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REMARKS ON THE RELATION BETWEEN QUANTUM DYNAMICS AND FRACTAL SPECTRA

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Abstract

We investigate the large time behaviour of various solutions of the Schrödinger equation in terms of some local and global dimensions of the spectral measure. We emphasize in particular the rôle of the Hausdorff and correlation dimensions for the growth exponents of position moments. We also discuss the stability of such exponents under local perturbations.

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1 Introduction

The connections between quantum dynamics and spectral properties of the Hamiltonians generating the time evolution has a long history going back to the celebrated RAGE theorem [RS]. The occurence of unusual types of spectra in various models from solid state physics, e.g. singular continuous or dense pure point, has stimulated a renewed interest for a deeper investigation of the transport properties associated to such spectra. A crucial step in this direction was accomplished by Guarneri [G] who investigated the rôle of a global scaling exponent for the spectral measures of the Floquet operators associated to pulsed systems for the time behaviour of various dynamical quantities such as correlation functions or mean square displacement. Those results were extended by one of us [C] to continuous dynamics on $\mathbb{R}^d$ using results of Strichartz [S] on the asymptotics of Fourier transforms of uniform γ-dimensional measures (see definition 2.2 below).

Soon after, Holschneider [H] using wavelet transform and Ketzmerick et al. [KPG] emphasized the rôle of the correlation dimension of the spectral measures for such asymptotics. More recently, new contributions were provided by [BMZ], [GM], [dRJLS] and [La]. One of the aim of this paper is to clarify the links between the whole set of results obtained so far and in particular to derive from them the fact that the information dimension of the spectral measures provides a universal lower bound on the growth exponents of the moments for the position operator.

Then using some kind of stability property for the growth exponent of moments, we will provide some explicit examples showing that in general the information dimension does not provide upper bounds. In particular it is easy to construct discrete Schrödinger operators with pure point spectrum having in some sense almost ballistic growth of high moments. This has already been obtained by del Rio et al. [dRJLS] on a specific example; we show here that this occurs for almost every one site perturbations of Schrödinger operators whose spectral measures have information dimension close to 1.

The plan of this paper is as follows; in chapter 2 we recall the definition of local and global exponents of spectral measures and give a rigourous proof that the correlation dimensions control time-decay of correlation functions. In chapter 3 we consider growth exponents for moments of position operators; our result in terms of information dimension is in fact a more elaborate form of a recent result of Last [La]. In chapter 4 we present a stability result under rank one perturbations which shows that dynamics are in a weak sense invariant under such perturbations. Finally in Appendix A we relate theorem 2.2 of chapter 2 to those obtained in [H] using wavelet transform and we prove that the mean quadratic dimensions are equal to the correlation dimensions; as this paper was written we learned of a preprint of Guerin and Holschneider showing similar estimates [GH].

2 Fractal dimension of measure and asymptotic behaviour of Fourier transform

Let us first recall the definition of local exponents for measures and their relation to Hausdorff and correlation dimensions.

Definition 2.1 *Local Exponents*

Consider a positive finite Borel measure μ on $I\!R$. The upper and lower scaling exponents are defined as follow:

$$\gamma^-(x) = \liminf_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon} \tag{1}$$

$$\gamma^+(x) = \limsup_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon} \tag{2}$$

where $B_\varepsilon(x)$ is the open ball of center x and radius $\varepsilon/2$.

Remarks. 1

(a) It is easy to see that $\gamma^-(x)$ is a measurable function; indeed, it is a consequence of standard arguments and of the fact that for all x, the limit in the definition of $\gamma^-(x)$ remain unchanged if the continuous variable ε is replaced by any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \searrow 0$ and $\log \varepsilon_{n+1}/\log \varepsilon_n \to 1$ (see [Y]). In particular for any α, the sets $\{x \mid \gamma^-(x) \geq \alpha\}$ are μ measurable.

(b) As a consequence of Radon-Nykodim decomposition of a measure, one obtains that:

$$\mu \; a.e.x : \quad 0 \leq \gamma^-(x) \leq \gamma^+(x) \leq 1$$

(c) It is well known that for μ a probability measure:

$$\mu\, \text{ess.inf}\, \gamma^-(x) \leq \dim_{\mathrm{H}}(\mu) \leq \mu\, \text{ess.sup}\, \gamma^+(x) \tag{3}$$

where $\dim_{\mathrm{H}}(\mu)$ is the Hausdorff dimension of μ (see [F] and [Pe]), also called information dimension.

(d) In the particular case where the scaling exponents are μ-almost everywhere constant, a theorem of Young asserts that if

$$\gamma^-(x) = \gamma^+(x) = \overline{\gamma} \;\; is\ constant\ for\ \mu\, a.e.x, \tag{4}$$

then

$$\dim_{\mathrm{H}}(\mu) = \overline{\gamma}$$

Relation (4) is exceptional; it is satisfied however for any absolutely continuous measure with $\overline{\gamma} = 1$. We will extend this result in chapter 3 by showing that

$$\dim_{\mathrm{H}}(\mu) = \mu\,\text{ess.sup}\,\gamma^{-}(x)$$

Definition 2.2 *Let $0 \le \alpha \le 1$; μ is said to be locally $\alpha-$ dimensional at x_0 if:*
$\exists c < +\infty$ *such that* $\forall\, 0 < \varepsilon < 1$ $\mu(B_\varepsilon(x_0)) \le c\,\varepsilon^\alpha$
μ is said to be locally uniformly $\alpha-$ dimensional $(L.U.\alpha)$ if:
$\exists c < +\infty$ *and* $0 < \varepsilon_0 < 1$ *such that for* μ *a.e.x* , $\forall \varepsilon,\ 0 < \varepsilon < \varepsilon_0\,,\quad \mu(B_\varepsilon(x)) \le c\,\varepsilon^\alpha$

In [S], Strichartz proves the following (see also [La] for a simple proof):

Theorem 2.1 *If μ is a measure on $I\!R$ which is $L.U.\alpha$, and $f \in L^2(d\mu)$, then there exists C_1 depending only on μ such that:*

$$\sup_{T \ge 1} T^{\alpha-1} \int_{-T}^{T} |(f\mu)\hat{}(t)|^2 dt \le C_1\, \|f\|_2{}^2 \tag{5}$$

where $\hat{}$ denotes the Fourier transform.

Remarks. 2 Each finite positive measure is L.U.0. A measure μ which has a pure point component on x_0 has a local exponent equal to zero at x_0. An absolutely continuous measure (w.r.t. Lebesgue measure) with a density $f(x)$ has a local exponent equal to 1 at x_0 if f is locally bounded near x_0. A measure which is locally of dimension α at x is locally of dimension β for any $\beta \le \alpha$, and has a scaling exponent $\gamma^{-}(x) \ge \alpha$

Theorem 2.1 has been used to study the time behaviour of moments $\langle\langle |X^m| \rangle\rangle_T$ (see (24) and (25) for the definition) and to extend the result of Guarneri [G] and Last [La] to $L^2(I\!R^d)$ (see [C]). An immediate consequence of this theorem is the following:

Corollary 2.1 *Let $\mathcal{H}$ be a Hilbert space and H a self-adjoint operator on $\mathcal{H}$. Let $E_\lambda(.)$ be the spectral family of the hamiltonian H, and define, for $\psi_0 \in \mathcal{D}(H)$, $d\mu_{\psi_0} = d\langle E_\lambda \psi_0, \psi_0\rangle$ where $\langle\ ,\ \rangle$ denotes the scalar product on $\mathcal{H}$.*

For fixed ψ_0 we denote by $\psi(t)$ the solution of the Schrödinger equation :

$$\begin{cases} \imath \dfrac{d}{dt}\psi(t) = H\,\psi(t) \\ \\ \psi(t=0) = \psi_0 \end{cases}$$

Assume that the spectral measure $d\mu_{\psi_0}$ is $L.U.\alpha$; then if

$$p(t) = |\langle \psi_0,\, \psi_t\rangle|^2$$

the correlation function

$$C(T) \equiv \frac{1}{T} \int_0^T p(t)\, dt \tag{6}$$

satisfies:

$$\exists C_1 < +\infty \; s.t. \;\; \forall T > 1, \;\; C(T) \leq \; C_1\, T^{-\alpha} \tag{7}$$

and for any $\varphi \in \mathcal{D}(H)$

$$\frac{1}{T} \int_0^T |\langle \varphi\,, \psi_t \rangle|^2\, dt \leq \; C_1\, T^{-\alpha} \|\varphi\| \tag{8}$$

The proof follows obviously from:

$$\langle \varphi,\, \psi(t) \rangle = \int_{I\!R} f(\lambda) e^{-\imath t \lambda} d\mu_{\psi_0}(\lambda) \equiv (f\, \mu_{\psi_0})\hat{}\,(t)$$

for some f in $L^2(d\mu_{\psi_0})$.

Unfortunately, it appears that the greatest α such that μ is L.U.α can be rather small in comparison for example with the μ-essential infimum of the lower local exponents $\gamma^-(x)$ (see example of appendix B).

One can try to improve the situation by considering other characteristic exponents of the measure μ. This has been done in particular by Geisel et al. [KPG] and Holschneider [H]. The first one emphasized the role of the so-called correlation dimension defined as follow:

Definition 2.3 *Let μ be any finite positive Borel Measure on $I\!R$ and*

$$\gamma_2(\varepsilon) = \int_R \mu((\omega - \frac{\varepsilon}{2}, \omega + \frac{\varepsilon}{2}])\, d\mu(\omega) \tag{9}$$

then:

$$D_2^- = \liminf_{\varepsilon \to 0} \frac{\log \gamma_2(\varepsilon)}{\log \varepsilon} \; \textit{is the lower correlation dimension} \tag{10}$$

$$D_2^+ = \limsup_{\varepsilon \to 0} \frac{\log \gamma_2(\varepsilon)}{\log \varepsilon} \; \textit{is the upper correlation dimension} \tag{11}$$

If $D_2^+ = D_2^- = D_2$, then D_2 is called the correlation dimension of μ.

Remarks. 3

(i) D_2^+ and D_2^- remain unchanged if one replaces $(\omega - \frac{\varepsilon}{2}, \omega + \frac{\varepsilon}{2}]$ by $B_\varepsilon(\omega)$ in the definition of $\gamma_2(\varepsilon)$. It is obvious if μ is a purely continuous measure; if the pure point part μ_p of μ is such that $\mu_p \neq 0$, then $D_2^+ = D_2^- = 0$ for γ_2 defined with $(\omega - \frac{\varepsilon}{2}, \omega + \frac{\varepsilon}{2}]$ or $B_\varepsilon(\omega)$.

(ii) In [KPG], Ketzmerick et al. infer from a formal calculation that:

$$\gamma_2(\varepsilon) \sim \varepsilon^{D_2} \Leftrightarrow C(T) \sim T^{-D_2} \tag{12}$$

where $C(T)$ is defined as in (6) and D_2 is associated to μ_{ψ_0}. This is confirmed by some numerical computations; an analytical proof will be given and generalized below in theorem 2.2. Our results are strongly related to those obtained by Holschneider in [H] with the wavelet transform. Comparison with Holschneider 's results is presented in Appendix A.

Proposition 2.1 *If μ is a finite Borel measure on $\mathbb{R}$, then:*

$$D_2^- \leq \mu\,\text{ess.inf}\,\gamma^-(x) \tag{13}$$

Proof: By the definition of D_2^-, for fixed η such that $D_2^- > \eta > 0$,

$$\begin{aligned} &\exists\varepsilon(\eta)\,,\ \forall\varepsilon < \varepsilon(\eta)\ :\ \int \frac{\mu(B_\varepsilon(x))}{\varepsilon^{D_2^- - 3\eta/4}}\, d\mu(x) \leq \varepsilon^{\eta/4} \\ &\Rightarrow \int d\mu(x) \int_0^{\varepsilon(\eta)} \frac{\mu(B_\varepsilon(x))}{\varepsilon^{D_2^- - 3\eta/4}} \frac{d\varepsilon}{\varepsilon} < +\infty \\ &\Rightarrow\ \ \mu - a.e.x\ \int_0^{\varepsilon(\eta)} \frac{\mu(B_\varepsilon(x))}{\varepsilon^{D_2^- - 3\eta/4}} \frac{d\varepsilon}{\varepsilon} < +\infty \end{aligned} \tag{14}$$

Assume that for some x one has $D_2^- - 2\eta > \gamma^-(x)$; then by the definition of $\gamma^-(x)$, $\exists\ (\varepsilon_n)_{n\in\ \mathbb{N}}, \varepsilon_n \searrow 0$, such that

$$\varepsilon_n < \frac{\varepsilon_{n-1}}{2}\ \ and\ \ \mu(B_{\varepsilon_n}(x)) > \varepsilon_n^{\gamma^-(x)+\eta/2} > \varepsilon_n^{D_2^- - 3\eta/2}$$

Then since $\mu(B_\varepsilon(x))$ is a non decreasing function of ε one has $\mu(B_\varepsilon(x)) > \varepsilon^{D_2^- - \eta}$ at least on the interval $[\varepsilon_n, \varepsilon_n'] = [\varepsilon_n, \varepsilon_n^{1-\eta/2}]$. Now let N be the smallest integer such that $\varepsilon_N < \varepsilon(\eta)$ and $\varepsilon_N^{1-\eta/2} > 2\varepsilon_N$; then for all $n > N$, $A_n \equiv [\varepsilon_n, 2\varepsilon_n] \subset [\varepsilon_n, \varepsilon_n^{1-\eta/2}]$, and all the A_n are disjoint, thus:

$$\int_0^{\varepsilon(\eta)} \frac{\mu(B_\varepsilon(x))}{\varepsilon^{D_2^- - 3\eta/4}} \frac{d\varepsilon}{\varepsilon} \geq \sum_{n>N} (2\varepsilon_n)^{-1} \frac{\varepsilon_n^{D_2^- - \eta}}{(2\varepsilon_n)^{D_2^- - 3\eta/4}}\, \varepsilon_n\ = +\infty \tag{15}$$

¿From (14) and (15) one obtains:

$$\mu - a.e.x :\ \ \gamma(x) \geq D_2^- - 2\eta$$

This proves the proposition since the above result is valid for all $\eta > 0$.

Remarks. 4 With the same arguments as before, one can prove that $\forall S \in \mathcal{B}(\mathbb{R})$, $\mu(S) > 0$:

$$\liminf_{\varepsilon \to 0} \frac{\log \int_S \mu(B_\varepsilon(x))\, d\mu(x)}{\log \varepsilon} \leq \mu \,\text{ess.inf}\{\gamma^-(x)\,,\ x \in S\}$$

We are now ready to prove the following:

Theorem 2.2 *Let $\psi_0 \in \mathcal{D}(H)$. With the notations of corollary 2.1, if D_2^+ and D_2^- are the upper and lower correlation dimensions of the measure μ_{ψ_0}, one has:*

$$\liminf_{T \to \infty} \frac{\log C(T)}{\log T} = -D_2^+ \tag{16}$$

$$\limsup_{T \to \infty} \frac{\log C(T)}{\log T} = -D_2^- \tag{17}$$

For the proof of this theorem, we need the following preliminary result:

Lemma 2.1 *For $\mu \equiv \mu_{\psi_0}$ and $\varepsilon > 0$, let*

$$A(\varepsilon) = \int_{\mathbb{R}} |F_\mu(x + \imath\varepsilon)|^2\, dx \tag{18}$$

where $F_\mu(z)$ is the Borel transform of μ: $F_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(y)}{y - z}$. Then, with the same notations as in corollary 2.1:

(i)

$$A(\frac{1}{T}) = 4\pi \int_0^\infty e^{-\frac{t}{T}} p(t)\, dt \tag{19}$$

(ii)

$$A(\varepsilon) = 4\pi\varepsilon \int_{\mathbb{R} \times \mathbb{R}} \frac{d\mu(y)\, d\mu(y')}{(y - y')^2 + 4\varepsilon^2} \tag{20}$$

Remarks. 5 A similar quantity is used in [dRJLS] for a rank one perturbation theory, and in [BMZ] to relate the behaviour of the Green function to singularities of the spectral measure.

Proof of lemma: (i) is well known from Kato's theory of smooth operator [RS] (ii) follows from a straightforward calculation using Cauchy formula.

Proof of theorem 2.2:

Let $c^- = \liminf_{\varepsilon\to 0} \dfrac{\log C(1/\varepsilon)}{\log \varepsilon}$ and $c^+ = \limsup_{\varepsilon\to 0} \dfrac{\log C(1/\varepsilon)}{\log \varepsilon}$; then for all $\nu > 0$:

$$\begin{aligned}
\frac{1}{T}\int_0^T |\hat\mu(t)|^2\, dt &\le \frac{e}{T}\int_{I\!R} e^{-t^2/T^2}\int e^{-\imath(x-y)t} d\mu(x)d\mu(y)\, dt \\
&= e\sqrt{\pi}\int d\mu(x)d\mu(y)e^{-(x-y)^2T^2/4} \\
&= e\sqrt{\pi}\int_{|x-y|<T^{-(1-\nu)}} d\mu(x)d\mu(y)e^{-(x-y)^2T^2/4} \\
&\quad + \int_{|x-y|\ge T^{-(1-\nu)}} d\mu(x)d\mu(y)e^{-(x-y)^2T^2/4} \\
&\le e\sqrt{\pi}\int \mu(B_{2T^{-(1-\nu)}}(x)\, d\mu(x) \; + e^{-T^{2\nu}/4}
\end{aligned}$$

which implies that

$$C(T) \le 2e\sqrt{\pi}\,\gamma_2(2T^{-(1-\nu)}) \tag{21}$$

from which one immediatly deduces that

$$c^+ \ge -D_2^- \text{ and } c^- \ge -D_2^+ \tag{22}$$

Now, using (20) one has:

$$\begin{aligned}
A(\varepsilon) &\ge \frac{\pi}{2\varepsilon}\int_{I\!R} \mu(B_{2\varepsilon}(y))d\mu(y) \\
&\ge \frac{\pi}{2\varepsilon}\gamma_2(\varepsilon)
\end{aligned}$$

Which implies together with (19) that:

$$\begin{aligned}
\frac{1}{8}\gamma_2(\frac{1}{T}) &\le \frac{1}{T}\int_0^\infty e^{-\frac{t}{T}}p(t)\, dt \\
&\le \frac{1}{T}\int_0^{T^{1+\nu}} e^{-\frac{t}{T}}p(t)\, dt + \frac{1}{T}\int_{T^{1+\nu}}^\infty e^{-\frac{t}{T}}p(t)\, dt
\end{aligned}$$

The first term on the right hand side is smaller than $T^\nu\, C(T^{1+\nu})$ and the second is smaller than T^{-n} at $+\infty$, for any $n \in I\!N$. Hence, for some constant a, one has: $\forall \nu > 0\,,\ \exists T(\nu) < \infty$ such that $\forall T > T(\nu)$:

$$\gamma_2(\frac{1}{T}) \le a\, T^\nu C(T^{1+\nu})$$

and then

$$c^+ \le -D_2^- \text{ and } c^- \le -D_2^+ \tag{23}$$

This inequality together with (22) gives the expected result.

Remarks. 6 One can also obtain the result (17) with the help of Wiener Tauberian theorem [W] and the mean quadratic dimensions (see appendix A).

3 Lower bounds on moments

In this chapter, we consider self-adjoint operator H on $\mathcal{H} = l^2(\mathbf{Z}^d)$ or $\mathcal{H} = L^2(\mathbb{R}^d)$ and the moments of ψ_0:

$$\langle |X|^m(t)\rangle = \langle \psi(t),\, |X|^m \psi(t)\rangle \tag{24}$$

Here ψ_0 and $\psi(t)$ are defined as in corollary 2.1, as well as the measure μ_{ψ_0}, and X is the position operator. The time averaged moments are:

$$\langle\langle |X|^m\rangle\rangle_T \equiv \frac{1}{T}\int_0^T \langle |X|^m(t)\rangle\, dt \tag{25}$$

The main result of this chapter is

Theorem 3.1 *With the same notations as above, if H is self-adjoint on $l^2(\mathbf{Z}^d)$ and $\psi_0 \in \mathcal{D}(H)$,*

$$\forall m > 0 \text{ and } \forall \nu > 0\,,\ \exists \tilde{C}(m, d, \nu, \psi_0) \ \text{ such that :}$$

$$\langle\langle |X|^m\rangle\rangle_T \geq \tilde{C}\, T^{\frac{m}{d}(\gamma_0 - \nu)} \tag{26}$$

where $\gamma_0 \equiv \mu_{\psi_0}\,\text{ess.sup}\,\gamma^-(x)$

Proof: The proof of the theorem consists of three parts

(1) Let (e_n) be an orthonormal basis of $l^2(\mathbf{Z}^d)$. If for some ψ_1, μ_{ψ_1} is $L.U.\alpha$, then corollary 2.1 tells us that for $p_n(T, \psi_1) \equiv \frac{1}{T}\int_0^T |\langle e_n,\, \psi_1(t)\rangle|^2\, dt$, $\exists C_1 < \infty$ such that $\forall N \in \mathbf{Z}$, $\forall T > 1$:

$$\sum_{|n|\leq N} p_n(T, \psi_1) \leq N^d C_1\, T^{-\alpha} \tag{27}$$

(2) Let $\gamma_0 = \mu_{\psi_0}\,\text{ess.sup}\gamma^-(x)$, for fixed $\nu > 0$, $\exists S \in \mathbb{R}$ such that $\mu_{\psi_0}(S) > 0$ and $\forall x \in S$, $\gamma_0 \geq \gamma^-(x) > \gamma_0 - \nu$. Moreover, Egoroff theorem implies that $\exists S' \subset S$, $\mu(S') > 0$, such that the measurable functions $f_\varepsilon(x) \equiv \inf\limits_{\varepsilon' < \varepsilon} \dfrac{\log \mu(B_{\varepsilon'}(x))}{\log \varepsilon'}$ converge uniformly on S' when $\varepsilon \to 0$. Let $\psi_1 = E(S')\psi_0$, where $E(\ )$ is the spectral projection for H; then μ_{ψ_1} is $L.U.\, \gamma_0 - \nu$ and $\|\psi_1\| > 0$.

(3) Now define $\psi_{1,t} = E(S')\psi_t$, $\psi_{2,t} = \psi_t - \psi_{1,t}$ and

$$n_0(\psi_1, T) = \max\Big\{k \mid \frac{1}{T}\int_0^T (\sum_{|n|<k} |\langle \psi_{1,t},\, e_n\rangle|^2) dt \leq \frac{\|\psi_1\|^4}{9}\Big\}$$

then for $\psi_{1,t}(n) = |\langle\psi_{1,t},\, e_n\rangle|$ and $\psi_{2,t}(n) = |\langle\psi_{2,t},\, e_n\rangle|$, one has:

$$\begin{aligned}\sum_{|n|<n_0} p_n(T,\psi_0) &\leq \frac{1}{T}\int_0^T \Big(\sum_{|n|<n_0} (\psi_{1,t}(n)+\psi_{2,t}(n))^2\Big)dt \\ &\leq \frac{1}{T}\int_0^T \Big(\sum_{|n|<n_0} \psi_{1,t}^2(n)\Big)\, dt + 2\|\psi_2\|\Big(\frac{1}{T}\int_0^T \Big(\sum_{|n|<n_0} \psi_{1,t}^2(n)\Big)\, dt\Big)^{\frac{1}{2}} \\ &\quad +\|\psi_2\|^2 \end{aligned} \tag{28}$$

By the definition of n_0, one obtains from (28) that

$$\sum_{|n|\geq n_0} p_n(T,\psi_0) \geq \frac{1}{6}\|\psi_1\|^2 \tag{29}$$

then

$$\begin{aligned}\langle\langle|X|^m\rangle\rangle_T &\equiv \sum_{n\in\, \mathbf{Z}^d} |n|^m p_n(T,\psi_0) \\ &\geq n_0(\psi_1,T)^m \sum_{|n|\geq n_0} p_n(T,\psi_0)\end{aligned} \tag{30}$$

Since μ_{ψ_1} is $L.U.\,\gamma_0-\nu$, one obtains from (27) and the definition of n_0:

$$n_0(T,\psi_1) \geq (T^{\gamma_0-\nu}\frac{\|\psi_1\|^4}{9C_1})^{1/d} \tag{31}$$

which implies together with (29) and (30) that

$$\langle\langle|X|^m\rangle\rangle_T \geq \tilde{C}\, T^{m(\gamma_0-\nu)/d} \tag{32}$$

where $\tilde{C}$ is a finite constant depending on ψ_1, m, d, and ν.

Remarks. 7 This result can also be deduced from Last's theorem, which is presented in term of most continuous component of the measure; indeed, the α-continuous part $\mu_{\psi_0}^{c,\alpha}$ of μ_{ψ_0} is supported on a set $T_{\alpha,c}$ (in the sense $\mu_{\psi_0}^{c,\alpha}(\mathbb{R}-T_{\alpha,c})=0$), such that $\{x\,|\,\gamma^-(x)>\alpha\}\subset T_{\alpha,c}\subset\{x\,|\,\gamma^-(x)\geq\alpha\}$ (see [R]). Then, $\forall\alpha<\gamma_0$, $\mu_{\psi_0}^{c,\alpha}\neq 0$.

If $\exists S$, $\mu(S)>0$ such that $\forall x\in S$, $\gamma^-(x)=\gamma_0$, then one can take $\nu=0$ in theorem 3.2 and corollary 3.1. This is the case for example if the absolutely continuous part of μ is not zero, and for which $\gamma_0=1$.

One can relate the dimension γ_0 to the information dimension as follows:

Theorem 3.2 *For μ_{ψ_0} and γ_0 as above:*

$$\gamma_0 = \dim_{\mathrm{H}}(\mu_{\psi_0}) \tag{33}$$

Proof Let $\mu \equiv \mu_{\psi_0}$ and $S \equiv S(\nu) = \{x \in \mathbb{R} \mid \gamma^-(x) \geq \gamma_0 - \nu\}$. Let $\mu|_S$ be the probability measure defined as: for all subset A of $\mathbb{R}$, $\mu|_S(A) = \mu(S \cap A)/\mu(S)$. For every A such that $\mu(A) = 1$, one has $\mu|_S(A) = 1$; then

$$\dim_{\mathrm{H}}(\mu|_S) \leq \dim_{\mathrm{H}}(\mu) \tag{34}$$

(3) and the definition of $\mu|_S$ gives:

$$\gamma_0 - \nu \leq \mu|_{\mathrm{S}}\,\mathrm{ess.inf}\gamma^-(x) \leq \dim_{\mathrm{H}}(\mu|_S) \tag{35}$$

Now, for fixed $\nu > 0$, let $S'(\nu) \equiv \{x \mid \gamma^-(x) < \gamma_0 + \nu\}$. One has $S'(\nu) \subset T_{s,\gamma_0+\nu} \equiv \{x \mid \limsup_{\varepsilon\to 0} \frac{\mu(B_\varepsilon(x))}{\varepsilon^{\gamma_0+\nu}} = \infty\}$, where $T_{s,\gamma_0+\nu}$ is a set that supports the singular part of μ with respect to $H^{\gamma_0+\nu}$ (see [R] and [La]), $H^{\gamma_0+\nu}$ beeing the $(\gamma_0+\nu)$-Hausdorff measure. Then

$$H^{\gamma_0+\nu}(S'(\nu)) = 0 \tag{36}$$

Now since $\mu(S'(\nu)) = 1$, (36) implies:

$$\gamma_0 + \nu \geq \dim_{\mathrm{H}}(S'(\nu)) \geq \dim_{\mathrm{H}}(\mu) \tag{37}$$

(33) follows from (35) and (37).

Remarks. 8 I.Guarneri informed us that he has obtained independently results similar to theorem 3.1 and 3.2 [G2].
One can also relate γ_0 to the correlation dimension introduced in chapter 2 (Notice that one already has from proposition 2.1 that $D_2^- \leq \gamma_0$). Define:

$$D_2^-(\psi_0) \equiv \sup\{\liminf_{\varepsilon\to 0} \frac{\log \int_A \mu_{\psi_0}(B_\varepsilon(x))\, d\mu_{\psi_0}(x)}{\log \varepsilon} \ ;\ A \in \mathcal{B}(\mathbb{R}) \textit{ and } \mu_{\psi_0}(A) > 0\ \} \tag{38}$$

(Here $\mathcal{B}(\mathbb{R})$ *is the Borel algebra of* $\mathbb{R}$*)*
Then:

$$D_2^- \leq \mu_{\psi_0}\,\mathrm{ess.inf}\gamma^-(x) \leq D_2^-(\psi_0) = \gamma_0 \tag{39}$$

γ_0 defined as above.

Corollary 3.1 *If H is self-adjoint on $l^2(\mathbb{Z}^d)$, then $\forall \psi_0 \in \mathcal{D}(H)$, $\forall m > 0$ and $\forall \nu > 0$, $\exists\, \tilde{C}(m,d,\nu,\psi_0) < \infty$ s.t.*

$$\forall T > 1, \quad \langle\langle |X|^m \rangle\rangle_T > \tilde{C}\, T^{\frac{m}{d}(\dim_{\mathrm{H}}(\mu_{\psi_0}) - \nu)} \tag{40}$$

Remarks. 9 The result (40) also holds in the continuous case when $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ if $V^+ \in K_d^{loc}$ and $V_- \in K_d$ ([La], [C]).

Corollary 3.2 *Consider a self-adjoint operator $H = -\Delta + V$ on $l^2(\mathbf{Z}^d)$, with V bounded. Let $\psi \in \mathcal{D}(H)$. If the information dimension $\gamma_0 = \dim_{\mathrm{H}}(\mu_\psi)$ verifies $\gamma_0 > 0$ then for $\varepsilon > 0$, $\varepsilon < \gamma_0$ and $\nu > 0$, there exists $C_1(\nu) > 0$, $C_2(\varepsilon, \nu) > 0$, $C_3(\nu) > 0$ and $T(\varepsilon, \nu)$ such that with the notations of corollary 2.1, $\forall T > T(\varepsilon, \nu)$:*

$$\frac{1}{T}\int_0^T \sum_{C_2 t^{\frac{\gamma_0-\varepsilon}{d}} < |n| < C_1 t} |\langle \psi_t, \delta_n \rangle|^2 \, dt \geq C_3 T^{-\nu} \tag{41}$$

Proof: First of all, remark that since H is bounded, it is easy to prove that $\forall m > 0$, $\exists C_1(m)$, $a(m,\nu) > 1$ and $t_0(m) < \infty$ such that $\forall t > t_0$:

$$\sum_{|n| > C_1 t} |n|^m \, |\langle \psi_t, \delta_n \rangle|^2 < e^{-at} \tag{42}$$

(see proposition C.1). This implies that for T large enough:

$$\begin{aligned}
\frac{1}{T}\int_0^T \sum_{n \in \mathbf{Z}^d} |n|^m \, |\langle \psi_t, \delta_n \rangle|^2 \, dt &= \mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_{t_0}^T \sum_{n \in \mathbf{Z}^d} |n|^m \, |\langle \psi_t, \delta_n \rangle|^2 \, dt \\
&\leq \mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_{t_0}^T (\sum_{|n| \leq C_1 t} |n|^m \, |\langle \psi_t, \delta_n \rangle|^2 + e^{-at}) \, dt \\
&\leq \frac{4}{T}\int_{t_0}^T \sum_{|n| \leq C_1 t} |n|^m \, |\langle \psi_t, \delta_n \rangle|^2 \, dt
\end{aligned} \tag{43}$$

On the other hand, for $\rho \equiv \frac{\gamma_0 - \varepsilon}{d}$, $0 < \varepsilon < \gamma_0$ and some fixed $C_2 > 0$:

$$\frac{1}{T}\int_0^T \sum_{|n| < C_2^{1/m} t^\rho} |n|^m |\langle \psi_t, \delta_n \rangle|^2 \, dt \leq C_2 \, T^{\frac{m(\gamma_0 - \varepsilon)}{d}} \tag{44}$$

Thus, for C_2 suitably chosen depending on m and ε, theorem 3.1 together with (43) and (44) implies for T large enough:

$$\frac{1}{T}\int_0^T \sum_{C_2^{1/m} t^\rho < |n| < C_1 t} |n|^m |\langle \psi_t, \delta_n \rangle|^2 \, dt \geq C_2 \, T^{\frac{m}{d}(\gamma_0 - \varepsilon)} \tag{45}$$

On the other hand,

$$\frac{1}{T}\int_0^T \sum_{C_2^{1/m} t^\rho < |n| < C_1 t} |n|^m |\langle \psi_t, \delta_n \rangle|^2 \, dt \leq C_1^m T^{m-1} \int_0^T \sum_{C_2^{1/m} t^\rho < |n| < C_1 t} |\langle \psi_t, \delta_n \rangle|^2 \, dt \tag{46}$$

Then (45) and (46) give

$$\frac{1}{T}\int_0^T \sum_{C_2^{1/m} t^\rho < |n| < C_1 t} |\langle \psi_t, \delta_n \rangle|^2 \, dt \geq C_2 C_1^{-m} T^{\frac{m}{d}(\gamma_0 - \varepsilon) - m}$$

For fixed $\nu > 0$, one obtains the result (41) for m small enough.

Remarks. 10 For H self-adjoint on $\mathcal{H}$, and some $\psi \in \mathcal{D}(H)$, define the minimal growth rate of $|X|^m$ for fixed $m > 0$ as:

$$\beta_m = \liminf_{T\to\infty} \frac{\log\langle\langle |X|^m\rangle\rangle_T}{\log T} \tag{47}$$

then relation (42) tells us that for $\mathcal{H} = l^2(\mathbf{Z}^d)$, $\beta_m \leq m$. Moreover, it is easy to see (proposition C.2) that $\frac{\beta_m}{m}$ is increasing with m. Thus $\frac{\beta_m}{m}$ admit a limit value when m tends to infinity. We will see later (theorem 4.1) that this value is in some sense stable under rank one perturbation.

Applications of corollary 3.2: The Fibonacci chain is a model for a quasicrystal in dimension 1. This model is defined by the discrete Schrödinger operator on $l^2(\mathbf{Z})$:

$$(H\psi)(n) \equiv \psi(n+1) + \psi(n-1) + (V\psi)(n) \tag{48}$$

where $(V\psi)(n) = V_n\psi(n)$ and $(V_n)_{n\in\mathbf{Z}}$ takes values $+V$ or $-V$ at site n, arranged in a Fibonacci sequence. This model has extensively been studied, and it has been proven that it's spectrum is singular continuous and is a Cantor set of Lebesgue measure zero [BIST], [Sü]. Numerical computations [ZM] show that for $\psi_0(n) = \delta_{n,0}$, $C(T)$ behaves like $c_0 T^\beta$ for large T, where c_0 is a constant and $0 < \beta < 1$, with β approaching 1 when the modulation strengh V tends to zero. In particular, $\forall\nu > 0$, $\exists V$ small enough such that the spectral measure μ_{ψ_0} has a lower correlation dimension $D_2^- \geq 1-\nu$ (see theorem 2.2); this gives a quasi-ballistic motion using corollary 3.2, since $\dim_{\mathrm{H}}(\mu_{\psi_0}) \geq D_2^-$.

Another example is the Harper model described by:

$$(H\psi)(n) \equiv \psi(n+1) + \psi(n-1) + \lambda\cos(2\pi n\sigma)\psi(n) \tag{49}$$

on $l^2(\mathbf{Z}^d)$, $\sigma = \dfrac{\sqrt{5}-1}{2}$ is the golden mean, and λ is a parameter. This model describes two dimensional electrons in a uniform magnetic field with periodic potential. The spectrum of this operator is a singular continuous Cantor set which has Lebesgue measure zero only if $\lambda = 2$. If $\lambda < 2$, the spectrum is absolutely continuous, and numerical computations obtained in [ZM] gives $C(T) \sim (c_1 \log T)T^{-1}$ so that $D_2^- = \dim_{\mathrm{H}}(\mu_{\psi_0}) = 1$, ($\psi_0$ as above). By corollary 3.2 the motion is ballistic (which is an expected result for an absolutely continuous spectrum with extended states).

In these two examples, the behaviour of $C(T)$ is closely linked with the one of $\langle\langle |X|^2\rangle\rangle_T$. This is due to the fact that in these examples D_2^- is close to $\dim_{\mathrm{H}}(\mu_{\psi_0})$; obviously for arbitrary H and ψ_0 this needs not be so. As a purely theoretic example consider $\mu = \mathcal{L}_{[1/2,1]} + \frac{1}{2}\delta_0$ where $\mathcal{L}_{[1/2,1]}$ is the Lebesgue measure on $[1/2, 1]$ and δ_0

is the Dirac measure. In this example, the best upper bound for the correlation function is given by $\limsup \dfrac{\log C(T)}{\log T} = 0$, whereas $\langle\langle |X|^2\rangle\rangle_T \geq T^{2-\varepsilon}$, $\forall \varepsilon > 0$. In general, D_2^- picks the "less continuous" part of the measure μ_{ψ_0} whereas $\dim_{\mathrm{H}}(\mu_{\psi_0})$ does the opposite (In the last example, one has $D_2^- = 0$ whereas $\dim_{\mathrm{H}}(\mu_{\psi_0}) = 1$).

Unfortunately, $\dim_{\mathrm{H}}(\mu_{\psi_0})$ is not yet the best exponent to describe moments. On the one hand, in [dRJLS], there is an example of a Schrödinger operator with pure point spectrum for which the maximal growth rate of $\langle\langle |X|^2\rangle\rangle_T$ is close to ballistic, that is $\forall \nu > 0$, $\exists (T_n)_{n\in \mathbb{N}}$, $T_n \to \infty$, such that $\langle\langle |X|^{)}\rangle\rangle_{T_n} \geq T_n^{2-\nu}$. Such a spectrum implies $D_2^-(\psi_0) = 0$, and the corollary 3.2 and (1) would give only $\langle\langle |X|^2\rangle\rangle_T \geq$ *constant*. This shows that the lower bounds obtained are strictly one sided.

Furthermore, if one recalls the example of [SW]: Let μ be the conventional Cantor measure. For the associate operator A and vector ψ_0: $\dim_{\mathrm{H}}(\mu_{\psi_0}) \geq D_2^- = \frac{\log 2}{\log 3}$, and the perturbed measure μ_λ associated to the perturbed operator $A_\lambda = A + \lambda\langle\psi_0, .\rangle\psi_0$ is pure point and then local exponents for A_λ are μ-almost everywhere zero. This example proves the instability of $\dim_{\mathrm{H}}(\mu_{\psi_0})$. On the other hand, one expects that the lower bound of $\langle\langle |X|^2\rangle\rangle_T$ does not move too much under small perturbation. It then appears trough this simple example that the lower bound obtained may be not sharp. Our purpose in the next chapter is to illustrate this point of view by a simple perturbative result.

4 Stability Theorem

We wish here to exhibit a stability theorem under rank one perturbation P for the moments $\langle\langle\ |X|^m\ \rangle\rangle_T$. One then obtains a relation between the dynamical quantities $\langle\langle\ |X|^m\ \rangle\rangle_T$ respectively associated to the operators H_0 and H_λ where $H_\lambda = H_0 + \lambda P$, $\lambda \in \mathbb{R}$. Then, using a rank one perturbation's theory [SW], one can show that in a one dimensional case, for the perturbed Fibonacci hamiltonian, the dynamical quantity $\langle\langle\ |X|^2\ \rangle\rangle_T$ behaves like $\langle\langle\ |X|^2\ \rangle\rangle_T \geq C\,T^{1-\varepsilon}$, $\varepsilon > 0$ small, although the spectral measure is pure point.

Let H_0 be a self-adjoint operator on $L^2(\mathbb{R}^d)$ or $l^2(\mathbf{Z}^d)$. For $\psi \in L^2(R^d)$ (resp. in $l^2(\mathbf{Z}^d)$) let $P = \langle\ ,\psi\rangle\psi$ be a rank one self-adjoint projection onto a vector ψ and $H_\lambda = H_0 + \lambda P$, $\lambda \in R$ a family of self-adjoint operator with the same domain $\mathcal{D}(H_0)$. We denote by ψ_t^0 and ψ_t^λ the respective solutions of the Schrödinger equation associated to H_0 and H_λ with the same initial conditions $\psi_0^0 = \psi$, $\psi_0^\lambda = \psi$, and by $\langle\langle\ |X|_0^m\ \rangle\rangle_T$, $\langle\langle\ |X|_\lambda^m\ \rangle\rangle_T$, $m > 0$, the corresponding moments:

$$\langle\langle\ |X|_0^m(t)\ \rangle\rangle_T = \frac{1}{T}\int_0^T \langle\psi_t^0, |X|^m\psi_t^0\rangle dt$$

$$\langle\langle\ |X|_\lambda^m\ \rangle\rangle_T = \frac{1}{T}\int_0^T \langle\psi_t^\lambda, |X|^m\psi_t^\lambda\rangle dt$$

One can prove:

Theorem 4.1 *$\forall\lambda$, $\forall\nu > 0$, there exists $T_0(\nu)$ and $C(\lambda,\nu)$ such that: $\forall m > 0$, $\forall\, T > T_0$:*

$$\langle\langle\ |X|_\lambda^m\ \rangle\rangle_T \geq C(\lambda,\nu)\, T^{D_2^- -2-\nu}\langle\langle\ |X|_0^m\ \rangle\rangle_T \tag{50}$$

where D_2^- is the lower correlation dimension of μ_{ψ_0} associated to ψ and H_0.

Proof By Duhamel's formula , one can write the solution ψ_t^0 as :

$$\psi_t^0 = \psi_t^\lambda + i\lambda\int_0^t \psi_{t-s}^\lambda\langle\psi_s^0,\psi\rangle ds \tag{51}$$

Thus , multiplying equation (51) by $\mid X\mid^{m/2}$ and taking the norm one has:

$$\|\mid X\mid^{m/2}\psi_t^0\|^2 \leq 2\Big\{\|\mid X\mid^{m/2}\psi_t^\lambda\|^2 + \mid\lambda\mid^2\Big\{\int_0^t \mid\langle\psi_s^0,\psi\rangle\mid\|\mid X\mid^{m/2}\psi_{t-s}^\lambda\|ds\Big\}^2\Big\}$$

The Cauchy-Schwarz inequality gives

$$\|\mid X\mid^{m/2}\psi_t^0\|^2 \leq 2\Big\{\|\mid X\mid^{m/2}\psi_t^\lambda\|^2 + \mid\lambda\mid^2\int_0^t \mid\langle\psi_s^0,\psi\rangle\mid^2 ds\int_0^t \|\mid X\mid^{m/2}\psi_{t-s}^\lambda\|^2 ds\Big\} \tag{52}$$

which implies that

$$\langle\langle\ |X|_0^m\ \rangle\rangle_T \leq 2\Big\{\langle\langle\ |X|_\lambda^m\ \rangle\rangle_T + \frac{\mid\lambda\mid^2}{T}\int_0^T\Big(\int_0^T \mid\langle\psi_s^0,\psi\rangle\mid^2 ds\int_0^T\|\mid X\mid^{m/2}\psi_{t-s}^\lambda\|^2 ds\Big)dt \tag{53}$$

Recalling that

$$\int_0^T \mid\langle\psi_s^0,\psi\rangle\mid^2 ds = TC(T)$$

and $\forall\, T > T_0$, $C(T) \leq c\, T^{-D_2^- +\nu}$ for some constants c and T_0 depending only on ν, as proved in theorem 2.2, one can write the inequality (53) as:

$$\langle\langle\ |X|_0^m\ \rangle\rangle_T \leq 2\Big\{\langle\langle\ |X|_\lambda^m\ \rangle\rangle_T + c\mid\lambda\mid^2 T^{2-D_2^-+\nu}\langle\langle\ |X|_\lambda^m\ \rangle\rangle_T\Big\}$$

which leads to write, for some constant $C(\lambda,\nu) > 0$ and $\forall\, T > T_0$:

$$\langle\langle\ |X|_\lambda^m\ \rangle\rangle_T \geq C(\lambda,\nu)\, T^{D_2^- -2-\nu}\langle\langle\ |X|_0^m\ \rangle\rangle_T \tag{54}$$

and proves the statement □

A direct consequence of this theorem is the following corollary:

Corollary 4.1 *If $d = 1$, for all $\nu > 0$, there exists $C(\lambda,\nu) > 0$ and $T_0 > 0$ such that $\forall T > T_0$,*

$$\langle\langle\, |X|^2_\lambda\, \rangle\rangle_T \geq C(\lambda,\nu) T^{3D_2^- -2-\nu} \tag{55}$$

Proof It suffices to apply the corollary 3.1 to H_0 and remark that the information dimension of μ_ψ is such that $\dim_{\mathrm{H}}(\mu_\psi) \geq D_2^-$. □

Definition 4.1 *Let H_0 be a discrete hamiltonian on $l^2(\boldsymbol{Z}^d)$; given C_1, C_2 two positive constants and $0 < \rho < 1$, define the operator $A_{t,m,\rho}$ as:*

$$A_{t,m,\rho} \equiv \sum_{C_2 t^\rho \leq |n| \leq C_1 t} |n|^{m/2} |\delta_n\rangle\langle\delta_n| \tag{56}$$

Corollary 4.2 *For any $\nu > 0$ and $\lambda \in \mathbb{R}$, $\exists T(\nu) < \infty$ and $C(\lambda,\nu,\rho)$ such that $\forall T > T(\nu)$, one has:*

$$\frac{1}{T}\int_0^T \|A_{t,m,\rho}\psi_t^\lambda\|^2 dt \geq C(\lambda,\nu,\rho)\, T^{D_2^- -2-\nu} \frac{1}{T}\int_0^T \|A_{t,m,\rho}\psi_t^0\|^2 dt \tag{57}$$

Proof: Same as for theorem 4.1.

As a consequence of this corollary and of corollary 3.2 one can easily prove the following stability result:

Corollary 4.3 *Let H_0 be a self-adjoint operator on $l^2(\boldsymbol{Z}^d)$ and H_λ defined as for theorem 4.1 , then $\forall \varepsilon > 0$, $\varepsilon < \gamma_0$ and $\nu > 0$, $\exists C_1(\nu) > 0$, $C_2(\varepsilon\,,\nu) > 0$, $C_3(\nu) > 0$ and $T(\varepsilon\,,\nu)$ such that $\forall T > T(\varepsilon\,,\nu)$:*

$$\frac{1}{T}\int_0^T \sum_{C_2\, t^{\frac{\gamma_0-\varepsilon}{d}} < |n| < C_1\, t} |\langle\psi_t^\lambda\,,\delta_n\rangle|^2\, dt \;\geq\; C_3\, T^{D_2^- -2-\nu} \tag{58}$$

where D_2^- is the lower correlation dimension of the spectral measure associated to ψ and H_0.

Remarks. 11

(i) It follows from theorem 4.1 that the lower bound (26) is far from beeing sharp as can be seen from the following example. Let H_0 be the Fibonacci chain model hamiltonian's as described in (48). It is known that the spectrum of H_0 is singular continuous (see e.g. [BG] and references therein). Let $\psi(n) = \delta_{n,0}$, then the correlation dimension D_2^- of μ_ψ, as pointed out in the previous chapter, depends on V and for all $\nu > 0$ there exists a small enough value of V such that $D_2^- \geq 1-\nu$. Thus for the perturbed hamiltonian $H_\lambda = H_0 + \lambda P$

(P defined as above), corollary 4.1 implies that for fixed $\lambda > 0$ and $\nu > 0$, $\exists\, T_0(\nu) < \infty$ and $C(\lambda\,,\nu)$ such that:

$$\forall T > T_0(\nu), \quad \langle\langle\ |X|^2_\lambda\ \rangle\rangle_T \geq C(\lambda,\nu)\, T^{1-\nu} \tag{59}$$

or equivalently

$$\liminf_{T\to\infty} \frac{\langle\langle |X|^2_\lambda \rangle\rangle_T}{T^{1-2\nu}} = \infty$$

One also knows that the spectrum of H_0 is a Cantor set $\mathcal{C}$ of Lebesgue measure 0 so that in the terminology of [SW]:

$$B(x) \equiv \int \frac{d\mu_\psi(y)}{(y-x)^2} < \infty \quad \forall x \notin \mathcal{C}$$

which implies that $d\mu_\psi^\lambda$ associated to ψ and H_λ is pure point for Lebesgue almost every λ (see [SW]). So $d\mu_\psi^\lambda$ has zero Hausdorff dimension but for $\nu < 1$ the lower bound given by (26) for $m = 2$ is strictly less than the one of (59). We refer to [La] and [dRJLS] for even more striking examples of hamiltonians with zero-Hausdorff dimensional spectral measures having

$$\limsup_{T\to\infty} \frac{\langle\langle |X|^2 \rangle\rangle_T}{T^2/\log T} = \infty$$

(ii) Let $\beta_{m,0}$ and $\beta_{m,\lambda}$ be the lower growing exponent defined by (47) for H_0 and H_λ respectively. Then it follows from (50) that

$$\beta_{m,\lambda} \geq \beta_{m,0} + D_2^- - 2$$

and in particular

$$\lim_{m\to\infty} \frac{\beta_{m,\lambda}}{m} \geq \lim_{m\to\infty} \frac{\beta_{m,0}}{m}$$

Reversing the rôles of H_0 and H_λ one obtains

$$\lim_{m\to\infty} \frac{\beta_{m,\lambda}}{m} = \lim_{m\to\infty} \frac{\beta_{m,0}}{m} \tag{60}$$

This is an interesting stability property although we don't know how to caracterize these limits in term of suitable dimensions of spectral measures. Notice that in the previous example (59) gives $\lim_{m\to\infty} \frac{\beta_{m,\lambda}}{m} \geq 1-\nu$ instead of the mere $\beta_{m,\lambda} \geq 0$ given by (26).

A Appendix

In this chapter, we relate correlation dimensions D_2^+ and D_2^- to mean quadratic dimension and wavelet dimensions. First of all, let us define mean quadratic dimensions already introduced in [L] to generalize the results of Strichartz (5).

Definition A.1

$$W(\varepsilon)=\frac{1}{\varepsilon}\int_R \mu^2((\omega-\frac{\varepsilon}{2},\omega+\frac{\varepsilon}{2}])\,d\omega$$

Then

$$w^+ = \limsup_{\varepsilon\to 0}\frac{\log W(\varepsilon)}{\log\varepsilon} \tag{61}$$

$$w^- = \liminf_{\varepsilon\to 0}\frac{\log W(\varepsilon)}{\log\varepsilon} \tag{62}$$

are respectively the upper and lower mean quadratic dimensions.

Proposition A.1

$$D_2^+ = w^+ \tag{63}$$

$$D_2^- = w^- \tag{64}$$

Proof

$$\begin{aligned}
W(\varepsilon) &= \frac{1}{\varepsilon}\int_R \mu^2((\omega-\frac{\varepsilon}{2},\omega+\frac{\varepsilon}{2}])\,d\omega \\
&= \frac{1}{\varepsilon}\sum_{k\in\mathbf{Z}}\int_{k\varepsilon}^{(k+1)\varepsilon} d\omega\int_{(\omega-\frac{\varepsilon}{2},\omega+\frac{\varepsilon}{2}]} d\mu(\omega')\mu((\omega-\frac{\varepsilon}{2},\omega+\frac{\varepsilon}{2}]) \\
&= \frac{1}{\varepsilon}\sum_{k\in\mathbf{Z}}\int_0^{\varepsilon} d\xi\int_{(\xi+k\varepsilon-\varepsilon/2,\xi+k\varepsilon+\varepsilon/2]} d\mu(\omega')\mu((\xi+k\varepsilon-\varepsilon/2,\xi+k\varepsilon+\varepsilon/2]) \qquad (65)\\
&\leq \frac{1}{\varepsilon}\int_0^{\varepsilon} d\xi\sum_{k\in\mathbf{Z}}\int_{(\xi+k\varepsilon-\varepsilon/2,\xi+k\varepsilon+\varepsilon/2]} d\mu(\omega')\mu((\omega'-\varepsilon,\omega'+\varepsilon]) \\
&\leq \frac{1}{\varepsilon}\int_0^{\varepsilon} d\xi\,\gamma_2(2\varepsilon) \\
&\leq \gamma_2(2\varepsilon) \qquad (66)
\end{aligned}$$

On the other hand, one has from (65):

$$W(\varepsilon) \geq \frac{1}{\varepsilon}\int_0^{\varepsilon} d\xi\sum_{k\in\mathbf{Z}}\int_{(\xi+k\varepsilon-\varepsilon/4,\xi+k\varepsilon+\varepsilon/4]} d\mu(\omega')\mu((\omega'-\varepsilon/4,\omega'+\varepsilon/4]) \tag{67}$$

and

$$
\begin{aligned}
W(\varepsilon) &= \frac{1}{\varepsilon}\sum_{k\in\mathbf{Z}}\int_{k\varepsilon-\varepsilon/2}^{(k+1)\varepsilon-\varepsilon/2} d\omega\int_{(\omega-\varepsilon/2,\omega+\varepsilon/2]} d\mu(\omega')\mu((\omega-\varepsilon/2,\omega+\varepsilon/2]) \\
&= \frac{1}{\varepsilon}\int_0^{\varepsilon}\sum_{k\in\mathbf{Z}}\int_{(\xi+k\varepsilon-\varepsilon,\xi+k\varepsilon]} d\mu(\omega')\,\mu((\xi+k\varepsilon-\varepsilon,\xi+k\varepsilon]) \\
&\geq \frac{1}{\varepsilon}\int_0^{\varepsilon}\sum_{k\in\mathbf{Z}}\int_{(\xi+k\varepsilon-3\varepsilon/4,\xi+k\varepsilon-\varepsilon/4]} d\mu(\omega')\,\mu((\omega'-\varepsilon/4,\omega'+\varepsilon/4]) \qquad (68)
\end{aligned}
$$

(67) and (68) imply that :

$$W(\varepsilon)\geq\frac{1}{2}\gamma_2(\frac{\varepsilon}{2}) \tag{69}$$

Finally, one obtains from (66) and (69):

$$\frac{1}{2}\gamma_2(\frac{\varepsilon}{2})\leq W(\varepsilon)\leq\gamma_2(2\varepsilon)$$

which proves (63) and (64).

Let us now recall some recent results of Holschneider [H], and give their connections with our results. The main tool used here is the wavelet transform which is powerfull in analyzing micro-local properties of functions or measures. Let us first recall some definitions:

Let $\mathcal{S}_+(\mathbb{R})$ be the subspace of Schwartz functions whose Fourier transform is supported by the positive frequencies. For any positive Radon measure μ in $\mathcal{S}'_+(\mathbb{R})$ and any g in $\mathcal{S}_+(\mathbb{R})$, we define the wavelet transform w.r.t. g in the Fourier space as:

$$\mathcal{W}_g\mu(b,a)=\frac{1}{2\pi}\int_R\overline{\hat{g}}(a\omega)e^{ib\omega}\hat{\mu}(\omega)d\omega \tag{70}$$

Where $\hat{}$ denotes the Fourier transform.

Definition A.2

Let $\Gamma(\varepsilon,2)=min\{\int_\varepsilon^1\frac{d\alpha}{\alpha}\int_R db\mid\mathcal{W}_g\mu(b,\alpha)\mid^2,\int_0^\varepsilon\frac{d\alpha}{\alpha}\int_R db\mid\mathcal{W}_g\mu(b,\alpha)\mid^2\}$ *and:*

$$\kappa^- = \liminf_{\varepsilon\to 0}\frac{\log\Gamma(\varepsilon,2)}{\log a} \tag{71}$$

$$\kappa^+ = \limsup_{\varepsilon\to 0}\frac{\log\Gamma(\varepsilon,2)}{\log a} \tag{72}$$

The last two quantities are called lower and upper 2-wavelet dimensions.

Remarks. 12 κ^+ and κ^- do not depend on the choice of the wavelet g, provided $g \neq 0$ (see[H]).

The second integral in the definition of $\Gamma(\varepsilon, 2)$ is used when the first converges when $\varepsilon \to 0$. This is the case only if μ, as an element of $\mathcal{S}'_+(\mathbb{R})$ is in $L^2(\mathbb{R})$.

Theorem A.1 *With the notations of corollary 2.1, let $\eta \in S'(\mathbb{R})$, $\eta \neq 0$ satisfying $s \star \eta \in S(\mathbb{R})$ for all $s \in S(\mathbb{R})$. Assume that $\eta \notin L^2(\mathbb{R})$. Then $\kappa^-(2) \leq 0$ and it follows that*

$$-\kappa^+(2) - 1 = \liminf_{T\to\infty} \frac{\log C(T)}{\log T} \leq \limsup_{T\to\infty} \frac{\log C(T)}{\log T} = -\kappa^-(2) - 1 \tag{73}$$

We want here to relate the quantities κ^+ and $\kappa^-$to the correlation dimensions D_2^+ and D_2^- introduced in definition 2.3. An immediate consequence of theorem A1 and our theorem 2.2 is that:

$$D_2^+ = \kappa^+ + 1 \quad \text{and} \quad D_2^- = \kappa^- + 1$$

We give here another proof of similar relations independently of $C(T)$ in the following proposition.

Proposition A.2 *Let μ be a Borel probability measure on $\mathbb{R}$, such that it is not in $L^2(\mathbb{R})$, then:*

$$D_2^+ \leq \kappa^+ + 1 \tag{74}$$

$$D_2^- = \kappa^- + 1 \tag{75}$$

Proof: By parseval equation, one has

$$\begin{aligned} \Gamma(\varepsilon, 2) &= \int_\varepsilon^\infty \frac{da}{a} \int_{\mathbb{R}} d\omega \mid \mathcal{W}_g\mu(b,a) \mid^2 \\ &= \int_\varepsilon^\infty \frac{da}{a} \int_{\mathbb{R}} d\omega \mid \hat{g}(a\omega) \mid^2 \mid \hat{\mu}(\omega) \mid^2 \end{aligned}$$

We choose $g \in \mathcal{S}_+(\mathbb{R})$ such that $\text{supp}(\hat{g}) \subset [\delta, 1]$ for $\delta > 0$, and $\int_{\mathbb{R}^+} \mid \hat{g}(a) \mid^2 \dfrac{da}{a} = c_0 > 0$

For $F(\varepsilon) = \int_\varepsilon^\infty \mid \hat{g}(a) \mid^2 \dfrac{da}{a}$, we have $\Gamma_g(\varepsilon, 2) = \int_{\mathbb{R}^+} d\omega\, F(\varepsilon\omega) \mid \hat{\mu}(\omega) \mid^2$
and

$$\begin{aligned} c_0 \int_0^{\frac{\delta}{\varepsilon}} \mid \hat{\mu}(\omega) \mid^2 \, d\omega &\leq \Gamma_g(\varepsilon, 2) \\ &\leq c_0 \int_0^{\frac{1}{\varepsilon}} \mid \hat{\mu}(\omega) \mid^2 \, d\omega \end{aligned} \tag{76}$$

On the other hand, one has :

$$\begin{aligned} W(\varepsilon) &= \frac{1}{\varepsilon}\int_{I\!R}\mu^2(\omega-\frac{\varepsilon}{2},\omega+\frac{\varepsilon}{2})\,d\omega \\ &= \frac{1}{\varepsilon}\int_{I\!R}\mid(\chi_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\star\mu)(\omega)\mid^2\,d\omega \\ &= \frac{1}{2\pi}\frac{\varepsilon}{2}\int_0^\infty\frac{sin^2(\frac{\varepsilon\omega}{2})}{(\frac{\varepsilon\omega}{2})^2}\mid\hat{\mu}(\omega)\mid^2\,d\omega \qquad (77) \end{aligned}$$

¿From (76) and (77), one easily has that for some $c_0' > 0$:

$$W(\varepsilon) \geq c_0'\,\varepsilon\Gamma_g(\varepsilon,2) \Rightarrow \quad D_2^+ \leq \kappa^+ + 1 \qquad (78)$$

$$D_2^- \leq \kappa^- + 1 \qquad (79)$$

Moreover, one has:

$$\begin{aligned} \varepsilon\int_{R^+}\frac{sin^2(\frac{\varepsilon\omega}{2})}{(\frac{\varepsilon\omega}{2})^2}\mid\hat{\mu}(\omega)\mid^2\,d\omega &\leq \varepsilon\sum_n\int_0^{\frac{2\pi n}{\varepsilon}}[(\frac{1}{2\pi})^n-(\frac{1}{2\pi})^{n+1}]\mid\hat{\mu}(\omega)\mid^2\,d\omega \\ \Rightarrow\quad W(\varepsilon) &\leq \frac{2\pi-1}{2\sqrt{2\pi}}\varepsilon\sum_n\frac{1}{2\pi^{n-1}}\,\Gamma_g(\frac{\varepsilon}{2\pi\delta n}) \qquad (80) \end{aligned}$$

By the definition of κ^-, we have for $\varepsilon < \varepsilon_0$ small enough :

$$\forall\nu\,,0<\nu<\kappa^-+1\,,\ \exists\, c(\nu)<\infty\ s.t.\ \Gamma_g(\varepsilon)\leq c(\nu)\varepsilon^{\kappa^--\nu}$$

So (80) implies that:

$$\begin{aligned} W(\varepsilon) &\leq c_1(\nu)\sum_n\Big((2\pi n)^{(\nu-\kappa^-}\frac{1}{(2\pi)^{n-1}}\Big)\,\varepsilon^{\kappa^--\nu+1} \\ &\leq c_2(\nu)\,\varepsilon^{\kappa^--\nu+1} \\ &\Rightarrow\quad D_2^- \geq \kappa^- + 1 \qquad (81) \end{aligned}$$

The result follws from (78),(79) and (81).

B Appendix

Our goal here is essentially to present some results or examples with some specific measures that permit to understand the link between the different dimensions introduced in this paper. Let us first introduce a somewhat surprizing example of measure for which the largest α such that it is L.U.α is smaller than any local exponents $\gamma^-(x)$.

Consider the absolutely continuous measure μ defined on $(0,1]$ as follow:

$$d\mu = \nu\, x^{\nu-1} dx \quad for\ fixed\ \nu \in (0,\frac{1}{2}) \tag{82}$$

Then $\mu\ a.e.x.:\ \gamma^-(x) = \gamma^+(x) = 1$, but the supremum over all the α such that μ is L.U.α is less than ν. That is, $\forall x_0 \in (0,1]$, $\exists \varepsilon_0$ small enough and a constant $\tilde{c} = \frac{\nu^2}{x_0^{1-\nu}}$ such that $\forall \varepsilon < \varepsilon_0,\ \mu(B_\varepsilon(x_0)) \leq \tilde{c}\,\varepsilon$, but one can not hope to find ε_0 and $\tilde{c}$ uniform in x. Indeed, let $\alpha_0 = \nu + \delta$ for some $\delta > 0$, and assume $\exists\, c > 0$ and $\varepsilon_0 > 0$ such that

$$\forall \varepsilon < \varepsilon_0\,,\ \mu\, a.e.x\,,\ \ \mu(B_\varepsilon(x)) < c\,\varepsilon^{\alpha_0} \tag{83}$$

For any $0 < x_0 < 1/2\inf(\varepsilon_0, (\frac{c}{2\nu})^\delta))$ and $1/2\inf(\varepsilon_0,(\frac{c}{2\nu})^\delta)) < \varepsilon < \inf(\varepsilon_0,(\frac{c}{2\nu})^\delta))$, one has $\mu(B_\varepsilon(x)) \geq c\varepsilon^{\alpha_0}$ which is in contradiction with (83).

It follows from theorem 2.2 and an explicit calculation of $\hat{\mu}$ that $D_2^- = D_2^+ = 2\nu$ ([La]), whereas $\mu\,$ess.inf$\gamma^-(x)$ is equal to 1 for every $\nu > 0$. Thus this example shows that (13) cannot in general be turned into an equality.

C Appendix

Proposition C.1 *Let H be defined on $l^2(\mathbb{Z}^d)$ as:*

$$H\psi(n) = (-\Delta + V)\psi(n) = \sum_{|n-k|=1} \psi(k) \ + V(n)\psi(n)$$

Assume that V is bounded; then $\forall \psi \in \mathcal{D}(H)$ with compact support, $\forall m \geq 0$, $\exists c_1 > 0$, $a > 1$ and $t_0 < +\infty$, all depending on m, $\|V\|_\infty$ and ψ such that $\forall t > t_0$:

$$\sum_{|n|>c_1 t} |n|^m |\langle \psi_t\,, \delta_n\rangle|^2 < e^{-at} \tag{84}$$

where ψ_t is defined as in corollary 2.1.

Proof: Assume for simplicity that $\psi(n) = \delta_0(n)$; for general ψ of compact support the proof extends easily. Let $m \geq 0$ and $c_1 > 2e(2d + \|V\|_\infty)$. For $n \in \mathbb{N}$, $H^n\psi$ is supported on $\{k \in \mathbb{Z}^d\,,\ |k| \leq n\}$, one has:

$$\begin{aligned}
\Big(\sum_{|k|>c_1 t} |k|^m |\langle \psi_t\,,\ \delta_k\rangle|^2\Big)^{\frac{1}{2}} &\leq \ \|\sum_{|k|>c_1 t}\ \sum_{n>|k|} \frac{|k|^{\frac{m}{2}} t^n}{n!} |\langle H^n\psi\,,\ \delta_k\rangle|\,\delta_k\| \\
&\leq \ \sum_{n>c_1 t} \frac{n^{\frac{m}{2}} t^n}{n!}\,(2d + \|V\|_\infty)^n \\
&\leq \ 2\sqrt{2\pi} \sum_{n>c_1 t} n^{\frac{m-1}{2}-n} (t(2d + \|V\|_\infty)e)^n \\
&\leq \ 4\sqrt{2\pi}(c_1 t)^{\frac{m-1}{2}} e^{-c_1 \ln(2) t}
\end{aligned}$$

In the third inequality, we have used Stirling formula.

Proposition C.2 *Let H be any self-adjoint operator on $\mathcal{H} = l^2(\mathbb{Z}^d)$ (or $L^2(\mathbb{R}^d)$), and $\psi \in \mathcal{D}(H)$, then for β_m defined as in (47), $(\frac{\beta_m}{m})_{m>0}$ is an increasing sequence. Moreover, if $\mathcal{H} = l^2(\mathbb{Z}^d)$, $\exists \beta_\infty \leq 1$ such that:*

$$\lim_{m\to\infty} \frac{\beta_m}{m} = \beta_\infty \tag{85}$$

Proof: Assume that $\forall t$, $\psi_t \in \mathcal{D}(|X|^m)$, then, for $0 < m' < m$, $\psi_t \in \mathcal{D}(|X|^{m'})$. Applying Jensen's inequality to the probability measure $d\nu_t(x) \equiv |\psi_t(x)|^2\, dx$, one obtains:

$$\begin{aligned} \langle |X|^m \rangle_t &= \langle |X|^m \psi_t \,, \psi_t \rangle \\ &= \int_{\mathbb{R}} (|X|^{m'})^{m/m'}\, d\nu_t(x) \\ &\geq \langle |X|^{m'} \rangle_t{}^{m/m'} \end{aligned} \tag{86}$$

One applies once again Jensen's inequality to the probability measure $\chi_{[0,T]}\, \frac{dt}{T}$, where $\chi_{[0,T]}$ is the characteristic function of $[0,T]$, which gives immediatly:

$$dt\Big)^{m/m'}$$

which proves that $(\frac{\beta_m}{m})_{m>0}$ is an increasing sequence. As a consequence of proposition C.1, if $\mathcal{H} = l^2(\mathbf{Z}^d)$, one has $\beta_m \leq m$, which implies the last statement.

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