STRATIFIED LOW TEMPERATURE PHASES OF STRATIFIED SPIN MODELS. A GENERAL PIROGOV – SINAI APPROACH

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Abstract. We adapt and improve the existing Pirogov – Sinai technology to obtain a general and unifying approach to the study of low temperature, "stratified" phases for classical spin models whose hamiltonian may not even be translation invariant but is "stratified" i.e. invariant with respect to all "horizontal" shifts (not changing the last coordinate). Examples are "stratified" versions of classical models like the Ising model with "vertically dependent" external field; models in halfspaces or layers and also those translation invariant models where Dobrushin's phases with rigid interfaces (one or more) appear. Our method brings some clarification and sharpening even when applied to the ordinary situations of the Pirogov – Sinai theory [S], [Z]. Our main result transcripts the question of characterizing the "stratified" Gibbs states of the given model to the question of finding the *ground states* of some auxiliary *one dimensional* model with infinite range but quickly decaying interactions.

I. Introduction, Notes on the Development of the Problem and Some Examples

The rigorous study of Gibbs states having translation noninvariant structure with a "rigid interface" goes back to the pioneering Dobrushin's paper [1]. Several authors continued this study; we note e.g. articles [HKZ] where an attempt to combine basic Dobrushin's ideas with the power of Pirogov – Sinai theory was made.

The leading idea in these investigations was to transcript the problem of description of the structure of the rigid interface (between the two translation invariant phases above and below) to a suitable *lower dimensional* problem. In more concrete terms, using the expansion of the partition sums above and below the interface, the behaviour of the "walls" of the interface between the + and − phases in the three

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dimensional Ising model can be viewed as a behaviour of contours of some auxiliary two dimensional perturbed Ising model.

Recently, we applied a similar approach based on the reduction to a lower dimensional problem to the study of wetting phenomena and entropic repulsion in the Ising model in halfspace [HZ]. During our attempt to pursue the method to other interesting situations, like the study of "wetting layers" emerging in some phases of the Blume – Capel model and also in the order – disorder – (other)order phases appering in the Potts model below the critical temperature (the article [MZ] is under preparation) we found that the additional technical problems are forcing us to look for a more appropriate method. Finally we were lead to a conclusion that the "dimensional reduction method" based on this particular kind of a partial exansion of the considered model should be abandoned. Instead, we found a modification of the Pirogov – Sinai theory which applies *directly* to these "stratified" situations. We hope that the fact that our new version of the Pirogov – Sinai theory gives even some new insight and simplifications into the traditional "translation invariant" Pirogov – Sinai theory confirms that the method developed by us is adequate.

Methodologically, our approach is based on the version [Z] of the Pirogov – Sinai theory but the concept of a "stable" ("small") contour and of a "metastable ensemble" is now investigated in a greater depth. Moreover, the concept of a contour ensemble now *disappeared* from our version of Pirogov – Sinai theory completely! The concept of a, suitably defined, "contour functional" $F(\mathbf{\Gamma})$ (as compared to the contour energy $E(\mathbf{\Gamma})$) remains as a very important *testing quantity* (allowing one to decide whether the contour is "small" or not) but instead of the construction of auxiliary contour models a central point of our approach is the idea of a succesive partial *expansion* of the model based on an important new technical step which is called *recoloring* of the contour here. Recoloring of a contour $\mathbf{\Gamma}$ in a partially expanded model means that a new, "more expanded" model with the *same* partition functions is constructed where $\mathbf{\Gamma}$ does not appear yet as a contour . We will show that the "metastable" submodels of the given model (constructed for any stratified boundary condition) can be *expanded completely* and that for the "stable" boundary conditions, the corresponding metastable model will be identical with the original "physical" model. Roughly speaking, all the external contours will be "small" resp. "recolorable" (in the sense of [Z] resp. of this paper) in such a situation. The organization of our expansions will make unnecessary estimates like "Main Lemma" of [Z].

Instead, we have now a more powerful method based on our Theorems 5 and 6. To summarize, we converted the Pirogov – Sinai theory just to a carefully organized *method* of (succesive) *expansion of some partition functions*. The use of expansion techniques is absolutely crucial in our situation and the construction of the expansions is a more delicate task than in the translation invariant situations studied before. Namely, contours of the models studied so far were "crusted" in the sense that the events outside and inside the contour were independent. This is *not* valid here in our new situation where contours can be also interpreted as "walls" (the terminology of [D]) of the interface and the events happening "inside" resp. "outside" of the wall cannot be tracted as independent ones.

This problem was solved in [HKZ] by taking expansions "above" and "below" the interface and by replacing the walls by more complex "aggregates" of walls and clusters[1] but for complicated phases with *several* interfaces such an approach is too complicated.

Now we treat both the "crusted" contours and the "noncrusted" ones (walls) in the *same way*. However, the fact that some contours "are not crusted" implies that the testing quantity $F(\mathbf{\Gamma})$ called the "contour functional" of a given contour $\mathbf{\Gamma}$ must be now defined much more carefully[2]. We construct (succesively, by induction) the expansion of the *whole* metastable model, leaving out the previous idea (of [HKZ]) of the expansion in *two different steps* (first the expansion of the ensemble of contours and then of the ensemble of the walls resp. aggregates).

Let us mention some typical examples which can be treated by our method.

(1) a) Models in halfspace $\mathbb{Z}^\nu_+ = \{t \in \mathbb{Z}^\nu; t_\nu \geq 0\}$ with "unstable" boundary condition on the bottom (like the $-$ boundary condition for the ferromagnetic Ising model with a negative external field (making $+$ the only ground state of the model)) Than the "Basuev states" (terminology of R. L. Dobrushin) with a weeting layer of minuses appear.
b) Models in layers (like in [MS], [MDS])

(2) a) Models of the Blume-Capel type with spins belonging to some finite set $Q \in \mathbb{R}$ and with the hamiltonian consisting of a quadratic (e.g.) pair interaction and a potential V :

$$H(x) = \sum_{(t,s)} (x_t - x_s)^2 + \sum_t V(x_t) \tag{1.1}$$

where V has several "potential wells" (of the approximately same depth). If x_+, x_0, x_- mark the bottoms of three adjancent wells of V then it may happen, for suitable choice of V, that both x_+ and x_- give rise to a stable phase while x_0 is unstable. Then one should expect also the existence of a phase which "goes vertically from x_+ to x_- through a layer of a metastable 0-th phase". The question is about the determination of the width of the 0-th layer.
b) Such a situation appears, in the Fortuin – Kasteleyn representation, also for the Potts model with large number of spins below the critical temperature, where phases of the type order – layer of disorder – another order exist. (The paper [HMZ] which is under preparation will be devoted to these questions.)

(3) a) "Sedimentary Ising rock". Consider some ordinary translation invariant Pirogov – Sinai type model and add to it a small perturbative hamiltonian which is invariant with respect to the $\mathbb{Z}^{\nu-1}$ shifts (we identify here $\mathbb{Z}^{\nu-1}$ with the subspace $\mathbb{Z}^{\nu-1}x\{0\}$ of $\mathbb{Z}^\nu$) i.e. depends on the last ("vertical") coordinate t_ν of $t = (t_1, \dots, t_\nu) \in \mathbb{Z}^\nu$ only. Then, one should expect phases with a rich structure of (many) layers (of "stable or slightly instable translation invariant phases of the unperturbed hamiltonian "). For example if one adds, to a ferromagnetic Ising model, a small "horizontally invariant" external field with approximately zero mean over the vertical shifts, one

[1]The whole situation was then projected to $\mathbb{Z}_{\nu-1}$ which is the main idea of the paper [D].
[2]Retaining its meaning, vaguely speaking, of the "work needed to install the given contour".

should expect phases with infinitely many layers (of changing $\pm$ phases), and the problem is to compute the exact positions of the layers.
b) We will see later that the class of "horizontally invariant models" fitting our scheme is much broader and many examples which are not small perturbations of translation invariant models can be constructed.

Our main result is given in part III, section 8:
In the translation invariant Pirogov – Sinai theory, one constructs, for any reference configuration (i.e. for any "local ground state") a quantity called the "metastable free energy". If the minimum of this quantity is attained in some configuration y, then the Gibbs state characterized as the "local perturbation of y" exists ([Z]). Here, our "reference configurations" are (all!) stratified configurations; instead of quantities mentioned above we construct some auxiliary *one dimensional* model of the Ising type whose configurations correspond to various "horizontally invariant regimes" of the original model. The *ground states* of this one dimensional model correspond to the different *stratified* Gibbs states which we are looking for! This is our Main Theorem (section 8); the quantities $h_t(y) = h_{t_\nu}(y),\ t \in \mathbb{Z}^\nu$ constructed there give all the essential information about the model.

These quantities are in principle computable as they are given by cluster expansion series (with complicated, but very quickly decaying terms). In the case when y is the ground state of the corresponding one dimensional model ("stability of y") the quantities $h_t(y)$ have the physical interpretation of the "density of free energy of the y–th Gibbs state at the vertical level t_ν" .

Note. We are concentrated, in this paper, in the investigation of a phase picture for a *fixed* hamiltonian. The investigation of phase diagrams of particular models (notice that there are in principle infinite parameters in the models like (3) above!) should be based on the study of the mapping

$$\{ \text{ hamiltonian} \longmapsto \text{the ground states of } \{h_t(y)\} \ \} \tag{1.2}$$

using theorems from the differential calculus of (infinitely) many variables (like the implicit function theorem). This may require a suitable technical modification of the definition of the contour functional F and the quantity $h_t(y)$[3] to obtain as nice differentiability (even local analyticity) properties of (1.2) as possible.

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II. General Description of the Considered Model. Transcription to an Abstract Pirogov – Sinai Type Model

Given a configuration space

$$\mathbb{X} = S^{\mathbb{Z}^\nu}\ , \quad \nu \geq 3 \tag{2.1}$$

[3] Such a modification could act on the nonground values of y only; the ground values of $h_t(y)$ have nontrivial physical interpretation and there can be no arbitrariness in their definition!

where S is a finite set(of "spins") we will consider a general "horizontally invariant" ("stratified") hamiltonian on X: The hamiltonian will be a finite range one (in what follows we consider a suitable norm on $\mathbb{Z}^\nu$ e.g. the l_∞ one)

$$H_\Lambda(x_\Lambda|x_{\Lambda^c}) = \sum_{\substack{A\cap\Lambda\neq\emptyset \\ \operatorname{diam} A\leq r}} \Phi_A(x_A) \tag{2.2}$$

where Φ_A are some "interactions", i.e. functions on S^A with values in $\mathbb{R}\cup+\infty$, which are"stratified" in the sense explained below.

We will study the structure of (stratified) *Gibbs states* of the model, more precisely of the probabilities[4] which are given in finite volumes Λ by formulas

$$P_\Lambda^{x_{\Lambda^c}}(x_\Lambda) = Z(\Lambda, x_{\Lambda^c})^{-1}\exp(-\frac{1}{T}H_\Lambda(x_\Lambda|x_{\Lambda^c})) \tag{2.3}$$

where T is the "temperature" and the partition function $Z(\Lambda, x_{\Lambda^c})$ is

$$Z(\Lambda, x_{\Lambda^c}) = \sum_{x_\Lambda}\exp(-\frac{1}{T}H_\Lambda(x_\Lambda|x_{\Lambda^c})). \tag{2.4}$$

Suitable *infinite volume limits* will be constructed from these finite volume Gibbs states.

Notes. 1. In fact, some other (more special than (2.4)) partition functions – namely so called (strictly) diluted partition functions will be important later and the recurrent structure of the measures (2.3) – which is formulated by the DLR equations – will not be used explicitly in our later approach. More adequate for our later approach is the idea that Λ is some (large) volume which will be *fixed* in the main part of our future considerations. (Only at the very end of the paper – when proving and interpreting our Main Theorem, section 3.8 – this "playground" Λ will be expanded to the whole $\mathbb{Z}^\nu$ and the limit Gibbs states thus obtained will be investigated.)
2. In the following we will always put $T = 1$ i.e. we include the term $\frac{1}{T}$ into the definition of Φ_A and H. Thus the temperature will be just one of the parameters in the hamiltonian. We emphasize that in this paper we are interested only in the clarification of the phase picture for a given *fixed* hamiltonian. Doing this, one can study the *change* of this picture (and of relevant quantities like the free energies) when the parameters are changing. Our approach gives some basic tools for doing that: namely we define useful quantities called *metastable free energies* which really govern the behaviour of the phase diagram – see our Main Theorem. However, the relevant result on the behaviour of the phase *diagram* is not even formulated in our paper!
3. One could be interested in the structure of Gibbs states, under suitable boundary conditions, also for other infinite volumes like the halfspace $\mathbb{Z}^\nu_+ = \{t \in \mathbb{Z}^\nu; t_\nu \geq 0\}$ or in a layer. It is not hard to see that such a situation could be modeled on $\Lambda = \mathbb{Z}^\nu$ by choosing a suitable modification of the hamiltonian: for example if we put

$$\Phi_A(x_A) = +\infty$$

[4]The probabilities $P_\Lambda^{x_{\Lambda^c}}(\cdot)$ are called *finite volume Gibbs states* under boundary condition x_{Λ^c}.

whenever $A \not\subset \mathbb{Z}^\nu_+$ and $x_A \neq \{x_t = \bar{x}_t, t \in A\}$ we obtain a limit Gibbs state on $\mathbb{Z}^\nu_+$ under the boundary condition $\bar{x}$ on $\mathbb{Z}^\nu_- = \mathbb{Z}^\nu \setminus \mathbb{Z}^\nu_+$.

4. In fact, sensible and nontrivial results (requiring the full strength of all the forthcoming constructions) can be formulated even for a fixed *finite* volume Λ (imagine the cardinality $|\Lambda| = 10^{27}!$), with suitable boundary conditions. However, in this case it is of course natural to study also a torus with periodic boundary conditions. Though we do not work out here the (topological) modifications needed to carry our study from the case $\mathbb{Z}^\nu$ to the case of periodic boundary conditions, we expect that only minor parts of the text should be adapted or replaced by another arguments (for example the parts of the text using the lexicographic ordering of $\mathbb{Z}^\nu$).

1. Stratified configurations, hamiltonians and states

For any $u \in \mathbb{Z}^{\nu-1}$ consider the shifts $U \equiv \{t \mapsto t + (u, 0)\} : \mathbb{Z}^\nu \to \mathbb{Z}^\nu$ and correspondingly define the shifts

$$\{x \mapsto U(x)\} \,:\, S^A \to S^{U(A)}$$

where $U(x) = \tilde{x}$ has coordinates $\tilde{x}_{t+u} = x_t$, $t \in A$ and

$$\Phi_A \mapsto U\Phi_{U(A)}$$

where $U\Phi_{U(A)}(U(x_A)) = \Phi_A(x_A)$. Say that a configuration x is *stratified* (or *horizontally invariant* if

$$U(x) = x \quad \text{for each } u \in \mathbb{Z}^{\nu-1} = \{(t_1, \dots, t_{\nu-1}, 0)\} \subset \mathbb{Z}^\nu\,.$$

Definition. We denote by $\mathbb{S} \subset \mathbb{X}$ the collection of *all stratified configurations*. Analogously we define the notion of a *stratified hamiltonian* $H = \{\Phi_A\}$ and a *stratified* (Gibbs) *measure* μ by requiring

$$\{U\Phi_A\} = \{\Phi_A\}\,,\;\; U(\mu) = \mu$$

for each $u \in \mathbb{Z}^{\nu-1}$.

Notes. **1.** These will be the "local ground states" of our model. Of course only *some* of these configurations will "deserve" this name. Analogously, in the traditional Pirogov – Sinai situation, only some of the constant (resp. periodic) configurations "deserve" the name of "local ground state". However, it is often problematic to separate these "true local ground states" from the other horizontally invariant (analogously, translation invariant resp. periodic) configurations. We will see below that a nonexistence of a substantial energetic barrier between the "true local ground states" and the remaining elements of $\mathbb{S}$ would deteriorate the validity of the Peierls condition. The best solution in such a situation seems to be to choose the reference family $\mathbb{S}$ as big as above and to look for energetical barriers between $\mathbb{S}$ and configurations which are *not* stratified.[5]

[5] This is an interesting methodological point even for the ordinary Pirogov – Sinai theory. We now suggest to consider the family of *all* translation invariant configurations as the "reference family" of configurations in the ordinary Pirogov –Sinai setting. Such an approach leads, in fact, to a sharper and clearer formulation of the Peierls condition – see below.

2. The framework when all local ground states of the model are assumed to be stratified (analogously: translation invariant in the ordinary Pirogov – Sinai theory) seems at first sight to be too narrow in the situations (like Ising antiferromagnet) where *periodical* (local) ground states occur. However, it is easily seen that periodical resp. horizontally periodical configurations can be converted to constant resp. stratified ones by taking *blockspin transformation* and so the setting we introduce here is sufficiently general.
3. The fact that we are selecting *several* configurations (in fact the whole family $\mathbb{S}$) as the "reference" ones – expecting that some of these configurations may (possibly, under suitable adjustment of the hamiltonian) give rise to corresponding Gibbs states – suggests that our interest lies in the situations where phase transitions *may occur*. Thus, the possible "degeneracy of the ground state"[6] is the situation of our interest. Though in most situations we will have to deal with only *one* Gibbs state corresponding to a given hamiltonian we want to have a theory dealing at the same time with the situations of phase coexistence. This requirement distinguishes the Pirogov – Sinai theory from the methods focused on the study of the *unicity* region. Recall that in the region of phase *unicity* other, well developed methods of study (based essentially on the Dobrushin' s unicity theorem and later investigations of the "complete analyticity" properties by [DSA]) are available.

On the other hand, in the regions where phase coexistence is expected no serious alternative to the Pirogov – Sinai theory exists reaching a comparable level of generality and universality of its applications.

2. Precontours and Admissible Systems of Precontours

Given a configuration $x \in \mathbb{X}$ say that a point $t \in \mathbb{Z}^\nu$ is *stratified* point of x, more precisely *y-stratified* of x (where $y \in \mathbb{S}$) if

$$x_{\tilde{t}} = y_{\tilde{t}}$$

holds for each $\tilde{t} \in \mathbb{Z}^\nu$ such that $|\tilde{t} - t| \le r$.

Note. This is an analogy of the notion of a correct point of the ordinary Pirogov – Sinai theory; r is the range of interactions.

Say that $\Lambda \subset \mathbb{Z}^\nu$ is a *standard volume* if any $t, t' \in \Lambda^c$ with the same last coordinate $t_\nu = t'_\nu$ can be connected by a "horizontal" (keeping the last coordinate intact) connected path in Λ^c , i.e. if all the sets $C_n = \{t \in \mathbb{Z}^\nu ; t_\nu = n\} \setminus \Lambda$ are connected. A configuration $x \in \mathbb{X}$ will be called *y-diluted*, for $y \in \mathbb{S}$, if there is some $\mathbb{E} \subset \mathbb{Z}^\nu$ with standard finite connected components such that all points of $\mathbb{E}^c$ are y-stratified.

This value y will be called the *external colour* of x. We will use the notation $y = x^{\text{ext}}$.

[6] By a *degeneracy* of a (local) ground state one usually means the fact that *several* (local) ground states exist for a given hamiltonian. Here, we are looking for (local) ground states among the elements of $\mathbb{S}$ i.e. for the configurations $y \in \mathbb{S}$ such that $\sum_A (\Phi_A(x_A) - \Phi_A(y_A)) > 0$ whenever x differs from y on a finite set whose vertical size is "not too big").

Precontours. For any diluted x denote by $B(x)$ the collection of all its nonstratified (i.e. stratified for *no* $y \in \mathbb{S}$) points. If C is a connected component of $B(x)$ then the pair

$$\Gamma = (C, x_C)$$

will be called the *precontour* of x and we will write

$$C = \operatorname{supp} \Gamma \, .$$

The prefix pre- suggests that the notion of a precontour is a provisional one. It will be replaced below by a more elaborate notion of a contour, with more convenient properties:

Admissible systems of precontours. By an *admissible system of precontours* we will mean any system $\mathcal{D} = \{\Gamma_i\}$ of precontours which is a collection of *all* precontours of *some* diluted configuration x. The configuration x is uniquely determined by $\mathcal{D}$ only in all horizontal levels intersecting $\mathcal{D}$, otherwise it will be given by the context (typically by the boundary conditions outside of the given finite volume, in which we will be normally working) and it will be denoted by $x_{\mathcal{D}}$.

Of course, this is a slight abuse of notations but we will see later that the quantities defined below as functions of $x_{\mathcal{D}}$ will not depend, in fact, on this ambiguity in the choice of $x_{\mathcal{D}}$.

We will also use the notion of an admissible system of precontours $\mathcal{D}$ in a given (standard or nonstandard) *finite* volume Λ. In such a case, we will assume that a configuration $x = x^{\Lambda}_{\mathcal{D}}$ can be defined in the whole lattice $\mathbb{Z}^{\nu}$ having the following properties : i) $\mathcal{D}$ is the collection of all precontours of $x = x^{\Lambda}_{\mathcal{D}}$ and ii) all the points of Λ^c are stratified points of the configuration $x = x^{\Lambda}_{\mathcal{D}}$.

3. Contours

In the ordinary Pirogov – Sinai theory, where elements of the "reference family" $\mathbb{S}$ are constant configurations, one immediately realizes that precontours are "crusted" in the sense that the (only!) infinite component of $(\operatorname{supp}\Gamma)^c$ – called the exterior of $\mathbf{\Gamma}$ – satisfies the property that all its points are "y-correct" where y is the external colour of Γ. Even more importantly, the interior of $\mathbf{\Gamma}$ (the union of the remaining components) is "disconnected" from the exterior of $\mathbf{\Gamma}$. This enables to construct "telescopic equations" relating the diluted partition function in a given volume Λ to the "crystallic" (see e.g. [S]) partition functions of the external contours appearing in Λ, and therefore again to the diluted partition functions of the *interiors* of these contours.

This is *not* so here, where precontours can have the shape of Dobrushin's "walls" (see below) separating various types of " ceilings " (i.e. various horizontally invariant configurations in our case) and the possible sense of the very notion of an exterior resp. interior of a precontour surely deserves a clarification.

With such "noncrusted" objects, one has to be more careful when defining suitable *hierarchy* between them; the usual concepts from the ordinary Pirogov – Sinai theory like the concept of an "external contour" or the notion of a contour "inside of the other one" must be defined more cautiously and precontours are *not* the suitable objects to do that. Imagine various precontours like "fingers" or "bumerangs" intersecting "interiors" of other precontours having the shape of Dobrushin's "walls" – for example the case of two walls each having an "appendix, a finger" touching the "interior" of the other wall – to be assured that the notion of an external or internal contour requires a careful definition here.

We should warn the reader that there will be *no* telescopic equations in our approach. However, the quantity $A(\mathbf{\Gamma})$ constructed in section III.3 substitutes these telescopic equations, in some sense.

The definition of a wall suggested by Dobrushin (precontours "of the interface" are called "walls" in [D], and correspondingly also in [HKZ] and [HZ]) resolves these problems in the special situation of one interface, by considering the projection of the situation appearing at the interface to the sublattice $\mathbb{Z}^{\nu-1}$. However, this construction can be hardly transferred to the situations where two or more parallel ceilings appear.

Thus, the proper definition of a contour in our situation (where general stratified phases appear and where possibly *many* rigid interfaces appear in the considered phases) cannot follow literally the concept of the above mentioned "wall". We will choose another aproach, which does *not* use an auxiliary transcription of the situation to the dimension $\nu - 1$:

The crucial notion in our approach will be that of the "exterior" (in $\mathbb{Z}^\nu$!) of an admissible system of precontours ; alternatively the complementary notion of an interior i.e. the volume "swallowed" by the given admissible system of precontours: (Notice that the definition below still follows essentially the original Dobrushin's approach.)

Definition. Let $\mathbf{\Gamma} = \{\Gamma_i\}$ be an admissible system of precontours in $\mathbb{Z}^\nu$ (or in a given standard volume Λ). We denote by $\operatorname{ext}\mathbf{\Gamma}$ the collection of all points of $(\operatorname{supp}\mathbf{\Gamma})^c$ which can be accessed from infinity by some vertically constant (i.e. keeping the last coordinate intact), correct (in the sense that each point of the path is a stratified one) connected path. We denote by

$$V(\mathbf{\Gamma}) = (\operatorname{ext}\mathbf{\Gamma})^c\,. \tag{2.5}$$

Note. This is just the intersection of all standard volumes containing $\operatorname{supp}\mathbf{\Gamma}$. In the rest of the paper – the exception will be the proof of our Main Theorem (in fact, this exception will be really relevant only for some explanatory notes interpreting in some detail the structure of the phases constructed by Main Theorem) we could work mainly with *finite standard volumes* Λ and always with a boundary condition $y \in \mathbb{S}$ given on the boundary of the volume Λ^c ; i.e. we will work with diluted configurations having a finite number of precontours only.

Now we come to the definition of contours:

Definition. Say that an admissible subcollection $\mathbf{\Gamma}' \subset \mathbf{\Gamma}$ is *removable* from the admissible collection $\mathbf{\Gamma}$ (we will also alternatively say that $\mathbf{\Gamma}'$ is *interior* in $\mathbf{\Gamma}$– though this characterization has not such a selfexplanatory meaning as in the ordinary, translation invariant, Pirogov – Sinai theory) if

$$V(\mathbf{\Gamma}') \cap \operatorname{supp}(\mathbf{\Gamma} \setminus \mathbf{\Gamma}') = \emptyset. \tag{2.6}$$

and moreover if $\mathbf{\Gamma} \setminus \mathbf{\Gamma}'$ is again an admissible system.

Note. Imagine that the subsystem $\mathbf{\Gamma}'$ was "replaced by its external colour" (induced by $\mathbf{\Gamma}'$ inside $V(\mathbf{\Gamma}')$). Removability of $\mathbf{\Gamma}'$ means just the *possibility* of such a replacement .

Definition. An admissible collection $\mathbf{\Gamma}$ of precontours in a given volume Λ, with $V(\mathbf{\Gamma}) \subset \Lambda$ and with *no* removable subfamilies will be called a *contour in* Λ.

Note. Thus, contours are some "minimal " – in the sense that no subsystem can be removed from them – collections of precontours which exist as admissible systems in the given volume Λ. If all the horizontal sections of Λ are even simply connected (i.e. have exactly one component) then any contour in Λ is also a contour in the whole lattice $\mathbb{Z}^\nu$. Otherwise, the property "being a contour" depends on the volume Λ and the boundary condition, too . For example, imagine a precontour having a shape of a "bumerang" with different colours on its corners which are assumed to have the same vertical level. This can be a contour in a suitable bigger "bumerang" Λ – which will be typically the interior of some other, "exterior" contour. However, it is not the contour in the whole lattice $\mathbb{Z}^\nu$.

4. Representation theorem

Definition. We introduce the structure of an *oriented graph* on the collection of all contours more precisely the following "hierarchy" between the contours: Write $\mathbf{\Gamma} \to \mathbf{\Gamma}'$ whenever $\mathbf{\Gamma} \cup \mathbf{\Gamma}'$ is admissible, $V(\mathbf{\Gamma}') \cap \operatorname{supp} \mathbf{\Gamma} \neq \emptyset$ however $\mathbf{\Gamma}$ is removable from $\mathbf{\Gamma} \cup \mathbf{\Gamma}'$.

Note. The notation $\mathbf{\Gamma} \to \mathbf{\Gamma}'$ will be frequently used, in an analogous sense, also for general *admissible systems* (not only for contours) $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$.

To say that $\mathbf{\Gamma}$ is a contour of a *given configuration* will not be so straightforward as for the precontours (and as in the ordinary translation invariant situations of the Pirogov – Sinai theory). However, the following representation theorem is still valid:

Theorem 1. Any diluted configuration $x \in \mathbb{X}$ having a finite $B(x)$ is *uniquely* represented by a graph on some finite subcollection of the collection of all contours of the model; such a graph is always a "forest" of trees. By a tree we mean an oriented connected graph *without loops* i.e. without cycles of bonds (cycles of "arrows") of the type $\mathbf{\Gamma}_1 \to \mathbf{\Gamma}_2 \to \ldots \mathbf{\Gamma}_n \to \mathbf{\Gamma}_1$. The external contours of the forest (namely those contours which are not the starting points of some arrows of the graph) have mutually disjoint volumes, and more generally any subsystem of the forest constructed by the rule "if the end of the arrow is removed then the beginning of the arrow is removed, too" is an admissible system in $\mathbb{Z}^\nu$. The mapping $\{$ configuration $\to$ forest of trees $\}$ is one to one; in particular any forest corresponds to some configuration of the original model. An analogous statement is also true for any diluted configuration in any finite standard volume.[7]

Notes. 0) The statement that *any* forest corresponds to some configuration of the original model will not be used in the following. What will be used only is the property that the considered collection of forests is *horizontally translation invariant*.

[7] Once again: any x is uniquely determined by the family of its contours. This family is a "forest of trees" in the relation $\to$. Conversely, any such "forest of trees of contours" is an admissible system i.e. it determines (uniquely) some configuration.

1) The generalization of the above result to general diluted configurations in infinite volumes with infinite $B(x)$ will not be considered here. When formulating the properties of typical configurations of infinite volume Gibbs states (constructed as the consequence of our Main Theorem at the very end of the paper), this could be done by saying that any such configuration can be interpreted as an *infinite* forest of *finite* trees (connected graphs of contours without loops). However, one really needs such formulations only for some explanatory notes commenting in more detail the structure of infinite volume Gibbs states constructed by our Main Theorem. Otherwise, there will be no need for the consideration of configurations in infinite volumes in the rest of the paper.
2) To reconstruct the configuration x *uniquely* from the forest, we need to know also the *external* colour x^{ext}. Namely, the external colour of the forest can be only partially recovered from the contours of the forest. However, its value elsewhere will be usually given by the context (by the boundary conditions outside of the finite volume, in which we will be actually working).
Definition.
Removable contours of the forest will be called the *internal contours* of x. A contour which can appear as a single remaining contour after some succession of removals will be called the *external contour* of x. By a *removal* of $\mathbf{\Gamma}'$ from $\mathbf{\Gamma}$ we mean the replacement of $x_{\mathbf{\Gamma}}$ by $x_{\mathbf{\Gamma}\setminus\mathbf{\Gamma}'}$.

Proof of Theorem 1. It is based on the following

Lemma 1. If $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ are two different internal contours of an admissible system $\mathbb{D}$ then

$$V(\mathbf{\Gamma}) \cap V(\mathbf{\Gamma}') = \emptyset \,. \tag{2.7}$$

Note. By an internal contour of $\mathbb{D}$ we mean here (we have not yet proven the theorem!) any *minimal possible* removable subsystem of $\mathbb{D}$.

Proof of Lemma 1. Denote by

$$C = V(\mathbf{\Gamma}) \cap V(\mathbf{\Gamma}') \,.$$

We will show that the assumption $C \neq \emptyset$ would lead to the removability of the system of all precontours of $\mathbf{\Gamma}$ contained in C and this would be in contradiction with the fact that $\mathbf{\Gamma}$ is contour. Denote by y resp. y' the external colours $\mathbf{\Gamma}$ resp. $\mathbf{\Gamma}'$: these configurations are defined locally for any nonempty slice

$$C^n_{\mathbf{\Gamma}} = V(\mathbf{\Gamma}) \cap \mathbb{Z}^\nu_n$$

resp. $C^n_{\mathbf{\Gamma}'}$; here we denote by $\mathbb{Z}^\nu_n = \{t \in \mathbb{Z}^\nu; t_\nu = n\}$. For any $n \in \mathbb{N}$ such that

$$C^n = C^n_{\mathbf{\Gamma}} \cap C^n_{\mathbf{\Gamma}'} \neq \emptyset$$

define y^* as the external colour of C^n. Notice that all the points of ∂C^c are stratified (because they belong either to $V(\mathbf{\Gamma})^c$ or $V(\mathbf{\Gamma}')^c$) and thus we have either $y^* = y$ or $y^* = y'$ for any level n such that $C^n \neq \emptyset$. Notice also that all points of

∂C^c can be accessed "from the infinity" by a connected horizontal path belonging to C^c. (We will not prove here this obvious fact, saying that the intersection of two simply connected sets is again simply connected.) Now there are two possible situations:

(1) If there is some $n \in \mathbb{Z}$ such that

$$y_n^* = y_n' \neq y_n$$

then we have the obvious relation (look at the level n!)

$$\operatorname{supp} \mathbf{\Gamma} \setminus C \neq \emptyset$$

and therefore the admissible system $\mathbf{\Gamma}^*$ of all precontours of $\mathbb{D}$ contained in C must *not* contain all precontours of $\mathbf{\Gamma}$. However $\mathbf{\Gamma}^*$ is removable (just replace $\mathbf{\Gamma}^*$ by its external colour y^* in C!) which is a contradiction with the fact that $\mathbf{\Gamma}$ is a contour.

(2) If $y_n = y_n'$ for all n (where both colours are uniquely defined) then it is easy to see also that $y^* = y = y'$ on ∂C^c. Then we follow an analogous argument as in (1): The condition $C \neq \emptyset$ would mean that the collection $\mathbf{\Gamma}^*$ is also a contour – and this is not a contradiction (with the fact that both $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ are contours i.e. with no removable subsystem $\mathbf{\Gamma}^*$) only if $C = V(\mathbf{\Gamma}) = V(\mathbf{\Gamma}')$. However, then e.g. $\mathbf{\Gamma} \setminus \mathbf{\Gamma}^*$ must be empty because otherwise the intersection of $\mathbf{\Gamma} \setminus \mathbf{\Gamma}^*$ and $V(\mathbf{\Gamma}')$ would be nonempty and $\mathbf{\Gamma}'$ would not be an interior contour of the system.

Proof of Theorem 1. We proved in Lemma 1 that interior contours of x are uniquely defined, with mutually disjoint volumes $V(\mathbf{\Gamma})$. We can remove all these internal contours (in an arbitrary order!) thus obtaining some new configuration $\bar{x}$. Then we determine all the internal contours of $\bar{x}$; after removing them from $\bar{x}$ we obtain another configuration $\bar{\bar{x}}$ etc. At the final step some collection of contours which are both internal and external remains; their removal has as its result the stratified configuration x_{ext}. The bonds of the forest representing x are all the pairs of the type $\mathbf{\Gamma} \to \mathbf{\Gamma}'$ where $\mathbf{\Gamma}$ is an internal contour of some intermediate configuration $(x, \bar{x}, \bar{\bar{x}}, \dots)$ and $\mathbf{\Gamma}'$ appears as an internal contour after successive removal of all $\mathbf{\Gamma}$ such that $\operatorname{supp} \mathbf{\Gamma} \cap V(\mathbf{\Gamma}') \neq \emptyset$.

Notes.
1. Of course one can easily construct examples of "loops of contours" $\mathbf{\Gamma}_1, \dots, \mathbf{\Gamma}_{n+1} = \mathbf{\Gamma}_1$ such that $\mathbf{\Gamma}_i \to \mathbf{\Gamma}_{i+1}$ for each $i = 1, \dots, n$ but *no* such cycles can appear in the representation by Theorem 1. They form a *single contour* there!
2. Analogously, one can formulate an analogous representation theorem for configurations in finite standard volumes.

5. Connectivity of contours

Theorem 2. Contours are "halfconnected" in the sense that

$$|\operatorname{supp} \mathbf{\Gamma}| \geq \frac{64}{81} \operatorname{con} \mathbf{\Gamma} \tag{2.8}$$

where $\operatorname{con} \mathbf{\Gamma}$ denotes the minimal possible cardinality of a connected set containing $\operatorname{supp} \mathbf{\Gamma}$.(Compare also Theorem 2' in Section II.6.)

Note. The notion of $\operatorname{con} \mathbf{\Gamma}$ will be *modified* in later sections, its new value (denoted by $\operatorname{conn} \mathbf{\Gamma}$) being actually smaller than $\operatorname{con} \mathbf{\Gamma}$ (however without having a simplifying effect on the proof below.)

Proof. We have to understand the structure of contours of our model – which are defined as some complicated "conglomerates" of (connected!) precontours. We present here only an outline of the proof leaving out the "topological" details.
Definition. Let $\{\Gamma_i\}$ be an (unadmissible) family of precontours. Say that the two points $t, s \in \partial(\cup_i \operatorname{supp}\Gamma_i)^c$ with the same height $t_\nu = s_\nu$ are in a conflict in $\{\Gamma_i\}$ if their colours (induced by neighboring precontours) are different and at the same time t and s *can* be connected by some connected horizontal (keeping t_ν constant) path not intersecting $\cup_i \operatorname{supp}\Gamma_i$. Say that Γ *cures* the conflict between t, s if t, s are no more in conflict in $\{\Gamma_i\}$ & $\{\Gamma\}$.

Lemma. Let $\Gamma_1, \Gamma_2, \dots \Gamma_n$ be a sequence of precontours such that each Γ_k cures some conflict in $\{\Gamma_1, \Gamma_2, \dots \Gamma_{k-1}\}$. Then $\{\Gamma_1, \Gamma_2, \dots \Gamma_n\}$ is $\frac{8}{9}$connected.

Proof of Lemma. We will construct for each $\{\Gamma_1, \Gamma_2, \dots \Gamma_k\}$ a connected set $M_k \supset \cup_{i=1}^{k} \operatorname{supp}\Gamma_i$ by the following inductive procedure: Connect each Γ_k to the (yet constructed) "connected conglomerate" $M_{k-1} \supset \cup_{i=1}^{k-1} \operatorname{supp}\Gamma_i$ in the following way: there must be some horizontal section

$$C_k^n = \operatorname{supp}\Gamma_k \cap \mathbb{Z}_n^\nu$$

such that in some internal component of $(C_k^n)^c$ (taken in the lattice $\mathbb{Z}_n^\nu$), some points of $\cup_{i=1}^{k-1}\Gamma_i$ can be found. (Otherwise Γ_k would cure no conflict.) Now take M_k as the union of M_{k-1} and $\operatorname{supp}\Gamma_k$ and of some shortest path connecting C_k^n and M_{k-1}. Clearly, the lenght of such a shortest path is no more than $\frac{1}{8}$ of the cardinality of C^n.
Definition. Say that an admissible collection $\{\Gamma_1, \Gamma_2, \dots \Gamma_n\}$ cures Γ_1 if each Γ_k, $k \le n$ cures some conflict $\{\Gamma_1, \Gamma_2, \dots \Gamma_{k-1}\}$.
(Recall that an admissible family of precontours in Λ is a family of precontours whose "outside colours" are not in mutual conflict.)

Proof of Theorem 2. Take some precontour $\Gamma_1 \in \mathbf{\Gamma}$ and cure it successively to obtain some $\frac{8}{9}$ connected subcontour $\tilde{\mathbf{\Gamma}}_1$ of $\mathbf{\Gamma}$. Denote by $\mathbf{\Gamma}_1$ some maximal $\frac{8}{9}$ connected supersystem of $\tilde{\mathbf{\Gamma}}_1$ in $\mathbf{\Gamma}$. Asssume that $\mathbf{\Gamma}^1 = \mathbf{\Gamma} \setminus \mathbf{\Gamma}_1$ is nonempty. (Otherwise, the proof of the theorem would be complete.) Take some $\Gamma_2 \in \mathbf{\Gamma}^1$ and cure its conflicts analogously as $\mathbf{\Gamma}_1$ was constructed, and so on.(Notice that the connecting paths starting from the precontours curing some conflicts in Γ_2 etc. do not touch $\mathbf{\Gamma}_1$ yet!)

Thus we obtain some decomposition

$$\mathbf{\Gamma} = \bigcup_{i=1}^{N} \mathbf{\Gamma}_i \tag{2.9}$$

where $\mathbf{\Gamma}_i$ are $\frac{8}{9}$ connected contours such that $\cup_{i=k+1}^{N}\mathbf{\Gamma}_i$ is nonremovable from $\cup_{i=k}^{N}\mathbf{\Gamma}_i$. Take the admissible collection $\cup_{i=1}^{N-1}\mathbf{\Gamma}_i$ i.e. $\mathbf{\Gamma} \setminus \mathbf{\Gamma}_N$ and denote by $\mathbb{D}_1, \dots, \mathbb{D}_m$ the internal contours of this truncated system. Notice that *all* $V(\mathbb{D}_i)$ must be intersected by $\mathbf{\Gamma}_N$ (those nonintersected by $\mathbf{\Gamma}_N$ would be *removable* from $\mathbf{\Gamma}$!) and therefore $\mathbf{\Gamma}_N$ looks like a "bumerang" such that $\mathcal{D}_i$ are like some "rings" entwining it.

Clearly, we can connect $\mathbf{\Gamma}_N$ to any $\mathbb{D}_i$ "at the expense of $\mathbb{D}_i$" by some shortest path from $\mathbf{\Gamma}_N \cap V(\mathbb{D}_i)$ having a lenght at most $\frac{1}{8}$ of the cardinality of $\mathbb{D}_i$. For any $\mathbb{D}_i$ only *one* path will be constructed. This procedure of "making connections above" can be replaced also for the bigger "bumerang" $\mathbf{\Gamma}_N \cup \cup_i \mathbb{D}_i$ "at the expense of the internal contours" of the system $\mathbf{\Gamma} \setminus (\mathbf{\Gamma}_N \cup \cup_i \mathbb{D}_i)$ etc. Assuming yet that we have submerged $\operatorname{supp} \mathbf{\Gamma}_i$ into some connected set C_i^* such that

$$\operatorname{card} C_i^* \le (1 + \frac{1}{8}) \operatorname{card} \operatorname{supp} \mathbf{\Gamma}_i \tag{2.10}$$

we can submerge $\operatorname{supp} \mathbf{\Gamma}$ into some connected set C^{**} such that $C^{**} = C^* \cup P$, where P is the support of connected paths constructed above and

$$\operatorname{card} C^{**} \le (1 + \frac{1}{8}) \operatorname{card} C^* \le \frac{81}{64} \operatorname{card} \operatorname{supp} \mathbf{\Gamma}.$$

6. Supercontours

The contours defined so far would still have some inconvenient features later. Remember that they are "not crusted" – in the sense that the internal contours $\mathbf{\Gamma}$ intersecting some other $V(\mathbf{\Gamma}')$ need not to satisfy the condition $V(\mathbf{\Gamma}) \subset V(\mathbf{\Gamma}')$.[8]

This is a fundamental obstacle – which apparently can *not* be remedied by some "better" definition of a contour. However, even the fact that the external contour $\mathbf{\Gamma}'$ can be much "smaller " than $\mathbf{\Gamma}$ would be rather inconvenient in our following considerations. (This will be so for pure technical reasons ; see part III, Theorem 5 and also Theorem 7.) The latter inconveniency can be, however, remedied; one can redefine contours such that $\mathbf{\Gamma}'$ is always "substantially bigger" than $\mathbf{\Gamma}$ if $\mathbf{\Gamma} \to \mathbf{\Gamma}'$: Below, we will "glue together some contours" to achieve this property, keeping still the validity of Theorem 2 (with a slightly smaller constant) for the newly defined conglomerates of contours. These conglomerates will be called supercontours and below (in Part III) we will work exclusively with them (instead of working with contours).

Definition. For any system of contours $\mathcal{D}$ introduce its "diameter"

$$\operatorname{diam} \mathcal{D} = \max_{\mathbf{\Gamma} \in \mathcal{D}} \operatorname{diam} \mathbf{\Gamma}. \tag{2.11}$$

Consider some (total) ordering $\prec$ on the set of all systems of contours satisfying the following requirements : 1) if $\operatorname{diam} \mathcal{D} < \operatorname{diam} \mathcal{D}'$ then $\mathcal{D} \prec \mathcal{D}'$; 2) if $\mathcal{D} \prec \mathcal{D}'$ then $\operatorname{diam} \mathcal{D} \le \operatorname{diam} \mathcal{D}'$; 3) if $\mathcal{D}$ is a subsystem of $\mathcal{D}'$ then $\mathcal{D} \prec \mathcal{D}'$; 4) if $\mathcal{D} = \mathcal{D}' + t$ where $t \prec 0$ in the lexicographic order on $\mathbb{Z}^\nu$ then $\mathcal{D} \prec \mathcal{D}'$.

Notes. The particular choice of a norm on $\mathbb{Z}^\nu$ is not relevant here. However, in view of the further usage of the notion of a diameter in part III let us make the agreement that from now on the l_∞ norm $|t| = \max\{t_i\}$ will be used everywhere in what follows.

The lexicographic order on $\mathbb{Z}^\nu$ is assumed to be *fixed throughout the paper*. The requirements 1), 2), 3) and 4) clearly define a *partial* order and we simply extend it to some total ordering of the family of all systems of contours.

[8] Such a property was valid, it seems to us, in all the previous applications of the Pirogov – Sinai theory.

Definition. Say that a contour of the "forest" of Theorem 1 has an index $n > 0$ if it is an internal contour of the forest remaining after i) the removal of all the *internal* contours of the forest (these contours will be said to have the index 1) and then after ii) the successive removal of all the contours having the index $2, 3, ..., n-1$.

For any contour $\mathbf{\Gamma}'$ having the index 2 find the *biggest* (in $\prec$) contour $\mathbf{\Gamma}$ such that $\mathbf{\Gamma} \to \mathbf{\Gamma}'$. If $\mathbf{\Gamma}$ is bigger than $\mathbf{\Gamma}'$ or the cardinality of $\operatorname{supp} \mathbf{\Gamma}$ is bigger than the cardinality of $\operatorname{supp} \mathbf{\Gamma}'$ connect the contour $\mathbf{\Gamma}$, by a shortest connected path, to $\mathbf{\Gamma}'$. Moreover, if the cardinality of the support of the conglomerate thus formed by $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ (and their connecting path) is still not at least twice bigger than the cardinality of $\operatorname{supp} \mathbf{\Gamma}'$ for any other interior $\mathbf{\Gamma}'' \to \mathbf{\Gamma}$ repeat the procedure once again with $\mathbf{\Gamma}''$ instead of $\mathbf{\Gamma}$.

Repeat the same process with the collections of all remaining interior contours (kept intact after the glueing procedures above) $\mathbf{\Gamma} \to \mathbf{\Gamma}'$ which are bigger (in the sense above) than some contour $\mathbf{\Gamma}'$ having the index 3 . Then repeat the same glueing procedure with contours having the index 4 etc.

Finally remove all the remaining interior contours (not used in the glueing procedures above for $n = 2, 3, \dots$) and repeate the whole above contruction again and again.

Supercontours of the original forest are then defined as the connected (by paths constructed above) conglomerates of contours of the original forest. The notion of a supercontour includes also all the contours left intact by the above procedure.

We will use the short name supercontour for a supercontour of *some* forest. Notice that the relation $\to$ between supercontours is defined in a natural way because glueing cancels the arrow $\mathbf{\Gamma} \to \mathbf{\Gamma}'$ but does *not* effect the remaining arrows of the forest.

Theorem 2'. The new forest obtained by the construction above is uniquely determined by the old forest. Any forest of supercontours whose external supercontours form an admissible system can appear as the result of the construction above. The relation $\mathbf{\Gamma} \to \mathbf{\Gamma}'$ is a *subset* of the relation $\mathbf{\Gamma} \prec \mathbf{\Gamma}'$ for all pairs of supercontours of the given model. Moreover, the cardinality of $\operatorname{supp} \mathbf{\Gamma}'$ is always *at least twice bigger* than the cardinality of $\operatorname{supp} \mathbf{\Gamma}$. Supercontours satisfy again the statement of Theorem 2, with the constant 64/81 replaced by some smaller constant, e.g.

$$|\operatorname{supp} \mathbf{\Gamma}| \geq 1/2 \ \operatorname{con} \mathbf{\Gamma}. \tag{2.8'}$$

Note. These properties will have no special importance in the rest of part II – but they will be quite convenient later in Theorem 5 and also in Theorem 7.

The *proof* is quite straightforward. To see that Theorem 2 remains valid notice that the connecting path constructed in any step of the construction of a supercontour is done "at the expense" of the external contour $\mathbf{\Gamma}'$; notice that any contour $\mathbf{\Gamma}'$ of the original forest is used at most once in such a construction, and to construct a connected path "inside" $\mathbf{\Gamma}'$, one does not need more than $1/2 \operatorname{con} \mathbf{\Gamma}'$ points (and $\frac{81 \cdot 2}{64 \cdot 3} < \frac{1}{2}$).

In the rest of the paper, we will work exclusively with *supercontours* (of a given admissible system of precontours). We will *omit* the prefix "super" in the following. (However, this will be important only later, in part III.)

7. Expression of the hamiltonian

For any stratified configuration $y \in \mathbb{S}$ define its "density of energy" at $t \in \mathbb{Z}^\nu$:

$$e_t(y) = \sum_{A \ni t} \Phi_A(y_A)|A|^{-1}. \tag{2.12}$$

Given a contour $\mathbf{\Gamma}$ one would like to define also a quantity having the meaning of the "contour energy". One could think for example about the "energy excess" of $H(x_{\mathbf{\Gamma}})$, where $x_{\mathbf{\Gamma}}$ denotes the configuration having $\mathbf{\Gamma}$ as its *only* contour, with respect to "something like $H(x_{\text{ext}})$" where x_{ext} denotes the "external colour of $\mathbf{\Gamma}$". However, such a straightforward approach to the definition of an energy of a contour is reasonable only in the very special cases – when the density of energy inside $\mathbf{\Gamma}$ is the *same* as outside of $\mathbf{\Gamma}$, at *any* horizontal level.

Otherwise, it will be necessary (to keep the interpretation of a contour energy as a quantity which is "localized on $\operatorname{supp}\mathbf{\Gamma}$ ") to replace the quantity $H(x_{\mathbf{\Gamma}}) - H(x_{\text{ext}})$ (which will be, of course, also very important later – see (2.19)) by the following, perhaps too formally defined at first sight, quantity: first extend the notation $e_t(x)$ for any (even *nonstratified* in t) x by putting

$$e_t(x) = e_t(\hat{x})\,; \quad t = (t_1, t_2, \ldots, t_\nu) \tag{2.13}$$

where $\hat{x}$ is the stratified continuation of the vertical section $\{x_{(t_1,\ldots,t_{\nu-1},(\cdot))}\}$. Now define the configuration $x = x_{\mathbf{\Gamma}}^{\text{best}}$ *minimizing* the sum (notice that the terms of this infinite sum are fixed outside $\operatorname{supp}\mathbf{\Gamma}$!)

$$\sum_{t \in \mathbb{Z}^\nu} e_t(x) \tag{2.14}$$

under the condition that $x = x_{\mathbf{\Gamma}}$ on $(\operatorname{supp}\mathbf{\Gamma})^c$ and also on the set

$$\partial \operatorname{supp}\mathbf{\Gamma} = \{t, \operatorname{dist}(t, (\operatorname{supp}\mathbf{\Gamma})^c) \leq r\}.$$

Put[9]

$$E(\mathbf{\Gamma}) = H(x_{\mathbf{\Gamma}}) - \sum_{t \in \mathbb{Z}^\nu} e_t(x_{\mathbf{\Gamma}}^{\text{best}}) \quad \text{where} \quad H(x) = \sum_{A \subset \mathbb{Z}^\nu} \Phi_A(x_A). \tag{2.15}$$

Then we have the following expression of the hamiltonian (Λ will be always a finite set in the sequel) :

Theorem 3. *Let x be a diluted configuration in Λ. Then*

$$H(x_\Lambda | x_{\Lambda^c}) = \sum_{t \in \Lambda} e_t(x_{\mathcal{D}}^{\text{best}}) + \sum_{\mathbf{\Gamma} \in \mathcal{D}} E(\mathbf{\Gamma}) \tag{2.16}$$

where $\mathcal{D}$ denotes the system of all contours of x and $x_{\mathcal{D}}^{\text{best}}$ is defined as above (with $\mathcal{D}$ instead of $\mathbf{\Gamma}$), by (2.14).

[9] Of course this is again only a formal expression – but obviously the terms in the sums on the right hand side of the equation for $E(\mathbf{\Gamma})$ can be reorganized such that a sum with only a finite number of nonzero terms is obtained.

Proof. Immediate, if we notice that $E(\mathbf{\Gamma})$ is a local quantity (depending on $\mathbf{\Gamma}$ only) and also an additive one:

$$E(\mathbf{\Gamma} \cup \mathbf{\Gamma}') = E(\mathbf{\Gamma}) + E(\mathbf{\Gamma}').$$

Notice that whenever $\mathbf{\Gamma}$ is the unique contour of x then (2.16) follows directly from (2.15).

8. The Peierls condition. The abstract Pirogov – Sinai model

In the following we will assume that for any set G and any stratified configuration $y \in \mathbb{S}$ the following (Peierls type) inequality holds with a sufficiently large constant $\tau > 0$:

$$\sum_{\mathbf{\Gamma}:\ \operatorname{supp}\mathbf{\Gamma}=G\ \&\ x^{\mathrm{ext}}_{\mathbf{\Gamma}}=y} \exp(-E(\mathbf{\Gamma})) \le \exp(-\tau|G|) \tag{2.17}$$

We recall that we include the inverse temperature into the hamiltonian and therefore τ is of the order of the inverse temperature.

Notes. 1.In practice, one usually establishes (2.17) through the inequalities

$$E(\mathbf{\Gamma}) \ge \tau^*|G| \tag{2.17*}$$

with suitable $\tau^* \ge \tau$.

2. In the following we will work *exclusively with the expression* (2.16). It is generally advisable to develop the Pirogov – Sinai theory in an abstract setting (2.16) – with the Peierls condition (2.17) established. The reformulation of the original model to the language (2.16) can be considered as a suitable "preparation" of the given "physical model" – and the Pirogov – Sinai theory actually only *starts* with the setting (2.16) & (2.17).

3. Such a preparation of the model (i.e. a conversion to the form (2.16) by a suitable definition of the notion of a contour) is often not unique i.e. it is not "naturally determined" by the given model . One can adapt it in various ways for various concrete situations – by modifying the concept of a "stratified point", for example, or even by considering contours as objects different from those we constructed here. Remember, for example, that in the ordinary theory of the low temperature Ising model, contours are commonly defined as selfavoiding *paths* in the dual lattice.

9. Diluted and strictly diluted partition functions

In addition to the volume $V(\mathbf{\Gamma})$ there will be another (actually more important in the following) modification of this notion, denoted by $V_{\updownarrow}(\mathbf{\Gamma})$. The details of its definition will be important only later. See also (3.19) in part III; $V_{\updownarrow}(\mathbf{\Gamma})$ will be defined here as a suitable "cone over $V(\mathbf{\Gamma})$", more precisely as follows:[10]

[10] The "protecting zone" $V_{\updownarrow}(\mathbf{\Gamma})$ over $V(\mathbf{\Gamma})$ constructed below is apparently unnecessarily big. This will make no harm, however. In later sections of part III, we will be more strict when tackling analogous difficulties as in the definition (3.19). The logarithmic height of the "caps" of $V_{\updownarrow}(\mathbf{\Gamma})$ over $V(\mathbf{\Gamma})$ would be quite sufficient, in fact.

Conoidal sets. Say that a volume Λ is a conoidal volume (or conoid) if it contains, with each "horizontal set" $B \subset \mathbb{Z}^\nu_m \cap \Lambda$ (where $\mathbb{Z}^\nu_m$ is the collection of points of $\mathbb{Z}^\nu$ with the fixed last coordinate $t_\nu = m$) the whole "cone" $\{t \in \mathbb{Z}^\nu : \operatorname{dist}(t, B) \leq \frac{1}{2} \operatorname{dist}(t, \partial B)\}$ where ∂B denotes the boundary of B taken in $\mathbb{Z}^\nu_m$. If $\mathbf{\Gamma}$ is a contour or an admissible system we define $V_\updownarrow(\mathbf{\Gamma})$ as the union of $\operatorname{supp} \mathbf{\Gamma}$ and the smallest conoid containing $V(\mathbf{\Gamma}) \setminus \operatorname{supp} \mathbf{\Gamma}$. Equivalently, this is the union of $V(\mathbf{\Gamma})$ and of the smallest conoid containing all the "upper and lower ceilings of $\mathbf{\Gamma}$" i.e. the flat parts of the boundary of $V(\mathbf{\Gamma})$ which are outside of $\operatorname{supp} \mathbf{\Gamma}$.

We denote, from now on, by symbols $Z^y(\Lambda)$ resp. $Z^y_\updownarrow(\Lambda)$ the partition functions

$$Z^y(\Lambda) = \sum \exp(-H(x_\Lambda | y_{\Lambda^c})) \tag{2.18}$$

where the sum is over all diluted configurations whose all contours satisfy the condition

$$\operatorname{dist}(V(\mathbf{\Gamma}), \Lambda^c) \geq 2 \tag{2.18'}$$

resp. analogously for $Z^y_\updownarrow(\Lambda)$,

$$\operatorname{dist}(V_\updownarrow(\mathbf{\Gamma})), \Lambda^c) \geq 2\,. \tag{2.18''}$$

Of course, for conoidal Λ the both partition functions (2.18') and (2.18") are the *same*.

The rest of this paper (and the essence of the Pirogov – Sinai theory in its presented version) consists of the effort to *expand* the considered diluted partition functions (3.18) (more precisely (3.18")) as far as it is possible or reasonable – in order to deduce some useful corollaries from these expansions.

The attempt to expand partition functions (2.18) can be based on the older idea of a contour model [PS] (or of a metastable contour model [Z]). Though this notion in fact *disappeared* from the presented version of Pirogov – Sinai theory – instead of speaking about suitable "metastable contour models" we will work, in fact, only with *expansions* of their partition functions (i.e. the partition functions of the metastable submodels of the given model) – it is perhaps useful to start with some intuitive arguments suggesting the introduction of the basic notion of a *contour functional*. This is just the introduction to the later, more technical constructions. We will see later that the very notion of a contour functional ([S]) "survived" in our approach (in contrary to the idea of a contour model) and it is still of a central importance!

10. The idea of a contour functional

The basic task of the Pirogov – Sinai theory is to determine those configurations y among the "reference" ones (reference means stratified in our case) which are stable in the sense that they give rise to Gibbs states whose almost all configurations are some "local" perturbations of the considered reference (stratified) configuration. More precisely, they are "y-diluted". A useful and intuitively appealing tool to determine whether a given configuration is "stable" is the construction of a "metastable model" [Z] (around the given reference configuration). To define such a metastable model one introduces ([S], [Z]) an auxiliary quantity called the "contour functional" $F(\mathbf{\Gamma})$ (the "work needed to install the given contour") which "tests" those contours whose appearance as of external contours of the metastable model is allowed. To get an idea of such a testing quantity let us start with its "zero temperature version": Put

$$F_0(\mathbf{\Gamma}) = H(x_{\mathbf{\Gamma}}) - H(x_{\mathbf{\Gamma}}^{\text{ext}}) = E(\mathbf{\Gamma}) - A_0(\mathbf{\Gamma}) \tag{2.19}$$

where

$$A_0(\mathbf{\Gamma}) = \sum_{t \in V(\mathbf{\Gamma})} \left(e_t(x_{\mathbf{\Gamma}}^{\text{ext}}) - e_t(x_{\mathbf{\Gamma}}^{\text{best}})\right) \tag{2.20}$$

This quantity is just a first approximation to the more relevant quantity given at this moment only *formally* by

$$F_{\text{formal}}(\mathbf{\Gamma}) = \log Z^y(\mathbb{Z}^\nu) - \log Z^{\mathbf{\Gamma}}(\mathbb{Z}^\nu) \tag{2.21}$$

where y is the external colour of $\mathbf{\Gamma}(y = x_{\mathbf{\Gamma}}^{\text{ext}})$ and $Z^{\mathbf{\Gamma}}$ denotes the partition function "over all configurations on $\mathbb{Z}^\nu$ containing the contour $\mathbf{\Gamma}$". Below we will define, by relations (3.21) & (3.22), a *rigorous counterpart* of this quantity, which will play a very important role in the sequel.

For contours which are "not very big" the quantity F_0 is a good approximation to F_{formal}. It enlightens somehow the concept of a small contour used below; the term $A_0(\mathbf{\Gamma})$ typically satisfies an estimate like

$$A_0(\mathbf{\Gamma}) \leq C|V(\mathbf{\Gamma})| \tag{2.22}$$

with a constant C which is sufficiently smaller than τ (imagine the Ising model with a small external field) and therefore if e.g.

$$C|V(\mathbf{\Gamma})| \leq \frac{\tau}{2}|\operatorname{supp}\mathbf{\Gamma}|$$

(this will hold for contours which are "not too big") we have, from the Peierls condition (2.17*), the inequality

$$F(\mathbf{\Gamma}) \geq E(\mathbf{\Gamma}) - C|V(\mathbf{\Gamma})| \geq \frac{\tau}{2}|\operatorname{supp}\mathbf{\Gamma}|\,. \tag{2.23}$$

We see that $y = x_{\mathbf{\Gamma}}^{\text{ext}}$ is really a "local ground state" because installing of a "not too big" contour increases its energy.

Unfortunately, it is not trivial to define quantities like F_{formal} rigorously. While such a task is solved rather straightforwardly in other situations of the Pirogov – Sinai theory (where contours are "crusted" – in the sense that there is no dependence between events inside and outside $\mathbf{\Gamma}$), here the presence of "ceilings" (flat horizontal parts of boundaries of $V(\mathbf{\Gamma})$ which do *not* belong to $\operatorname{supp}\mathbf{\Gamma}$) causes problems! These problems lead to the necessity of considering of suitable *expansions*, and this is the main subject of the forthcoming part of the paper.

III. The Concept of a Mixed (Partially Expanded) Model. Recoloring.

This, the main and the final part of the paper is devoted to the construction of suitable *expansions* of partition functions of models considered in part II. Here we introduce the important technical notion of a "mixed" (or, partially expanded) model which serves as an *intermediate construction* between the original concept of an "abstract Pirogov – Sinai model" and our final aim which is an utmost expansion of the partition functions of the considered model.

Cluster expansions were always an important tool in the Pirogov – Sinai theory. However, in previous versions of this theory, the expansions were viewed merely as some auxiliary technique applied to the study of special polymer models (contour models) which were constructed first. One could think that the cluster expansion method could be replaced by "something else" giving "comparably nice" expressions (or, possibly, suitable bounds only) for the partition functions of the contour models.

This is not so here where the idea of a partial expansion enters even our basic terminology, namely the concept of the mixed model. For example an analogy of the notion of a "metastable model" (see [Z]) can *not* be (apparently) defined here without the language of expansions; even the formulation of our Main Theorem uses this language.

Of course the very idea of a "partial expansion" is not at all new. It was used (in various context also in situations close to the subject of the presented paper - see [I], [HKZ], [B], ...) by many authors but mostly as an important but auxiliary tool while in our formulation it is really the cornerstone of the theory.

Our basic expansion step (Theorem 5, Theorem 4 and the Lemma preceding it) – called recoloring by us *incorporates* some of the usual cluster expansion ideology (based on expansion by power series and on the use of equations e.g. of Kirkwood – Salsburg type) into the very construction of the contour functional.

A consequence of our approach is that our use of cluster expansion technique is *selfcontained* and we need no references to the literature. We can, however, mention [M], [KP], [DZ] (as the papers having direct influence on the present paper) from the numerous literature on the subject of cluster expansions.

The construction of one "recoloring step" (Theorem 4) will not yet give the required expansion of the model. It must be *repeated* (infinitely) many times. The iterative nature of our constructions cannot be hidden "somewhere into the proofs" but appears already at the level of the basic notions.

We organize this part of the paper as follows. In section 1 we analyze the notion of a "cluster" (of supports of contours or, more generally, of another clusters); the clusters are then identified with suitable graphs (without cycles) on $\mathbb{Z}^\nu$. Then, in section 2, we define the central concept of a *mixed* model. This notion corresponds to an idea of a "partially expanded model" ; however it is useful to consider such a concept in a broader sense.

Section 3 describes an important procedure – the "recoloring" (i.e. removing of $\mathbf{\Gamma}$ from the model & adjusting of the new cluster series such that the partition functions would not change) of a single interior contour $\mathbf{\Gamma}$; in the context of a general mixed model. The important concept of a recolorable contour (or admissible subsystem) is introduced here: it corresponds, roughly speaking, to the validity of the Peierls condition for the contour functional $F(\mathbf{\Gamma})$. Section 4 applies the result of section 3 in such a way that recoloring of all the shifts of $\mathbf{\Gamma}$ is obtained; the resulting new mixed model is again a translation invariant one if the original mixed model satisfied this property. Technically, sections 3 and 4 (and later section 7) form the core of our paper.

Later sections 5,6,7 are then devoted to the problems of the succesive construction of "more expanded" mixed models: An important intermediate result is Theorem 6 (section 6) giving a sufficient condition for the recolorability of an interior system of contours in a general mixed model. Namely, to have more specific examples of recolorable systems we introduce there a related but better controllable notion of a small resp. *extremally small* system of contours (which is more useful than mere notion of a recolorable system).

The message of the sections 5 to 7 is roughly speaking the following: once there are some small contours in the mixed model then there is still "something left to recolor" i.e there are still also some recolorable subsystems in the model.

The notion of an extremally small system is an elaboration of the older idea of a "small" or "stable" contour ([Z]). Notice that small resp. extremally small systems can contain the "large" ("not extremally small") contours or admissible systems as its *internal* subsystems.

Theorem 6 is proved with the help of Theorem 7; the latter is already some general statement about the "connectivity constant" of some special ("tight") sets appearing in the study of extremally small systems.

Only after finishing the sequence of all the expansions (recolorings) organized by us we will be able to say what the *metastable* model is – in Section 8. This will be the submodel of the original abstract Pirogov – Sinaimodel where only those configurations containing *no* "redundant" (i.e. surviving in the "fully expanded" model: by the fully expanded model we mean the final mixed model remaining at the moment when the inductive procedure of its partial expansions was completed) external contour resp. admissible system will be admitted!

Section 8 formulates then our main result, using the quantities called "metastable free energies" just constructed by expansions. It turns out that that the minimality of the metastable free energy of some $y \in \mathbb{S}$ really means that there are *no contours at all* in the fully expanded model under such a boundary condition i.e. the metastable model corresponding to y gives an appropriate $y-$ th Gibbs state.

The fact that under "stable" boundary conditions, "everything is recolorable" (i. e. the complete expansion of the partition functions is obtained) is the core of the proof of the main theorem. Having proved the preparatory Theorems 5,6, this is now almost a tautology.

Our new method based on Theorem 5 and Theorem 6 replaces the previous coarser arguments from [Z] (which moreover cannot be used in these new situations). However, even in the situation of [Z] our new method is simpler (at least conceptionally: inequalities for the partition functions employed in [Z] are now systematically replaced by corresponding expansions whenever possible) and more powerful. We plan to show the advantages of this new approach in the study of further situations which are not covered by the usual variants of the Pirogov – Sinai theory.

1. Clusters

This section prepares some technical notions and constructions needed for the proper formulation of the expansions which are used below.

Cluster expansions of partition functions of polymer models are often written, in the literature, in the following form:

$$\log Z^y(\Lambda) = \sum_{T\subset\Lambda} k_T \tag{3.0}$$

where $Z^y(\Lambda)$ is the considered (diluted) partition function in volume Λ under a boundary condition $y \in \mathbb{S}$ and $k_T = k_T^y, T \subset \mathbb{Z}^\nu$ are some local quantities (indexed by "connected clusters" T – see below for more details about this notion) which are "quickly decaying" e.g. like (this will be the form used below by us)

$$|k_T| \leq \varepsilon^{\operatorname{conn} T} \tag{3.1}$$

where $\operatorname{conn} T$ is something like the "cardinality of a minimal connected set containing the cluster T". (See below in (3.6) for the definition of a quantity $\operatorname{conn} T$ which will be used in our later considerations.)

This will be our *final goal*: establishing of such expansions for a collection – as large as possible – of diluted partition functions of the given model.

More complete information says that the quantities k_T are in fact sums of quantities indexed by some "clusters of sets (resp. of contours) $\{\mathbf{\Gamma}_i\}$" (and having a value which is a $\pm$ product, over the cluster, of contour functionals $\exp(-F(\mathbf{\Gamma}_i)))$ having the given support T. While one can ignore the detailed description of the structure of k_T when applying the above expressions (e.g. in order to obtain useful *bounds* for partition functions – which was the typical application of the cluster expansions in most previous variants of the P. S. theory) here it will be necessary to retain the more precise information because these expansions *will be iterated* repeatedly many times.

Before defining the notion of a cluster formally, we start with the explanation of the notion of $\operatorname{conn} T$ for the case when T is a *set*. Our definition relates such a notion to finding of some " shortest commensurately connected" superset containing T; this will be important later in this section when analogous construction will be applied also to a general cluster. We start in fact with the definition of a slightly changed quantity denoted by $\operatorname{Conn} T$ which is defined in a more direct way.

Note. The value of $\operatorname{con} T$ used in the definition of a contour in part II is inconvenient here and cannot be reasonably used in what follows. However, our new value denoted by $\operatorname{conn} T$ will be actually *smaller* than the corresponding quantity of section II and this will enable to transfer immediately the estimates of the type (2.8),(2.8') which were established in part II (and later combined with the Peierls condition) to our present context.

In the definition of $\operatorname{conn} T$ (see Definition 2 below) we will use the notion of an abstract *tree*, often also with a specified *root*:

Definition. An abstract *tree* is defined as an equivalence class, with respect to the isomorphisms of graphs, of unoriented graphs without cycles. (By a cycle of a graph G we mean a collection of the type $\{t_1, t_2\}, \dots, \{t_n, t_1\}$ composed of bonds of G.) If we want to specify also the *root* of such a tree (i.e. mark one point of the graph) then such an object can be defined also in a recursive way, just by specifying the collection of all subtrees, with marked roots, emerging if the root of the tree is removed.

Note. The identification of a cluster with a suitable tree – which will be given below – suggests that the following idea of the summation of cluster expansion series will be developed: instead of estimating the number of various clusters with the same length we rather employ here the idea of the summation over the trees (based on the recursive summation over the outer bonds of the tree).

It seems that this method gives good estimates. Therefore, we are following this method here, in spite of the fact that the treatment given below is maybe too much general for the purposes of the forthcoming text. Namely, a weaker version could be apparently also made which would be closer to our later approach of section 7 – which is based on the notion of a tight set; see the proof of Theorem 7. Nevertheless we keep the method of summation over trees here, also as a suitable reference for possible further applications of the method (like the paper [COZ] which is under preparation).

Definition 1. By a *commensurate tree on* $\mathbb{Z}^\nu$ we mean the following object:

(1) It is a pair $\mathcal{T} = (G, \phi)$ consisting of *an abstract tree* G and a *mapping* ϕ of this abstract tree G to $\mathbb{Z}^\nu$; the mapping can be constructed, after fixing of some root of the given abstract tree, also recursively (according to the recursive definition of an abstract tree given above): the image of the newly added root is specified at each stage of the construction. The vertices of the abstract tree G are mapped (generally not one to one) to some subset of $\mathbb{Z}^\nu$ which will be called the *support* of the tree (denoted by $\operatorname{supp}\mathcal{T}$). Notice that possibly several vertices of the given abstract tree G can be mapped to the same $t \in \mathbb{Z}^\nu$. Then these vertices of the tree $\mathcal{T}$ will be sometimes denoted by symbols t', t'', t''' ... to distinguish them.

(2) The *bonds* of the tree $\mathcal{T}$ constructed in (1) are (unordered) pairs of the type

$$\{t, s\}\,; s = t + 2^k \vec{e}_i$$

where $k \in \mathbb{N}, t \in 2^k\mathbb{Z}^\nu$ and where $\vec{e}_i$ is either zero or a vector of the canonical base of $\mathbb{Z}^\nu$. More precisely we consider bonds of the type $\{\phi(a), \phi(b)\}$ (where $\phi(a) \in \{t', t'', \dots\}$ and $\phi(b) \in \{s', s'' \dots\}$) which are images under ϕ of the corresponding bonds of the abstract tree G. (We put no limitations on the number of such bonds per a given pair $\{t, s\}$.)

(3) The *commensurability* is meant here in the sense that if $\{A = \phi(a), B = \phi(b)\}$ and $\{A = \phi(a), C = \phi(c)\}$ are two bonds then

$$\frac{1}{2}\rho(A, B) \le \rho(A, C) \le 2\rho(A, B) \tag{3.2}$$

where the distance $\rho(A, B)$ between $A \in \{t', t'', \dots\}$ and $B \in \{s', s'', \dots\}$ is defined as 2^k resp 1 according to whether $s = t + 2^k\vec{e}_i$ or $s = t$ in the above relation.

(4) we define the length of such a tree as the number of its bonds *excluding* all the bonds ("loops") of the type $\{t', t''\}$.

Note. The usage of the lattices $2^k\mathbb{Z}^\nu$ and our very notion of a commensurability will be quite important in the following. The choice of the factor 2 in (3.2) is more or less arbitrary but convenient later. We should notice that later, in the proof of Theorem 6 below, the notion of commensurability will be transcripted to an *alternate* language based on the usage of the unit *cubes* from lattices $2^k\mathbb{Z}^\nu$ (considered as cubes from the original lattice $\mathbb{Z}^\nu$) instead of the employment of the *bonds* of the type above.

Definition 2. Given any set $T \subset \mathbb{Z}^\nu$ we assign to it a shortest possible commensurate tree containing for any $t \in T$ *at least one bond* of the type $\{t', t''\}$. We will denote such a tree (it is often not determined uniquely, even if its root is already selected) as $\mathcal{T} = \mathcal{T}(T)$. We recall that the length of the tree was defined by (4) above and therefore the loops $\{t', t''\}$ are not contributing to the length of the tree; the condition that all such loops are in the considered tree can be replaced by requiring that any $t \in T$ belongs to some bond of the tree having the length *at most* 2.

Define the auxiliary quantity $\operatorname{Conn} T$ as the *length* of the tree $\mathcal{T}(T)$. In the following, it will be more useful to have a modified version of this quantity, denoted by $\operatorname{conn} T$ and defined as follows:

$$\operatorname{conn} T = \operatorname{Conn} T + [3\nu \log_2 \operatorname{diam} T]. \tag{3.4}$$

Note. (3.4) will be a more adequate quantity than $\operatorname{Conn} T$ in what follows; see Proposition below. Namely, the clusters of sets will be defined below in a recursive way as collections of objects (sets or contours) "whose diameters are not smaller than their distance to other objects of the collection"; and when constructing additional commensurate path connecting a given set T with a point in distance $\operatorname{diam} T$ one requires an additional amount of $\approx \log \operatorname{diam} T$ steps:

Lemma 1. *Let $\rho(t, s) = d$. Then there is a commensurate path starting in the loop $\{t, t\}$ and ending in the loop $\{s, s\}$ having the length at most equal to $[\,3\nu \log_2 d]$.*

Proof of Lemma 1. It follows easily from the following considerations: first notice that it suffices to consider the case of the dimension $\nu = 1$. Consider now the path on $\mathbb{Z}$ with steps having the lengths

$$1, 2, 4, \ldots, 2^k, \ldots, 4, 2, 1$$

which overcomes the distance $d = 3 \cdot 2^k - 2$. The length of this path is $2k - 1 \leq 2 \log_2 d$. If

$$3\ 2^k - 3 \leq d' \leq 6\ 2^k - 3$$

then it is possible to construct a commensurate path overcoming the distance d' simply by doubling *some* of the steps in the sequence above or possibly by tripling the middle step. We need at most $2k - 1 + k + 2 \ \leq\ 3 \log_2 d'$ steps which completes the proof.

Now we come to the definition of a *cluster* :

Definition 3. The notion of a *cluster* of sets (only *some* sets will be employed in the construction of clusters, see below) is defined recursively, retaining the letter T for the notation of clusters, as follows:

i) Any set

$$T = \operatorname{supp} \mathbf{\Gamma}$$

where $\mathbf{\Gamma}$ is a contour or an admissible system (to be specified below; we will consider below only some special, "recolorable" systems $\mathbf{\Gamma}$ which will be defined later) is a cluster.

ii) If T_i are some clusters and T_0 is from i) such that

$$\operatorname{dist}(\operatorname{supp} T_0, \operatorname{supp} T_i) \leq \min\{\operatorname{diam} \operatorname{supp} T_0, 2\ \operatorname{diam}(\operatorname{supp} T_i)\} \tag{3.5}$$

holds for each $i \geq 1$ then the collection

$$T = (T_0, \{T_i\})$$

is again a cluster. We denote

$$\operatorname{supp} T = \operatorname{supp} T_0 \cup \cup_i \operatorname{supp} T_i\ .$$

The set T_0 will be called the *core* of the cluster T.

Note. The condition (3.5) is a technical one; its adequateness (with respect to our actual constructions) will be seen later in Theorem 4. The appearance of the additional "logdiam" term (compared to $\operatorname{Conn} T$) in the definition of $\operatorname{conn} T$ will be seen to be related to our formulation of the condition (3.5). See Proposition below.

Definition 4. We assign, to any cluster T, a commensurate tree $\mathcal{T}$ as follows: If the trees $\mathcal{T}_0$ and $\mathcal{T}_i$ are already constructed – by Definition 2 and the induction assumption for T_i (recall that $\operatorname{Conn} T_0 = |\mathcal{T}_0|$ where $\mathcal{T}_0$ is a shortest commensurately connected tree whose support contains T_0) then we define $\mathcal{T}$ as the shortest possible commensurately connected tree containing (as mutually disjoint subtrees) all the trees $\mathcal{T}_0$ and $\mathcal{T}_i$ and such that all branches of $\mathcal{T} \setminus \mathcal{T}_0$ start with some loop of the type $\{t', t''\}$. To have an idea about the length of $\mathcal{T}$ consider a tree

$$\mathcal{T}' = \mathcal{T}_0 \cup \cup_i (\mathcal{T}_i \cup P_i)$$

where P_i are some shortest possible commensurate paths, each of them starting in some loop $\{s_i, s_i'\}$ of $\mathcal{T}_0$ and ending in some loop $\{t_i, t_i'\}$ of $\mathcal{T}_i$.
In analogy to Definition 2, the quantity $\operatorname{Conn} T$ is now defined as the length of the tree $\mathcal{T}$ and we put

$$\operatorname{conn} T = \operatorname{Conn} T + [3\nu \log_2 \operatorname{diam} \operatorname{supp} T]. \tag{3.6}$$

In order to reconstruct back the original cluster T from a given tree $\mathcal{T}$ the following notion will be useful:

Definition 5. Assume that some total ordering $\prec$ on the collection of all subsets of $\mathbb{Z}^\nu$ is defined, extending both the lexicographic order between the shifts of a given set as well as the relation $A \prec B$ if $A \subset B$. Say that a cluster $T = (T_0, \{T_i\})$ is a *standard* one if $S \prec T_0$ holds for any set S used in the recurrent definition of the clusters T_i.

Notes. 1) Notice that $\operatorname{Conn} T$ is not greater than $|\mathcal{T}'|$ i.e.

$$\text{(3.6')} \qquad \operatorname{conn} T \leq |\operatorname{Conn} T_0| + \sum_i |\operatorname{Conn} T_i| + \sum_i l_i + [3\nu \log_2 \operatorname{diam} \operatorname{supp} T].$$

In fact, in Theorem 6 we will show that for all clusters considered later by us, the quantities $\operatorname{conn} T$ and $|\operatorname{supp} T|$ will be of the *same order*. Moreover, one could rewrite the present section for this (narrower) setting in the spirit analogous to that of the later Theorem 6, without employing the bothering (but small!) logdiam terms. We prefer the more general exposition here in view of wider applicability of the estimates obtained here also to other situations.
2) The mapping from clusters to trees constructed above is not one to one. However, if we assign to any bond of $\mathcal{T}$ a "flag" i.e. mark it by a value 0 or 1 and interpret the components of $\mathcal{T}$ marked by 0 resp. 1 as the sets used in the definition of T resp. as the connecting paths (connecting T_0 with T_i etc.) then the core of the cluster T (and the cores of T_i etc.) can be recognized just by looking at the biggest (at $\prec$) 0 – component of $\mathcal{T}$. Thus, one has a crude bound 2^n for the number of standard clusters T with the same tree $\mathcal{T}$ of the cardinality n. In the forthcoming applications, all clusters constructed by us will be standard ones and so we will not discuss the possible modifications of the estimates discussed below which would be needed if also nonstandard clusters would appear.

The following estimate will be used later in Theorems 5 and 5' (though in a slightly changed form). It says that having established a slightly stronger version of the estimate (3.1) for the *sets* T one obtains (3.1) also for all *clusters* T if the quantities k_T are given by the recurrent formulas below. The quantity $\operatorname{Conn} T$ does not seem to have comparably nice properties ; the additional " logdiam" term in our definition of $\operatorname{conn} T$ seems to be essential here.

Next we formulate two auxiliary results: The first one will be directly used later (in a slightly different form not changing its essence – see the proof of Theorem 5). On the other hand, the second result is its corollary which we formulate in a *more general* setting – which will be possibly interesting also in other situations where our method can be applied. This latter result resembles the classical Meyer method (see Ruelle's book [R1]).

Proposition 1. *Assume that the quantities k_T are defined recursively by formulas*

$$\text{(3.7)} \qquad k_T = k_{T_0} \prod_i k_{T_i}.$$

Assume that for the sets T_0 the following stronger variant of (3.1) is valid:

$$\text{(3.8)} \qquad |k_{T_0}| \leq \varepsilon^{\operatorname{conn} T_0 + 6\nu \log_2 (\operatorname{diam} T_0 + 6)}.$$

Then the estimate

$$|k_T| \leq \varepsilon^{\operatorname{conn} T} \tag{3.9}$$

holds also for all the clusters $T = \{T_0, \{T_i\}\}$, *with* $\operatorname{conn} T$ *defined by the preceding definition, assuming that it is already valid for all clusters* $T_i; i \neq 0$ *in (3.7) from which the clusters* T *were formed.*

Proof. It suffices to prove that

$$\operatorname{Conn} T_0 + 6\nu \log_2(\operatorname{diam} T_0 + 6) + \sum_i \operatorname{conn} T_i \geq \operatorname{conn} T. \tag{3.10}$$

Notice first the following simple estimate. Define the support $\operatorname{supp} T$ and the diameter $\operatorname{diam} T$ of a cluster $T = \{T_i\}$ recursively by putting $\operatorname{supp} T = \operatorname{supp} T_0 \cap \operatorname{supp} T_i$ and $\operatorname{diam} T = \operatorname{diam} \operatorname{supp} T$.

Lemma 2. *Let* T_j *be the longest of all clusters* T_i *(maximizing its diameter). Then*

$$\log_2 \operatorname{diam} T \leq \log_2(\operatorname{diam} T_0 + 6) + \log_2 \operatorname{diam} T_j$$

assuming that the right hand side is greater or equal to 6.

Proof of Lemma 2. It is straightforward: notice that the condition (3.5) implies the bound

$$\operatorname{diam} T \leq \operatorname{diam} T_0 + (1 + 2 + 2 + 1) \operatorname{diam} T_j.$$

Then we use the inequality $\log_2(x + 6y) \leq \log_2(x + 6) + \log_2 y$ which is surely valid if $x \geq 1$ and $y \geq 1$.

Notice that it suffices now to establish the following bound (from which the required bound (3.9) is obtained by summing with the bound of Lemma 2):

$$\operatorname{Conn} T_0 + 3\nu \log_2(\operatorname{diam} T_0 + 6) + \sum_{i \neq j} \operatorname{conn} T_i + \operatorname{Conn} T_j \geq \operatorname{Conn} T$$

which is surely valid because then we can rewrite it (notice that $l_j \leq 3\nu \log_2(\operatorname{diam} T_0 + 6)$ by Lemma 1 and omit the number 6!) in a stronger form

$$\operatorname{Conn} T_0 + \sum_i (\operatorname{Conn} T_i + l_i) \geq \operatorname{Conn} T$$

where l_i denotes the length of the path P_i used in the definition of the auxiliary tree $\mathcal{T}'$ (see Definition 4). Namely, the bound $l_j \leq 3\nu \log_2 \operatorname{diam} T_0$ follows from the condition (3.5). The validity of the last inequality follows from the very definition of $\operatorname{Conn} T$ (see (3.6')) and this completes the proof of Proposition.

Notations. In the following we will usually write, for clusters T,

$$t \in T, T \subset \Lambda, \operatorname{dist}(T, \Lambda), \ldots$$

instead of the more precise notations

$$t \in \operatorname{supp} T, \operatorname{supp} T \subset \Lambda, \operatorname{dist}(\operatorname{supp} T, \Lambda) \ldots.$$

Writing $G \in T$ we will mark the situation when the set G was used in the recursive definition of T (as the "core" of some intermediate cluster used in the construction) of the cluster T.

Finally we formulate one consequence of the condition (3.1), to be used later in the formulation of our main result.

Proposition 2. *If there is a small ε such that for each cluster T,*

$$|k_T| \leq \varepsilon^{\operatorname{conn} T} \tag{3.11}$$

then the cluster series with the terms k_T quickly converge in the following sense: for any $t \in \mathbb{Z}$ and for any $d \in \mathbb{N}$ we have

$$\sum_{T; t \in T, \operatorname{conn} T \geq d} |k_T| \leq (C\varepsilon)^d \tag{3.12}$$

and analogously for $\operatorname{conn} T \geq d$ *replaced by* $|T| \geq d$.

First notice that instead of $\operatorname{conn} T$ it suffices to prove an analogous result for the smaller and "more natural" quantity $\operatorname{Conn} T$. We recall that our introduction of the quantity $\operatorname{conn} T$ was motivated by the necessity to derive (3.1) for all *clusters* from something like (3.9) which should be assumed to be valid for all *sets* T. Once we have (3.12) for *all* clusters we can forget the quantity $\operatorname{conn} T$ and replace it by $\operatorname{Conn} T$ if the convergence of the cluster expansion is investigated.

We will prove Proposition 2 in the following broader setting (which is closely related to usual estimates in the theory of the Mayer expansions – see the book [R1]). It is easy to understand that Proposition 3 below actually *generalizes* the statement of Proposition 2: Recall that we identified any cluster T with some commensurate tree $\mathcal{T}$ on $\mathbb{Z}^\nu$ and the number of standard clusters corresponding to $\mathcal{T}$ is at most $2^{|\mathcal{T}|}$. Having this in mind the forthcoming result can be formulated for quantities $k_{\mathcal{T}}$ indexed by commensurate *trees* $\mathcal{T}$ on $\mathbb{Z}^\nu$.

Proposition 3. *Let the quantities $k_{\mathcal{T}}$ be given as products of some quantities denoted by $k_{\{t,s\}}$ or k_b (see the commentary below)*

$$k_{\mathcal{T}} = \prod_b k_b$$

where the product is over all the "bonds" $b = \{\{A, B\}, \phi\}$ of the commensurate tree $\mathcal{T} = \{G, \phi\}$. Recall that the "bonds" $\{A, B\}$ of an abstract tree G are mapped by ϕ to unoriented pairs $\{t = \phi(A), s = \phi(B)\}$ of points of $\mathbb{Z}^\nu$. The notation k_b is used instead of a more explicit notation $k_{\{t,s\}}$ for $b = \{\{A, B\}, \phi\}$ such that $\{t = \phi(A), s = \phi(B)\}$. Assume that these quantities k_b are nonnegative and $k_{\{t,t\}} = 1$ for each t. Let for any unordered pair $b = (t, s)$ (a slight abuse of notations) we have the estimate

$$\sum k_{b'} \leq q \tag{3.13}$$

where the summation in (3.13) is over all unordered pairs $b' = (s, u), s \neq u$ which are commensurate with b and q is some small (e.g. $q < 1/4$) positive constant. Then for any pair b we have also the bound

$$\sum k_{\mathcal{T}} \leq k_b \cdot q' \tag{3.14}$$

where the summation in (3.14) is over all commensurate trees $\mathcal{T}$ containing the "bond" b as its "extremal bond" and having a length at least 2. By the extremality of $b = \{t = \phi(A), s = \phi(B)\}$ we mean here that one vertice of the pair $\{A, B\}$ remains "free", "endvertice" in the original abstract tree; this must hold at least for one of the bonds $\{A, B\}$ which are mapped to the given pair $\{t, s\}$. The quantity q' can be chosen like $3q$.

Note. More precisely, the optimal value of q' can be found from the equation (see the end of the proof below)

$$\exp(q'') = 1 + q'$$

where q'' denotes the supremum, over all bonds b, of the sums analogous to that in the left hand side of (3.13) but with modified terms

$$k''_{b'} = \frac{k_{b'}(1+q')}{1 - k_{b'}(1+q')}$$

where q' has its previously established value. Namely if all k_b are small and we already have established the smallness of q' then we have the crude bound $k''_{b'} < 2k_b(1+q')$; hence we have also the inequality $q'' < 2q(1+q')$. See the end of the proof below.

Proof. Apply the method of induction. Denote by $\sum_{\mathcal{T}}^{<n} k_{\mathcal{T}}$ the subsum of $\sum_{\mathcal{T}} k_{\mathcal{T}}$ taken over all trees $\mathcal{T}$ which "have the level at most n" in the sense that the longest path to be found in $\mathcal{T}$ starting in b contains at most n bonds. Removing of a bond b from such a tree and cutting of all the connections going through the "nonfree" vertex of b results in an appearance of certain (nonzero) number – say m – of subtrees of the level at most $n-1$. (These subtrees are "sticked" to that vertex of the bond b which is not "free").

Now, considering the induction assumption for $\sum_{\mathcal{T}}^{<n-1} k_{\mathcal{T}}$ i.e. the inequality $\sum_{\mathcal{T}}^{<n-1} k_{\mathcal{T}} \leq q'$ one gets the following estimate. Let us start with the situation where only one bond b' (possibly a "multiple one" if looking at the image ϕ of the original abstract tree G in $\mathbb{Z}^\nu$ i.e. several $\{\phi(A), \phi(B)\}$ with the same $\{t = \phi(A), s = \phi(B)\}$ are possible) sticks to b at a given vertice t of b. Let $\mathcal{T}'$ denote the subtree "continuing" the bond b'. Summation over all possible $\mathcal{T}'$ gives the following contribution (the subtree $\mathcal{T}''$ resulting from the removal of b' from $\mathcal{T}'$ may be already empty or may possibly contain some more bonds; for the sum over the latter situation we already have the induction asssumption with the bound q'): $k_{b'}(1+q')$ if only one subtree sticks to t. If we are counting also the cases when *several* ($k \geq 1$) subtrees starting with the same bond b' stick to b we have the contribution

$$\sum (k_{b'}(1+q'))^k = \frac{k_{b'}(1+q')}{1 - k_{b'}(1+q')}.$$

Now perform the summation over all more complicated possibilities when $m \geq 1$ different types of bonds b' stick to b; each of them can have some multiplicity $k \geq 1$ and each such bond b' is the starting bond of some – empty or nonempty – subtree. (Contributions of these situations were just counted above.) We get the final estimate

$$\sum_{\mathcal{T}}^{<n} k_{\mathcal{T}} < \sum_m (q'')^m/m! \leq \exp(q'')$$

which concludes the inductive step of the proof of Proposition 3.

Finally, it is easy to see that the condition of the type (3.1) implies the validity of the conditions (3.13),(3.13') for suitably chosen *new* $k_T = k_{\mathcal{T}} = \prod_b k_b$ (namely with $k_b \approx \varepsilon$) *majorizing* the original values k_T. Then also the constant q in (3.13) can be taken as $q \approx N\varepsilon$ where N denotes the upper bound for the number of bonds commensurate to b. The constant C in Proposition 2 depends then only on the dimension (and on our choice of the constant 2 in the definition of commensurate bonds) if ε is sufficiently small.

2. The Concept of a Mixed Model.

The *configuration space* of the mixed model in a volume Λ (typically a standard one) under a boundary condition $y \in \mathbb{S}$ given on Λ^c, will be given as a suitable *subset* of the configuration space of (the strictly diluted configurations of) the original, abstract Pirogov – Sinai model. Thus, we consider a collection of pairs

$$(x_\Lambda, \{\mathbf{\Gamma}_i\})$$

where $x_\Lambda \cup y_{\Lambda^c}$ is assumed to be *stratified* for any given point of $\Lambda \setminus (\cup_i \operatorname{supp} \mathbf{\Gamma}_i)$ and has a "vertically best extension" to $\cup_i \operatorname{supp} \mathbf{\Gamma}_i$ (in the sense of minimal possible value of $\sum e_t(x)$). Here, $\mathbf{\Gamma} = \{\mathbf{\Gamma}_i\}$ is an admissible collection of contours satisfying the condition $\operatorname{dist}(V_\updownarrow(\mathbf{\Gamma}), \Lambda^c) \geq 2$ and giving a nonconflicting precription, together with y, for the value of x_Λ.

Later, we will often assume (but notable exception is the forthcoming section 3!) that the configuration space is horizontally translation invariant in the sense that a horizontal (& strictly diluted in a given volume Λ) shift of an allowed configuration is again an allowed configuration.

Notes.
0) The volume Λ may be often considered to be a standard and even a conoidal one, e.g. a large cube. We will not consider at all any possible relations between the models in *different* volumes! In particular, we will not derive any analogy of the "telescopic equations" for diluted partition functions (which are usually formulated in such situations). In what follows, Λ is just a suitable, fixed, finite substitute for the whole lattice $\mathbb{Z}^\nu$; the thermodynamic limit $\Lambda \to \mathbb{Z}^\nu$ being performed only at the very end of our considerations, in the last section of the paper.
1) In all the configurations $(x, \{\mathbf{\Gamma}_i\})$ considered below (in the forthcoming sections preceding the statement of Main Theorem) we will have $x = x^{\text{best}}_{\{\mathbf{\Gamma}_i\}}$. We will usually omit the superscript "best" in the following.
2) Sometimes (like in the study of the probabilities of external contours in our Main Theorem) we will be interested also in nonstandard volumes Λ (like the complement, in some standard Λ', of some set $\cup \operatorname{supp} \mathbf{\Gamma}_i'$ (where $\{\mathbf{\Gamma}_i'\}$ is an admissible system

of contours) and we could in principle consider also those boundary conditions y which are only locally from S. However, we will not need this.[11]

The *hamiltonian* of the mixed model will be given (compare with (2.16)!) as follows: Consider any configuration $(x_{\tilde{\Lambda}}, \mathcal{D})$ of the abstract Pirogov – Sinai model which is given on a set $\Lambda \subset \mathbb{Z}^\nu$ such that $\mathcal{D} = \{\mathbf{\Gamma}_i\}$ is an admissible system in Λ and x_Λ is compatible with $\mathcal{D}$ i.e. stratified outside of the set $\operatorname{supp} \mathcal{D} = \cup_i \operatorname{supp} \mathbf{\Gamma}_i$ and "best" inside $\operatorname{supp} \mathcal{D}$. (See (2.14). Denote the "external" value of x as x^{ext}. (For nonstandard Λ it is defined at least on $\partial \Lambda^c$). It is from $\mathbb{S}$ (maybe only locally if Λ is a nonstandard volume). Define, for $(x_{\tilde{\Lambda}}, \mathcal{D})$ where $\mathcal{D} = \{\mathbf{\Gamma}_i\}$, the quantity $H(\mathcal{D}, x^{\mathrm{ext}})$, more precisely denoted by

$$H((x_\Lambda, \mathcal{D}), x^{\mathrm{ext}}) = \sum_i E(\mathbf{\Gamma}_i) + \sum_{t \in \Lambda} e_t(x) - \sum_{T \subset \Lambda \setminus \operatorname{supp} \mathcal{D}} k_T(x) \tag{3.15}$$

where $E(\mathbf{\Gamma})$ and e_t are as before (we recall that both $E(\mathbf{\Gamma})$ and $e_t(y)$ are quantities which are horizontally translation invariant only) and where $k_T(x)$ are some new, quickly decaying quantities which can be viewed as an "additional cluster field" and which are required to satisfy the following conditions:

(1) (Horizontal translation invariance of k_T) The quantities $k_T = k_T(x)$ are local in the sense that they depend only on $x_t, \operatorname{dist}(t, T) \le \operatorname{diam} T$. They moreover fulfil the relation

$$k_T(x) = k_{UT}(Ux) \tag{3.16}$$

whenever both quantities are nonzero (see Note 3 below), for any horizontal shift U such that both T and UT sufficiently "inside" $\Lambda \setminus \operatorname{supp} \mathcal{D}$:

$$\operatorname{dist}(T, \operatorname{supp} \mathcal{D} \cup \Lambda^c_{\mathrm{nonflat}}) \ge \operatorname{diam} T$$
$$\operatorname{dist}(UT, \operatorname{supp} \mathcal{D} \cup \Lambda^c_{\mathrm{nonflat}}) \ge \operatorname{diam} T\,.$$

where $\Lambda^c_{\mathrm{nonflat}}$ denotes the collection of those points $t \in \Lambda^c$ which do *not* satisfy the property ("horizontal translation invariancy of Λ at s") that the vertical sections $\Lambda_{\{t\}} = \Lambda \cap \{(t_1, \ldots, t_{\nu-1}, (.)\}$ commute with U $(U(\Lambda_{\{t\}}) = \Lambda_{\{Ut\}})$for horizontal shifts U of unit length.

(2) (Quick decay of k_T) There is a small $\varepsilon > 0$ such that for any T,

$$|k_T| \le \varepsilon^{\operatorname{conn} T}. \tag{3.17}$$

Notes.

1. We will later glue together – in our "recoloring procedures" – contours $\mathbf{\Gamma}$ and clusters T such that $\operatorname{dist}(T, \operatorname{supp} \mathbf{\Gamma}) \le 2 \operatorname{diam} T$ – to form new clusters (of some new mixed model); this is one of the reasons why we added, in the preceding section, the

[11] If the volume Λ is a standard one i.e. if all the "horizontal sections" of Λ have simply connected components then the phrases "y is locally resp. globally from $\mathbb{S}$ on $\partial\Lambda^c$ " have the *same* meaning.

"safety constant" $3\ \log_2 \operatorname{diam} T$ to the quantity $\operatorname{Conn} T$ in the definition of $\operatorname{conn} T$ to keep the control over the connectivity properties of the new clusters formed by such (recursive) procedures.

2. The collection of allowed contours (and of allowed configurations, see below) will vary from one mixed model to another. Typically the allowed set of contours will be some subset of the original collection of contours (of some given abstract Pirogov – Sinai model) – and this subset will become even *smaller* after applying further expansions (recolorings) to the given model.

On the contrary, the collection of nonzero k_T will always *grow* with such an expansion. See the forthcoming section for more details.

3. The restriction of the assumption (3.16) to nonzero products $k_T\, k_{UT}$ is related to the fact that, in the forthcoming section, we will work with ("slightly") translation noninvariant models; the new cluster quantities k_T will be constructed successively in the lexicographic order, through an infinite sequence of intermediate (noninvariant) mixed models. However, if the condition (3.16) is complemented by the assumption that *both* k_T and k_{UT} are nonzero if at least *one* of them is nonzero and if the configuration space is horizontally invariant (in the sense of what admissible systems are allowed in $\mathbb{Z}^\nu$) we will speak about the *translation invariant* mixed model.

4. Having specified the collection of allowed *contours* of the mixed model we do not even require that all admissible collections of allowed contours are allowed configurations of the mixed model. At this moment we impose no special requirements on what collections of contours are really allowed in our model; see the forthcoming sections 3,4,7 for a more concrete information about the actual choice of the configuration space.

The *partition functions* of the mixed model in a given volume Λ (*strictly* diluted ones; we can forget the notion of a diluted partition function for most of part III; however our volumes Λ will be often the standard and moreover the conoidal ones) will be given as

$$Z^\alpha_\updownarrow(\Lambda) = \sum_{\mathcal{D} \subset\subset \Lambda} Z^\alpha_{\mathcal{D}}(\Lambda)$$

where α is a boundary condition on $\partial\Lambda^c$ (which is locally from $\mathbb{S}$) and $\mathcal{D}$ is an admissible family of contours; the notation $\mathcal{D} \subset\subset \Lambda$ will mean, everywhere in the following, that $\operatorname{dist}(V_\updownarrow(\mathbf{\Gamma})), \Lambda^c) \geq 2$ for any contour $\mathbf{\Gamma}$ on $\mathcal{D}$. By (3.15) we define

$$Z^\alpha_{\mathcal{D}}(\Lambda) = \exp(-\sum_i E(\mathbf{\Gamma}_i)) \exp(-\sum_{t\in\Lambda} e_t(\delta)) \exp(\sum_{T \subset \Lambda \setminus \operatorname{supp} \mathcal{D}} k_T(\delta)) \tag{3.18}$$

where δ denotes the configuration$(\alpha \cup \partial\mathcal{D})^{\mathrm{best}}$.

Note. It seems unnatural to use the symbol Z for the "mere Gibbs factor" (3.18). However, the case $k_T = 0$ is not a typical example here. In a more general case, the considered mixed model corresponds actually to some partial expansion of the model (2.18). Then (3.18) is really some partition function, corresponding to an event "$\mathcal{D}$ is the collection of (still) nonexpanded contours of the original model (2.18)".

3. Recoloring of a single internal admissible subsystem $\mathbf{\Gamma}$

This is a central construction of our approach, *replacing* (together with the forthcoming constructions of later sections) the concept of a (metastable) *contour model* used in the previous versions of the Pirogov – Sinai theory.

In this section we describe some abstract, "algebraic" aspects of *one recoloring step* (of an arbitrary mixed model). An invariant (conserving the horizontal invariancy) modification of this construction will be given in the forthcoming section 4.

A suitable *sequence* of (translation invariant) recoloring steps – yielding as its final result the "total" expansion of a given Pirogov – Sinai abstract model – will be discussed later, starting from section 5.

Recoloring will be just one step towards the desired "total expansion" of the model, and this step is described in detail in Lemma and Theorem 4 below.

Roughly speaking, recoloring of $\mathbf{\Gamma}$ will just mean a *replacement* of a given mixed model by another mixed model where $\mathbf{\Gamma}$ will *not* already be allowed as a contour and where some *new* quantities k_T (for some *new* clusters T containing G as its core) will appear. The remainder of the model will be kept intact and the crucial fact will be that the diluted partition functions of both the original and recolored model will be required to be the *same* for all finite volumes.

Let us start with the definition of the following important quantity $A(\mathbf{\Gamma})$ which "measures the instability of $x^{\mathrm{ext}}_{\mathbf{\Gamma}}$ in $V(\mathbf{\Gamma})$" and which will play a key role later when defining a rigorous substitute to (2.21); see also the remark below Definition 2.

There will be several variants of the quantity $A(\mathbf{\Gamma})$ – see below – and the technical difference between their definitions (essentially the decision what volume will be used instead of $V(\mathbf{\Gamma})$) will be quite important in this section, in spite of the fact that the values of all these quantities will be roughly the same. Let us introduce again (it will be quite indispensable in what follows) the definition of a suitable "protecting zone" of the volume $V(\mathbf{\Gamma})$:

Definition 1. Define first the "conoidal" volume

$$V_{\updownarrow}(\mathbf{\Gamma}) = V(\mathbf{\Gamma}) \cup V'(\mathbf{\Gamma}) \tag{3.19}$$

where $V'(\mathbf{\Gamma})$ is the collection of all points $t \in (V(\mathbf{\Gamma}))^c$ satisfying the bound

$$\operatorname{dist}(t, V(\mathbf{\Gamma})) \leq \frac{1}{2} \operatorname{dist}(t, \operatorname{supp} \mathbf{\Gamma}) \,. \tag{3.20}$$

Denote by $A(\mathbf{\Gamma})$, more precisely by $A(\mathbf{\Gamma}, x_{V_{\updownarrow}(\mathbf{\Gamma})})$ the quantity

$$A(\mathbf{\Gamma}, x_{V_{\updownarrow}(\mathbf{\Gamma})}) = \sum_{t \in V(\mathbf{\Gamma})} (e_t(y) - e_t(x)) + \sum_{\substack{T \subset V_{\updownarrow}(\mathbf{\Gamma}) \\ T \cap \operatorname{supp} \mathbf{\Gamma} = \emptyset}} k_T(x_{V_{\updownarrow}(\mathbf{\Gamma})}) - \sum_{T \subset V_{\updownarrow}(\mathbf{\Gamma})} k_T(y) \tag{3.21}$$

where $x = (\partial \mathbf{\Gamma})^{\mathrm{best}}$ (the configuration minimizing the hamiltonian under the condition $\partial \mathbf{\Gamma}$) and $y = x^{\mathrm{ext}}_{V_{\updownarrow}(\mathbf{\Gamma})}$.

Note. Thus, to be able to define $A(\mathbf{\Gamma})$ we must know the configuration x on the whole volume $V_{\updownarrow}(\mathbf{\Gamma})$ (because $k_T(x)$ resp. $k_T(y)$ depend on the values of x resp. y

on T and we do not assume $T \subset V(\mathbf{\Gamma})$). However, the value of x on $V_{\updownarrow}(\mathbf{\Gamma})$ will be normally determined by the context in which the contour (or admissible system) $\mathbf{\Gamma}$ will appear and so we will use the shorter notation $A(\mathbf{\Gamma})$ instead of the more precise notation $A(\mathbf{\Gamma}, x_{V_{\updownarrow}(\mathbf{\Gamma})})$ without any ambiguity.

We will see that the quantity

$$F(\mathbf{\Gamma}) = E(\mathbf{\Gamma}) - A(\mathbf{\Gamma}) \tag{3.22}$$

is a useful exact substitute for the formal quantity F_{formal} from (2.21). The choice of the set $V_{\updownarrow}(\mathbf{\Gamma})$ will guarantee (among other convenient properties) that the clusters constructed below having the "core" $\mathbf{\Gamma}$ will be sufficiently "tight"(which would not be the case if we would take mere $V(\mathbf{\Gamma})$ here).

However, if $\mathbf{\Gamma}$ is an interior subsystem of some bigger admissible collection $\mathbf{\Gamma}\&\mathcal{D}$ of all contours of some configuration $(x_\Lambda, \mathbf{\Gamma}\&\mathcal{D})$ in a finite volume Λ, the following modifications of the quantity $A(\mathbf{\Gamma})$ will be sometimes considered later, too: (We will be interested below only in the cases when moreover $V_{\updownarrow}(\mathbf{\Gamma}) \cap \mathcal{D} = \emptyset$ i.e when $\mathbf{\Gamma}$ is not "too tightly attached" to $\mathcal{D}$.)

Definition 2. In analogy to (3.21) define also the modified quantities $A_{\text{loc}}(\mathbf{\Gamma})$, $A_{\text{full}}(\mathbf{\Gamma})$, $A_{\text{full},\mathcal{D},\Lambda}(\mathbf{\Gamma})$ as in the relation (3.21) but with the volume $V_{\updownarrow}(\mathbf{\Gamma})$ in the second and the third sum on the right hand side of (3.21) being replaced successively by volumes $V(\mathbf{\Gamma})$, $\mathbb{Z}^\nu$, $\Lambda \setminus \operatorname{supp} \mathcal{D}$. Corespondingly define, by (3.22), the quantities $F_{\text{loc}}(\mathbf{\Gamma})$, $F_{\text{full}}(\mathbf{\Gamma})$, $F_{\text{full},\mathcal{D},\Lambda}(\mathbf{\Gamma})$ and also $F_0(\mathbf{\Gamma})$) (taking $A_0(\mathbf{\Gamma})$ from (2.20)).

Note. Assuming the existence of cluster expansion for the partition functions in the expression (2.21) for F_{formal} one sees that the true analogy of the quantity F_{formal} is $F_{\text{full}}(\mathbf{\Gamma})$, not $F(\mathbf{\Gamma})$. However, $F_{\text{full}}(\mathbf{\Gamma})$ is a *nonlocal* quantity (though a very quickly converging sum of local quantities k_T) and this would be inpractical in the expansions constructed below. However, we return to the value $F_{\text{full}}(\mathbf{\Gamma})$ in later sections. On the other hand, $F_{\text{loc}}(\mathbf{\Gamma})$ is a perfectly local quantity, but sometimes it is a "too crude" (and therefore never used below) approximation to $F_{\text{formal}}(\mathbf{\Gamma})$; such a situation happens in the cases where there are *very* big[12] flat "ceilings" on the boundary of $V(\mathbf{\Gamma})$; a situation having no analogy in the translation invariant situation where the choice of $F_{\text{loc}}(\mathbf{\Gamma})$ would be O.K. because contours are "crusted" in that case.

The quantity $F(\mathbf{\Gamma})$ is a reasonable compromise because it approximates $F_{\text{full}}(\mathbf{\Gamma})$ with a great accuracy ($\sim \varepsilon |\operatorname{supp} \mathbf{\Gamma}|$) and at the same time it is "local" in a reasonable sense. One could take even smaller sets $V'(\mathbf{\Gamma})$ to retain this accuracy but our choice will have additional advantages below in Theorem 5. The quantity $F_{\text{full},\mathcal{D},\Lambda}(\mathbf{\Gamma})$ is just a temporary notation used in the proof of Lemma below.

The forthcoming lemma is an *essential step* in the procedure called "recoloring of an internal contour". This is further developed by Theorem 4, concluding the effort of Section 3.

[12]A general "philosophical" remark: sometimes, one is fighting severe technical problems in the Pirogov – Sinai theory which however start to be relevant only in volumes which are really *astronomically large*; for example the problem mentioned above is hardly of much relevance in volumes of a size, say 10^{27}!

Lemma. *Assume that we have a mixed model satisfying (3.1). Let* $\Lambda \subset \mathbb{Z}^\nu$. *Let* $\mathcal{D}$ & $\mathbf{\Gamma} \subset\subset \Lambda$ *be an admissible system such that* $\mathbf{\Gamma}$ *is its removable subsystem, satisfying moreover the condition* $V_\updownarrow(\mathbf{\Gamma}) \cap \mathcal{D} \cap \Lambda^c = \emptyset$. *Let* α *be a boundary condition on* $\partial\Lambda^c$ *which is in conformity with* $\mathcal{D}$ & $\mathbf{\Gamma}$ *(such that there exist a configuration* $x_{\mathcal{D}\&\mathbf{\Gamma}} = (\alpha \cup \partial\mathcal{D} \cup \partial\mathbf{\Gamma})^{\text{best}}$ *for which all points of* $(\operatorname{supp}\mathcal{D} \cup \operatorname{supp}\mathbf{\Gamma})^c \cap \Lambda$ *are stratified). Then*

$$Z^\alpha_{\mathcal{D}\&\mathbf{\Gamma}}(\Lambda) = Z^\alpha_{\mathcal{D}}(\Lambda)\exp(-F(\mathbf{\Gamma}))\exp\Big(\sum k_T^{\text{new}}\Big) \tag{3.23}$$

where k_T^{new} *depends on both* $x_{\mathcal{D}\&\mathbf{\Gamma}}/T$ *and* $x_{\mathcal{D}}/T$ *and the summation is over all* $T \subset \Lambda$ *such that* $T \not\subset V_\updownarrow(\mathbf{\Gamma}), T \cap V(\Gamma) \neq \emptyset, T \cap \operatorname{supp}\mathcal{D} = \emptyset$ *and* $\operatorname{dist}(T, \operatorname{supp}\mathbf{\Gamma}) \le 2\operatorname{diam} T$. *The quantities* k_T^{new} *are given by formulas (3.26), (3.27) below and they satisfy the estimate*

$$|k_T^{\text{new}}| \le 2\,\varepsilon^{\operatorname{conn} T} \tag{3.24}$$

for each T.

Proof. Write $\gamma\delta$ resp. δ instead of $x_{\mathcal{D}\&\mathbf{\Gamma}}$ resp. $x_{\mathcal{D}}$. Write $Z^\alpha_{\mathcal{D}\&\mathbf{\Gamma}}(\Lambda)$ as

$$\begin{aligned}
&\exp(-E(\mathbf{\Gamma}) - E(\mathcal{D}))\exp\Big(-\sum_{t\in\Lambda} e_t(\gamma\delta)\Big)\exp\Big(\sum_{\substack{T\subset\Lambda\\ T\cap(\operatorname{supp}\mathcal{D}\cup\operatorname{supp}\mathbf{\Gamma})=\emptyset}} k_T(\gamma\delta)\Big) = \\
&= \exp(-E(\mathcal{D}))\exp\Big(-\sum_{t\in\Lambda} e_t(\delta)\Big)\exp(-F_{\text{full},\mathcal{D},\Lambda}(\mathbf{\Gamma}))\exp\Big(\sum_{\substack{T\subset\Lambda\\ T\cap(\operatorname{supp}\mathcal{D})=\emptyset}} k_T(\delta)\Big) = \\
&= Z^\alpha_{\mathcal{D}}(\Lambda)\exp(-F(\mathbf{\Gamma}))\exp\Big(\sum_{\substack{T\subset\Lambda\\ T\not\subset V_\updownarrow(\mathbf{\Gamma})\\ T\cap(\operatorname{supp}\mathcal{D}\cup\operatorname{supp}\mathbf{\Gamma})=\emptyset}} k_T(\gamma\delta) - \sum_{\substack{T\subset\Lambda\\ T\not\subset V_\updownarrow(\mathbf{\Gamma})\\ T\cap\operatorname{supp}\mathcal{D}=\emptyset}} k_T(\delta)\Big)
\end{aligned} \tag{3.25}$$

which proves (3.23) with the following choice of the quantities k_T^{new} (recall that the "old" quantities k_T satisfy (3.1), hence we have (3.24)):

$$k_T^{\text{new}} = k_T(\gamma\delta) - k_T(\delta) \tag{3.26}$$

if $T \subset \Lambda$, $T \not\subset V_\updownarrow(\mathbf{\Gamma})$, $T \cap (\operatorname{supp}\mathcal{D} \cup \operatorname{supp}\mathbf{\Gamma}) = \emptyset, T \cap V(\mathbf{\Gamma}) \neq \emptyset$, resp.

$$k_T^{\text{new}} = -k_T(\delta) \tag{3.27}$$

if $T \subset \Lambda$, $T \not\subset V_\updownarrow(\mathbf{\Gamma})$, $T \cap \mathcal{D} = \emptyset$, $T \cap \operatorname{supp}\mathbf{\Gamma} \neq \emptyset$. The condition $T \cap V(\mathbf{\Gamma}) \neq \emptyset$ (in (3.26); contrast it to the condition $T \not\subset V_\updownarrow(\mathbf{\Gamma})$!) follows from the observation that

$$\begin{aligned}
&T \not\subset V_\updownarrow(\mathbf{\Gamma})\ \&\ \operatorname{dist}(T, \operatorname{supp}\mathbf{\Gamma}) \ge 2\operatorname{diam} T \Rightarrow T \cap V(\mathbf{\Gamma}) = \emptyset \\
&\Rightarrow (\gamma\delta)_T = \delta_T \Rightarrow k_T^{\text{new}} = 0\,.
\end{aligned} \tag{3.28}$$

(Compare the definition of $V_\updownarrow(\mathbf{\Gamma})$; the condition $T \cap V(\mathbf{\Gamma}) \neq \emptyset$ & $T \not\subset V_\updownarrow(\mathbf{\Gamma})$ would imply that there is some $t \in T \setminus V_\updownarrow(\mathbf{\Gamma})$ with $\operatorname{dist}(t, V(\mathbf{\Gamma})) \le \frac{1}{2}\operatorname{dist}(T, \operatorname{supp}\mathbf{\Gamma})$.)

Notation. An interior subsystem $\mathbf{\Gamma}$ of an admissible system $\mathbf{\Gamma}'$, satisfying also the condition $V_{\updownarrow}(\mathbf{\Gamma}) \cap (\mathbf{\Gamma}' \setminus \mathbf{\Gamma}) = \emptyset$ will be called a *strictly interior* subsystem of $\mathbf{\Gamma}'$.

Theorem 4. *Assume that we have a mixed model satisfying the condition (3.1). Consider the partition functions ((3.18))*

$$Z^{\alpha}_{\mathcal{D},G}(\Lambda) = \sum_{\mathbf{\Gamma}:\operatorname{supp}\mathbf{\Gamma}=G} Z^{\alpha}_{\mathcal{D}\&\mathbf{\Gamma}}(\Lambda)$$

and

$$Z^{\alpha}_{\mathcal{D},[\,G\,]}(\Lambda) = Z^{\alpha}_{\mathcal{D}}(\Lambda) + Z^{\alpha}_{\mathcal{D},G}(\Lambda). \tag{3.29}$$

where $\mathcal{D}\&\mathbf{\Gamma}$ is an admissible system such that $\mathbf{\Gamma}$ is its strictly interior subsystem, satisfying moreover the condition $V_{\updownarrow}(\mathbf{\Gamma}) \cap \Lambda^c = \emptyset$. (The partition functions above correspond to the events "$\mathcal{D}$ & $\mathbf{\Gamma}$ appears" resp. "$\mathcal{D}$ & possibly $\mathbf{\Gamma}$ appears"; $\mathbf{\Gamma}$ is a contour such that $\operatorname{supp}\mathbf{\Gamma} = G$.*) These partition functions can be expressed as*

$$Z^{\alpha}_{\mathcal{D},G}(\Lambda) = (\sum_{T:G\in T} k^{+}_{T}) Z^{\alpha}_{\mathcal{D}}(\Lambda) \tag{3.30}$$

resp.

$$Z^{\alpha}_{\mathcal{D},[G]}(\Lambda) = \exp(\sum_{T:G\in T} k^{*}_{T}) Z^{\alpha}_{\mathcal{D}}(\Lambda) \tag{3.31}$$

where $G \in T$ means that G is the core of the cluster T and k^{+}_{T} resp. k^{*}_{T} are some new cluster terms, described in detail in the proof. The leading new quantity $k^{*}_{G} = k^{+}_{G}$ is equal to

$$k^{+}_{G} = k^{+}_{G,\alpha} = \sum_{\mathbf{\Gamma}:\operatorname{supp}\mathbf{\Gamma}=G} \exp(-F(\mathbf{\Gamma})). \tag{3.32}$$

The remaining quantities k^{+}_{T} obey the bounds, for any cluster of the type $T = (G, \{T_i\})$
(where T_i ; $i = 1, ..., m$ are clusters of the given mixed model)

$$|k^{+}_{T}| \le k^{+}_{G}\ \varepsilon^{\ \sum_{i=1}^{m} \operatorname{conn} T_i} \tag{3.33}$$

and analogous bounds are valid for k^{*}_{T}. See the proof ((3.41)) for the more precise form of the estimate on the right hand side; also for more complicated clusters T. The quantities k^{+}_{G}, k^{+}_{T}, k^{*}_{T} do not already depend on $\mathbf{\Gamma}$; they depend only on G and on the values δ_G[13] *and are translation invariant in the sense of (3.15).*

[13]See the preceding Lemma; the quantity k^{+}_{G} does not already depend on the interior of $\mathbf{\Gamma}$ in contrast to the quantity $\exp(-F(\mathbf{\Gamma}))$.

Moreover, if k_G^+ satisfies (3.7) then the validity of the bound (3.1) in the given mixed model implies its validity also for the new quantities k_T^+ and k_T^*.[14]

The procedure called "recoloring of a contour" described by formula (3.31) will be used later repeatedly many times for all the "smallest possible" strictly interior subsystems $\mathbf{\Gamma}$. The new cluster quantities k_T^* will play later the same role as the "old" quantities k_T); and therefore it will be crucial to ensure the validity of the estimate (3.1) for them: Comparing (3.33),(3.32) with (3.7) (Proposition of Section 1) one sees that the sufficient smallness of k_G^* will be guaranteed by the following estimate :

Definition. We will say that an admissible system $\mathbf{\Gamma}$ is *recolorable* if

$$\sum_{\mathbf{\Gamma}:\operatorname{supp}\mathbf{\Gamma}=G} \exp(-F(\mathbf{\Gamma})) \leq \exp(-\tau' \operatorname{conn} G) \tag{3.34}$$

holds with a sufficiently large τ' (to be specified in the Corollary below; see (3.35)). The constant τ' will be chosen below roughly as $\frac{\tau}{12\nu}$.

Notes. 1. Notice that this is the requirement on the *set* G, not on a particular contour $\mathbf{\Gamma}$ – though practically this is closely related to saying that for each $\mathbf{\Gamma}$ with the same support G, one has a bound

$$F(\mathbf{\Gamma}) \geq \tau'' \operatorname{conn} \mathbf{\Gamma} \tag{3.34"}$$

with some other large τ''.
2. One should always have in mind that the property "to be recolorable" is rather sensitive (for contours of *very* large size, of course) with respect to the boundary conditions. The mere knowledge of the external colour of $\mathbf{\Gamma}$ (at the horizontal level of $V(\mathbf{\Gamma})$) may be insufficient to decide the recolorability. In general, the external colour of the whole $V_{\updownarrow}(\mathbf{\Gamma})$ must be known.

Corollary. *Let $\mathbf{\Gamma}$ be recolorable, with τ' satisfying the inequality*

$$\exp(-\tau' \operatorname{conn} G) \leq \varepsilon^{\operatorname{Conn} T + 6\nu \log_2(\operatorname{diam} T + 6)}. \tag{3.35}$$

Then the validity of the bound (3.1) (with $\operatorname{conn} T$ defined by (3.6)) in the given mixed model implies its validity also for the new quantities k_T^+ and k_T^*.

Notes. This means that the new mixed model constructed by the formula (3.31) (which has a new, richer family of cluster fields $\{k_T\}\&\{k_T^*\}$ but which does not yet allow $\mathbf{\Gamma}$ as an interior subsystem of its configuration) is of the *same type* as before, satisfying again the estimate (3.1). The bound (3.35) relates the appropriate choices of constants ε, τ' in (3.1) and (3.34). The latter condition is a Peierls type condition for the quantity $F(\mathbf{\Gamma})$, and one has to choose a suitable $\tau' \leq \tau$ there, to obtain a useful notion. (τ is from (2.17).) This will be discussed later and we will see that when taking $\tau' = c\tau$ the convenient choice of the constant will be $c = \frac{1}{12\nu}$.

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[14] In the next section, we will formulate a "horizontally invariant" modification of Theorem 4, namely Theorem 5.

One should also notice that later, when performing our successive process of "recoloring of all (recolorable) $\mathbf{\Gamma}$", our procedure will be organized in such a way (see the forthcoming section) that quantities k^*_T with *new* clusters T (nonexistent with nonzero k_T in the previous mixed model) will appear at each stage of the construction.

Proof of Theorem 4. By (3.23) we have

$$Z^\alpha_{\mathcal{D},G}(\Lambda) = \sum_{\mathbf{\Gamma}:\operatorname{supp}\mathbf{\Gamma}=G} Z^\alpha_{\mathcal{D}\&\mathbf{\Gamma}}(\Lambda) = Z^\alpha_{\mathcal{D}}(\Lambda)\sum_{\mathbf{\Gamma}}\exp(-F(\mathbf{\Gamma}))\exp(\sum_T k^{\mathrm{new}}_T)\,.$$

Writing $\exp(-F(\mathbf{\Gamma})) = \xi_{\mathbf{\Gamma}} k^+_G$ and expanding the exponential this can be written as (3.36)

$$Z^\alpha_{\mathcal{D},G}(\Lambda) = Z^\alpha_{\mathcal{D}}(\Lambda)\Big(\sum_{\mathbf{\Gamma}}\xi_{\mathbf{\Gamma}}\sum_{k=0}^{\infty}\sum_{\substack{(T_1,\dots,T_k)\\(n_1,\dots,n_k)}}\prod_{i=1}^{k}\frac{1}{n_i!}(k^{\mathrm{new}}_{T_i})^{n_i}\Big)k^+_G = Z^\alpha_{\mathcal{D}}(\Lambda)(\sum_T k^+_T)$$

where the new values of k^{new}_T are denoted here as[15]

$$k^+_T = \sum_{\mathbf{\Gamma}:\operatorname{supp}\mathbf{\Gamma}=G}\xi_{\mathbf{\Gamma}} k^+_G\prod_{i=1}^{k}\frac{1}{n_i!}(k^{\mathrm{new}}_{T_i})^{n_i}. \tag{3.37}$$

Here, the clusters T are defined, for $n_i = 1$, as

$$T = (G, \{T_i\}). \tag{3.38}$$

(including also the empty collection $\{T_i\}$); otherwise the cluster T contains the corresponding number n_i of copies of T_i.

Notice that the expression of k^+_T by values k^{new}_T is not exactly as (3.7), Proposition of Section 1. However, it is straightforward to adapt the corresponding estimates noticing that $\sum_{\mathbf{\Gamma}}\xi_{\mathbf{\Gamma}} = 1$. Thus, one obtains (3.33) and then, from (3.34) and (3.35), also (3.1) for the quantity k^+_T. (If some $n_i > 1$ then the estimate is even a stronger one).

The expansion (3.31) is obtained by taking logarithms:

$$\log Z^\alpha_{\mathcal{D},[G]}(\Lambda) = \log Z^\alpha_{\mathcal{D}}(\Lambda) + \log(1 + \sum_{T:G\in T} k^+_T) = \log Z^\alpha_{\mathcal{D}}(\Lambda) + \sum_{T:G\in T} k^*_T \tag{3.39}$$

where

$$k^*_T = \frac{(-1)^{n-1}}{n}\prod_{i=1}^{n} k^+_{T_i} \tag{3.40}$$

[15] It is perhaps worth mentioning that whereas $F(\mathbf{\Gamma})$ depend on the values $\mathbf{\Gamma}$, in particular on the "interior colour of $\mathbf{\Gamma}$", the quantity k^+_T already depends only on G and on the "external colour" of $\mathbf{\Gamma}$.

for any cluster $T = (T_1, T_2, \ldots, T_n)$. (Again,one has to modify correspondingly this formula if multiple copies of one cluster T_i appear in T).

Strictly speaking, clusters of such a type were not defined yet. However, if e. g. three clusters T_1 , T_2 and T_2 have the same core G then we identify the ordered triple (T_1, T_2, T_3) for the clusters T_i given as $T_i = \{G, T_i^*\}$ (where T_i^* denotes some collection of clusters) with the cluster

$$\{G, T_1^*, \{G, T_2^*, \{G, T_3^*\}\}\}.$$

Now, for any such cluster $T = (T_1, T_2, \ldots, T_n)$ (recall that $G \in T_i$ for each i!) one obtains, after some inspection, the desired bound

$$|k_T^*| \leq (k_G^+)^n \ \varepsilon^{\sum_{i=1}^n \operatorname{conn}^+(T_i)} \leq \varepsilon^{\operatorname{conn} T} \tag{3.41}$$

where $\operatorname{conn}^+(T)$ denotes the quantity $\sum_j \operatorname{conn} T_j$ for $T = (G, \{T_j\})$; the last inequality follows from (3.34) and (3.35) by a similar argument as we used in the estimate of (3.37) above. This concludes the proof of Theorem 4 and of its Corollary.

4. Recoloring: towards a new stratified mixed model

The aim of this section is to formulate a procedure (based on Theorem 4) which converts a given *horizontally translation invariant* mixed model into a new horizontally translation invariant model, having the same "diluted "partition functions but with a *smaller* set of allowed configurations (and with a *richer* set of clusters T having nonzero contributions k_T; the "old" nonzero values k_T being kept at the *same value* as before). Such a transformation of the model could be characterized as the "removal, from the model, of all configurations which have a shift of $\mathbf{\Gamma}$ among its strictly interior subsystems".

Recall the ordering $\prec$ of systems of contours we introduced in section II (see (2.11) and the text below it). We will say that a recolorable system $\mathbf{\Gamma}$ of contours is *smallest recolorable* system of the given mixed model if it is strictly interior (in the considered volume Λ), recolorable and moreover there is no smaller strictly interior recolorable (smaller in $\prec$) $\mathbf{\Gamma}'$ which would appear in some configuration of the given mixed model.

Note. Below we will use the recoloring step formulated by Theorem 5 *successively*, according to the growing "size" of the smallest recolorable contours $\mathbf{\Gamma}$ which have to be recolored. Moreover, the mixed models studied by us later will appear as the result of successive recolorings applied to some given P. S. abstract model; the configuration spaces of the mixed models thus obtained will be defined in terms of requirements on the size of the smallest interior recolorable subsystems of the configuration. See the forthcoming sections for details.

Equivalent mixed models. Two mixed models will be said to be equivalent if all their strictly diluted partition functions are the *same*. (Usually, the configuration space of one of these two mixed models will be a suitable *subset* of the configuration space of the other model.)

Theorem 5. *Assume that we have a horizontally translation invariant mixed model satisfying the condition (3.1). Let* $\mathbf{\Gamma}$ *be a recolorable*[16] *subsystem whose horizontal shift can appear as a strictly interior subsystem of some configuration of the model. Assume that* $k_T = 0$ *holds for all clusters* T *containing a shift of* $\operatorname{supp} \mathbf{\Gamma}$ *("containing" in the sense that* $\operatorname{supp} \mathbf{\Gamma}$ *was used in the recursive construction of the cluster* T *).*

Then there is an equivalent mixed model having the following properties :

(1) *Its configuration space is the collection of all configurations of the original mixed model which do not contain a horizontal shift of* $\mathbf{\Gamma}$ *among its (smallest possible) strictly interior recolorable subsystems.*
(2) *If* T *is a cluster not containing* $\mathbf{\Gamma}$ *then the value of* k_T *in the new mixed model is the same as before.*
(3) *If* T *contains* $\mathbf{\Gamma}$ *then the new value of* k_T *satisfies the condition (3.1), too, assuming that* ε *and* τ' *are such that (3.35) holds.*

Proof. This follows from Theorem 4 if we use it successively in the following way: take the (completely ordered in $\prec$) sequence of all shifts of $\mathbf{\Gamma}$. Given $t \in \mathbb{Z}^{\nu}$ consider an intermediate "t – th model" which has the configuration space defined by the requirement that exactly those congigurations of the original mixed model are allowed for which *no* interior subsystems $\mathbf{\Gamma} + t'$ such that $t' \prec t$ exist. If s is the nearest greater point to t ($t \prec s$) in the given volume Λ (remember that we always work in finite volumes) then we define the transition to the "s – th model" by the very procedure described in Theorem 4.

It is straightforward to check the translation invariance (3.16) and unicity of the definition (not depending on the actual volume Λ if T is distant from its complement – in the sense of (3.16)) of the new quantities k_T thus obtained.

Note. There are also other methods to prove Theorem 5. Namely, it is possible (and it is, in fact, apparently a more standard way how to deal with these cluster expansions) to reformulate the Lemma and Theorem 4 above for a *simultaneous recoloring* of all the shifts of $\mathbf{\Gamma}$ *at once*. We do not follow such a (more direct, but with slightly more complicated formulas) approach here, in this paper. Such an approach is used also in the recent lectures [ZRO] (in a simplest possible form, we believe) and in future, we plan to replace the arguments based on the successsive use of the lexicographic order by this more standard approach.

5. Small and extremally small systems of contours

We are still working with a general mixed model. Only later we will explain the relevant choice of a mixed model in various concrete situations ; this choice will always be given as some partial expansion of the original abstract Pirogov – Sinai model which we are investigating.

The successive application of the recoloring procedures constructed in the preceding chapter will finally lead to a family (indexed, in any volume, by elements of $\mathbb{S}$) of mixed models where *no* recolorable systems will be left! The reader is advised

[16] smallest possible (in a geometrical sense); this will be the case needed in the applications of Theorem 5 below

to skip briefly to the section 8 and to look at the formulation and the proof of Main Theorem – to see the important consequences of this fact.

In the meantime, in sections 5 to 7, we will investigate the notion of a recolorable subsystem – and the related notion of a small resp. extremally small subsystem introduced in this section – in more depth, to obtain a useful supplementary "topological" result (Theorems 6 and 7) needed in the proof of the Main Theorem. Our discussion of the forthcoming notions of a "small" and an "extremally small" system, and the notion of a "skeleton" introduced below (notice that we are introducing there another testing quantity $A^*(\Box)$ – as a more careful alternative to $A(\mathbf{\Gamma})$) is perhaps slightly more detailed than absolutely necessary (if a shortest possible proof of Main Theorem is required). However, we are keeping it here as we expect our more detailed exposition to be useful not only here (giving more information and estimates with better constants) but also in the investigation of the "metastability" problem and in the study of the completeness of the phase picture constructed by our Main Theorem.The reader interested in acquiring the idea of the proof of Main Theorem can skip now directly to section 8 – omitting even the very notion of a small contour (given below) but finally realizing that some variant of such a notion (and the topological Theorem 7) is needed there!

Recall the definition of $A(\mathbf{\Gamma})$, $A_{\text{full}}(\mathbf{\Gamma})$, $A_{\text{loc}}(\mathbf{\Gamma})$ and of the corresponding quantities $F(\mathbf{\Gamma})$ from section 3, definition 2. When checking the condition (3.34) (through (3.34")) one needs inequalities of the type

$$A(\mathbf{\Gamma}) \le \tilde{\tau}|\operatorname{supp}\mathbf{\Gamma}|$$

where $\tilde{\tau}$ is a suitable, "not too large" constant (like $\frac{\tau}{2}$).

In fact, there is quite a freedom in the choice of the constants $\tilde{\tau}$, τ', τ'' (from (3.34) resp. from the relation (3.49) below) and the difference between the various variants of $A(\mathbf{\Gamma})$ resp. $F(\mathbf{\Gamma})$ (which was so important in Section 3) will be quite irrelevant here[17]. Namely, we obviously have the following bound.

Proposition.

$$|A(\mathbf{\Gamma}) - A_{\text{full}}(\mathbf{\Gamma})| \le \varepsilon'|\operatorname{supp}\mathbf{\Gamma}| \tag{3.42}$$

where the constant ε' is of the order ε^n, n being the cardinality of a smallest possible cluster appearing in (3.21) and $\varepsilon = C\exp(-\tau)$ for a suitable constant $C = C(\nu)$.

The quantities $A_{\text{full}}(\mathbf{\Gamma})$ (and, even more importantly, the quantities $A^*_{\text{full}}(\Box)$ introduced below) will be more convenient in these final sections than $A(\mathbf{\Gamma})$ and the bounds of the type $A(\mathbf{\Gamma}) \le \tilde{\tau}|\operatorname{supp}\mathbf{\Gamma}|$ will be studied for them instead of $A(\mathbf{\Gamma})$. Then we supplement these bounds by (3.42).[18]

The meaning of the quantities $A(\mathbf{\Gamma})$ is that they give some information about the "volume gain of the free energy" caused by the fact that inside $\mathbf{\Gamma}$, possibly some "more stable" regime is found. One could ask this question in a more precise way:

[17] except of the choice of $A_{\text{loc}}(\mathbf{\Gamma})$ – which would be too rough in some situations where some "reallly big" ceilings appear.

[18] The quantities $A(\mathbf{\Gamma})$ remain, of course, in (3.22) but their test "whether they are dangerously big" will now be done through the closely related but "more nicely looking" quantities $A_{\text{full}}(\mathbf{\Gamma})$.

whether the regime which resides inside $\mathbf{\Gamma}$ is the "best" possible and also what is the "energetically optimal realization" of such a contour. Fortunately, one does not need to investigate these questions in more detail, in particular the question "what is the optimal shape of a contour" is quite irrelevant here.

On the other hand, the question "what is the best regime to be found inside $\mathbf{\Gamma}$" *will* be important in the investigations below and we will approach it as follows: We rewrite, from now on, the quantity $A_{\text{full}}(\mathbf{\Gamma})$ in a more concise way, replacing the sum over *cluster* quantities k_T by a more nicely looking (and more flexible) sum of suitable *point* quantities. These latter quantities are however nonlocal (but very quickly converging limits of local quantities). Introduce the following notations.

Definition. For any mixed model and any stratified configuration y define the quantity

$$f_t(y) = e_t(y) - \sum_{T: t \in T} \frac{k_T(y)}{|T|}. \tag{3.43}$$

For an arbitrary nonstratified configuration x, define $f_t(x) = f_t(x_t^{\text{hor}})$ where x_t^{hor} is the horizontally invariant extension of the configuration $x_{(t_1, \ldots, t_{\nu-1}, (\cdot))}$.

Note. These quantities will be very important in the sequel. However, in spite of their "physical" meaning which we discover below (they will be interpreted as the "density, at t, of the free energy of the metastable state constructed around y") there is still some arbitrariness in their definition: For example, the modified quantity

$$\hat{f}_t(y) = e_t(y) - \sum_{t \in T} k_T(y) \tag{3.44}$$

where the sum is over all clusters T such that t is the first point of $\operatorname{supp} T$ in the lexicographic order could be used in the same way.[19]

Agreement. Here and below we need to work with configurations $y \in \mathbb{S}$ defined on the whole lattice $\mathbb{Z}^\nu$. Let us make an agreement that whenever we have a stratified configuration defined at the moment only in a partial way (typical situation: the external colour of some configuration defined in some finite volume, or in the interior of some bigger contour) then we extend it[20] in some prescribed (fixed for once) way to a configuration on the whole $\mathbb{Z}^\nu$. The details of the extension will be irrelevant.

Proposition. *The quantity $A_{\text{full}}(\mathbf{\Gamma})$ can be expressed also by the formula*

$$A_{\text{full}}(\mathbf{\Gamma}) = \sum_{t \in \mathbb{Z}^\nu} \left(f_t(x_{\mathbf{\Gamma}}^{\text{ext}}) - f_t(x_{\mathbf{\Gamma}})\right) + \Delta(\mathbf{\Gamma}) \quad \textit{where} \quad |\Delta(\mathbf{\Gamma})| \le \varepsilon' |\operatorname{supp} \mathbf{\Gamma}|. \tag{3.45}$$

Proof. This follows from the observation that any T "not touching $\operatorname{supp} \mathbf{\Gamma}$" which is counted in the definition of $A_{\text{full}}(\mathbf{\Gamma})$ is counted also (exactly once!) in the above

[19] Namely, the physically important quantities like $\sum_t (f_t(z) - f_t(y))$ are the *same* for both alternatives, whenever y and z are stratified and differing only on some layer of a finite width.

[20] In fact, this is a comparable act of arbitrariness like that we used in our choice of the sets $V_{\updownarrow}(\mathbf{\Gamma})$.

sum $\sum_{t\in\mathbb{Z}^\nu}(f_t(x_{\mathbf{\Gamma}}^{\mathrm{ext}})-f_t(x_{\mathbf{\Gamma}}))$. The corrections to this observation (they are needed only for clusters touching $\operatorname{supp}\mathbf{\Gamma}$) form, when summed together, the very term $\Delta(\mathbf{\Gamma})$ which therefore satisfies (being the sum of small and quickly decaying quantities $k_T|T|^{-1}$) the bound above.

The forthcoming notion will be useful for understanding what would be the "best possible gain in free energy" inside a given contour $\mathbf{\Gamma}$:

Definition. Given a configuration x which is stratified outside of some volume V denote by x_V^{best} the configuration minimizing, at each vertical section $(t_1,\dots,t_{\nu-1},(.))$, the value $\sum_{t_\nu\in\mathbb{Z}} f_t(x')$ under the condition that $x'=x$ outside of V. We will usually consider such a configuration x in a volume $V=V(\mathbf{\Gamma})$, where $\mathbf{\Gamma}$ is a contour or an admissible system.

Notice that here, in comparison to the formulation of the Peierls condition in Part II, we use the quantity $f_t(x')$ instead of $e_t(x')$. However, f_t is roughly equal to e_t and the sum of the terms (f_t-e_t) over $\operatorname{supp}\mathbf{\Gamma}$ is again (like in (3.42)) of the order $\varepsilon'|\operatorname{supp}\mathbf{\Gamma}|$ which is a quantity quite negligible when checking the validity of the Peierls condition.

Now we are able to define an alternative (with the same intuitive meaning) to $A(\mathbf{\Gamma})$ – which will be more flexible in the fortcoming estimates.

Definition. Given any finite volume[21] V and any $y\in\mathbb{S}$ introduce the quantity $A^*(V)$, more precisely $A^*_{\mathrm{full}}(V,y)$:

$$A^*_{\mathrm{full}}(V,y)=\max\{\sum_{t\in\mathbb{Z}^\nu}(f_t(y)-f_t(z))\}=\sum_{t\in\mathbb{Z}^\nu}(f_t(y)-f_t(y_V^{\mathrm{best}})) \tag{3.47}$$

where the maximum is taken over all z which are equal to y outside of V and where y_V^{best} is the configuration realizing this maximum.

Notes. 0. Compare (3.47) with (3.45) (for $V=V(\mathbf{\Gamma})$). We have, of course, the relation

$$|A^*_{\mathrm{full}}(V,y)-A(\mathbf{\Gamma})|\le\varepsilon'|\operatorname{supp}\mathbf{\Gamma}| \tag{3.48}$$

whenever $V=V(\mathbf{\Gamma})$ and y is the external colour of $\mathbf{\Gamma}$.
1. Our preference of this notion (to an alternative $A_{\mathrm{full}}(V)$ which was defined with the help of summation over *clusters* k_T) is mainly an aesthetical one. Namely, the sums of *point* quantities will be more convenient in later estimates. Notice that we do *not* require any stratification of z (and of y_V^{best}) inside V. Later (see the section Skeleton) we will introduce, for technical reasons, some new, artificial "contours" of the model. These new contours will have the shape of a (large) cube $\square$ living inside of some stratified regime of the actual "physical" configuration; their "energy" will be defined (typically, this energy will be of the order $\operatorname{diam}\square$ only) just to compensate the "volume gain inside $\square$" (steming from the fact that the configuration inside of such an artificial contour will be assumed to "jump freely to some better regime inside $\square$"). The interplay between these formal notions and

[21] We will use later this quantity not only for volumes $V=V(\mathbf{\Gamma})$ but also for *cubes* V.

between the behaviour of the real contours of the model can be best studied in the language of the quantities A^*_{full} .
2. Recall that the quantity A^*_{full} requires the knowledge of y in the whole $\mathbb{Z}^\nu$. Thus, there is still some arbitrariness in the definition of $A^*_{\text{full}}(V,y)$ because our y is usually given only in some finite volume. This arbitrariness is compensated by the more transparent form of the right hand side of (3.47) (compared to (3.21)).

Thus, when estimating the size of the quantities $A(\mathbf{\Gamma})$ we will work, from now on, with the more convenient quantities $A_{\text{full}}(\mathbf{\Gamma})$. or even $A^*_{\text{full}}(V(\mathbf{\Gamma}))$. In fact, it is advisable[22] to restrict the discussion of the "dangerously big value of $A(\mathbf{\Gamma})$" to the superordinated *cubes* only:

Definition. A cube $\Box$ will be called *small* with respect to a configuration $y \in \mathbb{S}$ if the following inequality[23] holds :

$$A^*(\Box, y) \leq \tau' \operatorname{diam} \Box \tag{3.49}$$

where τ' is something like $\tau' = \tau - \varepsilon'$ and ε' is from (3.45),(3.48). If y is given only partially (on some neighborhood of Λ; y is, in fact, *always* given in some finite volume only) then $\Box$ will be called small with respect to y if it is small for *some* stratified extension of y.

Say that a cube $\Box$ in $\mathbb{Z}^\nu$ is the *covering cube* of a volume $S \subset \mathbb{Z}^\nu$ resp. of a contour $\mathbf{\Gamma}$ if $\Box$ is the smallest possible cube (smallest in the usual ordering on the collection of cubes which is defined as an extension of *both* the lexicographic order of all the shifts of a single cube as well as the inclusion relation between cubes) which is a superset of S resp. of $V_\updownarrow(\mathbf{\Gamma})$. (Notice that we take $V_\updownarrow(\mathbf{\Gamma})$ instead of mere $V(\mathbf{\Gamma})$ here, the latter being equivalent to $\operatorname{supp} \mathbf{\Gamma}$). We will denote the covering cube $\Box$ by a symbol $\Box(S)$ resp. $\Box(\mathbf{\Gamma})$. If $\Box$ is the covering cube of $\mathbf{\Gamma}$ and y is the external colour of $\mathbf{\Gamma}$ then we will say that $\mathbf{\Gamma}$ is *small* if $\Box$ is small with respect to y. We will say that a strictly interior subsystem $\mathbf{\Gamma}$ of some admissible system $\mathcal{D}$ in Λ, under some boundary condition y given of $\partial\Lambda^c$, is small in Λ if $\mathbf{\Gamma}$ is small for the exterior colour of $\mathbf{\Gamma}$ induced by $\mathcal{D}$ and y.

Notes. 1. Recall that we take, here and everywhere in part III, the norm $|t| = \max\{|t_i|\}$.
2. The property "to be small" is formulated with the help of an (arbitrarily chosen, but fixed) extension of the external colour y of $\Box$ (extension to the whole $\mathbb{Z}^\nu$). It will *not* be formulated for subsystems which are not strictly interior in Λ. It is easy to see that for any small $\mathbf{\Gamma}$ we have the inequality, with y denoting the external colour of $\mathbf{\Gamma}$ (extended to the whole $\mathbb{Z}^\nu$, as mentioned above)

$$A^*_{\text{full}}(V(\mathbf{\Gamma}), y) \leq A^*_{\text{full}}(V_\updownarrow(\mathbf{\Gamma}), y) \leq A^*_{\text{full}}(\Box(\mathbf{\Gamma})) \leq \tau' \operatorname{diam} \mathbf{\Gamma}. \tag{3.50}$$

[22] They are some technical subtleties in this recommendation. They will be more clear later, after defining the notion of an extremally small system, in the proof of Theorem 6. See (3.55).

[23] Do not care about the particular choice of the constant $\approx \tau$ here. *Any* sufficiently big constant would do the job. On the other hand, the advisibility of our very choice of $\operatorname{diam} \mathbf{\Gamma}$ will be clear only later. We mention that the choice of $\partial\Box$ instead of $\operatorname{diam} \mathbf{\Gamma}$ here (such an alternative could maybe look more natural as the quantities $|\operatorname{supp} \mathbf{\Gamma}|$ appear otherwise anywhere whenever the energy of a contour is considered) would cause difficulties later.

Complement this with (3.42) and (3.45)! The idea now is, roughly speaking, that all small contours resp. admissible systems should be *recolorable*. This is obviously true for *contours* because we have from (3.50) and (3.42) the following inequalities (see (2.17*), (2.8) and the Proposition in part II, section 6)

$$F(\mathbf{\Gamma}) = E(\mathbf{\Gamma}) - A(\mathbf{\Gamma}) \geq (\tau^* - \varepsilon)|\operatorname{supp}\mathbf{\Gamma}| - \tau' \operatorname{diam}\mathbf{\Gamma} \geq \frac{\tau^*}{2}|\operatorname{supp}\mathbf{\Gamma}| \tag{3.51}$$

and the last term is greater than, say $\tau^*/4 \ \operatorname{con}\mathbf{\Gamma}$ because contours are (as we know from Theorem 2, part II) halfconnected.

However, we will often need to "recolor" also some *more complicated* interior admissible systems $\mathbf{\Gamma}$ with unclear apriori relation between $|\operatorname{supp}\mathbf{\Gamma}|$ and $\operatorname{con}\mathbf{\Gamma}$ (resp. $\operatorname{conn}\mathbf{\Gamma}$). In such a case, the corresponding more general argument (valid for any admissible system $\mathbf{\Gamma}$) will be developed in (3.54) below. However, the notion of "smallness" has to be modified here and the arguments (3.51) should be replaced by the more detailed bounds given below (in (3.60)).

Definition. We will say that a small, strictly interior subsystem $\mathbf{\Gamma}$ of a configuration $(x, \mathcal{D})$ is *extremally small* in $\mathcal{D}$ (more precisely in $(x, \mathcal{D})$) if it is small and moreover if the following recursive requirement for $\mathbf{\Gamma}$ is satisfied: for *no* strictly interior subsystem $\mathbf{\Gamma}' \subsetneqq \mathbf{\Gamma}$, $\mathbf{\Gamma}'$ is already extremally small. The recursion starts for those $\mathbf{\Gamma}$ for which there are no strictly interior $\mathbf{\Gamma}' \subsetneqq \mathbf{\Gamma}$ at all.

Notes. **1.** This is the point where our interpretation of contours as supercontours (recall that we replaced the original notion of a contour by a more elaborate notion of a supercontour already in part II, section 7) finally becomes useful. We can claim now that any admissible system $\mathcal{D}$ has the following property: after the removal of a removable subsystem $\mathbf{\Gamma}$ from $\mathcal{D}$, no removable subsystem $\mathcal{D}'$ of $\mathcal{D}$ remains such that $\mathcal{D}' \prec \mathbf{\Gamma}$ but $\mathbf{\Gamma} \to \mathcal{D}'$. This property will be highly desirable when recoloring the extremally small subsystems (which is something which we will do later, when proving that extremal smallness implies recolorability). Otherwise we could not use Theorem 5!
2. The adjective "small" resp. "extremally small" has only a loose relation to the actual *size* of $\operatorname{supp}\mathbf{\Gamma}$. It is in accordance with the usage of this term (and also of the related, perhaps even more confusingly sounding term "stable") in [Z]. There are other adjectives used to describe such a property in the literature – like "damped" in [K].
3. A typical example of an extremally small system $\mathcal{D}$ is a collection of the type $\mathcal{D} = \mathbf{\Gamma}_{\text{ext}} \& \{\mathbf{\Gamma}_i : \mathbf{\Gamma}_i \to \mathbf{\Gamma}_{ext}\}$ where the contours $\mathbf{\Gamma}_i$ are *not* small but the whole system $\mathcal{D}$ is small. The case $\{\mathbf{\Gamma}_i\} = \emptyset$ is the most common one, of course.
4. One should again emphasize that there is always some freedom in the definitions of these notions. For example the large quantities τ, τ' in the definition of a small resp. recolorable $\mathbf{\Gamma}$ can be changed – they can have even an " individual value " $\tau_{\mathbf{\Gamma}}$ ($\gg 1$) for any particular system $\mathbf{\Gamma}$ etc. These ambiguitites are more important than the arbitrariness of the choice of y in (3.49) but still have no "physical" meaning as they will *not* affect the (physically meaningful) notion of a stable phase used in Main Theorem.

Proposition. *A configuration $(x, \mathcal{D})$ of a mixed model whose all strictly external subsystems are small contains at least one extremally small subsystem.*

Proof. Consider the decomposition of $\mathcal{D}$ into minimal strictly external (by a strict externality of $\mathbf{\Gamma}$ in $\mathcal{D}$ we mean that $V_{\updownarrow}(\mathbf{\Gamma}) \cap \operatorname{supp}(\mathcal{D} \setminus \mathbf{\Gamma} = \emptyset)$ subsystems. Denote them as $\{\mathbf{\Gamma}_i \ ; i = 1, 2, \dots\}$. Take the smallest, in the geometrical sense of $\prec$, external subsystem, say $\mathbf{\Gamma}_1$, of this configuration. Now, if the collection of extremally small subsystems (equivalently, by induction, the collection of small subsystems) $\mathbf{\Gamma}' \subsetneqq \mathbf{\Gamma}_1$ of $\mathbf{\Gamma}_1$ would be empty then $\mathbf{\Gamma}_1$ itself would be extremally small!

The importance of the notion of a small resp. extremally small subsystem stems from the fact that extremally small subsystems provide practically the only relevant *example* of recolorable subsystems.[24] The following result gives such a statement. It is a crucial step (together with Theorem 5 above) in the proof of the forthcoming Main Theorem.

Theorem 6. *If* $\tau' = \frac{\tau}{12\nu}$ *then any extremally small subsystem is recolorable.*

(We mean the values from (2.17) and (3.34')).

Theorem 6 will imply that after the completion of the recoloring procedures of Theorem 5 (applied to the original P.S. model), *no* small contours or subsystems will be left in the final mixed model. This leads to the Main Theorem, see section 8.

6. The proof of Theorem 6

Say that $\mathbf{\Gamma}$ is *tight* if any its removable subcollection $\mathcal{D}$ satisfies the bound

$$\operatorname{dist}(\square(\mathcal{D}), \mathbf{\Gamma} \setminus \mathcal{D}) \leq \operatorname{diam} \square(\mathcal{D}). \tag{3.52}$$

(Recall that we take the l_∞ norm everyhere.)

We will show that the proof of Theorem 6 can be reduced to the case of extremally small tight systems $\mathbf{\Gamma}$. Then the quantity $A(\mathbf{\Gamma})$ in $F(\mathbf{\Gamma}) = E(\mathbf{\Gamma}) - A(\mathbf{\Gamma})$ will become "safely small" with respect to $E(\mathbf{\Gamma})$ (the quantity $A^*(\mathbf{\Gamma})$ will be even *nonpositive* in the most important case of the "stable" external colour of $\mathbf{\Gamma}$!) and Theorem 6 will be simply some *combinatorial statement* relating the values $|\operatorname{supp} \mathbf{\Gamma}|$ and $\operatorname{conn} \mathbf{\Gamma}$. See Theorem 7 below.

However, the case of systems $\mathbf{\Gamma}$ which are *not tight* is the more characteristic and important for the proof. The quantities $A(\mathbf{\Gamma})$ (more specifically, the quantities $A^*_{\text{full}}(\square)$) will then play an important role in this reduction to tight systems.

The skeleton of $\mathcal{D}$.

Consider the following auxiliary construction in any volume Λ, for any (locally) stratified configuration y on Λ. We will actually use it below with the special choice $\Lambda = V_{\updownarrow}(\mathcal{D}) \setminus \operatorname{supp} \mathcal{D}$ (for a contour or admissible system $\mathcal{D}$) and with y being the local colour induced by $\mathcal{D}$ on $V(\mathcal{D})$.

[24] One can construct, of course, examples of recolorable but not extremally small systems. They correspond, however, just to some marginal cases not covered by the particular choice of the constants $\tau, \tau', \tau(\mathbf{\Gamma})$ in the definition of a small contour $\mathbf{\Gamma}$.

Definition. Given a stratified configuration y and a cube $\square$ say that $\square$ is *minimal nonsmall* if $A^*(\square, y) > \tau'(\operatorname{diam}\square)$ (see (3.49)) and no smaller cube which is at the same time a subcube of $\square$ satisfies such a condition.[25]

Given a volume Λ, let us find some smallest possible (in $\prec$) minimal nonsmall cube $\square \subset \Lambda$ (if there is one). Take all the adjacent (having distance 1 to $\square$) cubes $\square' \subset \Lambda$ which are horizontal shifts of $\square$, then take all the adjacent, horizontally shifted cubes to the cubes just constructed etc. Thus we obtain some "layer" (only partially filled in Λ; cubes which would go outside of Λ are excluded!) of cubes inside Λ.

Construct also other possible partial layers *not touching* those constructed before, according to the rule that a layer with a *smallest possible* diameter of its "paving blocks" is constructed in each step.

The exact meaning of the statement that the layers would not mutually "touch" is that (e.g.) the vertical distance between any two adjacent layers is bigger than the *logarithm* of the thickness of both layers. Why we require this will be expained below.[26]

The collection of all minimal cubes of Λ thus constructed will be called the *skeleton* of Λ. The same construction can be defined, in analogous way, in any (generally nonstandard) volume with a given *locally stratified* configuration. In particular, a *skeleton* of the interior $V(\mathcal{D}) \setminus \operatorname{supp}\mathcal{D}$ of any *extremally small system* $\mathcal{D}$ can be thus constructed.

Note. The fact that skeleton has a "smallest possible grain" is maybe slightly superfluous here but it will surely be useful not only below but also in some other, more detailed estimates (like those used in the study of the completeness of the phase picture constructed by Main Theorem). In the situation where $\Lambda = \Lambda' \setminus \operatorname{supp}\mathcal{D}$, $\mathcal{D}$ extremally small, the smallest possible grain of the skeleton of $\mathcal{D}$ guarantees that the size of any interior (and therefore nonsmall) $\mathbf{\Gamma} \subset \mathcal{D}$ is *at least as big* as the size of the nearest neighboring cube from the skeleton (if there is one).

We conclude: for any extremally small $\mathcal{D}$ in a volume Λ with a stratified boundary condition y given on the boundary of the complement of Λ, we constructed the "skeleton" of the volume $V(\mathbf{\Gamma}) \setminus \operatorname{supp}\mathbf{\Gamma}$ which is composed of nonsmall *cubes*. These cubes are "densely packed" as formulated above.

Rearrangement of the energy of $\mathcal{D}$.

Let us define the following "rearrangement" of the energy of a given extremally small configuration $\mathcal{D}$: Imagine that the cubes of the skeleton are just some new "contours".

The idea is to show that such an "enrichened" system $\mathcal{D}^*$ of "contours" is *tight* in the sense (3.52), and the value of its contour functional $F(\mathcal{D}^*)$ (see (3.54) below) is *smaller* than that of $F(\mathcal{D})$; however $\operatorname{conn}\mathcal{D}^*$ is apparently bigger than $\operatorname{conn}\mathcal{D}$. Therefore, by checking the recolorability of the enriched system we will also prove

[25] Sometimes, for "stable" y (defined below in Main Theorem) such a cube will not exist; however this is not the case of a typical nontrivial situation below.

[26] The reason is to keep A^*_{full} roughly additive as a function defined on the union of cubes of the skeleton. We will see below that the logarithmic distance will assure this – because of the exponential decay of the terms k_T in the sums (3.43) used in the definition of the quantities f_t. See Lemma below (3.54).

the recolorability of the *original* system. This will give the desired generalization of the argument already given in (3.51) for the case when $\mathcal{D} = \mathbf{\Gamma}$ is a single contour.

Let us show this in more detail: Denote, as announced, by $\mathcal{D}^*$ the collection of all contours of $\mathcal{D}$ and also of *all the cubes* of the skeleton of $\mathcal{D}$. Let us make an agreement that for any cube $\square = \mathbf{\Gamma}$ we put (just to unify the notations in the formula (3.54) below!)

$$|\operatorname{supp}\mathbf{\Gamma}| = \operatorname{diam}\square. \tag{3.53}$$

With this notation, using the Peierls condition (2.17) and the definition (3.22) of $F(\mathcal{D})$ we can prove (3.34') by showing the following inequalities: Recall that

$$A^*(\square, x) \geq \tau' \operatorname{diam}\square \tag{3.49'}$$

holds for any cube of the skeleton. See (3.49); x denotes here the configuration induced by $\mathcal{D}$ on $\square$ (and extended somehow to the whole $\mathbb{Z}^\nu$). Now we have the following relation between the contour functionals of the original extremally small system $\mathcal{D}$ and the ("enrichened", by cubes of the skeleton) system $\mathcal{D}^*$:

Proposition. *We have the relation*

$$\tau|\operatorname{supp}\mathcal{D}| - A(\mathcal{D}) \geq \tau'|\operatorname{supp}\mathcal{D}| - A^*_{\text{full}}(\mathcal{D}) \geq \tau'|\operatorname{supp}\mathcal{D}^*| - A^*_{\text{full}}(\mathcal{D}^*). \tag{3.54}$$

To prove this, notice that the first inequality just informs us about the approximation of A by A^*_{full} (compare (3.42),(3.45) and (3.47)) while the second inequality will be shown now to be a *consequence of the very definition of the skeleton*. Write (3.49') as

$$\tau'|\operatorname{supp}\mathbf{\Gamma}| - A^*_{\text{full}}(\mathbf{\Gamma}) \leq 0 \tag{3.49"}$$

for any new "contour" $\mathbf{\Gamma} = \square$ of the skeleton of $\mathcal{D}$.[27]

The idea of the proof of (3.54) is that the terms $F(\mathcal{D})$ resp. more precisely $\tau'|\operatorname{supp}\mathcal{D}| - A^*_{\text{full}}(\mathcal{D})$ are essentially *additive* (as functions of the components $\mathbf{\Gamma}$ of $\mathcal{D}$). The additivity of the functions of the type $E(\mathcal{D}) = \sum E(\mathbf{\Gamma})$ (where the sum is over all contours of $\mathcal{D}$) is of course trivial. Concerning the approximate additivity of the function A^*_{full} we have the following auxiliary result.[28]

Lemma. *Let $\mathcal{D}^* = \mathcal{C} \cup \mathcal{S}$ be a compatible collection of contours $\mathcal{C} = \{\mathbf{\Gamma}_i\}$ and mutually disjoint cubes $\mathcal{S} = \{\square_j\}$ such that the cubes from $\mathcal{S}$ are not intersecting the contours of the system $\mathcal{C}$. Let the horizontal distance between any two cubes from $\mathcal{S}$ be greater or equal than the logarithm of the diameter of the smaller cube.*

[27] What follows will be just a suitable play with the quantities $A^*_{\text{full}}|\square|$ and $\tau'|\operatorname{diam}\square|$ (where the first quantity is replaced by the second one for any cube of the skeleton). We can interpret this replacement as an "installing of an artificial contour $\square$".

[28] The quantity $A(V)$ is of course *exactly* additive for disjoint volumes V but it would have some other, more severe disadvantages (than A^*_{full}) when the "surface tension" along the vertical sides of the cubes would be discussed.

Then (we take all the quantities A^*_{full} with respect to the corresponding external colour induced by $\mathcal{C}$)

$$A^*_{\text{full}}(\mathcal{D}^*) = A^*_{\text{full}}(\mathcal{C}) + \sum_{\Box \in \mathcal{S}} A^*_{\text{full}}(\Box S) + D(\mathcal{D}^*) \tag{3.55}$$

where the correction term D satisfies the bound (with some large $\tilde{\tau}$)

$$|D(\mathcal{D}^*)| \le \sum_{\Box \in \mathcal{S}} (\operatorname{diam} \Box)^{-\tilde{\tau}}. \tag{3.55D}$$

Proof. Consider two such collections $\mathcal{D}^*$ which differ just by *one cube* $\Box$ i.e. let $\mathcal{D}^{**} = \mathcal{D}^* \& \Box$. Assume that the cube $\Box$ is no greater than any cube of $\mathcal{D}^*$.

It is now sufficient to prove the bound

$$|D| = |A^*_{\text{full}}(\mathcal{D}^{**}) - A^*_{\text{full}}(\mathcal{D}^*) - A^*_{\text{full}}(\Box)| \le (\operatorname{diam} \Box)^{-\tilde{\tau}}. \tag{3.55D'}$$

Notice that (3.43) can be written also as the sum over intervals I

$$f_t(y) = e_t(y) - \sum_{I \subset \mathbb{Z} \,:\, t \in I} k_I \tag{3.43I}$$

where k_I is the sum of all the contributions to (3.43) having a fixed projection I of T to the last coordinate axis. Of course, we have a bound, for suitable large $\hat{\tau}$

$$|k_I| \le \exp(-\hat{\tau}|I|).$$

Imagine that in the expression on the left hand side of (3.55I) we *ignore* (when substituting these quantities, for any vertical section of $\Box$ and any t, into (3.47)) *all* I intersecting *both* $\Box$ and some *other* cube resp. contour of D^*. Then the relation

$$A^*_{\text{ignore}}(\mathcal{D}^{**}) - A^*_{\text{ignore}}(\mathcal{D}^*) - A^*_{\text{ignore}}(\Box) = 0 \tag{3.55I}$$

is *exact*, as simple inspection shows. The correction (due to the quantities k_I just ignored) is then obviously of the order $\exp(-\hat{\tau} d)$ where d is the distance of $\Box$ and the cubes from $\mathcal{D}^*$. This proves (3.55I), and therefore also (3.55D').

Let us continue now the investigation of the right hand side of (3.54): By (3.50), the right hand side of the relation (3.54) is *greater* than (we use here the very[29] smallness of the cube $\Box(\mathcal{D}^*)$!)

$$\tau' |\operatorname{supp} \mathcal{D}^*| - A^*_{\text{full}}(\Box(\mathbf{\Gamma}^*)) \ge \tau'(|\operatorname{supp} \mathcal{D}^*| - \operatorname{diam} \Box(\mathcal{D}^*)) \tag{3.56}$$

which is surely *greater* (notice that $\operatorname{diam} \Box(\mathcal{D}^*) = \operatorname{diam} \Box(\mathcal{D})!$) than, say, $\frac{\tau}{2}|\operatorname{supp} \mathcal{D}^*|$.

Thus it suffices to show now that the quantity $\frac{\tau}{2}|\operatorname{supp} \mathbf{\Gamma}^*|$ (and, therefore, also the right hand side of (3.57)) is greater than, say, $\frac{\tau}{12\nu} \operatorname{conn} \mathbf{\Gamma}$.

We will prove this by proving the following result (Theorem 7). First generalize the notion of a tight system $\mathbf{\Gamma}$ to any subset of $\mathbb{Z}^\nu$:

[29] Notice that our use of the *squares* in the definition of smallness is rather important technically. Namely, our method of the proof relies quite heavily here on the fact that the enriched (by cubes of the skeleton) system $\mathcal{D}^*$ of any small $\mathcal{D}$ is again a small system!

Tight sets.

Say that $S \subset T$ is isolated in T $(T \subset \mathbb{Z}^\nu)$ if

$$\operatorname{dist}(\square(V_{\updownarrow}(S)), T \setminus S) \geq \operatorname{diam} \square(V_{\updownarrow}(S)). \tag{3.52'}$$

Say that T is *tight* if there are no isolated subsets of T.

Notes. 1. The choice of $\square(V_{\updownarrow}(S))$ above is of course motivated by our definition of $\square(\mathbf{\Gamma})$ and our emphasis on the notion of a *strict* interiority and diluteness. "Topologically", there is not much difference between the choice above or the choice of the cube $\square(S)$.
2. Notice that the system $\operatorname{supp} \mathbf{\Gamma}^*$ is already tight because the definition of a skeleton of $\mathbf{\Gamma}$ gives no room for subsets of the type (3.52) for the enriched set $T = \operatorname{supp} \mathbf{\Gamma}^*$!

Theorem 7. *If T is tight then*

$$\operatorname{conn} T \leq 6\nu |T|. \tag{3.57}$$

Let us start the proof of Theorem 7 by the simple observation (used in Lemma 1, section 7 below) that any contour of $\mathbf{\Gamma}^*$ can be assumed to have a a cardinality at least $1024 = 2^{10}$. Really, it is rather straightforward to see the validity of a bound

$$\operatorname{conn} \mathcal{D} \leq 9 |\operatorname{supp} \mathcal{D}| \tag{3.57'}$$

for any interior tight subsystem $\mathcal{D}$ of $\mathbf{\Gamma}$ whose cardinality is $d \leq 1024$. To check this bound remember that our contours (supercontours in the sense of Chapter 2; this is the second moment – after the proof of Theorem 5 – where we profit from their properties) are such that $\mathbf{\Gamma} \to \mathbf{\Gamma}'$ implies that the cardinality of $\operatorname{supp} \mathbf{\Gamma}'$ is at least twice bigger than, say the cardinality of $\operatorname{supp} \mathbf{\Gamma}$. Thus, the longest branch of the forest $\mathcal{D}$ has at most 9 sites.
Now take any tree of $\mathcal{D}$. Retain the notation $\mathcal{D}$ for it, and for any subtree $\mathcal{E} \subset \mathcal{D}$ construct a suitable commensurate path connecting $\mathcal{E}$ to some superordinated contour of the remainder of $\mathcal{D}$. The length of any such path can be surely chosen smaller than, say, the cardinality of the support of $\mathcal{E}$:
Namely, consider the projection to $\mathbb{Z}^{\nu-1}$: then all the projections of contours of $\mathcal{D}$ are connected. Construct first such a connecting a path in the projection to $\mathbb{Z}^{\nu-1}$, with "horizontal" steps of length 1 (surely, less than $d = \operatorname{diam} \mathcal{E}$ steps are needed), and then add suitable vertical components to the already constructed horizontal steps to keep the successive steps commensurate and to guarantee that the path (not only its projection) really starts in $\mathcal{E}$ and ends in $\mathcal{D} \setminus \mathcal{E}$. The vertical distance to be overcomed is also smaller than $d = \operatorname{diam} \mathcal{E}$ and clearly, a suitable choice of d commensurate vertical steps (which can be understood as vertical components of the horizontal unit steps constructed above) overcoming the given vertical distance is also possible.

Now, when comparing the total length $\sum l(P)$ of the these paths $P = P(\mathcal{E})$, connecting any interior subtree $\mathcal{E} \subset \mathcal{D}$ to the remainder of the corresponding tree of $\mathcal{D}$, with the sum of the cardinalities $|\mathcal{E}|$ of the subtrees (which surely have at least as many points as is the length of the path $P(\mathcal{E})!$) we see that $\sum l(P(\mathcal{E})) < 9|\operatorname{supp}\mathcal{D}|$ because each point of $\mathcal{D}$ was used at most 9 times in the above consideration. (This statement is actually some "weaker" analogy of Theorem 7 for systems having contours with a cardinality less than 1024.)

Moreover, by the same argument one can connect any such "small sized" $\mathcal{D}$ to the superordinated tree of the remainder of $\mathbf{\Gamma}$ by a path containing no more than $|\operatorname{supp}\mathcal{D}|$ of additional steps. The conclusion is that $10|\operatorname{supp}\mathcal{D}|$ is surely the upper bound for the number of points needed to make each such $\mathcal{D}$ commensurately connected and connected also to remainder $\mathbf{\Gamma}\setminus\mathcal{D}$. Thus we can really restrict ourselves to the case when the smallest interior components of $\mathbf{\Gamma}$ have a cardinality at least 1024.

Note. Notice that after the removal of any such "small sized" subtree $\mathcal{D}$, the remainder is still extremally small (if the original admissible system was).

Thus it suffices to prove the desired inequality

$$\operatorname{conn}\mathbf{\Gamma}^* \leq 6\nu|\operatorname{supp}\mathbf{\Gamma}^*| \tag{3.58}$$

at the assumption that all the contours of $\mathbf{\Gamma}^*$ have already a (halfconnected: recall Theorem 2 and Proposition in section 2.7) support having a cardinality at least 1024. If we assume contours to be moreover *connected* then it suffices to show (in slightly more general setting) that the inequality

$$\operatorname{conn} S \leq 3\nu|S| \tag{3.59}$$

holds for any $S\subset\mathbb{Z}^\nu$ at the assumption that the components of S have cardinality at least 1024.

Apparently, the proof of such a geometrical statement will finish also the proof of Theorem 6. Namely, then we can conclude the arguments of (3.54) and (3.55) as follows:

$$\tau|\operatorname{supp}\mathcal{D}| - A(\mathcal{D}) \geq \tau'|\operatorname{supp}\mathcal{D}^*| - A^*_{\text{full}}(\mathcal{D}^*) \geq \tau'|\operatorname{supp}\mathcal{D}^*| - \tau'\operatorname{diam}\square(\mathcal{D}^*)$$

i.e.

$$\tau|\operatorname{supp}\mathcal{D}| - A(\mathcal{D}) \geq \frac{\tau}{2}|\operatorname{supp}\mathcal{D}^*| \geq \frac{\tau}{12\nu}\operatorname{conn}\mathcal{D}. \tag{3.60}$$

Proof of (3.59). Commensurately connected collections of cubes.

It will be useful to *reformulate* first the notion of a commensurately connected graph in an alternate language – based on the employment of *cubes* from $\mathbb{Z}^\nu$ instead of the bonds from $2^k\mathbb{Z}^\nu$:

Definition. Say that the two cubes $\square,\square'\subset\mathbb{Z}^\nu$ are commensurate if

$$\square\cap\square'\neq\emptyset \quad\text{and}\quad |\log_2\operatorname{diam}\square - \log_2\operatorname{diam}\square'| \;\leq\; 2. \tag{3.61}$$

(Notice that the constant 2^1 in (3.2) was replaced by 2^2 in (3.61). This is for purely technical reasons and will be convenient below.)

Proposition 1. *If G is a commensurately connected graph then the collection $\{\square(b), b \in G\}$ of covering cubes of bonds of G is commensurately connected in the sense above.*

This is immediate, by comparing (3.2) and (3.61).

The opposite relation (that any commensurately connected collection of cubes can be "approximated" by a commensurately connected graph) can be also established: Introduce first another auxiliary geometrical notion.

Definition. If $\square$ is a cube with a diameter $2^k \leq \operatorname{diam} \square < 2^{k+1}$ then the lexicographically first point of $\square \cap 2^k \mathbb{Z}^\nu$ will be called the anchor of $\square$, denoted by $a(\square)$.

Proposition 2. *Let $\mathcal{S} = \{\square_i\}$ be a commensurately connected collection of cubes from $\mathbb{Z}^\nu$. Then there is a commensurately connected tree $\mathcal{T}$ such that all the anchors $a(\square_i)$ are among the (possibly multiple) vertices of $\mathcal{T}$ and*

$$|\mathcal{T}| \leq 3\nu |\mathcal{S}|. \tag{3.62}$$

Proof. We may assume that $\mathcal{S}$ is already a tree. Take any commensurate bond $\{\square, \square'\} \in \mathcal{S}$. Write $[\log_2 \operatorname{diam} \square] = k, [\log_2 \operatorname{diam} \square'] = k'$; we may assume that $k' \in \{k, k+1, k+2\}$. A straightforward inspection shows that it suffices to consider the case $\nu = 1, k = 1$, and $a(\square') = 0$. Notice that then the following path from $a = a(\square)$ to $a' = a(\square')$ can always be constructed :

$$a' = a + v_1 + v_2 + v_3 \tag{3.62}$$

where the vectors v_i having the lengths $2^{l_i}; l_i \in \mathbb{N}$ satisfy the following requirements:

$$k \leq l_1 \leq k+1,\ l_1 - 1 \leq l_2 \leq l_1 + 1,\ l_2 - 1 \leq l_3 \leq l_2 + 1,\ k' \leq l_3 \leq k' + 1.$$

It is clear that the tree defined by all the bonds $\{a, a+v_1\}, \{a+v_1, a+v_1+v_2\}, \{a+v_1+v_2, a'\}$ (where a, a' vary over all commensurate pairs $\square, \square'$ and the triple of the type above is actually repeated in the direction of any coordinate axis for $\nu \geq 1$) is commensurately connected.

Corollary. *Let $S \subset \mathbb{Z}^\nu$. Let $\operatorname{conn}_\square$ denote the cardinality of a smallest commensurately connected collection of cubes satisfying the following requirement: if all points of S are added (we identify the points of $\mathbb{Z}^\nu$ as cubes of diameter 1) then the whole collection is commensurately connected. Then*

$$\operatorname{conn}_\square S \geq \frac{1}{3\nu} (\operatorname{conn} S). \tag{3.63}$$

We will now prove Theorem 7 by showing that the inequality

$$\operatorname{conn}_\square S \leq |S| \tag{3.64}$$

holds for any tight set S whose components have a cardinality at least 1024.

Second covering cube.

Define now a suitable collection of cubes having a size 2^k where $k = 1, 2, 3, \ldots$ such that any cube in $\mathbb{Z}^\nu$ can be "packed", with a reasonable "accuracy", by some cube of the collection:

Definition. Denote by $\mathcal{K}_k$ the collection of all cubes in $\mathbb{Z}^\nu$ which are shifts, by suitable values from the lattice $2^{k-1}\mathbb{Z}^\nu$, of the unit cube $[0, 2^k]^\nu$ in the lattice $2^k\mathbb{Z}^\nu$. Write $\mathcal{K} = \cup_k \mathcal{K}_k$ where $k = 1, 2, \ldots$. We have the natural ordering $\prec$ on $\mathcal{K}$ extending the ordering by size resp. the lexicographic ordering of shifts of one particular cube; this ordering can be extended to suitable total ordering of *all* cubes in $\mathbb{Z}^\nu$ which is in accordance with the inclusion relation as well as with the lexicographic order of mutually shifted cubes, and we denote by $\widehat{\square}$ the (lexicographically first) cube from $\mathcal{K}_k$, k smallest possible, containing $\square$. This will be called the *second covering cube* of $\square$ resp. of a set S such that $\square = \square(S)$.

Notice the following fact : if $\square$ is the covering cube of S then the second covering cube of S contains $\square$ and has a diameter at most four times bigger than $\square$.

7. The Proof of (3.64)

Let us make an agreement that an explicit choice of the appropriate constants here will be given below only for the case of the dimension $\nu = 3$. Apparently, for $\nu > 3$ the final constant in Theorem 7 is even better – but we do not care yet.

Black and grey cubes of a set $S \subset \mathbb{Z}^\nu$. Say that a cube $\square$ is a black cube of $S \subset \mathbb{Z}^\nu$ if $\square \cap S$ contains at least $4(\operatorname{diam}\square)^{\frac{1}{2}}$ points, resp. at least $\operatorname{diam}\square$ points if its diameter is smaller than 16. Any cube *from* $\mathcal{K}$ which is the second covering cube of some black cube of S will be called the grey cube of S. (It obviously contains at least $2(d')^{\frac{1}{2}}$ points of S where $d' \leq 4d$ denotes the diameter of the corresponding grey resp. black cube.) We will show the following statements :

Lemma 1. *If T is tight and its connected components have a diameter at least* 16 *then the collection of its black cubes is commensurately connected.*

Lemma 2. *The number of grey cubes of a size at least* 1024 *of any set $S \subset \mathbb{Z}^\nu$ is no greater than* $1/2\ |S|$.

Lemma 3. *If $\mathcal{S} = \{\square_i\}$ is a commensurately collected collection (of black cubes of some set S) then the collection $\widehat{\mathcal{S}} = \{\widehat{\square_i}\}$ (of grey cubes of S) is contained in some commensurately connected collection $\mathcal{S}'$ such that $|\mathcal{S}'| < 2|\widehat{\mathcal{S}}|$.*

Proof of Lemma 1. We will proceed by the induction over the number of points in T. Say that S is a nice subset of T if it has the following property : for any $t \in S$ there is a commensurately connected collection of black cubes $\{\square_i\}$ of S which is concentric (i.e. $t \in \square_i$ for each i), starts in the covering cube of S and ends in t.

Take some maximal nice subset S of T. We will show that $S = T$ if T is tight. Really, if $N = T \setminus S$ is nonempty then either there is some isolated subset M of N or N is tight. In the former case take M as the smallest possible isolated (and therefore tight) subset of N. Then M (or N itself, in the latter case) is also nice by the induction assumption. Take the covering cube $\square(M)$ of M.

We claim now that there is some black cube $\Box'$ of S such that

$$\text{(3.65)}\quad \operatorname{dist}(\Box(M),\Box') < \operatorname{diam}\Box(M) \quad\text{and}\quad |\log_2 \operatorname{diam}\Box(M) - \log_2 \operatorname{diam}\Box'| \leq 1.$$

This follows from the fact that M can*not* be isolated in $S \cup M$ (otherwise M would be isolated also in T). Therefore, there is some $t \in S$ whose distance from $\Box(M)$ is no greater than $\operatorname{diam}\Box(M)$ and we take an appropriately large black cube $\Box' \ni t$. (Its existence is guaranteed by the "nicety" of S.)

Now, if $\Box^*$ is any black supercube of $\Box'$ (black in S) then the supercube of $\Box(M) \cup \Box^*$ – denoted as $\Box^{**}$–will be shown to be again a black cube (of the whole set T) and this would mean that $S \cup N$ would be nice, as simple inspection shows.(Check that there is now a commensurate path from t to $\Box(M)$ and any commensurate chain of cubes going "up" from $\Box(M)$ through cubes of the type $\Box^*$ can be modified by going through corresponding cubes $\Box^{**}$. Thus $S = T$. The modified chain is clearly also a commensurate one, containing $\Box(M)$.)

The observation that $\Box^{**}$ is black in T follows from the following more general statement :

Lemma. *If $\Box', \Box''$ ($\Box'' = \Box(M)$ in the above application)are two cubes which are black cubes of some sets T', T'' ; $T' \cap T'' = \emptyset$ and such that* $\operatorname{dist}(\Box',\Box'') \leq \operatorname{diam}\Box'$ *and* $|\log_2 \operatorname{diam}\Box'' - \log_2 \operatorname{diam}\Box'| \leq 1$ *then the covering cube $\Box(\Box' \cup \Box'')$ is the black cube of the set $T = T' \cup T''$.*

This is easily seen (it suffices to consider the case of the dimension $\nu = 1!$) from the inequality (d denotes the distance between two cubes of diameters 1 ($\Box'$) resp. x ($\Box''$))

$$\text{(3.66)}\qquad d \leq 1 \;\&\; 1/2 \leq x \leq 2 \Rightarrow (1 + x + d)^{\frac{1}{2}} \leq 1 + x^{\frac{1}{2}}.$$

Proof of Lemma 2. We will give the proof only for the case $\nu = 3$. For the purposes of this proof modify the cubes $a + [0, 2^k]^\nu$ from $\mathcal{K}_k$ to the following form: $a + [0, 2^k - 1]^\nu$. Then the system $\mathcal{K}_k$ can be decomposed into 8 pavings of $\mathbb{Z}^\nu$ by disjoint sets. Take the sum

$$\text{(3.70)}\qquad \frac{8}{2}\sum_{10}^{\infty} \frac{1}{2^{\frac{k}{2}}} < 7/16$$

and imagine that any point of $t \in S$ transfers the $\frac{1}{2}2^{\frac{-k}{2}}$ – th portion of its "unit mass" (3.70) to any cube of $\mathcal{K}_k$ containing t. By the definition of a grey cube, the total mass thus transferred to any grey cube of S is at least 1 and, therefore, the cardinality of the set of all grey cubes of S (of a diameter bigger than 1024) is smaller than $7/16\ |S| \leq 1/2\ |S|$.

Proof of Lemma 3. This easily follows from the following observation : if $\Box, \Box'$ are commensurate cubes then either their second covering cubes $\widehat{\Box}, \widehat{\Box'}$ are also commensurate or one of these latter two cubes can be replaced by an auxiliary, twice bigger supercube from $\mathcal{K}$ such that the first statement is true.

This concludes the proof of Theorem 7 if we moreover notice, that to connect all the components of S to their black supercubes we can construct commensurate paths with less than, say, $1/2\ |S|$ steps.

Thus, also Theorem 6 is proven.

Note. Theorem 6 is a stronger and better replacement for the "Main Lemma" of [Z].One could use its analogy also in the translation invariant situation of [Z]. Then it can have (e.g.) the following form:

If $\mathbf{\Gamma}_i$ are mutually external "large contours" such that $a|V(\mathbf{\Gamma}_i) > \tau|\operatorname{supp}\mathbf{\Gamma}_i|$ and if we denote by $\mathrm{ext} = \Lambda \setminus \cup V(\mathbf{\Gamma}_i)$ then

$$a|\,\mathrm{ext}\,| > \tau\,\mathrm{Conn}(\Lambda, \{\mathbf{\Gamma}_i\}) \tag{3.71}$$

where the integer on the right hand side denotes the cardinality of a smallest possible set whose union with Λ^c and all $\operatorname{supp}\mathbf{\Gamma}_i$ is connected.

With this lemma, one can rewrite the ordinary P.S. theory in a way analogous to that used here *without* an explicit construction of the contour models. See the lectures notes [ZRO].

8. The Metastable Model. The Main Theorem.

Theorems 5 and 6 imply that the process of recoloring does not stop up to the very moment when there are *no* recolorable removable subsystems available in the final mixed model. This final mixed model will be called the (totally) *expanded Pirogov – Sinai model*, corresponding to the original abstract Pirogov – Sinai model with the hamiltonian (2.16) satisfying (2.17). Thus we have the following result.

Corollary. *In the expanded model, only those configurations of contours remain which contain no strictly interior recolorable subsystems. In other words, which are represented (Theorem 1) by a graph (without loops) whose any complete subgraph (complete in the relation $\mathbf{\Gamma} \to \mathbf{\Gamma}'$: if the end of the arrow is in the subgraph then the whole arrow is also in the subgraph) which is moreover strictly interior is nonrecolorable (therefore also nonsmall).*

Definition. Configurations of contours of the type above will be called *redundant*.

Note. An equivalent characterization of redundancy in terms of the related notion of metastability will be given below. Notice that the extremal smallness of $\mathbf{\Gamma}$ means just the *smallness of* $\mathbf{\Gamma}$ & *redundancy of all strictly interior subsystems of* $\mathbf{\Gamma}$. Speaking about the nonsmallness of some $\mathbf{\Gamma}$ in Corollary above one has in mind the nonsmallness of $\mathbf{\Gamma}$ in the "provisional" mixed model, constructed up to the moment when the extremal smallness of $\mathbf{\Gamma}$ is checked before applying Theorem 5. In this provisional mixed model, there are already *no* extremally small systems which would be "geometrically smaller than $\mathbf{\Gamma}$" (in $\prec$).

By the way, nonsmallness of a contour (or admissible system) $\mathbf{\Gamma}$ in the final, fully expanded model and in the provisional one would mean almost the same. More precisely we note that for cubes $\square$ of a comparable (or smaller) size as $\mathbf{\Gamma}$ the quantities $A^*(\square)$ and $A^*(\square)$ are practically the same for both the provisional model (expanded "up to the size of $\mathbf{\Gamma}$") and the final expanded model. (The quantities $A(V)$ are *exactly* the same for volumes $V \subset V(\mathbf{\Gamma})$, in both models.) Namely, the clusters T having a different value in both considered mixed models (the provisional one and the final one) have a size at least $\operatorname{diam}\mathbf{\Gamma}$ and the difference between the corresponding values of $A(\mathbf{\Gamma})$ is thus of the order $\varepsilon^{\operatorname{diam}\mathbf{\Gamma}}$ (a tiny quantity compared to $\tau|\operatorname{supp}\mathbf{\Gamma}|$).

This fact (together with the fact that that A^* and A are almost the same in any mixed model) will be used later in the proof of Main Theorem (relation (3.81) below) and also in (3.90).

The metastable models. Say that a subcollection $\mathcal{D}$ of an admissible system $\tilde{\mathcal{D}}$ is an *external* one if $\tilde{\mathcal{D}} \setminus \mathcal{D}$ can be decomposed into disjoint collections $\mathcal{D}'$, $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$, $\dots$ such that $(V(\mathcal{D}) \cup_i V(\mathbf{\Gamma}_i)) \cap V(\mathcal{D}') = \emptyset$ and moreover $\mathbf{\Gamma}_i \to \mathcal{D}$ for any $i \geq 1$. A configuration $x = (x_{\text{best}}, \mathbf{\Gamma})$ which is y – diluted (i.e. equal to $y \in \mathcal{S}$ outside of some set having finite components which are standard sets) will be called *y–th metastable* (shortly, metastable) if no redundant external subsystem $\mathcal{D}$ of $\mathbf{\Gamma}$ exists. The restriction of the original abstract Pirogov – Sinai model (with the hamiltonian (2.16)) to all y–th metastable configurations will be called the y–th metastable model. If the abstract Pirogov – Sinai model was constructed as a representation of an original model (2.2) then we define the metastable y– th submodel of (2.2) as the restriction of the hamiltonian (2.2) to the configuration space $\mathbb{X}^y_{\text{meta}}$ of all diluted configurations $x \in \mathbb{X}$ whose representations $x = (x_{\text{best}}, \mathbf{\Gamma})$ in terms of their contours are y–th metastable in the sense above. Now we can also say that an admissible system $\mathcal{D}$ is redundant if and only if it has *no removable metastable subsystems*.

Notation. Recall the quantities $f_t(y)$, $y \in \mathcal{S}$ – see (3.43) – which were constructed for any mixed model. Consider now these quantities for the case of the totally expanded model whose construction was just finished. For the expanded model corresponding to the original Pirogov – Sinai abstract model, the quantity $f_t(y)$ will be denoted as

$$f_t(y) \;=\; h_t(y). \tag{3.72}$$

Now we are able to formulate the basic result of the paper:

Stable elements of $\mathcal{S}$. A stratified configuration $y \in \mathcal{S}$ for which there is *no* redundant contour or admissible system $\mathbf{\Gamma}$ such that $(x_{\mathbf{\Gamma}})^{\text{ext}} = y$ will be called *stable*.
In other words, y is stable if the collection of configurations $x \neq y$ having the value y outside Λ is empty in the fully expanded model.

Main Theorem. *Consider an abstract Pirogov – Sinai model defined by (2.16) and (2.17), with τ sufficiently large. Then the quantities $h_t(y)$ constructed by (3.72) are the free energies of the corresponding y– th metastable models. The configurations $y \in \mathbb{S}$ which are stable correspond precisely to those configurations from $\mathbb{S}$ which are the ground states of the quantity*

$$\sum_{t' \in [t]} h_{t'}(y) \tag{3.74}$$

where $[\,t\,]$ *denotes the collection of all* $t' = t + (0, 0, \ldots, 0, t_\nu); t_\nu \in \mathbb{Z}$. *The "ground state" is understood in the sense that we always have*

$$\sum_{t' \in [\,t\,]} (h_{t'}(\tilde{y}) - h_{t'}(y)) \geq 0 \tag{3.75}$$

if $\tilde{y} \in \mathbb{S}$ *differs from* y *on a layer of a finite width. For any stable* y *there exists a probability measure* P^y *on the configuration space of the given abstract Pirogov – Sinaimodel whose almost all configurations are* y*-th metastable and moreover, the conditioned probabilities* P^q_Λ *of* P^y*, being taken with respect to all configurations which are* y *– diluted resp. strongly diluted in* Λ *correspond to the diluted resp. strictly diluted ensembles (2.18).*

Corollary. *If the considered abstract Pirogov – Sinai model represents a "physical" model given by hamiltonian (2.2) then for any stable* y *there exists a Gibbs state (of the model (2.2)) on* $\mathbb{X} = S^{\mathbb{Z}^\nu}$ *whose support can be identified as a suitable subset of the collection* $\mathbb{X}^q_{\mathrm{meta}}$ *of all* y*-th metastable configurations.*

By a support of a probability measure we mean a Borel (more precisely countably compact) set having measure 1.

Notes. **0.** Clearly, the families of configurations $\mathbb{X}^y_{\mathrm{meta}}$ are mutually disjoint for different $y \in \mathbb{S}$. We do not study here in much detail the structure of a typical configuration (of a q – th Gibbs state) here. See, however, the final section 9 for some information. (This problem deserves a more full treatment. However, it seems reasonable to do this in connection with some future investigation of other related questions – like the completeness of the phase picture constructed here. We plan to devote a separate paper to these questions.)

1. There are no other stratified Gibbs states of such an abstract model. We are not giving here the proof of such a completeness of our phase picture (characterized by the stable values of $y \in \mathcal{S}$).It can be done similarly as in [Z]. See also some comments in section Concluding Notes below. However, we plan a more systematic treatment of this and related questions in a separate paper.

2. By the phrase "the y – th Gibbs state can be identified with the corresponding y – th metastable model" we mean that almost all configurations of this Gibbs state are y – diluted and moreover the "islands"(the components of $V_\updownarrow(\mathbf{\Gamma})$ where $\mathbf{\Gamma}$ is the collection of all contours of the the considered configuration x ; this covers the set of all points of x which are *not* y – stratified) are typically "small" and "rare"(but distributed with a uniform density throughout $\mathbb{Z}^\nu$). For a more complete statement, see the section 9 below.

3. Having in mind that the quantities $h_t(y)$ can be effectively computed from expansions (3.43) (within a given precision; of course this is in full a horribly complicated sum – but its terms are converging *very* quickly, indeed), our Main Theorem gives in fact a *constructive criterion* for finding the stable values $y \in \mathcal{S}$.

Practically, one may suggest an "approximate finding" of stable values of y from some "M–expanded" model (M is some square, for example) where only those extremally small subsystems whose size does *not* exceed the size of M are already recolored.

In fact, even for squares M quite small some useful approximations can be found, often enabling already to distinguish between the stable and nonstable y. This is because the series (3.43) are really very quickly converging and moreover we often have some additional symmetry in the special cases of interest – like the $+/-$ symmetry in some special cases of the Blume Capel models. (For Blume Capel models, even the smallest size 2×2 of the square M can be useful – see [BS]. Namely, considering only first two or three terms in (3.43) a correct conclusion about the stability of 0 on one side and $+/-$ on the other side can be made.)

4. In fact, in finite volumes Λ there is no noticeable difference between the behaviour of the stable y –th phase and another $\tilde{y}$ –th phase if the quantity

$$a = \sum_{t\in[\,t\,]} h_t(\tilde{y}) - h_t(y)$$

is such that, say, $a^{-1} > (\operatorname{diam}\Lambda)^{\nu-1}$. Quite straightforward estimate of the quantities

$$A(\Box) \le p(V(\Box)) \le \tau \operatorname{diam}\Box \le \tau(\operatorname{diam}\Lambda)^{\nu-1} p(V(\Box)) \tag{3.76}$$

where p denotes the orthogonal projection on $\mathbb{Z}^{\nu-1} \subset \mathbb{Z}^{\nu}$ shows that equations of the type (3.49) cannot be violated in such a small volume.("Small" can have a meaning "having a diameter of the order 10^{27}" here if a is very small; namely the difference between various h^y is only of the order $\exp(-F(\mathbf{\Gamma}))$) if e^y are the same and $\mathbf{\Gamma}$ is the smallest contour "which makes the difference" between two different y.)

5. Though the question of the existence of *at least one* stable $y \in \mathcal{S}$ is not the absolutely crucial one (as the preceding note shows) one should mention, nevertheless, that at least one stable $y \in \mathcal{S}$ really *does exist*:

Say that $y \in \mathcal{S}$ is N-ground if for any $\tilde{y} \in \mathcal{S}$ such that $\tilde{y} = y$ for $|t_\nu| \ge N$ we have the inequality (3.75). Now, if the configurations $y^N \in \mathcal{S}$ are N-ground then a suitable subsequence of $\{y^N\}$ must converge to some ground ("stable") value $y \in \mathcal{S}$. (We use the compactness of the space $\mathcal{S}$ in this argument as well as the quick convergence of the cluster series for the quantities $h_t(y)$).

6. For some models, like the Ising model with stratified random external field, the collection of all (almost) stable $y \in \mathcal{S}$ can be *very rich* and the phase diagram – as the function of all (vertically dependent) values of the field – extremely complicated. We plan to study this particular case in some later paper.

7. The latter example case shows that it is not very reasonable to try to formulate results about the shape of the *phase diagram* in full generality here – because possibly infinite parameters are present in the hamiltonians of the stratified type. However, if we call by a phase diagram of the model the very *mapping*

$$\{\ \ y \in \mathcal{S} \ \longmapsto \{h_t(y)\}\ \ \} \tag{3.79}$$

then the information about the actual phase diagram, its dependence on the parameters in the Hamiltonian etc. can be deduced from (3.79) ; just by using suitable variants of the implicit function theorem (possibly with infinitely many variables).

However, this is not a paper on analysis of manifolds and so we omit these questions completely.

It is worth noticing here that, in order to get a best possible smoothness of the mapping (3.79) (and of the mappings derived from it by implicit function theorems), it may be reasonable to *modify* suitably the definitions of extremally small contours etc. – to obtain the best available differentiability (even local analyticity) properties of this mapping. This question also deserves a separate study, like in [ZA].

Proof of Main Theorem. The key statements were already Theorems 5 and 6 above – which guarantee the very existence of the fully expanded model and therefore the possibility of the very formulation (based on the existence of the quantities h_t) of our result. Noticing this, one has to add now only a few additional observations :

1) If y satisfies (3.75) then for any admissible system $\mathbf{\Gamma}$ such that $(x_{\mathbf{\Gamma}})^{\text{ext}} = y$ we have, from the very definition of the quantity $A^*(\square)$, $\square = \square(\mathbf{\Gamma})$ (see (3.37) and the commentary below) the inequality

$$A^*(\square) < \varepsilon \operatorname{diam} \square. \tag{3.80}$$

Really, one could take even *zero* on the right hand side if the appropriate partially expanded model (namely the mixed model studied at the moment when the smallness of $\mathbf{\Gamma}$ was discussed)could be taken here. However, we have a different mixed model now (at the end of the recoloring procedures) – the fully expanded one – and we have to use (3.36)) to notice that

$$|A^*_{\text{partially expanded}}(\square) - A^*_{\text{expanded}}\,(\square)| \leq \varepsilon' |\operatorname{diam} \square| \tag{3.81}$$

where ε' is *very* small, of the size $\varepsilon^{\operatorname{diam} \square}$! Thus we see, that the smallness of any $\mathbf{\Gamma}$ with the external colour y satisfying (3.75) is almost a tautology and therefore, under such boundary condition y, there is *no* difference between the original Pirogov – Sinaimodel and the metastable one.

Moreover one has quickly converging expansions of partition functions with the boundary condition y in any volume, and this implies the validity of the properties of the "y – th Gibbs state" stated in the Main Theorem. One could prove also the exponential decay of correlations in any Gibbs state thus constructed. See the last section 9 below for some relations (namely (3.90)) proving (or, at least, preparing a ground for the proof) of these facts.

2) On the other hand, if y is *not* a ground state then there is some $\tilde{y}$ differing from y on some layer L of a finite width – say d – such that the vertical sum is

$$\sum_{t' \in [t]} h_{t'}(y) - h_{t'}(\tilde{y}) \geq \delta \tag{3.82}$$

for suitable $\delta > 0$. Take a very large box $B \subset \mathbb{Z}^{\nu-1}$ such that

$$\tau |\partial B| << \delta |B| \tag{3.83}$$

and consider the volume (which has the form of a "desk")

$$\Lambda = \{t \in \tilde{L}, \hat{t} \in B\}$$

where $\tilde{L}$ is another (thicker) layer containing L "sufficiently inside". Take the configuration $x = y$ outside Λ, $x = \tilde{y}$ inside Λ. Then, if we compute the quantity $A^*(\Lambda)$ for the volume

$$\Lambda = (\partial B \times \mathbb{Z}) \cap L$$

we have, according to (3.45), the bound from below:

$$(3.84)\quad A^*(\mathbf{\Gamma}) = \sum_{t\in\Lambda}(h_t(y) - h_t(\tilde{y})) \geq \delta|B| - \varepsilon|\operatorname{supp}\mathbf{\Gamma}| >> \tau d|\partial B| >> \tau|\operatorname{diam}\Lambda|$$

according to (3.49) which shows (compute $A^*(\Box)$ for $\Box \supset \Lambda$: this is even bigger than $A^*(\mathbf{\Gamma})$!) that the "contour $\mathbf{\Gamma}$ encircling the cylinder Λ" is *not* a small contour!

Strictly speaking, this argument requires some commentary – because we do *not* know that such a $\mathbf{\Gamma}$ is a contour of the model. This can be done precise simply by *allowing also contours with the weight* $+\infty$ in our abstract Pirogov – Sinai model; such an assumption does not change its "physical" properties but allows to work with a richer family of contours (which is quite indispensable here, as we see in (3.84)).

Note. This is only one of the arguments why it is generally advisable to work with an abstract Pirogov – Sinai model instead of the original spin model.

9. Properties of Typical Configurations. Concluding Notes

We are still in a situation of an abstract Pirogov – Sinai model. Denote by X^y_{meta} the configuration space of all configurations of the y -th metastable model. More precisely denote by $X^y_{\text{meta}}(\Lambda)$ the configuration space of all $x = (x^\Lambda_{\text{best}}, \mathbf{\Gamma})$ such that $\mathbf{\Gamma}$ is strictly diluted in Λ i.e. $V_\updownarrow(\mathbf{\Gamma}) \subset \Lambda$. We have the probability measure $P^y_{\text{meta},\Lambda}$ on $X^y_{\text{meta}}(\Lambda)$, the corresponding partition function $Z_{\text{meta}}(\Lambda, y)$ being given by the Gibbs density $\exp(-H(x_\Lambda|y))$ (and $H(x_\Lambda|y)$ being given by (2.16)). Write P^y_Λ instead of $P^y_{\text{meta},\Lambda}$ if y is stable. Having defined these "strictly diluted" Gibbs conditioned probabilities $P^y_{\text{meta},\Lambda}$, the question now is whether a suitable limit over $\Lambda \to \mathbb{Z}^\nu$

$$(3.85)\qquad P^y_{\text{meta}}(\mathcal{E}) = \lim_{\Lambda\to\mathbb{Z}^\nu} P^y_{\text{meta},\Lambda}(\mathcal{E})$$

exists for a sufficiently rich collection of events $\mathcal{E}$ ("sufficiently rich" would mean, in the spin model, e.g. the collection of all cylindrical events i.e. events depending on a finite number of coordinates; in the abstract Pirogov – Sinaimodel a simplest example of such an event is as below in (3.86)) and gives, for y *stable*, a Gibbs measure P^y on the whole $\mathbb{X}$. Let us show for example that the limit

$$(3.86)\qquad P^y(\mathcal{E}(t)) = \lim_{\Lambda\to\mathbb{Z}^\nu} P^y_\Lambda(\mathcal{E}(t))$$

exists for any (even nonstable) y and for any $t \in \mathbb{Z}^\nu$, for the event $\mathcal{E}(t)$ defined as follows: "$t \notin V_\updownarrow(\mathbf{\Gamma})$, where $\mathbf{\Gamma}$ denotes the collection of all contours of x" i.e. for the event "t is stricly exterior point of the given configuration x".

We have obviously the formula

$$P^y_\Lambda(\mathcal{E}(t)) = (Z^y_\updownarrow(\Lambda))^{-1} Z^y_\updownarrow(\Lambda \setminus t)\exp(-e_t). \tag{3.87}$$

We can expand, by (3.0) (in the fully expanded model) both the partition functions on the right hand side of (3.87). The possibility of such an expansion follows from the very notion of stability of y! (resp. from the definition of the metastable ensemble, if y is unstable and we compute the probability $P^y_{\Lambda,\mathrm{meta}}$). We have, for $\mathrm{dist}(\Lambda, t) \to \infty$, the following expression: (Analogous, slightly more cumbersome expressions obtained in any finite volume Λ are omitted here.)

$$\log P^y(\mathcal{E}(t)) = \sum (k^{\mathrm{ext}}_T - k_T) \tag{3.88}$$

where the quantities k_T resp. k^{ext}_T correspond to the expansion of $Z^y_\updownarrow(\Lambda))$ resp. $Z^y_\updownarrow(\Lambda \setminus t))$ and the sum is over those (quickly decaying by (3.1)!) quantities k_T resp. k^{ext}_T only which *touch* t (contain or have a distance at most 1).

Clearly, for sufficiently small ε we have from (3.1) and (3.88) the approximate relation

$$1 - P^y(\mathcal{E}(t)) \asymp \varepsilon. \tag{3.89}$$

This suggests that the "islands" of a typical configuration $(x, \mathbf{\Gamma})$ (interpreted here as connected components of $V_\updownarrow(\mathbf{\Gamma})$, but see below for a more elaborate notion of an "island") are really typically "small and rare" because they do not typically intersect a given (arbitrarily chosen) point $t \in \mathbb{Z}^\nu$. To make this intuitive description of a typical configuration (which is quite characteristic for the Pirogov – Sinai theory and the phase picture this theory gives) more detailed, define below the "islands" of a given configuration $(x, \mathbf{\Gamma})$ in a different, more detailed way grasping also some important features (namely the appearance of redundant contours) of the regime appearing *inside* $V_\updownarrow(\mathbf{\Gamma})$ and thus allowing also precise expansion formulas for the event "a given island appears":

Definition. An admissible subsystem $\mathbf{\Gamma} \subset \mathcal{D}$ of a configuration $(x, \mathcal{D})$ is called an *island* if there is a strictly external (in $\mathcal{D}$) superset $\mathbf{\Gamma}' \supset \mathbf{\Gamma}$ such that $V(\mathbf{\Gamma}' \setminus \mathbf{\Gamma}) \cap \mathbf{\Gamma} = \emptyset$ and $\mathbf{\Gamma}' \setminus \mathbf{\Gamma}$ contains no redundant subsystems. (In other words, a successive recoloring "around $\mathbf{\Gamma}$" can be applied, deleting all the elements of $\mathbf{\Gamma}' \setminus \mathbf{\Gamma}$. The case $\mathbf{\Gamma}' = \mathbf{\Gamma}$ is typical, of course, and $\mathbf{\Gamma}$ is most commonly either a single contour or a collection of one external contour and some interior redundant ones.) Denote by $P^y[\mathbf{\Gamma}]$ the probability (in P^y) of the event "$\mathbf{\Gamma}$ is an island of a given configuration".

Proposition.

$$P^y[\mathbf{\Gamma}] = \exp(-F_\infty(\mathbf{\Gamma}))\exp(\sum_T (k_T - k^{\mathrm{ext}}_T)) \tag{3.90}$$

where F_∞ is defined as in (3.22),(3.21) but with respect to the fully expanded model (not the temporary one, used in the moment when $\mathbf{\Gamma}$ was recolored!) and the sum *is over those T only which touch* $\operatorname{supp}\mathbf{\Gamma}$. *The quantities k_T resp. k_T^{ext} corrrespond to the quantities $k_T(\gamma\delta)$ resp. $k_T(\delta)$ in (3.25).*

Note. 1. Notice that F_∞ is a horizontally translation invariant quantity.
2. Notice that we do not formulate here the probabilities of the events depending also on the *interior* of $\mathbf{\Gamma}$. Having determined all possible $P^y_{\mathbf{\Gamma}}$ this is already (in the strictly diluted ensemble) a straightforward task, using the properties of conditioned Gibbs distributions in finite volumes.

Proof. This is just an application of (3.25), for $\mathcal{D} = \emptyset$. It is not *exactly* the same argument as the one which was used in section 3 for the recoloring of $\mathbf{\Gamma}$. Namely, a (slightly) different mixed model is used here, also with clusters "bigger than $\mathbf{\Gamma}$"; however the difference between $F(\mathbf{\Gamma})$ and $F_\infty(\mathbf{\Gamma})$ is extremely small, of the order $\varepsilon^{\operatorname{diam}\mathbf{\Gamma}}$.

Corollary. *The probability $P^y[\mathbf{\Gamma}]$ of an island $\mathbf{\Gamma}$ satisfies an estimate, with $\varepsilon = \exp(-\tau')$ where τ' is something like $\frac{\tau}{13\nu}$*

$$P^y[\mathbf{\Gamma}] \leq \varepsilon^{\operatorname{conn}\mathbf{\Gamma}} . \tag{3.92}$$

The probability of the event "there is an island around t having a diameter $\leq d$" can be estimated as const ε^d. The mean relative area, occupied by the islands $\mathbf{\Gamma}$ resp. by the "interiors" $V_\updownarrow(\mathbf{\Gamma})$ of a typical configuration is smaller than some $\varepsilon' = \varepsilon'(\varepsilon)$ resp. $\varepsilon'' = \varepsilon''(\varepsilon)$, and independent of the particular choice of the configuration. There is an exponential decay of correlations in the probability P^y.

We do not prove these general (but more or less straighforward once (3.89),(3.90) was established) facts here.[30]

Definition. Say that a volume $\Lambda \subset \mathbb{Z}^\nu$ is *balanced* or conoidal if for any admissible system in Λ,

$$V(\mathbf{\Gamma}) \subset \Lambda \Rightarrow V_\updownarrow(\mathbf{\Gamma}) \subset \Lambda.$$

To have examples of balanced sets, take the sets ("rectangular cup & cap glued together"; compare (3.19) and (3.20))

$$\{t \in \mathbb{Z}^\nu : \operatorname{dist}(t, \partial\Box) \leq 1/2 \ \operatorname{diam}\Box\}$$

where $\Box$ is a cube in $\mathbb{Z}^{\nu-1}$ and $\partial\Box$ denotes its boundary in $\mathbb{Z}^{\nu-1} \subset \mathbb{Z}^\nu$.

For balanced volumes, diluted and strictly diluted partition functions (and corresponding Gibbs measures) are obviously the *same* and the fact that the limit of *diluted* Gibbs measures is (for $\Lambda \to \mathbb{Z}^\nu$) a Gibbs measure on $\mathbb{X}$ is obvious.[31]

[30] The above arguments give also only an outline of the full proof that P^y constructed by (3.86) gives really a probability measure on $\mathbb{X}$ with the properties stated above. To interpret further this measure even as a Gibbs measure on $\mathbb{X}$ i.e. as a Gibbs measure of the *original hamiltonian (2.2)* it would be useful to have another general statement, whose proof is omitted in this paper (however, we do not know a suitable reference): A limit, for $\Lambda \to \mathbb{Z}^\nu$, of strictly diluted Gibbs measures in Λ is a Gibbs measure on $\mathbb{X}$. Instead of such a general statement we consider below only the more special limit over conoidal $\Lambda \to \mathbb{Z}^\nu$.

[31] The general case of an arbirary volume Λ is not considered here. It requires some more care concerning the "stability" of considered finite volume measures with respect to various boundary conditions. Anyway, these and related questions must be investigated in more detail also whenever a completeness of the phase picture constructed by Main Theorem is studied. We postpone this discussion to a forthcoming paper.

We expect that the new method presented in this paper will be applicable also in other situations (even nonstratified ones) where "noncrusted" contours appear. Notice also that the method is applicable to situations where one starts (after suitable preparation of the given "physical" model) with some *mixed model* instead of the abstract P.S. model. This is the case of models with continuous spins (studied in [DZ]) having several "potential wells", for example. In these models, the expansion around positive mass gaussians (approximating the regimes of the potential wells) of restricted ensembles (of configurations living in the vicinity of the potential wells) yields a mixed model of the type studied here, and then the analysis developed in part III of this paper could be applied to these models, possibly also for the wells which are not so "deep" such that the previous analysis (like [DZ]) could be applied to them.

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