}_j(t)$, which are obviously zero for $t=0$. Then in the sense of distributions $v(t)$ satisfies

$$\frac{\partial v(t)}{\partial t} = \Delta v(t) + \sum_{l=1}^{N} \bar{q}_l(t)\delta(\cdot - y_l(t)). \tag{3.12}$$

The unique solution of (3.12) is

$$v(t) = \sum_{l=1}^{N} \int_0^t ds \bar{q}_l(s) P_{t-s}(\cdot - y_l(s)). \tag{3.13}$$

Moreover (3.13) must satisfy the boundary condition at $y_j(t)$. By a straightforward computation, this implies that $\bar{q}_j$ is the solution in the sense of distributions of the system (1.8) with datum $f_j = 0$. Then $\bar{q}_j = 0$ and also $v(t) = 0$.
This concludes the proof of theorem 1. $\square$

4. The case of uniform motion for N=1

In this Section we give the explicit solution of the equation for the charge $q(t)$, and then of the evolution problem (1.4), (1.5), in the special case $N=1$ and $y(t) = vt$, $v \in R^3$. This is possible because the integral kernels $A(t,s)$, $B(t,s)$ now depends only on the difference $t-s$ and then the equation can be solved using the convolution properties of the Laplace transform.
¿From (1.11), (1.12) we easily compute

$$A(t,s) = \frac{v^2}{8\pi^{3/2}} \int_0^1 d\xi \xi^2 e^{-\frac{v^2\xi^2}{4}(t-s)}, \tag{4.1}$$

$$B(t,s) = \frac{1}{4\pi^{3/2}} \int_0^1 d\xi e^{-\frac{v^2\xi^2}{4}(t-s)}. \tag{4.2}$$

Denoting by $\hat{\eta}$ the Laplace transform of η and

$$a(t) = -\int_0^t ds q(s) \frac{A(t,s)}{\sqrt{t-s}}, \quad b(t) = -\int_0^t ds \dot{q}(s) \frac{B(t,s)}{\sqrt{t-s}}, \tag{4.3}$$

we have (see e.g. [8])

$$\hat{a}(p) = -\frac{\hat{q}(p)}{4\pi}\sqrt{p + \frac{v^2}{4}} + \frac{p\hat{q}(p)}{2\pi v}\sinh^{-1}\left(\frac{v}{2\sqrt{p}}\right), \tag{4.4}$$

$$\hat{b}(p) = -\frac{p\hat{q}(p)}{2\pi v}\sinh^{-1}\left(\frac{v}{2\sqrt{p}}\right) + \frac{q(0)}{2\pi v}\sinh^{-1}\left(\frac{v}{2\sqrt{p}}\right). \tag{4.5}$$

Imposing the boundary condition at vt, using (3.9), (4.4), (4.5) and evaluating an inverse Laplace transform ([8]), we find

$$q(t) = \int_0^t ds\ K(t-s) P_s u^0(vs), \tag{4.6}$$

$$K(t) = 4\pi e^{-\frac{v^2 t}{4}} \left(\frac{1}{\sqrt{\pi t}} - 4\pi\alpha\ e^{(4\pi\alpha)^2 t}\ \mathrm{Erfc}(4\pi\alpha\sqrt{t}) \right), \tag{4.7}$$

where Erfc$(\cdot)$ is the complementary error function ([10]).
¿From (3.1) and (4.6), (4.7) one also obtains an explicit formula for the solution of the evolution problem.
Note that this solution can be equivalently found via a change of coordinates. If one defines $V\psi(x) = \psi(x - vt)$ as a unitary operator in $L^2(R^3)$, one easily finds that the evolution problem for u is equivalent to the following evolution problem for $\zeta \equiv V^{-1}u$

$$\frac{\partial \zeta}{\partial t} = \Delta_{\alpha,0}\zeta + v \cdot \nabla\zeta, \tag{4.8}$$

where the point source is fixed at the origin and a new drift term appears. Using Fourier transform and the knowledge of domain and action of $\Delta_{\alpha,0}$, one can also solve equation (4.8) directly.

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