LARGE DEVIATIONS AND CONTINUUM LIMIT IN THE 2D ISING MODEL

C.-E. Pfister[a], Y. Velenik[b i]

[a]Département de Mathématiques, EPF-L
CH-1015 Lausanne, Switzerland
e-mail: cpfister@eldp.epfl.ch

[b]Département de Physique, EPF-L
CH-1015 Lausanne, Switzerland
e-mail: velenik@eldp.epfl.ch

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Abstract: We study the 2D Ising model in a rectangular box Λ_L of linear size $O(L)$. We determine the exact asymptotic behaviour of the large deviations of the magnetization $\sum_{t\in\Lambda_L}\sigma(t)$ when $L\to\infty$ for values of the parameters of the model corresponding to the phase coexistence region, where the order parameter m^* is strictly positive. We study in particular boundary effects due to an arbitrary real–valued boundary magnetic field. Using the self–duality of the model a large part of the analysis consists in deriving properties of the covariance function $\langle\,\sigma(0)\sigma(t)\,\rangle$, as $\|t\|\to\infty$, at dual values of the parameters of the model. To do this analysis we establish new results about the high–temperature representation of the model. These results are valid for dimensions $D\geq 2$ and up to the critical temperature. We then study the Gibbs measure conditionned by $\{\,|\sum_{t\in\Lambda_L}\sigma(t)-m|\Lambda_L|\,|\leq|\Lambda_L|L^{-c}\,\}$, with $0<c<1/4$ and $-m^*<m<m^*$. We construct the continuum limit of the model and describe the limit by the solutions of a variational problem of isoperimetric type.

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1 Introduction

We analyze the large deviations of the magnetization of the two–dimensional Ising model in the phase coexistence region, paying attention to boundary conditions. Our new results lead to a new approach of the wetting phenomenon, an important surface phenomenon which can be described in the Ising model [FP1], [FP2]. The theoretical physical aspects of the problem (wetting phenomenon) are discussed in a separate publication [PV].

1.1 Historical remarks

When there is a unique Gibbs measure the rate function describing the large deviations of the magnetization is given by the specific free energy of the model. It is therefore sufficient to control the *bulk* thermodynamical properties of the model in order to compute large deviations bounds. The situation is different when two Gibbs states coexist (phase coexistence region) because the large deviations of the magnetization are now driven by *boundary* effects. Consequently we must control the surface tension and surface free energies in order to get sharp large deviations bounds.

In their famous papers [MS1] and [MS2] Minlos and Sinai started the analysis of the large deviations of the magnetization for the two–dimensional Ising model in the phase coexistence region. They showed that the phenomenon of phase segregation is at the origin of the large deviations behaviour of the magnetization. In the eighties the problem was considered again. First Schonmann [S] (see also [CCS]) established lower and upper bounds for the large deviations with a completely different approach, which is non–perturbative as opposed to the work of Minlos and Sinai. Also, with different techniques, we have the works of Föllmer and Ort [FO] and [O]. A breakthrough was then made by Dobrushin, Kotecký and Shlosman in the late eighties [DKS]. They were able to get *exact* large deviations bounds for the magnetization and to get a detailed description of the typical configurations associated with large deviations in terms of the Wulff shape. Their results are valid at low temperature and for periodic boundary condition. After the announcement of these results Pfister [Pf2] obtained similar results valid at low temperature for + boundary condition. His method works as well for periodic boundary condition. Notice that the results depend on the choice of the boundary condition; see [Sh] for a study of some effects due to boundary conditions. More importantly, new tools are developed and several crucial estimates are done non–perturbatively. In particular sharp upper bounds for the probability of long contours are derived using moment inequalities (GKS–inequalities) and the self–duality of the model. These new techniques allow to considerably shorten some parts of analysis of [DKS]. Similar ideas appear independently in [ACC], where similar questions are studied for the percolation model. Recently substantial improvements have been obtained by Ioffe [I1], [I2], who obtained exact lower and upper bounds for the large deviations of the magnetization *for all temperatures below the critical one.* Deuschel and Pisztora [DPi] and [Pi] studied large deviations for percolation, Ising and Potts 3D models.

1.2 Isoperimetric inequality and large deviations

Consider an Ising model in the finite box

$$\Lambda_L := \{t = (t(1), t(2)) \in \mathbb{Z}^2 : \; -r_1 L \leq t(1) \leq r_1 L \, ; \; 0 \leq t(2) \leq 2r_2 L\} \, , \tag{1.1}$$

where $r_1, r_2 \in \mathbb{N}$ are two fixed numbers. Let $\sigma(t) = \pm 1$, $t \in \Lambda_L$, and define

$$H_{\Lambda_L} = - \sum_{\substack{\langle t,t'\rangle : \\ t,t' \in \Lambda_L}} \sigma(t)\sigma(t') - \sum_{\substack{t \in \Lambda_L : \\ t(2)=0}} h\sigma(t) - \sum_{\substack{t \in \Lambda_L : t(2)=2r_2 L \\ \text{or } t(1) = \pm r_1 L}} \sigma(t) \, . \tag{1.2}$$

Here $\langle t, t'\rangle$ denotes a pair of nearest neighbours points of the lattice $\mathbb{Z}^2$. The last two sums prescribed the boundary condition; h is a real parameter, the boundary magnetic field. The Gibbs probability measure associated with the energy function H_{Λ_L} and inverse temperature β is

$$\Xi(\Lambda_L)^{-1} \exp(-\beta H_{\Lambda_L}) \, , \tag{1.3}$$

with $\Xi(\Lambda_L)$ the normalization constant,

$$\Xi(\Lambda_L) := \sum_{\substack{\sigma(t) = \pm 1 : \\ t \in \Lambda_L}} \exp(-\beta H_{\Lambda_L}) \, . \tag{1.4}$$

We denote the Gibbs measure by μ_L^{h}. Probability with respect to that measure is also denoted by $P_L^{\mathrm{h}}[\,\cdot\,]$.

We study the asymptotic behaviour of $P_L^{\mathrm{h}}[\, A(m; c)\,]$ when $A(m; c)$ is the event

$$A(m; c) := \{ | \sum_{t \in \Lambda_L} \sigma(t) - m|\Lambda_L| \, | \leq |\Lambda_L| \cdot L^{-c} \} \, . \tag{1.5}$$

The solution to this problem is given in terms of the solution of a variational problem, which is the following isoperimetric problem with constraints defined in the rectangle

$$Q := \{x = (x(1), x(2)) : \; -r_1 \leq x(1) \leq r_1 \, ; \; 0 \leq x(2) \leq 2r_2\} \, . \tag{1.6}$$

The horizontal bottom part of the boundary of Q plays a special role; we set

$$w_Q := \{x \in Q : \; x(2) = 0\} \, . \tag{1.7}$$

Suppose that $\widehat{\tau} : \mathbb{R}^2 \to \mathbb{R}$ is a positive, convex function, which is positively homogeneous of degree one and such that $\widehat{\tau}(x) = \widehat{\tau}(-x)$. $\widehat{\tau}$ is a function of the parameter β, $\widehat{\tau}(x) = \widehat{\tau}(x; \beta)$. Suppose also that $\widehat{\tau}_{\mathrm{bd}}$ is such that

$$0 \leq |\widehat{\tau}_{\mathrm{bd}}| \leq \widehat{\tau}((1, 0)) \, . \tag{1.8}$$

$\widehat{\tau}_{\mathrm{bd}}$ is a function of the parameters β and h, $\widehat{\tau}_{\mathrm{bd}} = \widehat{\tau}_{\mathrm{bd}}(\beta, h)$. On the space of rectifiables curves in Q we introduce the functional

$$\mathtt{W}(\mathcal{C}) := \int_{\mathcal{C}} \widehat{\tau}(n(s)) \, ds + \Big[\widehat{\tau}_{\mathrm{bd}} - \widehat{\tau}((1, 0))\Big] |\mathcal{C} \cap w_Q| \, , \tag{1.9}$$

where $|\mathcal{C} \cap w_Q|$ is the Lebesgue measure of the subset $\mathcal{C} \cap w_Q$ of w_Q. If

$$[0, |\mathcal{C}|] \to \mathbb{R}^2 \quad , \quad a \mapsto (u(a), v(a)) \tag{1.10}$$

is a unit–speed parametrization of the curve $\mathcal{C}$, then

$$\int_{\mathcal{C}} \widehat{\tau}(n(s))\, ds = \int_0^{|\mathcal{C}|} \widehat{\tau}((u(a), v(a)); (\dot{u}(a), \dot{v}(a)))\, da\,. \tag{1.11}$$

We define in the standard way the interior and exterior of $\mathcal{C}$; $\mathrm{vol}\,\mathcal{C}$ is the area of the interior of $\mathcal{C}$.

Isoperimetric problem: *Find the minimum of functional* $\mathtt{W}$ *among all closed curves* $\mathcal{C}$ *inside* Q*, with* $\mathrm{vol}\,\mathcal{C}$ *fixed.*

There are two important differences with the isoperimetric problem considered by Wulff [Wu] in his theory of crystal: 1. the curve $\mathcal{C}$ is constraint to be inside the square Q; 2. the value of the functional depends not only on the tangent to the curve, but also on its location. The isoperimetric problem of Wulff was solved by [D]; a detailed study is done in [DKS]; see also [DP] for a recent original proof. The solution of the problem depends on the choice of $\widehat{\tau}_{\mathrm{bd}}$ and of the shape of the box Q, see [KP].

It is convenient to introduce $m^* = m^*(\beta)$, the order parameter of the Ising model (spontaneous magnetization). Suppose that $m^*(\beta) > 0$ i.e. $\beta > \beta_c$, the critical inverse temperature, and write the volume of $\mathcal{C}$,

$$\mathrm{vol}\,\mathcal{C} := 4r_1 r_2 \frac{m^* - m}{2m^*} \quad , \quad -m^* < m < m^*\,. \tag{1.12}$$

(The parameter m has the interpretation of a mean magnetization: inside $\mathcal{C}$ we have the phase with magnetization $-m^*$ and outside with magnetization m^*.) We set

$$\mathtt{W}^*(m) := \inf\Big\{\mathtt{W}(\mathcal{C}) : \mathcal{C} \subset Q\,,\, \mathrm{vol}\,\mathcal{C} = 4r_1 r_2 \frac{m^* - m}{2m^*}\Big\}\,. \tag{1.13}$$

An important property is that the infimum can be computed with $\mathcal{C}$ the boundary of a convex body (use Jensen's inequality and the convexity of $\widehat{\tau}$).

Theorem 1.1 *Let* $h \in \mathbb{R}$*,* $\beta > \beta_c$*,* $-m^* < m < m^*$ *and* $c = 1/4 - \delta$ *with* $0 < \delta < 1/4$*. There exists* $\widehat{\tau}_{\mathrm{bd}} = \widehat{\tau}_{\mathrm{bd}}(\beta, h)$ *and a function* $\widehat{\tau} : \mathbb{R}^2 \to \mathbb{R}$*, which is positive, convex and positively homogeneous of degree one, such that* $\widehat{\tau}(x|\beta) = \widehat{\tau}(-x|\beta)$ *and*

$$|\widehat{\tau}_{\mathrm{bd}}(\beta, h)| \le \widehat{\tau}((1,0)|\beta)\,. \tag{1.14}$$

If $\mathtt{W}$ *is defined by (1.9) and* $0 < \eta < \delta$*, then for* L *large enough*

$$\Big|\, \frac{1}{L} \ln P_L^{\mathrm{h}}[A(m; c)] + \mathtt{W}^*(m) \Big\} \,\Big| \le O(L^{\eta - \delta})\,. \tag{1.15}$$

We prove even stronger results, similar to those of [Pf2]. In particular, we have information about typical configurations of the conditionned measure $P_L^{\mathrm{h}}[\,\cdot\,|\,A(m, c)\,]$

(Theorems 11.1 and 11.2). This allows us to take the continuum limit in which we scale every lengths by $1/L$, so that we express the results in the fixed box Q.

We define the set of macroscopic droplets at equilibrium as

$$\mathcal{D}(m) := \{ \mathcal{V} \subset Q : |\mathcal{V}| = \frac{m^* - m}{2m^*}|Q| \, , \, \mathtt{W}_{\mathtt{t}}(\partial\mathcal{V}) = \mathtt{W}^*_{\mathtt{t}}(m) \} \, . \tag{1.16}$$

For each $\mathcal{V} \in \mathcal{D}(m)$ we have a magnetization profile,

$$\rho_{\mathcal{V}}(x) := \begin{cases} m^* & \text{if } x \in Q\backslash\mathcal{V}, \\ -m^* & x \in \mathcal{V}. \end{cases} \tag{1.17}$$

Let f be a real–valued function on Q; we set

$$d_1(f, \mathcal{D}(m)) := \inf_{\mathcal{V}\in\mathcal{D}(m)} \int_Q dx \, | \, f(x) - \rho_{\mathcal{V}}(x) \, | \, . \tag{1.18}$$

For each ω we define a magnetization profile $\rho_L(x;\omega)$ on Q. We subdivide the box Λ_L by the cells of a grid of lattice spacing $2[L^a]$. In each cell C of the grid we define the empirical magnetization $m_C(\omega)$ in the cell C,

$$\frac{1}{|C|}\sum_{t\in C} \sigma(t)(w) \, . \tag{1.19}$$

Then we set, for each $x \in Q$,

$$\rho_L(x;\omega) := m_C(\omega) \text{ if } Lx \in C \tag{1.20}$$

where Lx is the point $x \in Q$ scaled by L.

Theorem 1.2 *Let $\beta > \beta_c$, $h \in \mathrm{I\!R}$, $-m^* < m < m^*$ and $c = 1/4 - \delta > 0$. Then there exist a positive function $\overline{\varepsilon}(L)$ such that* $\lim_{L\to\infty} \overline{\varepsilon}(L) = 0$ *and two real numbers $\kappa > 0$ (see 12.40) and $1 > a > 0$ such that for L large enough*

$$P^h_L[\, \{ \, d_1(\rho_L(\,\cdot\,;\omega), \mathcal{D}(m)) \leq \overline{\varepsilon}(L) \, \} \, | A(m;c) \,] \geq 1 - \exp\{-O(L^\kappa)\} \, . \tag{1.21}$$

1.3 Outline of the paper

The proof of Theorems 1.1 and 1.2 are long. The basic strategy is taken from [Pf2]. To understand the large deviations in presence of two Gibbs measures we must study the phase boundaries, which in dimension two are random lines. A large part of the paper is devoted to that question. We use a special feature of the model, self–duality, to identify problems concerning the phase boundaries (at low temperatures) with problems concerning the two–point function (at high temperatures). The two–point function is nothing but the covariance of the random field described by the free boundary Gibbs measure. The two–point function is a fundamental quantity; we can identify the functions $\widehat{\tau}$ and $\widehat{\tau}_{\mathrm{bd}}$ with decay–rates of the two–point function, respectively the boundary two–point function. The first part of the paper is written in an independent way and is entirely devoted to a detailed study of the two–point function. In the second part we prove our main theorems.

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2 Ising model, contours, duality and inequalities

We give a list of the main definitions. The notion of contour of subsection 2.2 is essential for the whole paper. Additional definitions are given in parts I and II, when they are more specifically related to these parts. Throughout the paper $O(x)$ denotes a non–negative function of $x \in \mathbb{R}^+$, such that there exists a constant C with $O(x) \leq Cx$. The function $O(x)$ may be different at different places.

2.1 Ising model

We use the following notation and terminology.

The lattice is $\mathbb{Z}^2 := \{t = (t(1), t(2)) : t(i) \in \mathbb{Z}, i = 1, 2\}$. Its elements are called **sites**. We set $\mathbb{L} := \{t \in \mathbb{Z}^2 : t(2) \geq 0\}$ and $\Sigma_0 := \{t \in \mathbb{Z}^2 : t(2) = 0\}$. An **edge**, $e = \langle t, t' \rangle$, is an unordered pair of elements $t, t' \in \mathbb{Z}^2$ such that $|t(1) - t'(1)| + |t(2) - t'(2)| = 1$. We sometimes identify the edge $e = \langle t, t' \rangle$ with the unit length segment in $\mathbb{R}^2$ with end–points t, t'. The set of all edges is $\mathcal{E}$. An edge is **adjacent** to $t \in \mathbb{Z}^2$ if $e = \langle t, t' \rangle$. Let $B \subset \mathcal{E}$; the **index** of a site t **in** B is the number of edges of B, which are adjacent to t. A **configuration** ω is an element of the product space $\Omega := \{-1, 1\}^{\mathbb{Z}^2}$. The value of ω at $t \in \mathbb{Z}^2$ is $\omega(t)$; $\sigma(t)$ is the random variable $\sigma(t)(\omega) := \omega(t)$. Let $\Lambda \subset \mathbb{Z}^2$; $\mathcal{F}_\Lambda$ is the σ–algebra generated by $\sigma(t)$, $t \in \Lambda$. We set $\mathcal{F} := \mathcal{F}_{\mathbb{Z}^2}$. A function f is Λ–**local** if it is $\mathcal{F}_\Lambda$–measurable and Λ finite.

Let $\Lambda \subset \mathbb{Z}^2$; a configuration ω **satisfies the** Λ^+–**boundary condition** if $\omega(t) = 1$, $t \notin \Lambda$. For each edge e we introduce a non–negative number $J(e)$, called **coupling constant**. We set

$$H_\Lambda(\omega) := - \sum_{\substack{e=\langle t,t' \rangle: \\ e \cap \Lambda \neq \emptyset}} J(e)[\sigma(t)(\omega)\sigma(t')(\omega) - 1] \,, \tag{2.1}$$

which is interpreted as the energy inside Λ. The **Gibbs measure in** Λ **with** $+$ **boundary condition** is by definition the measure on $(\Omega, \mathcal{F})$ given by the formula

$$\mu_\Lambda^+(\omega) := \begin{cases} \Xi^+(\Lambda)^{-1} \exp(-H_\Lambda(\omega)) & \text{if } \omega \text{ satisfies the } \Lambda^+\text{–bd. cond.,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

$\Xi^+(\Lambda)$ is the normalization constant so that μ_Λ^+ is a probability measure. Expectation value with respect to μ_Λ^+ is denoted by $P_\Lambda^+[\,\cdot\,]$, $\langle \cdot \rangle_\Lambda^+$ or $\langle \cdot \rangle_\Lambda^{+,J}$. In a similar way we define the **Gibbs measure in** Λ **with** $-$ **boundary condition**. The **free boundary Gibbs measure in** Λ is by definition the probability measure on $\{-1, 1\}^\Lambda$ defined by

$$\mu_\Lambda := \Xi(\Lambda)^{-1} \prod_{e=\langle t,t' \rangle \subset \Lambda} \exp(J(e)\sigma(t)\sigma(t')) \,. \tag{2.3}$$

$\Xi(\Lambda)$ is the normalization constant, called **partition function**.

$$\Xi(\Lambda) := \sum_{\sigma(t),\, t \in \Lambda} \prod_{e=\langle t,t' \rangle \subset \Lambda} \exp(J(e)\sigma(t)\sigma(t')) \,. \tag{2.4}$$

Expectation value with respect to μ_Λ is denoted by $\langle\,\cdot\,\rangle_\Lambda$ or $\langle\,\cdot\,\rangle_\Lambda^J$.

Let $\Lambda'_L := \{t \in \mathbb{Z}^2 : -L \leq t(i) \leq L,\, i = 1,2\}$. There exists a limiting measure μ^+ on $(\Omega,\mathcal{F})$, $\mu^+ := \lim_{L\to\infty} \mu^+_{\Lambda'_L}$. Expectation value with respect to μ^+ is denoted by $P^+[\,\cdot\,]$, $\langle\,\cdot\,\rangle^+$ or $\langle\,\cdot\,\rangle^{+,J}$. The same construction is possible with Λ^-–boundary condition instead of Λ^+–boundary condition. The limiting measure is μ^-. Similarly, there exists a limiting measure μ on $(\Omega,\mathcal{F})$, $\mu := \lim_{L\to\infty} \mu_{\Lambda'_L}$. Expectation value with respect to μ is denoted by $\langle\,\cdot\,\rangle$ or $\langle\,\cdot\,\rangle^J$. Let $J(e) = \beta$ for every edge e. Then all measures defined above are translation–invariant. There exists $\beta_c := 1 + \sqrt{2}$, called **critical coupling**, which is characterized by the following properties (see subsection 2.3): the measures $\langle\,\cdot\,\rangle^{+,\beta}$ and $\langle\,\cdot\,\rangle^{-,\beta}$ are equal if and only if $\beta \leq \beta_c$; the **spontaneous magnetization** $m^* = m^*(\beta) = \langle\sigma(t)\rangle^{+,\beta}$ is strictly positive if and only if $\beta > \beta_c$. The **two–point function** $\langle\sigma(t_1)\sigma(t_2)\rangle$ is:

$$\langle\sigma(t_1)\sigma(t_2)\rangle := \lim_{L\to\infty} \langle\sigma(t_1)\sigma(t_2)\rangle_{\Lambda'_L}\,. \tag{2.5}$$

It is translation–invariant, $\langle\sigma(t_1)\sigma(t_2)\rangle = \langle\sigma(t_1+t)\sigma(t_2+t)\rangle$, $t \in \mathbb{Z}^2$. It is also invariant under axial symmetries with horizontal, vertical and diagonal axis. It is a non–trivial fact that

$$\langle\sigma(t_1)\sigma(t_2)\rangle = \langle\sigma(t_1)\sigma(t_2)\rangle^+ = \langle\sigma(t_1)\sigma(t_2)\rangle^-\,. \tag{2.6}$$

2.2 Contours

A **path** is an ordered sequence of sites and edges, $t_0, e_0, t_1, e_1, \ldots, t_n$, where $t_i \in \mathbb{Z}^2$ for all $i = 0,\ldots n$, and $e_j = \langle t_j, t_{j+1}\rangle \in \mathcal{E}$, $j = 0,\ldots,n-1$. By definition all edges of a path are different, but not necessarily all sites of the path. The initial point of the path is t_0 and the final point is t_n. The initial edge of the path is e_0 and the final edge is e_{n-1}. A path is **closed** if its final point coincides with its initial point; otherwise it is **open**. We say that a path is in a subset $A \subset \mathbb{Z}^2$ if $t_i \in A$, $\forall i = 0,\ldots n$; we say that it is in a subset $B \subset \mathcal{E}$ if $e_i \in B$, $\forall i = 0,\ldots n-1$. A subset $A \subset \mathbb{Z}^2$ is **connected** if for any pair of elements $t, t' \in A$ there is a path in A with initial point $t_0 = t$ and final point $t_n = t'$. A subset $B \subset \mathcal{E}$ is **connected** if for any pair of elements $e, e' \in B$ there is a path in B with initial edge $e_0 = e$ and final edge $e_{n-1} = e'$. Let $t \in \mathbb{Z}^2$; the **plaquette** $p(t)$ of center t is the subset of $\mathbb{R}^2$

$$p(t) := \{s = (s(1), s(2)) : |s(i) - t(i)| \leq 1/2\,, i = 1,2\}\,. \tag{2.7}$$

A subset $A \subset \mathbb{Z}^2$ is **simply connected** if the subset of $\mathbb{R}^2$, $\cup_{t\in A} p(t)$ is simply connected in $\mathbb{R}^2$. The **boundary** of B is the subset of $\mathbb{Z}^2$

$$\delta B := \{t \in \mathbb{Z}^2 : \text{index of } t \text{ in } B \text{ is odd}\}\,. \tag{2.8}$$

Let $B \subset \mathcal{E}$ be a non–empty subset. We decompose uniquely (up to orientation) B into a finite number of paths, such that they are disjoint two by two when considered as subsets of $\mathcal{E}$. (On the other hand sites may belong to two different paths.)

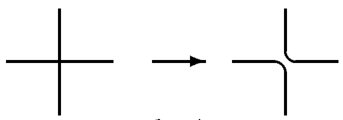
1. If $\delta B = \emptyset$, then choose an edge $e = \langle t, t'\rangle$ in B and set $t_0 := t$, $e_0 := e$ and $t_1 = t'$. The path is uniquely continued using rule A specified in the picture

below and by requiring that it is maximal and that its final point is t_0. We have thus defined a closed path. The unoriented corresponding path is called **closed contour**. Repeat this construction until all edges of B belong to some path.

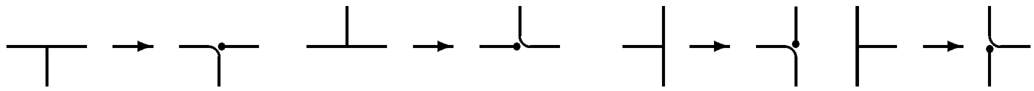
2. If $\delta B \neq \emptyset$, then choose first $t \in \delta B$, and set $t_0 := t$. Then choose e_0 among the adjacent edges to t_0 according to rules A' specified in the picture below. The path is uniquely continued using rules A and A' specified in the picture below and by requiring that it is maximal and its final point $t_n \in \delta B$. We have thus defined an open path. The unoriented corresponding path is called **open contour**. Initial points are marked by dots in the picture specifying the rules A'. Repeat this construction starting with a new point of δB; if all points of δB are already initial or final points of contours, and if there are still edges of B which do not belong to some contours, then do the construction (a) above.

Unoriented paths are called **contours**. The set of all contours of a configuration is denoted by $\underline{\gamma} = \{\gamma_1, \ldots, \gamma_n\}$. A contour is closed if and only if $\delta\gamma = \emptyset$. The **diameter** of a contour γ is

$$d(\gamma) := \max\{|t(1) - t'(1)| + |t(2) - t'(2)| : t, t' \in \gamma\}. \tag{2.9}$$



rule A



rules A'

the dots denote initial points of open paths

Let $\{\gamma_1, \ldots, \gamma_n\}$ be a family of contours. Let $\mathcal{E}(\gamma_1, \ldots, \gamma_n)$ be the set of all edges of these contours. We say that $\{\gamma_1, \ldots, \gamma_n\}$ is **compatible** if either $\mathcal{E}(\gamma_1, \ldots, \gamma_n) = \emptyset$, or $\{\gamma_1, \ldots, \gamma_n\}$ corresponds to the decomposition of the set $\mathcal{E}(\gamma_1, \ldots, \gamma_n)$ into contours. Let $\Lambda \subset \mathbb{Z}^2$; the family $\{\gamma_1, \ldots, \gamma_n\}$ is Λ–**compatible** if it is compatible and all edges of $\mathcal{E}(\gamma_1, \ldots, \gamma_n)$ are pairs of points of Λ. Compatible families of contours are denoted by $\underline{\gamma}$. A contour is an unoriented path; it is however useful to choose sometimes an orientation, and to consider a contour as a unit–speed parametrized curve in $\mathbb{R}^2$,

$$[0, |\lambda|] \ni s \mapsto \lambda(s) \in \mathbb{R}^2\,, \tag{2.10}$$

with initial point $\lambda(0) = t_1$ and final point t_2. The contour is closed iff $t_1 = t_2$; it is open if $\delta\lambda = \{t_1, t_2\}$. The **length** of a contour, $|\lambda|$, is the number of edges of λ. The length $|\underline{\gamma}|$ of a family of contours $\underline{\gamma}$ is the sum of the lengths of the contours of the family.

2.3 Duality

An important concept is the duality transformation [W]. It relates the properties of the Ising model for the couplings $J(e) = \beta < \beta_c$ to the properties of the dual model for the couplings $J^*(e^*) = \beta^* > \beta_c$. When the dimension is two the model is self–dual. The proper framework to study the duality transformation is the theory of cell–complexes. However we need only elementary facts, so that we define the duality transformation as follows. It consists of a geometric and an analytic part.

1. *Geometric part.* The **dual lattice** $(\mathbb{Z}^2)^*$ is

$$(\mathbb{Z}^2)^* := \{t = (t(1), t(2)) : t(i) + 1/2 \in \mathbb{Z}\,,\ i = 1, 2\}\,. \tag{2.11}$$

   To each edge e of $\mathbb{Z}^2$ we associate a **dual edge** e^* of $(\mathbb{Z}^2)^*$: it is the edge which crosses e in the middle, when both edges, e and e^*, are considered as unit length segments in $\mathbb{R}^2$.

2. *Analytic part.* The $*$–**transformation** is the transformation $x \mapsto x^*$ defined on $\{x : 0 \le x \le \infty\}$ into itself, given by the identity

$$\exp\{-2x\} = \tanh x^*\,. \tag{2.12}$$

   The $*$–transformation is such that $(x^*)^* = x$; it has a unique fixed–point $x_c := 1 + \sqrt{2}$.

The critical coupling of the 2–dimensional Ising model is $\beta_c = x_c$. It has been determined using the duality transformation in [KW]. Let $J(e)$ be a non–negative coupling constant. The **dual coupling constant** for the dual edge e^*, $J^*(e^*)$, is defined by the $*$–transformation, $\exp\{-2J(e)\} = \tanh J^*(e^*)$.

2.4 Correlation inequalities

The main tools for analysing the Ising model are correlations inequalities, also called moment inequalities. The Gibbs measures on Λ with Λ^+–boundary condition, Λ^-–boundary condition or free boundary condition are special cases of the probability measure

$$\nu_\Lambda := \frac{\exp\{\sum_{t,t'\in\Lambda} J(t,t')\sigma(t)\sigma(t') + \sum_{t\in\Lambda} k(t)\sigma(t)\}}{\text{normalization}} \tag{2.13}$$

Let $A \subset \Lambda$ and set

$$\sigma_A := \prod_{t\in A} \sigma(t)\,. \tag{2.14}$$

A function f is **increasing** if

$$\omega(t) \le \omega'(t) \quad \forall t \quad \Longrightarrow \quad f(\omega) \le f(\omega')\,. \tag{2.15}$$

Proposition 2.1 *Let Λ be finite and $J(t,t') \geq 0$ for all $t, t' \in \Lambda$.*

1. *If $k(t) \geq 0$ for all t, then GKS–inequalities hold* [Gr]

$$\langle \sigma_A \rangle_{\nu_\Lambda} \geq 0\,, \tag{2.16}$$

$$\frac{d \langle \sigma_A \rangle_{\nu_\Lambda}}{dJ(\{t,t'\})} \geq 0\,. \tag{2.17}$$

2. *If $k(t) \in \mathbb{R}$ and f and g are two increasing functions, then FKG–inequality hold* [FKG]

$$\langle f\, g \rangle_{\nu_\Lambda} \geq \langle f \rangle_{\nu_\Lambda} \langle g \rangle_{\nu_\Lambda}\,. \tag{2.18}$$

3. *If $k(t) \geq 0$ for all t, then GHS–inequalities hold* [GHS]

$$\frac{d^2 \langle \sigma(t) \rangle_{\nu_\Lambda}}{dk(t')dk(t'')} \leq 0\,. \tag{2.19}$$

Let $t_1, t_2 \in \mathbb{Z}^2$; a subset $B \subset \mathbb{Z}^2$ **separates** t_1 from t_2 if and only if any path from t_1 to t_2 contains an element of B.

Proposition 2.2 *Let $J(e)$ be non–negative for all edges e and $t_1, t_2 \in \mathbb{Z}^2$.*

1. *If $B \subset \mathbb{Z}^2$ is a finite subset which separates t_1 from t_2, then*

$$\langle \sigma(t_1)\sigma(t_2) \rangle \leq \sum_{t \in B} \langle \sigma(t_1)\sigma(t) \rangle \langle \sigma(t)\sigma(t_2) \rangle\,. \tag{2.20}$$

2. *$J(e) = \beta$, $\beta > 0$, for all edges e. Let $t = (t(1), t(2)) \in \mathbb{Z}^2$, such that $0 \leq t(1) \leq t(2)$. Then*

$$\langle \sigma(0)\sigma(t') \rangle < \langle \sigma(0)\sigma(t) \rangle\,, \tag{2.21}$$

   if either $t'(1) = t(1) + 1$ and $t'(2) = t(2)$, or $t'(1) = t(1)$ and $t'(2) = t(2) + 1$, or $t'(1) = t(1) - 1$ and $t'(2) = t(2) + 1$.

3. *The two–point function $\langle \sigma(0)\sigma(t) \rangle$ is invariant under symmetries with horizontal, vertical and diagonal axis.*

Proposition 2.2 1. is proven in [Sim] and 2. in [MM]. (To prove the strict inequality follow the proof of [MM] and apply the inequalities of section 3.5 in [FP1].)

Part I

Ising model at high temperature

We present here results concerning the Ising model on $\mathbb{Z}^2$ for $\beta < \beta_c$ (i.e. above the critical temperature). They are essential tools for the study of the large deviations estimates at low temperature and are of interest independently of the large deviations analysis. Our method is non-perturbative and the validity of these results is for all $\beta < \beta_c$. Some results of this part are not restricted to $d = 2$.

3 High–temperature representation

We recall the geometric description of the model, known as the high–temperature representation. The goal is to derive formula (3.13) which gives a representation of the two–point correlation function in terms of random lines. The correct point of view here is to consider the free boundary Gibbs measure on a graph. All graphs considered in this paper are subgraphs of the graph $(\mathbb{Z}^2, \mathcal{E})$. We use the following conventions. If a subgraph of $(\mathbb{Z}^2, \mathcal{E})$ is specified by its set of vertices $\Lambda \subset \mathbb{Z}^2$, then by definition the set $\mathcal{E}(\Lambda)$ of its edges is the set of all edges $e = \langle t, t' \rangle$ with $t, t' \in \Lambda$. If it is specified by its set of edges, then by definition the set of vertices is the set of all sites t of $\mathbb{Z}^2$ such that $t \in \Lambda$.

The partition function $\Xi(\Lambda)$ can be written as

$$\sum_{\sigma(t),\, t\in\Lambda} \prod_{e=\langle t,t'\rangle \in \mathcal{E}(\Lambda)} \cosh J(e)(1 + \sigma(t)\sigma(t') \tanh J(e)) \,. \tag{3.1}$$

We expand the product in (3.1). Each term of the expansion is labelled by a set of edges $\langle t, t' \rangle$: we specify the edges corresponding to factors $\tanh J(e)$. Then we sum over $\sigma(t)$, $t \in \Lambda$; after summation only terms labelled by sets of edges with empty boundary give a non–zero contribution. Any term of the expansion of (3.1), which gives a non–zero contribution, can be uniquely labelled by a Λ–compatible family $\underline{\gamma}$ of closed contours. Let e be an edge, γ a contour and $\underline{\gamma}'$ a compatible family of contours; we set

$$w(e) := \tanh J(e) \;\;,\;\; w(\gamma) := \prod_{e\in\gamma} w(e) \;\;,\;\; w(\underline{\gamma}') := \prod_{\gamma\in\underline{\gamma}'} w(\gamma) \,. \tag{3.2}$$

If $\underline{\gamma}' = \emptyset$, then $w(\underline{\gamma}') := 1$; $w(\gamma)$ is the **weight** of γ. The partition function is

$$\Xi(\Lambda) = 2^{|\Lambda|} \prod_{e\in\mathcal{E}(\Lambda)} \cosh J(e) \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \Lambda-\text{comp.}}} w(\underline{\gamma}) \,. \tag{3.3}$$

It is natural to introduce the **normalized partition function**

$$Z(\Lambda) := \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \Lambda-\text{comp.}}} w(\underline{\gamma}) \,. \tag{3.4}$$

More generally, given any Λ–compatible family $\underline{\gamma}'$ we set

$$Z(\Lambda|\underline{\gamma}') := \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \underline{\gamma}\cup\underline{\gamma}' \ \Lambda-\text{comp.}}} w(\underline{\gamma})\,. \tag{3.5}$$

In particular

$$Z(\Lambda) = Z(\Lambda|\emptyset)\,, \tag{3.6}$$

and

$$\Xi(\Lambda) = Z(\Lambda) 2^{|\Lambda|} \prod_{e\in\mathcal{E}(\Lambda)} \cosh J(e)\,. \tag{3.7}$$

Remark: For normalized partition functions we may have

$$Z(\Lambda_1) = Z(\Lambda_2) \tag{3.8}$$

with $\Lambda_1 \neq \Lambda_2$. Indeed, the condition for equality is that the graphs $(\Lambda_1, \mathcal{E}(\Lambda_1))$ and $(\Lambda_2, \mathcal{E}(\Lambda_2))$ have the same set of closed contours.

On the set of all families of Λ–compatible *closed* contours we define a probability measure

$$P_\Lambda[\,\underline{\gamma}\,] := \frac{w(\underline{\gamma})}{Z(\Lambda)}\,. \tag{3.9}$$

Let $\underline{\gamma}'$ be a Λ–compatible family of contours. We set

$$q_\Lambda(\underline{\gamma}') := \begin{cases} w(\underline{\gamma}')\dfrac{Z(\Lambda|\underline{\gamma}')}{Z(\Lambda)} & \text{if } \underline{\gamma}' \ \Lambda\text{–compatible,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

If $\underline{\gamma}'$ is a Λ–compatible family of *closed* contours, then

$$q_\Lambda(\underline{\gamma}') = P_\Lambda[\,\{\underline{\gamma} : \underline{\gamma}' \subset \underline{\gamma}\}\,]\,. \tag{3.11}$$

Let us consider the numerator of the two–point function $\langle\sigma(t_1)\sigma(t_2)\rangle_\Lambda$,

$$\sum_{\sigma(t),\, t\in\Lambda} \ \prod_{e=\langle t,t'\rangle\in\mathcal{E}(\Lambda)} \cosh J(e)(1+\sigma(t)\sigma(t')\tanh J(e))\sigma(t_1)\sigma(t_2)\,. \tag{3.12}$$

We expand the product as above. The presence of the variables $\sigma(t_1)$ and $\sigma(t_2)$ implies that the only terms in the expansion of the numerator of $\langle\sigma(t_1)\sigma(t_2)\rangle_\Lambda$, which give non–zero contributions, are those labelled by compatible families of contours containing one open contour λ such that $\delta\lambda = \{t_1, t_2\}$. The two–point function has the simple expression

$$\langle\sigma(t_1)\sigma(t_2)\rangle_\Lambda = Z(\Lambda)^{-1} \sum_{\substack{\lambda:\,\Lambda-\text{comp.} \\ \delta\lambda=\{t_1,t_2\}}} Z(\Lambda|\lambda)w(\lambda) = \sum_{\substack{\lambda:\,\Lambda-\text{comp.} \\ \delta\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda)\,. \tag{3.13}$$

Definition 3.1 *Let e be an edge and $B(e)$ the set formed by e and all edges adjacent to e. The* **edge–boundary** *of e is the contour $\Delta(e) \ni e$ of the decomposition of $B(e)$ into contours. Let $A \subset \mathcal{E}$; the* **edge–boundary** *$\Delta(A)$ of A is $\Delta(A) := \cup_{e \in A} \Delta(e)$.*

The **edge–boundary in** Λ *of an edge e is the set $\Delta_\Lambda(e)$ formed by $\Delta(e) \cap \mathcal{E}(\Lambda)$ and all edges $e' \in \mathcal{E}(\Lambda) \backslash \Delta(e)$ which are adjacent to e and such that $e \cap e'$ is a site of index one in $\mathcal{E}(\Lambda) \backslash \Delta(e)$. Let $A \subset \mathcal{E}$; the* **edge–boundary in** Λ *$\Delta_\Lambda(A)$ of A is $\Delta_\Lambda(A) := \cup_{e \in A} \Delta_\Lambda(e)$.*

A contour γ and its edge–boundary $\Delta(\gamma)$

Remark: Let $t \in \underline{\gamma}$. Then t has index 2 or 4 in $\Delta(\underline{\gamma})$. Therefore it has index 2 or 0 in $\mathcal{E} \backslash \Delta(\underline{\gamma})$. The same is true for $\Delta_\Lambda(\underline{\gamma})$: $t \in \underline{\gamma}$ has index 2 or 0 in $\mathcal{E}(\Lambda) \backslash \Delta_\Lambda(\underline{\gamma})$.

The notions of compatibility and edge-boundary are related.

Lemma 3.1 *Let $\underline{\gamma}'$ be a family of compatible contours (closed or open). Then a non–empty compatible family of n closed contours $\underline{\gamma} = \{\gamma_1, \ldots, \gamma_n\}$ is compatible with $\underline{\gamma}'$, that is $\underline{\gamma} \cup \underline{\gamma}'$ is compatible, if and only if no edge of γ_i is an edge of $\Delta(\underline{\gamma}')$, $\forall i = 1, \ldots, n$.*

A similar result holds for the edge–boundary in Λ if $\underline{\gamma}$ is Λ–compatible.

Proof. Suppose that there exists an edge $e \in \mathcal{E}(\gamma_i) \cap \Delta(\underline{\gamma}')$. If $e \in \mathcal{E}(\underline{\gamma}')$, then γ_i is not compatible with $\underline{\gamma}'$ because they are not disjoint as set of edges. If $e \in \Delta(\underline{\gamma}') \backslash \mathcal{E}(\underline{\gamma}')$ and $\mathcal{E}(\gamma_i) \cap \mathcal{E}(\underline{\gamma}') = \emptyset$, then one end–point t of e is of index at least three in $\Delta(\underline{\gamma}') \cup \mathcal{E}(\gamma_i)$: t has index 2 in γ_i and at least one in $\mathcal{E}(\underline{\gamma}')$. The decomposition of $\Delta(\underline{\gamma}') \cup \mathcal{E}(\gamma_i)$ into contours is not $(\gamma_i, \underline{\gamma}')$, hence γ_i and $\underline{\gamma}'$ are not compatible. On the other hand, if e_1, e_2 is a pair of edges adjacent to a site t of $\underline{\gamma}'$ such that $\{e_1, e_2\} \cap \Delta(\underline{\gamma}') = \emptyset$, then the decomposition of $\mathcal{E}(\underline{\gamma}') \cup \{e_1, e_2\}$ into contours is $\underline{\gamma}', \{e_1, e_2\}$. Therefore if $\mathcal{E}(\gamma_i) \cap \Delta(\underline{\gamma}') = \emptyset$, then γ_i is compatible with $\underline{\gamma}'$.

The same result holds for the edge–boundary in Λ if $\underline{\gamma}$ is Λ–compatible, because in that case the edges of $\Delta_\Lambda(\underline{\gamma}') \backslash \Delta(\underline{\gamma}')$ cannot be edges of *closed* contours in Λ. $\square$

Let Λ and $\underline{\gamma}'$, a family of Λ–compatible contours, be given. We define two graphs, $\Lambda(\underline{\gamma}')$ and $\Lambda_\#(\underline{\gamma}')$. The set of edges of $\Lambda(\underline{\gamma}')$ is

$$\mathcal{E}(\Lambda(\underline{\gamma}')) := \mathcal{E}(\Lambda) \backslash \Delta(\underline{\gamma}') \,. \tag{3.14}$$

The set of edges of $\Lambda_{\#}(\underline{\gamma}')$ is

$$\mathcal{E}(\Lambda_{\#}(\underline{\gamma}')) := \mathcal{E}(\Lambda)\backslash\Delta_{\Lambda}(\underline{\gamma}')\,. \tag{3.15}$$

On these graphs we consider Ising models defined by formula (2.4), taking the product over the edges of the graphs. The normalized partition functions are denoted by $Z(\Lambda(\underline{\gamma}'))$ and $Z(\Lambda_{\#}(\underline{\gamma}'))$.

Lemma 3.2 *Let Λ and $\underline{\gamma}'$, a family of Λ–compatible contours, be given. The normalized partition functions $Z(\Lambda(\underline{\gamma}'))$ and $Z(\Lambda_{\#}(\underline{\gamma}'))$ satisfy*

$$Z(\Lambda|\underline{\gamma}') = Z(\Lambda(\underline{\gamma}')) = Z(\Lambda_{\#}(\underline{\gamma}'))\,. \tag{3.16}$$

Proof. The edges of $\Delta_{\Lambda}(\underline{\gamma}')\backslash\Delta(\underline{\gamma}')$ cannot be edges of *closed* contours. Therefore the sets of closed contours of the graphs $(\Lambda_{\#}(\underline{\gamma}'), \mathcal{E}(\Lambda_{\#}(\underline{\gamma}'))$ and $(\Lambda(\underline{\gamma}'), \mathcal{E}(\Lambda(\underline{\gamma}'))$ are the same, hence

$$Z(\Lambda(\underline{\gamma}')) = Z(\Lambda_{\#}(\underline{\gamma}'))\,. \tag{3.17}$$

Lemma 3.1 implies that

$$Z(\Lambda|\underline{\gamma}') = Z(\Lambda(\underline{\gamma}'))\,. \tag{3.18}$$

$\square$

4 Exponential decay–rate

4.1 Two–point function

In this subsection we suppose that $J(e) = \beta \geq 0$ for all edges e of $\mathbb{Z}^2$.

Definition 4.1 *Let $t_1, t_2 \in \mathbb{Z}^2$ and $n \in \mathbb{N}$. The* **decay–rate** *of the two–point function is defined on $\mathbb{Z}^2$ by*

$$\tau(t_2 - t_1; \beta) := \lim_{n \to \infty} -\frac{1}{n} \ln \langle \sigma(nt_1)\sigma(nt_2)\rangle^{\beta} . \tag{4.1}$$

Proposition 4.1 *Let $J(e) = \beta \geq 0$ for all edges e of $\mathbb{Z}^2$. The decay–rate has the following properties.*

1. *The decay–rate is non–negative and is a decreasing function of β.*
2. *If $t_1, t_2 \in \mathbb{Z}^2$, then $\langle \sigma(t_1)\sigma(t_2)\rangle^{\beta} \leq \exp\{-\tau(t_2 - t_1; \beta)\}$.*
3. *If $\beta \geq \beta_c$, then $\tau(t; \beta) = 0$ for all t.*
4. *If $\beta < \beta_c$, then $\tau(t; \beta) > 0$ for all $t \neq 0$.*
5. *Let $|\cdot|$ be the Euclidean norm; the function $\tau(t;\beta)/|t|$, $t \neq 0$, can be extended by continuity to any $x \in \mathbb{R}^2$ with $|x| = 1$; it is defined on $\mathbb{R}^2$ by $\tau(x;\beta) := |x| \cdot \tau(x/|x|;\beta)$. It is invariant under the axial symmetries with horizontal, vertical and diagonal axis. There exists a constant K such that for any x and y, $|x| = 1$ and $|y| = 1$,*

$$|\tau(x;\beta) - \tau(y;\beta)| \leq K|x - y| . \tag{4.2}$$

Proof. Points 3. and 4. are consequences of the duality transformation and of [LP] (see also remark below). The continuity statement (4.2) is proved in [Pf2] section 6 (Lemmas 6.4 and 6.5). For the sake of completeness we prove the existence of the limit (4.1) and point 2. of the proposition. By GKS–inequalities and translation invariance we have with $t = t_2 - t_1$ and $n = n_1 + n_2$,

$$\langle \sigma(0)\sigma(nt)\rangle \geq \langle \sigma(0)\sigma(n_1 t)\rangle\langle \sigma(n_1 t)\sigma(nt)\rangle \geq \langle \sigma(0)\sigma(n_1 t)\rangle\langle \sigma(0)\sigma(n_2 t)\rangle . \tag{4.3}$$

Hence the standard subadditivity argument gives

$$\lim_{n \to \infty} -\frac{1}{n} \ln \langle \sigma(0)\sigma(nt)\rangle = \inf_{n} -\frac{1}{n} \ln \langle \sigma(0)\sigma(nt)\rangle . \tag{4.4}$$

□

Remark: The decay–rate has been computed; see [MW] chapters XI and XII. From this computation Proposition 4.1 follows; moreover $(1, 0)$ is a minimum of $\tau(x;\beta)$ on the unit sphere.

4.2 Boundary two–point function

In this section we consider the model on the semi–infinite lattice $\mathbb{L}$. We choose the coupling constants

$$J(e) := \begin{cases} \beta \geq 0 & \text{if } e = \langle t, t' \rangle \text{ with } t \notin \Sigma_0 \text{ or } t' \notin \Sigma_0, \\ h\beta \geq 0 & \text{if } e = \langle t, t' \rangle \text{ with } t \in \Sigma_0 \text{ and } t' \in \Sigma_0. \end{cases} \tag{4.5}$$

The **boundary two–point function** is defined for $t_1, t_2 \in \Sigma_0$ by

$$\langle \sigma(t_1)\sigma(t_2) \rangle_{\mathbb{L}} := \lim_{L \to \infty} \langle \sigma(t_1)\sigma(t_2) \rangle_{\Lambda'_L \cap \mathbb{L}} \,. \tag{4.6}$$

It is invariant under translations $t \in \Sigma_0$,

$$\langle \sigma(t_1 + t)\sigma(t_2 + t) \rangle_{\mathbb{L}} = \langle \sigma(t_1)\sigma(t_2) \rangle_{\mathbb{L}} \,. \tag{4.7}$$

Definition 4.2 *Let $t_1, t_2 \in \Sigma_0$ and $n \in \mathbb{N}$. The* **boundary decay–rate** *of the boundary two–point function is defined on Σ_0 by*

$$\tau_{\mathrm{bd}}(t_2 - t_1; \beta, h) := \lim_{n \to \infty} -\frac{1}{n} \ln \langle \sigma(nt_1)\sigma(nt_2) \rangle_{\mathbb{L}}^{\beta,h} \,. \tag{4.8}$$

Proposition 4.2 *Let $\beta \geq 0$ and $h \geq 0$. The boundary decay–rate has the following properties.*

1. *The boundary decay–rate is non–negative and decreasing in β and h.*
2. *If $t_1, t_2 \in \Sigma_0$, then $\langle \sigma(t_1)\sigma(t_2) \rangle_{\mathbb{L}}^{\beta,h} \leq \exp\{-\tau_{\mathrm{bd}}(t_2 - t_1; \beta, h)\}$.*
3. *For any $t \in \Sigma_0$, $\tau_{\mathrm{bd}}(t; \beta, h) \leq \tau(t; \beta)$.*
4. *If $\beta \geq \beta_c$, then $\tau_{\mathrm{bd}}(t; \beta, h) = 0$ for all $t \in \Sigma_0$.*
5. *If $\beta < \beta_c$, then $\tau_{\mathrm{bd}}(t; \beta, h) > 0$ for all $t \neq 0$. Moreover there exists a positive $h_c(\beta)$ so that for $t \in \Sigma_0$, $t \neq 0$,*

$$\begin{aligned} \tau(t; \beta) &= \tau_{\mathrm{bd}}(t; \beta, h) \text{ if } h \leq h_c(\beta), \\ \tau(t; \beta) &> \tau_{\mathrm{bd}}(t; \beta, h) \text{ if } h > h_c(\beta). \end{aligned} \tag{4.9}$$

Remark: The proof of the first part of Proposition 4.2 is the same as that of Proposition 4.1. Points 3. to 5. are proven using the duality transformation and results of [FP2]. The values of τ_{bd} and of $h_c(\beta)$ are known explicitly [A]; $h_c(\beta)$ is given by the following identities. Let $\widehat{\beta}$ be defined by

$$\exp\{-2\widehat{\beta}\} := \tanh \beta \,, \tag{4.10}$$

and $h_w(\widehat{\beta})$ by the equation

$$\exp\{2\widehat{\beta}\}\{\cosh 2\widehat{\beta} - \cosh 2\widehat{\beta} h_w(\widehat{\beta})\} = \sinh 2\widehat{\beta} \,. \tag{4.11}$$

Then

$$\exp\{-2\widehat{\beta} h_w(\widehat{\beta})\} = \tanh \beta h_c(\beta) \,. \tag{4.12}$$

The boundary decay–rate can be computed explicitely [P].

5 Basic estimates

We prove basic estimates for the high–temperature representation. These estimates are non–perturbative, valid for all $\beta < \beta_c$[1] and not restricted to dimension two. They are established for a finite graph Λ, which is always a subgraph of $\mathbb{Z}^2$ or $\mathbb{L}$. Implicitly all contours are defined on the graph Λ. The main ideas are from [Pf2] section 6. Lemmas 5.2 and 5.3 are new and crucial.

We do not always write explicitly the parameters β and h to simplify the notations. However, in cases we want to emphasize the coupling constants J in the partition functions we write for example $Z(\Lambda; J)$ instead of $Z(\Lambda)$.

Lemma 5.1 *Let $\beta < \beta_c$.*

1. *If $J(e) = \beta$ for all edges e, then for any $t_1, t_2 \in \Lambda$,*

$$\sum_{\substack{\lambda: \\ \delta\lambda = \{t_1,t_2\}}} q_\Lambda(\lambda) \leq \langle \sigma(t_2)\sigma(t_1)\rangle^\beta \leq \exp\{-\tau(t_2 - t_1)\}\,. \tag{5.1}$$

2. *If $\Lambda \subset \mathbb{L}$, $h \geq 0$ and $J(e)$ is defined by (4.5), then for any $t_1, t_2 \in \Sigma_0 \cap \Lambda$,*

$$\sum_{\substack{\lambda: \\ \delta\lambda = \{t_1,t_2\}}} q_\Lambda(\lambda) \leq \langle \sigma(t_2)\sigma(t_1)\rangle^J_{\mathbb{L}} \leq \exp\{-\tau_{\rm bd}(t_2 - t_1; \beta, h)\}\,. \tag{5.2}$$

3. *If $\Lambda \subset \mathbb{L}$, $h \geq 0$ and $J(e)$ is defined by (4.5), then for any $t_1, t_2 \in \Lambda$,*

$$\sum_{\substack{\lambda: \mathcal{E}(\lambda)\cap\mathcal{E}(\Sigma_0) = \emptyset \\ \delta\lambda = \{t_1,t_2\}}} q_\Lambda(\lambda) \leq \exp\{-\tau(t_2 - t_1)\}\,. \tag{5.3}$$

Proof. 1. follows from formula (3.13), GKS inequalities and Proposition 4.1.

2. is proved in the same manner.

3. The case $h \leq 1$ is easy. Indeed,

$$\begin{aligned}
\sum_{\substack{\lambda: \mathcal{E}(\lambda)\cap\mathcal{E}(\Sigma_0) = \emptyset \\ \delta\lambda = \{t_1,t_2\}}} q_\Lambda(\lambda) &\leq \sum_{\substack{\lambda: \\ \delta\lambda = \{t_1,t_2\}}} q_\Lambda(\lambda) \\
&= \langle \sigma(t_1)\sigma(t_2)\rangle^{\beta,h}_\Lambda \\
&\leq \langle \sigma(t_1)\sigma(t_2)\rangle^{\beta,1}_\Lambda \\
&\leq \langle \sigma(t_1)\sigma(t_2)\rangle^{\beta} \\
&\leq \exp\{-\tau(t_2 - t_1)\}\,.
\end{aligned} \tag{5.4}$$

Then the condition $\mathcal{E}(\lambda) \cap \mathcal{E}(\Sigma_0) = \emptyset$ implies that $w(\lambda)$ is independent of h; assume that (5.5) is true, which is a consequence of the next lemma,

$$\frac{Z(\Lambda|\lambda; h)}{Z(\Lambda; h)} \leq \frac{Z(\Lambda|\lambda; 1)}{Z(\Lambda; 1)} \quad \text{if } h \geq 1\,. \tag{5.5}$$

[1] β_c is defined as the smallest β such that $\tau(t;\beta)$ is equal to zero (see Def. 4.1).

Therefore

$$\sum_{\substack{\lambda:\mathcal{E}(\lambda)\cap\mathcal{E}(\Sigma_0)=\emptyset\\ \delta\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda;h) \le \sum_{\substack{\lambda:\mathcal{E}(\lambda)\cap\mathcal{E}(\Sigma_0)=\emptyset\\ \delta\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda;1)\,. \tag{5.6}$$

□

Lemma 5.2 *Let $\underline{\gamma}'$ be a Λ–compatible family of contours. Let $J'(e) \geq J(e) \geq 0$ for all edges of $\mathcal{E}(\Lambda)\backslash\Delta_\Lambda(\underline{\gamma}')$ and all edges $e \in \Delta_\Lambda(\underline{\gamma}')$ such that one end–point t of e has index 0 in $\mathcal{E}(\Lambda)\backslash\Delta_\Lambda(\underline{\gamma}')$. Otherwise $J(e) = J'(e)$. Then*

$$\frac{Z(\Lambda|\underline{\gamma}';J')}{Z(\Lambda;J')} \le \frac{Z(\Lambda|\underline{\gamma}';J)}{Z(\Lambda;J)}\,. \tag{5.7}$$

Proof. We recall that

$$\mathcal{E}(\Lambda_\#(\underline{\gamma}')) = \mathcal{E}(\Lambda)\backslash\Delta_\Lambda(\underline{\gamma}')\,. \tag{5.8}$$

We first assume that $J(e) = J'(e)$ for all edges $e \in \Delta_\Lambda(\underline{\gamma}')$. Let

$$J'_s(e) := \begin{cases} J'(e) & \text{if } e \notin \Delta_\Lambda(\underline{\gamma}'),\\ sJ'(e) & \text{if } e \in \Delta_\Lambda(\underline{\gamma}'). \end{cases} \tag{5.9}$$

If $s = 0$, then

$$Z(\Lambda;J'_0) = Z(\Lambda|\underline{\gamma}';J')\,. \tag{5.10}$$

Indeed,

$$Z(\Lambda;J'_0) = \sum_{\substack{\underline{\gamma}:\,\delta\underline{\gamma}=\emptyset\\ \underline{\gamma}\ \Lambda-\text{comp.}}} w(\underline{\gamma};J'_0)\,. \tag{5.11}$$

Since the coupling constants are J'_0, only terms with $\underline{\gamma} \cap \Delta_\Lambda(\underline{\gamma}') = \emptyset$ give a nonzero contribution in (5.11); therefore by (3.16) of Lemma 3.2

$$Z(\Lambda;J'_0) = \sum_{\substack{\underline{\gamma}:\,\delta\underline{\gamma}=\emptyset\\ \underline{\gamma}\ \Lambda_\#(\underline{\gamma}')-\text{comp.}}} w(\underline{\gamma};J') = Z(\Lambda_\#(\underline{\gamma}');J') = Z(\Lambda|\underline{\gamma}';J')\,. \tag{5.12}$$

If $s = 1$, then $Z(\Lambda;J'_1) = Z(\Lambda;J')$. Therefore (see (3.7))

$$\begin{aligned} \ln\frac{Z(\Lambda;J'_1)}{Z(\Lambda;J'_0)} &= \ln\frac{\Xi(\Lambda;J'_1)}{\Xi(\Lambda;J'_0)} - \ln\prod_{e\in\Delta_\Lambda(\underline{\gamma}')}\cosh J'(e) && (5.13)\\ &= \int_0^1 ds\,\frac{d}{ds}\ln\Xi(\Lambda;J'_s) - \ln\prod_{e\in\Delta_\Lambda(\underline{\gamma}')}\cosh J'(e)\,. && (5.14) \end{aligned}$$

We have a similar representation for J instead of J'. Let

$$J_s(e) := \begin{cases} J(e) & \text{if } e \notin \Delta_\Lambda(\underline{\gamma}'),\\ sJ(e) & \text{if } e \in \Delta_\Lambda(\underline{\gamma}'). \end{cases} \tag{5.15}$$

We have by GKS inequalities, $\langle\sigma(t)\sigma(t')\rangle_\Lambda^{J'_s} \geq \langle\sigma(t)\sigma(t')\rangle_\Lambda^{J_s} \geq 0$,

$$\begin{aligned}\int_0^1 ds\,\frac{d}{ds}\ln\Xi(\Lambda;J'_s) &= \sum_{e=\langle t,t'\rangle\in\Delta_\Lambda(\underline{\gamma}')} J'(e)\int_0^1 ds\,\langle\sigma(t)\sigma(t')\rangle_\Lambda^{J'_s} \\ &\geq \sum_{e=\langle t,t'\rangle\in\Delta_\Lambda(\underline{\gamma}')} J(e)\int_0^1 ds\,\langle\sigma(t)\sigma(t')\rangle_\Lambda^{J_s}\,.\end{aligned} \tag{5.16}$$

Since $\cosh J'(e) = \cosh J(e)$ on $\Delta_\Lambda(\underline{\gamma}')$,

$$\ln\frac{Z(\Lambda;J'_1)}{Z(\Lambda;J'_0)} \geq \ln\frac{Z(\Lambda;J_1)}{Z(\Lambda;J_0)} \tag{5.17}$$

or

$$\frac{Z(\Lambda|\underline{\gamma}';J')}{Z(\Lambda;J')} \leq \frac{Z(\Lambda|\underline{\gamma}';J)}{Z(\Lambda;J)}\,. \tag{5.18}$$

We consider now the general case. Using an iteration procedure it is enough to consider the case, when we modify only one coupling constant from $\Delta_\Lambda(\underline{\gamma}')$. Suppose that $e^*\in\Delta_\Lambda(\underline{\gamma}')$ and that one of its end–points has index 0 in $\mathcal{E}(\Lambda_\#(\underline{\gamma}'))$. No closed contours in $\Lambda_\#(\underline{\gamma}';J')\cup e^*$ contain e^*. So we don't have to put the corresponding coupling constant to zero for having (5.10); hence we can have $J'(e^*)\geq J(e^*)$ and

$$Z(\Lambda|\underline{\gamma}';J') = Z(\Lambda_\#(\underline{\gamma}');J') = Z(\Lambda_\#(\underline{\gamma}';J')\cup\{e^*\})\,. \tag{5.19}$$

We repeat the above proof by replacing $\Delta_\Lambda(\underline{\gamma}')$ by $\Delta_\Lambda(\underline{\gamma}')\backslash\{e^*\}$ and J'_s by

$$J''_s(e) := \begin{cases} J'(e) & \text{if } e\notin\Delta_\Lambda(\underline{\gamma}')\backslash\{e^*\}, \\ sJ'(e) & \text{if } e\in\Delta_\Lambda(\underline{\gamma}')\backslash\{e^*\}. \end{cases} \tag{5.20}$$

□

Lemma 5.3 *Let $J(e)\geq 0$. Let $\underline{\gamma}'$ be a Λ–compatible family of contours.*

1. *$q_\Lambda(\underline{\gamma}';J)$ is decreasing in $J(e)$ for all edges $e\in\mathcal{E}(\Lambda)\setminus\Delta_\Lambda(\underline{\gamma}')$ and all edges $e\in\Delta_\Lambda(\underline{\gamma}')\setminus\mathcal{E}(\underline{\gamma}')$ with one end–point of index 0 in $\mathcal{E}(\Lambda)\setminus\Delta_\Lambda(\underline{\gamma}')$.*

2. *$q_\Lambda(\underline{\gamma}';J)$ is decreasing in Λ, if $\Delta(\gamma')\subset\Lambda$. The limits*

$$q(\underline{\gamma}') := \lim_{L\to\infty} q_{\Lambda'_L}(\underline{\gamma}') \tag{5.21}$$

and

$$q_{\mathbb{L}}(\underline{\gamma}') := \lim_{L\to\infty} q_{\Lambda'_L\cap\mathbb{L}}(\underline{\gamma}') \tag{5.22}$$

exist; $q(\underline{\gamma}')\leq q_\Lambda(\underline{\gamma}')$ and if $\Lambda\subset\mathbb{L}$, then $q_{\mathbb{L}}(\underline{\gamma}')\leq q_\Lambda(\underline{\gamma}')$.

3. *Let J be given by (4.5) and $\Lambda\subset\mathbb{L}$. If $0\leq h\leq 1$, then $q_\Lambda(\underline{\gamma}';\beta,h)\geq q(\underline{\gamma}';\beta)$. If $1\leq h$, then $q_\Lambda(\underline{\gamma}';\beta,h)\geq q_{\mathbb{L}}(\underline{\gamma}';\beta,h)$.*

4. *Let J be given by (4.5) and $\Lambda \subset \mathbb{L}$. Assume that no edge of $\Delta(\underline{\gamma}')$ is adjacent to a site of Σ_0. Then*

$$q_\Lambda(\underline{\gamma}'; J) \geq \tag{5.23}$$
$$q(\underline{\gamma}'; \beta) \cdot \exp\Big\{-\beta^2 \sum_{e=\langle t,t'\rangle \in \Delta(\underline{\gamma}')} \sum_{\substack{t'':\\ t''(2)=1}} \mathrm{e}^{-[\tau(t-t''; \beta) + \tau(t'-t''; \beta)]}\Big\}.$$

Proof. Lemma 5.2 implies that $q_\Lambda(\underline{\gamma}'; J)$ is decreasing in J and therefore also in Λ. This proves 1., 2. and 3.. The proof of 4. is based on identities (5.13) and (5.16) with $\Delta(\underline{\gamma}')$ instead of $\Delta_\Lambda(\underline{\gamma}')$.

$$\ln \frac{Z(\Lambda; J_1)}{Z(\Lambda; J_0)} = \sum_{e=\langle t,t'\rangle \in \Delta(\underline{\gamma}')} J(e) \int_0^1 ds\, \langle \sigma(t)\sigma(t')\rangle_\Lambda^{J_s} - \ln \prod_{e \in \Delta(\underline{\gamma}')} \cosh J(e)\,, \tag{5.24}$$

where

$$J_s(e) := \begin{cases} J(e) & \text{if } e \notin \Delta(\underline{\gamma}'), \\ sJ(e) & \text{if } e \in \Delta(\underline{\gamma}'). \end{cases} \tag{5.25}$$

This identity shows that

$$\ln \frac{Z(\mathbb{L}; J_1)}{Z(\mathbb{L}; J_0)} = \sum_{e=\langle t,t'\rangle \in \Delta(\underline{\gamma}')} J(e) \int_0^1 ds\, \langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}}^{J_s} - \ln \prod_{e \in \Delta(\underline{\gamma}')} \cosh J(e)\,. \tag{5.26}$$

GKS–inequalities give

$$\langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} \leq \langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}}^{J_s} \leq \langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s}\,. \tag{5.27}$$

The $(\mathbb{L}\backslash\Sigma_0)^+$–boundary condition in (5.27) is obtained by introducing an external field on Σ_0 and then letting this field go to ∞. We compare the left–hand and right–hand sides. First, notice that

$$-\sigma(t)\sigma(t') + \sigma(t) + \sigma(t')\,, \tag{5.28}$$

is an increasing function, so that by FKG–inequalities

$$\langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} \leq \tag{5.29}$$
$$\langle \sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle \sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} + \langle \sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle \sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s}\,.$$

Define new coupling constants $J''(e)$,

$$J''(e) := \begin{cases} J_s(e) & \text{if } e \text{ not adjacent to } \Sigma_0, \\ aJ_s(e) & \text{otherwise.} \end{cases} \tag{5.30}$$

We can write using (5.29) and (4.5),

$$\langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} \leq \tag{5.31}$$
$$\sum_{\substack{t'':\\ t''(2)=1}} \beta \int_0^1 da \Big(\langle \sigma(t)\,;\, \sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} + \langle \sigma(t')\,;\, \sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} \Big)\,,$$

where

$$\langle \sigma(t)\,;\,\sigma(t'')\rangle^{+,J''}_{\mathbb{L}\backslash\Sigma_0} := \tag{5.32}$$
$$\langle \sigma(t)\,\sigma(t'')\rangle^{+,J''}_{\mathbb{L}\backslash\Sigma_0} - \langle \sigma(t)\rangle^{+,J''}_{\mathbb{L}\backslash\Sigma_0}\cdot\langle \sigma(t'')\rangle^{+,J''}_{\mathbb{L}\backslash\Sigma_0}\,.$$

GHS–inequalities imply

$$\langle \sigma(t)\,;\,\sigma(t'')\rangle^{+,J''}_{\mathbb{L}\backslash\Sigma_0} \leq \langle \sigma(t)\,;\,\sigma(t'')\rangle^{J''}_{\mathbb{L}\backslash\Sigma_0} = \langle \sigma(t)\,\sigma(t'')\rangle^{J_s}_{\mathbb{L}\backslash\Sigma_0}\,, \tag{5.33}$$

because the last expectation value is with respect to the Gibbs measure on $\mathbb{L}\backslash\Sigma_0$ with free boundary condition and consequently by symmetry

$$\langle \sigma(t)\rangle^{J_s}_{\mathbb{L}\backslash\Sigma_0} = 0\,. \tag{5.34}$$

GKS–inequalities imply now

$$\langle \sigma(t)\sigma(t'')\rangle^{J_s}_{\mathbb{L}\backslash\Sigma_0} \leq \langle \sigma(t)\sigma(t'')\rangle^{\beta} \leq \exp\{-\tau(t-t'';\beta)\}\,. \tag{5.35}$$

Therefore (with natural notation) we proved that

$$\begin{aligned} 1 \;&\geq\; \frac{q^{+}_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)}{q_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)} \\ &=\; \exp\Big\{-\beta\int_0^1 ds\,\Big(\langle\sigma(t)\sigma(t')\rangle^{+,J_s}_{\mathbb{L}\backslash\Sigma_0} - \langle\sigma(t)\sigma(t')\rangle^{J_s}_{\mathbb{L}\backslash\Sigma_0}\Big)\Big\} \\ &\geq\; \exp\Big\{-\beta^2\sum_{e=\langle t,t'\rangle\in\Delta(\underline{\gamma}')}\;\sum_{\substack{t''\,:\\ t''(2)=1}} \mathrm{e}^{-[\tau(t-t'';\beta)+\tau(t'-t'';\beta)]}\Big\}\,. \end{aligned} \tag{5.36}$$

The lemma now follows easily. We have, see (5.27)

$$\begin{aligned} q_{\mathbb{L}}(\underline{\gamma}';J) \;&=\; q_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)\frac{q_{\mathbb{L}}(\underline{\gamma}';J)}{q_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)} \\ &\geq\; q_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)\frac{q^{+}_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)}{q_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)} \\ &\geq\; q(\underline{\gamma}';\beta)\frac{q^{+}_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)}{q_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J)}\,, \end{aligned} \tag{5.37}$$

because $q_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J) \geq q(\underline{\gamma}';\beta)$ by the first part of 2. ($h=1$). □

Lemma 5.4 *Let $J(e)\geq 0$.*

1. *For any $t,t_1,t_2\in\Lambda$,*

$$\sum_{\substack{\lambda:\delta\lambda=\{t_1,t_2\}\\ t\in\lambda}} q_\Lambda(\lambda) \leq \langle\sigma(t_1)\sigma(t)\rangle_\Lambda\langle\sigma(t)\sigma(t_2)\rangle_\Lambda\,. \tag{5.38}$$

2. *Let $t,t_1,t_2\in\Lambda$. Let λ_1 be an open contour with $\delta\lambda_1=\{t_1,t\}$ and λ_2 be an open contour with $\delta\lambda_2=\{t,t_2\}$, so that $\lambda=\lambda_1\cup\lambda_2$ is an open contour with $\delta\lambda=\{t_1,t_2\}$. Then*

$$q_\Lambda(\lambda)\geq q_\Lambda(\lambda_1)q_\Lambda(\lambda_2)\,. \tag{5.39}$$

Proof. We prove 1.

Let λ be an open contour with $\delta\lambda = \{t_1, t_2\}$, considered as a unit–speed parametrized curve. Let s^* be the largest $s \in [0, |\lambda|]$ so that $\lambda(s^*) = t$. We decompose λ into $\{\lambda_1, \lambda_2\}$ by cutting λ at t and we set

$$\lambda_1 = \{\lambda(s) : s \in [0, s^*]\} \quad \text{and} \quad \lambda_2 = \{\lambda(s) : s \in [s^*, |\lambda|]\} . \tag{5.40}$$

Notice that by definition $\lambda_2(s) \neq t$ for any $s > s^*$, that is, λ_2 has exactly one adjacent edge to t. (The way we cut λ depends on the choice of the orientation of λ.) Consider the graph $\Lambda(\lambda_2)$ defined at the end of section 3; add to this graph the edge $e^{\#}$ of $\Delta(\lambda_2)$ which is adjacent to t, but does not belong to λ_2. We get a new graph $\Lambda^{\#}(\lambda_2)$; the normalized partition function of the Ising model on $\Lambda^{\#}(\lambda_2)$ is denoted by $Z(\Lambda^{\#}(\lambda_2))$. We claim that

$$Z(\Lambda|\lambda_1 \cup \lambda_2) = Z(\Lambda^{\#}(\lambda_2)|\lambda_1) . \tag{5.41}$$

First, by definition

$$\Delta(\lambda_1 \cup \lambda_2) = \Delta(\lambda_1) \cup \Delta(\lambda_2) . \tag{5.42}$$

Let $\underline{\gamma}$, $\delta\underline{\gamma} = \emptyset$, be Λ–compatible with $\lambda_1 \cup \lambda_2$. By Lemma 3.1

$$\mathcal{E}(\underline{\gamma}) \cap \Delta(\lambda_1 \cup \lambda_2) = \emptyset . \tag{5.43}$$

Therefore by (5.42)

$$\mathcal{E}(\underline{\gamma}) \subset \mathcal{E}(\Lambda^{\#}(\lambda_2)) , \tag{5.44}$$

and

$$\mathcal{E}(\underline{\gamma}) \cap \Delta(\lambda_1) = \emptyset . \tag{5.45}$$

This implies that $\underline{\gamma}$ is $\Lambda_{\#}$–compatible with λ_1. Conversely, if $\underline{\gamma}$, $\delta\underline{\gamma} = \emptyset$, is $\Lambda^{\#}(\lambda_2)$–compatible with λ_1, then

$$\mathcal{E}(\underline{\gamma}) \cap \Delta(\lambda_1) = \emptyset . \tag{5.46}$$

Suppose that e_1, e_2 are two edges of $\mathcal{E}(\underline{\gamma})$, which are adjacent to t. This is possible only if the edge e^* of λ_2, which is adjacent to t, belongs to $\Delta(\lambda_1)$. But this means that $\underline{\gamma}$ is Λ–compatible with $\lambda_1 \cup \lambda_2$. Using this result we get

$$\sum_{\substack{\lambda:\delta\lambda=\{t_1,t_2\} \\ t\in\lambda}} Z(\Lambda|\lambda)w(\lambda) = \sum_{\{\lambda_1,\lambda_2\}=\lambda} Z(\Lambda|\lambda)w(\lambda_1)w(\lambda_2) \tag{5.47}$$

$$= \sum_{\{\lambda_1,\lambda_2\}=\lambda} \frac{w(\lambda_1)Z(\Lambda|\lambda_1 \cup \lambda_2)}{Z(\Lambda^{\#}(\lambda_2))} Z(\Lambda^{\#}(\lambda_2))w(\lambda_2) .$$

If we sum in (5.47) over all λ_1, given λ_2, and use (5.41), then we obtain the two–point function of the Ising model on the graph $\Lambda^{\#}(\lambda_2)$:

$$\sum_{\lambda_1} w(\lambda_1) \frac{Z(\Lambda^{\#}(\lambda_2)|\lambda_1)}{Z(\Lambda^{\#}(\lambda_2))} = \langle\sigma(t_1)\sigma(t)\rangle_{\Lambda^{\#}(\lambda_2)} \leq \langle\sigma(t_1)\sigma(t)\rangle_{\Lambda} . \tag{5.48}$$

We can interpret in a similar way the remaining sum,

$$\sum_{\lambda_2} w(\lambda_2) \frac{Z(\Lambda^{\#}(\lambda_2))}{Z(\Lambda)} . \tag{5.49}$$

We have

$$Z(\Lambda^{\#}(\lambda_2)) = \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset \\ \Lambda^{\#}(\lambda_2)-\mathrm{comp}}} w(\underline{\gamma}) = \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset \\ \Lambda(\lambda_2)-\mathrm{comp}}} w(\underline{\gamma}) + \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset\,,\, e^{\#}\in\underline{\gamma} \\ \Lambda^{\#}(\lambda_2)-\mathrm{comp}}} w(\underline{\gamma}) \tag{5.50}$$

By construction, all open contours λ_2 have only one edge adjacent to t. In the first sum all closed contours of $\underline{\gamma}$ are compatible with λ_2, while in the second sum the closed contour containing $e^{\#}$ is glued together with λ_2 to give a new open contour of index 3 at t. Therefore

$$\sum_{\lambda_2} w(\lambda_2)\frac{Z(\Lambda^{\#}(\lambda_2))}{Z(\Lambda)} = \sum_{\substack{\lambda:\delta\lambda=\{t,t_2\} \\ \Lambda-\mathrm{comp}}} w(\lambda)\frac{Z(\Lambda|\lambda))}{Z(\Lambda)} = \langle\sigma(t)\sigma(t_2)\rangle_{\Lambda}\,. \tag{5.51}$$

We prove 2. Since $Z(\Lambda^{\#}(\lambda_2)) \geq Z(\Lambda(\lambda_2))$,

$$\begin{aligned} q_{\Lambda}(\lambda) &= w(\lambda_1)w(\lambda_2)\frac{Z(\Lambda|\lambda_1\cup\lambda_2)}{Z(\Lambda)} \\ &= w(\lambda_1)\frac{Z(\Lambda|\lambda_1\cup\lambda_2)}{Z(\Lambda(\lambda_2))}\cdot w(\lambda_2)\frac{Z(\Lambda(\lambda_2))}{Z(\Lambda)} \\ &\geq w(\lambda_1)\frac{Z(\Lambda|\lambda_1\cup\lambda_2)}{Z(\Lambda^{\#}(\lambda_2))}\cdot w(\lambda_2)\frac{Z(\Lambda(\lambda_2))}{Z(\Lambda)} \\ &= q_{\Lambda^{\#}(\lambda_2)}(\lambda_1)\cdot q_{\Lambda}(\lambda_2) \\ &\geq q_{\Lambda}(\lambda_1)\cdot q_{\Lambda}(\lambda_2) \end{aligned} \tag{5.52}$$

by Lemma 5.2. □

Lemma 5.5 *Let $\beta < \beta_c$.*

1. *If $J(e) = \beta$ for all edges e, then for any $t_0, \ldots, t_n \in \Lambda$, with $t_{n+1} \equiv t_0$,*

$$\sum_{\substack{\lambda:\delta\lambda=\emptyset \\ t_0,\ldots,t_n\in\lambda}} q_{\Lambda}(\lambda) \leq \exp\{-\sum_{i=0}^{n}\tau(t_{i+1}-t_i)\}\,. \tag{5.53}$$

2. *If $h \geq 0$ and $J(e)$ is defined by (4.5), then for $t, t_1, t_2 \in \Lambda \subset \mathbb{L}$,*

$$\sum_{\substack{\lambda:\delta\lambda=\{t_1,t_2\},t\in\lambda \\ \mathcal{E}(\lambda)\cap\mathcal{E}(\Sigma_0)=\emptyset}} q_{\Lambda}(\lambda) \leq \exp\{-\tau(t-t_1)-\tau(t_2-t)\}\,. \tag{5.54}$$

3. *We have similar results if we replace λ by a family of Λ–compatible contours.*

Proof. 1. follows from Lemma 5.4, GKS inequalities and Proposition 4.1. The proof of 2. is a consequence of Lemmas 5.1 and 5.4. □

Lemma 5.6 *Let $\beta < \beta_c$, $h \geq 0$ and $J(e)$ be defined by (4.5); let $\Lambda \subset \mathbb{L}$. The diameter of γ is $d(\gamma)$. There exists a positive constant $\alpha = \alpha(J)$ and a constant $C(\alpha)$, such that for small α, $C(\alpha) = O(\frac{1}{\alpha}\ln\frac{1}{\alpha})$, and for $l \geq C(\alpha)$*

$$\mathrm{Prob}[\,\exists\gamma\,,\, d(\gamma) \geq l\,] \leq |\Lambda| \cdot O(l^2)\exp\{-\alpha \cdot l\}\,. \tag{5.55}$$

Proof. We give to all closed contours an origin by choosing a total order on the lattice:

$$t < t' \Longleftrightarrow t(2) < t'(2) \quad \text{or} \quad t(2) = t'(2) \text{ and } t(1) < t'(1)\,. \tag{5.56}$$

The origin is also the initial point of the contour, viewed as a parametrized curve, which is counterclockwise oriented. To each γ with diameter $d(\gamma) > l$, we associate a sequence of points on the lattice as follows:

1. t_0' is the origin of γ. If $t_0'(2) = 0$, then s_0 is the last time such that $\gamma(s_0)(2) = 0$; we set $t_0 := \gamma(s_0)$. (t_0 is the largest point of γ such that $t_0(2) = 0$.) Otherwise $t_0 := t_0'$.
2. Let s_1 be the first time such that γ leaves the square of center t_0 and side $l/2$; we set $t_1 := \gamma(s_1)$.
3. Let s_2 be the first time greater than s_1 such that γ leaves the square of center t_1 and side $l/2$; we set $t_2 := \Gamma(s_2)$.
4. The procedure is iterated until it stops.

Thus for all closed γ we have a well–defined ordered sequence of points $(t_0', t_0, t_1, \ldots t_n)$.

For $x = (x(1), x(2)) \in \mathbb{R}^2$ let $|x|_1 := |x(1)| + |x(2)|$. Since $\beta < \beta_c$, $\tau(x) > 0$; we define α as the largest positive constant such that $\tau(x) \geq 2\alpha|x|_1$, $\forall x \in \mathbb{R}^2$. Clearly

$$\mathrm{Prob}[\,\exists\gamma\;\; d(\gamma) \geq l\,] \leq \sum_{t\in\Lambda}\sum_{\substack{d(\gamma)\geq l\\ t_0'(\gamma)=t}} q_\Lambda(\gamma)\,, \tag{5.57}$$

since $\mathrm{Prob}[\,\exists\gamma\,] = q_\Lambda(\gamma)$. Suppose that $(t_0', t_0, t_1, \ldots t_n)$ is fixed. Then

$$\sum_{\substack{\gamma:\,\delta\gamma=\emptyset\\ t_0',\ldots,t_{n+1}\in\gamma}} q_\Lambda(\gamma) \leq \exp\{-\sum_{i=0}^{n}\tau(t_{i+1}-t_i) - \tau_{\mathrm{bd}}(t_0'-t_0)\} \tag{5.58}$$

with $t_{n+1} \equiv t_0$. Therefore

$$\sum_{\substack{\gamma:\,\delta\gamma=\emptyset\,,d(\gamma)\geq l\\ t_0',\ldots,t_{n+1}\in\lambda}} q_\Lambda(\gamma) \leq \exp\{-\tau_{\mathrm{bd}}(t_0'-t_0)\}\exp\{-(\alpha/2)nl\}\,, \tag{5.59}$$

and

$$\begin{aligned}\sum_{\gamma:\,t_0'(\gamma)=t\,,d(\gamma)\geq l} q_\Lambda(\gamma) \quad &\leq \quad \sum_{t_0}\exp\{-\tau_{\mathrm{bd}}(t_0'-t_0)\} \\ &\qquad \cdot \sum_{n\geq 2}(2[l+2])^n\exp\{-(\alpha/2)nl\}\,.\end{aligned} \tag{5.60}$$

We can choose $C(\alpha)$ so that for $l \geq C(\alpha)$,

$$\sum_{\gamma:\, t_0'(\gamma)=t\,, d(\gamma)\geq l} q_\Lambda(\gamma) \leq O(l^2)\exp\{-\alpha \cdot l\}\,. \tag{5.61}$$

□

6 Random–line representation of the two–point function

6.1 Two–point function

Let $\beta < \beta_c$ and suppose that $J(e) = \beta$ for all edges e.

Let $\mathcal{L}$ be the set

$$\mathcal{L} := \{\lambda :\ \lambda = \emptyset \text{ or } \lambda \text{ is an open contour, } \delta\lambda = \{0, t\}\,,\, 0 \neq t \in \mathbb{Z}^2\}\,. \tag{6.1}$$

Let $q(\lambda)$ be the quantity of formula (5.21) when λ is an open contour; set $q(\lambda) = 1$ when $\lambda = \emptyset$. We have

$$\begin{aligned} \chi \;&:=\; \sum_{\lambda\in\mathcal{L}} q(\lambda) \\ &=\; 1 + \sum_{0\neq t\in\mathbb{Z}^2}\ \sum_{\lambda:\,\delta\lambda=\{0,t\}} q(\lambda) \\ &=\; \sum_{t\in\mathbb{Z}^2} \langle\sigma(0)\sigma(t)\rangle\,. \end{aligned} \tag{6.2}$$

The quantity χ is the **susceptibility** of the model. It is finite (see e.g. Lemma 5.1). On $\mathcal{L}$ we define a measure $\mathbb{M}$ with finite mass χ, by setting

$$\mathbb{M}(\lambda) := q(\lambda)\,. \tag{6.3}$$

Let $\{0 \to t\}$ denote the event $\{\lambda \in \mathcal{L} :\ \delta\lambda = \{0, t\}\}$. Then the two–point function is equal to

$$\langle\sigma(0)\sigma(t)\rangle = \mathbb{M}[\,\{0 \to t\}\,]\,. \tag{6.4}$$

Let $\lambda \in \mathcal{L}$ such that $\delta\lambda = \{0, t\}$ and $t(1) > 0$. The open contour λ is called **cylindrical** if $\lambda = \lambda' \cup e$ with e an horizontal edge and

$$\lambda' \subset \{s \in \mathbb{R}^2 :\ 0 \leq s(1) \leq t(1) - 1\}\,. \tag{6.5}$$

Let $\{0 \xrightarrow{c} t\}$ denote the event $\{\lambda \in \mathcal{L} :\ \delta\lambda = \{0, t\}\,,\, \lambda \text{ cylindrical}\}$. For $a \in \mathbb{N}$ we define

$$\chi(a, 0) := \sum_{\substack{t:\\ t(1)=a}} \langle\sigma(0)\sigma(t)\rangle = \sum_{\substack{t:\\ t(1)=a}} \mathbb{M}[\,\{0 \to t\}\,] \tag{6.6}$$

and

$$\chi_c(a,0) := \sum_{\substack{t:\\ t(1)=a}} \mathbb{M}[\,\{0 \xrightarrow{c} t\}\,]\,. \tag{6.7}$$

As a consequence of the estimates of section 5 we have the following lemma, which is inspired by [CC].

Lemma 6.1 *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges e; set $\tau^* := \tau((1,0))$ and*

$$\chi^* := \sum_{\substack{t:\\ t(1)=0}} \langle \sigma(0)\sigma(t)\rangle\,. \tag{6.8}$$

Then for any $a \in \mathbb{N}$

$$\chi_c(a,0) \le \chi(a,0) \le {\chi^*}^2 \chi_c(a,0)\,; \tag{6.9}$$

$$\lim_{a\to\infty} -\frac{1}{a}\ln\chi(a,0) = \lim_{a\to\infty} -\frac{1}{a}\ln\chi_c(a,0) = \tau^*\,; \tag{6.10}$$

$$\chi_c(a,0) \le \exp\{-\tau^*\cdot a\} \quad , \quad \chi(a,0) \ge \exp\{-\tau^*\cdot a\}\,. \tag{6.11}$$

Proof. The first inequalities (6.9) follow from Lemma 5.4 and GKS inequalities. We prove (6.10). Clearly,

$$\lim_{a\to\infty} -\frac{1}{a}\ln\chi(a,0) \le \lim_{a\to\infty} -\frac{1}{a}\ln\langle\sigma(0)\sigma((a,0))\rangle \le \tau^*\,. \tag{6.12}$$

Consider the two lines

$$l_1 := \{s \in \mathbb{R}^2 :\ s(2) = 2a - s(1)\} \quad , \quad l_2 := \{s \in \mathbb{R}^2 :\ s(2) = -2a + s(1)\}\,. \tag{6.13}$$

By Proposition 2.2 we have

$$\langle\sigma(0)\sigma(t')\rangle \le \langle\sigma(0)\sigma((a,0))\rangle \tag{6.14}$$

if $t' = (a,b)$, $b \in \mathbb{Z}$, or $t' \in l_1$ and $t'(1) \le a$, or $t' \in l_2$ and $t'(1) \le a$. If λ is a cylindrical contour with $\delta\lambda = \{0,t\}$, $t(1) = a$ and $|t(2)| \ge a$, then it must intersect l_1 or l_2. Applying Lemma 5.4 and (6.14)

$$\chi_c(a,0) = \sum_{\substack{t:\\ t(1)=a}} \mathbb{M}[\,\{0 \xrightarrow{c} t\}\,] \le 2a(1+\chi^*)\langle\sigma(0)\sigma((a,0))\rangle\,. \tag{6.15}$$

The last inequalities (6.11) follow from

$$\chi_c(a_1+a_2,0) \ge \chi_c(a_1,0)\chi_c(a_2,0) \tag{6.16}$$

and

$$\chi(a_1+a_2,0) \le \chi(a_1,0)\chi(a_2,0)\,. \tag{6.17}$$

$\square$

Remark: There are similar results for

$$\chi(a,a) := \sum_{\substack{t:\\ t(2)=a-t(1)}} \langle\sigma(0)\sigma(t)\rangle\,. \tag{6.18}$$

We study the influence of a wall on the two–point function. We do it in two steps. We consider the case of a vertical wall; similar results hold for horizontal walls.

Lemma 6.2 *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges e. Let l_a be the vertical line*

$$l_a := \{s \in \mathbb{R}^2 : s(1) = -a\}, \tag{6.19}$$

$t = (t(1), t(2))$, with $0 \le t(1)$, and $E_a(t)$ be the event

$$E_a(t) := \{\lambda \in \mathcal{L} : \delta\lambda = \{0, t\}, \lambda \cap l_a \neq \emptyset\}. \tag{6.20}$$

Then

$$\mathbb{M}[\, E_a(t)\,] \le 2|t(2)| \cdot \mathbb{M}[\,\{0 \to t + (2a, 0)\}\,] + \chi(a, 0) \cdot \mathbb{M}[\,\{0 \to t\}\,]. \tag{6.21}$$

Proof. λ is considered as a unit–speed parametrized curve with initial point 0. Let us assume that $t(2) \ge 0$. Let s be the first time λ touches the vertical line l_a; we set $u := \lambda(s)$. We have

$$\mathbb{M}[E_a(t)] \le \sum_{u \in l_a} \sum_{\substack{\lambda \in E_a(t): \\ u \in \lambda}} q(\lambda) \le \sum_{u \in l_a} \langle \sigma(0)\sigma(u)\rangle\langle\sigma(u)\sigma(t)\rangle. \tag{6.22}$$

If $u(2) \le 0$ or $u(2) \ge 2t(2)$, then by monotonicity and symmetry

$$\langle\sigma(u)\sigma(t)\rangle \le \langle\sigma(0)\sigma(t)\rangle. \tag{6.23}$$

Summing all these terms in (6.22), using (6.23), we get in (6.21) the term

$$\chi(a, 0) \cdot \mathbb{M}[\,\{0 \to t\}\,]. \tag{6.24}$$

Let $t_{2a} := (-2a, 0)$. If $0 \le u(2) \le 2t(2)$, then by symmetry and GKS inequalities

$$\begin{aligned} \langle\sigma(0)\sigma(u)\rangle\langle\sigma(u)\sigma(t)\rangle &= \langle\sigma(t_{2a})\sigma(u)\rangle\langle\sigma(u)\sigma(t)\rangle \\ &\le \langle\sigma(t_{2a})\sigma(t)\rangle \\ &= \langle\sigma(0)\sigma(t - t_{2a})\rangle. \end{aligned} \tag{6.25}$$

□

In the next lemma we estimate $\mathbb{M}[\,\{0 \to t + (2a, 0)\}\,]$ in terms of $\langle\sigma(0)\sigma(t)\rangle = \mathbb{M}[\,\{0 \to t\}\,]$, so that we get an upper bound for $\mathbb{M}[\,E_a(t)\,|\,\{0 \to t\}\,]$.

Lemma 6.3 *Let $\beta < \beta_c$ and $J(e) = \beta$ for all edges e. Let $t = (t(1), t(2))$ and l_a be the vertical line*

$$l_a := \{s \in \mathbb{R}^2 : s(1) = -a\}. \tag{6.26}$$

Let

$$\overline{a} := \min\{2a + t(1) - |t(2)|, 2a\} > 0. \tag{6.27}$$

and $E_a(t)$ be the event

$$E_a(t) := \{\lambda \in \mathcal{L} : \delta\lambda = \{0, t\}, \lambda \cap l_a \neq \emptyset\}. \tag{6.28}$$

Then

$$\mathbb{M}[\,E_a(t)\,|\,\{0 \to t\}\,] \le 8\chi(\overline{a}, \overline{a})|t(2)| + \chi(a, 0). \tag{6.29}$$

Proof. Let B be the set

$$B := \{t' : \ |t'(1) + 2a| + |t'(2)| = \overline{a}\}\,. \tag{6.30}$$

The set B separates $t_{2a} = (-2a, 0)$ and 0. In particular it separates $t_{2a} = (-2a, 0)$ and t. Applying Proposition 2.2

$$\begin{aligned} \mathbb{M}[\,\{0 \rightarrow t + (2a,0)\}\,] &= \langle \sigma(t_{2a})\sigma(t)\rangle \\ &\leq \sum_{t' \in B} \langle \sigma(t_{2a})\sigma(t')\rangle\langle \sigma(t')\sigma(t)\rangle\,. \end{aligned} \tag{6.31}$$

The definition of B has been chosen so that we have for all $t' \in B$

$$\langle \sigma(t')\sigma(t)\rangle \leq \langle \sigma(0)\sigma(t)\rangle\,. \tag{6.32}$$

Therefore

$$\mathbb{M}[\,\{0 \rightarrow t + (2a,0)\}\,] \leq 4\chi(\overline{a},\overline{a})\langle \sigma(0)\sigma(t)\rangle\,. \tag{6.33}$$

We conclude using Lemma 6.2. □

Lemma 6.3 allows to get upper and lower bounds for

$$\sum_{\substack{\lambda:\delta\lambda=\{0,t\} \\ \lambda\cap l_a=\emptyset}} q(\lambda) \tag{6.34}$$

in terms of $\langle \sigma(0)\sigma(t)\rangle$,

$$\langle \sigma(0)\sigma(t)\rangle \geq \sum_{\substack{\lambda:\delta\lambda=\{0,t\} \\ \lambda\cap l_a=\emptyset}} q(\lambda) = (1 - \mathbb{M}[\,E_a(t)\,|\,\{0 \rightarrow t\}\,])\langle \sigma(0)\sigma(t)\rangle\,. \tag{6.35}$$

The next proposition contains one of the main estimates of the paper.

Proposition 6.1 *Let $\beta < \beta_c$, $J(e) = \beta$ for all edges e, $t = (t(1), t(2)) \in \mathbb{Z}^2$, with $0 \leq t(2) \leq t(1)$ and $a \in \mathbb{N}$ with $2a < t(1)$. Let B_t be the square box*

$$B_t := \left\{ s \in \mathbb{Z}^2 : \ 0 \leq s(1) \leq t(1)\,;\ \frac{t(2) - t(1)}{2} \leq s(2) \leq \frac{t(1) + t(2)}{2} \right\}. \tag{6.36}$$

Then

$$\sum_{\substack{\lambda:\delta\lambda=\{0,t\} \\ \lambda \text{ inside } B_t}} q(\lambda) \geq \langle \sigma(0)\sigma(t)\rangle \Big[1 - O(|t(1)|\exp\{-O(a)\})\Big] \exp\{-O(a)\}\,. \tag{6.37}$$

λ inside B_t *means that $\lambda \subset B_t$ and no edge of λ, except the first and the last one, is adjacent to a site $t' \in B_t$ with*

$$t'(1) = 0 \text{ or } t'(1) = t(1) \text{ or } t'(2) = \frac{t(2) - t(1)}{2} \text{ or } t'(2) = \frac{t(1) + t(2)}{2}\,. \tag{6.38}$$

Proof. The proof is partly inspired by a result of [I1]; otherwise it follows a similar result proved in [Pf2]. We choose two points $u_a \in \mathbb{Z}^2$ and $v_a \in \mathbb{Z}^2$ such that

1. u_a is the point on the vertical line $\{t' : t'(1) = a\}$ with $u_a(2)$ minimal and $u_a(2) \geq a \cdot (t(2)/t(1))$.

2. v_a is the point on the vertical line $\{t' : t'(1) = t(1) - a\}$ with $v_a(2)$ maximal and $v_a(2) \leq t(2) - a \cdot (t(2)/t(1))$.

Then we choose two open contours λ_1 and λ_2 inside B_t with $\delta\lambda_1 = \{0, u_a\}$ and $\delta\lambda_2 = \{v_a, t\}$, such that λ_1 and λ_2 have minimal lengths.

Let λ' be an open contour with $\delta\lambda' = \{u_a, v_a\}$ which is inside B_t. We assume that u_a is the initial point. Let $s_1 \in [0, |\lambda'|]$ be the integer time defined by the condition that $t_1 := \lambda'(s_1)$ is on λ_1 so that $t_1(1)$ is minimal; similarly let $s_2 \in [0, |\lambda'|]$ be the integer time defined by the condition that $t_2 := \lambda'(s_2)$ is on λ_2 so that $t_2(1)$ is maximal.

By Lemma 5.4 we have

$$\sum_{\substack{\lambda':\delta\lambda'=\{u_a,v_a\} \\ \lambda' \text{ inside } B_t}} q(\lambda) \leq \sum_{t_1,t_2} \sum_{\substack{\lambda:\delta\lambda=\{t_1,t_2\} \\ \lambda \text{ inside } B_t}} q(\lambda) \langle\sigma(u_a)\sigma(t_1)\rangle\langle\sigma(t_2)\sigma(v_a)\rangle\,. \tag{6.39}$$

Let λ be an open contour of the sum (6.39); we extend λ, using the contours λ_1 and λ_2, to an open contour $\overline{\lambda}$, so that $\overline{\lambda}$ is inside B_t and $\delta\overline{\lambda} = \{0, t\}$: $\overline{\lambda}$ is the union of λ_1', λ and λ_2', with λ_1' the part of λ_1 from 0 to t_1 and λ_2' the part of λ_2 from t_2 to t. By Lemma 5.4 we have

$$q(\overline{\lambda}) \geq q(\lambda_1')q(\lambda)q(\lambda_2')\,. \tag{6.40}$$

Using Lemma 5.3 (replace $J(e)$ by $J'(e) = \infty$ for all $e \notin \Delta(\lambda_j')$), we can show for that

$$q(\lambda_j'; J) \geq \exp\{-O(|\lambda_j'|)\} \quad , \quad j = 1, 2\,. \tag{6.41}$$

Thus

$$q(\lambda) \leq q(\overline{\lambda}) \exp\{O(a)\}\,, \tag{6.42}$$

since $|\lambda_j| = O(a)$. Therefore

$$\sum_{\substack{\overline{\lambda}:\delta\overline{\lambda}=\{0,t\} \\ \overline{\lambda} \text{ inside } B_t}} q(\overline{\lambda}) \geq \sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\} \\ \lambda \text{ inside } B_t}} q(\lambda) \exp\{-O(a)\}\,. \tag{6.43}$$

The box B_t has been chosen so that we can apply Lemma 6.3 to estimate the right–hand side of (6.43). For every wall of the box we have $\overline{a} \leq 2a$ (see (6.27)). We apply the proposition by chosing $\overline{a} = a$. We get

$$\sum_{\substack{\overline{\lambda}:\delta\overline{\lambda}=\{0,t\} \\ \overline{\lambda} \text{ inside } B_t}} q(\overline{\lambda}) \geq \langle\sigma(u_a)\sigma(v_a)\rangle \Big[1 - O(|t(1)| \exp\{-O(a)\})\Big] \exp\{-O(a)\}\,. \tag{6.44}$$

In the last inequality we may replace $\langle\sigma(u_a)\sigma(v_a)\rangle$ by $\langle\sigma(0)\sigma(t)\rangle$, since

$$\langle\sigma(u_a)\sigma(v_a)\rangle \geq \langle\sigma(0)\sigma(t)\rangle\,. \tag{6.45}$$

$\square$

6.2 Boundary two–point function

There is a similar random–line representation for the boundary two–point function. The coupling constants are given by (4.5).

Let

$$\mathcal{L}_{\mathbb{L}} := \{\lambda \subset \mathbb{L} : \ \lambda = \emptyset \text{ or } \lambda \text{ is an open contour}, \delta\lambda = \{0, t\}\,, \, 0 \neq t \in \mathbb{L}\}\,. \tag{6.46}$$

We define a measure on $\mathcal{L}_{\mathbb{L}}$ by setting

$$\mathbb{M}_{\mathbb{L}}(\lambda) := q_{\mathbb{L}}(\lambda)\,. \tag{6.47}$$

The total mass of $\mathbb{M}_{\mathbb{L}}$ is

$$\chi_{\mathbb{L}} := \sum_{\lambda \in \mathcal{L}_{\mathbb{L}}} q_{\mathbb{L}}(\lambda) = \sum_{t \in \mathbb{L}} \langle \sigma(0)\sigma(t) \rangle_{\mathbb{L}}\,, \tag{6.48}$$

and by GKS–inequalities $\chi_{\mathbb{L}} \leq \chi$. We have, for $x \in \Sigma_0$,

$$\langle \sigma(0)\sigma(t) \rangle_{\mathbb{L}} = \mathbb{M}_{\mathbb{L}}[\,\{0 \to t\}\,]\,. \tag{6.49}$$

Notice the following interesting inequalities for $h = 1$, which are consequences of Lemma 5.2.

Lemma 6.4 *Let $\beta < \beta_c$, $J(e) = \beta$ for all edges. Then*

$$\mathbb{M}_{\mathbb{L}}[\,\{0 \to t\}\,] \geq \mathbb{M}[\{\lambda \subset \mathbb{L}\} \cap \{0 \to t\}] \tag{6.50}$$

and

$$\langle \sigma(0)\sigma(t) \rangle_{\mathbb{L}} \geq \mathbb{M}[\{\lambda \subset \mathbb{L}\}\,|\,\{0 \to t\}] \langle \sigma(0)\sigma(t) \rangle\,. \tag{6.51}$$

Proof. We have

$$\mathbb{M}_{\mathbb{L}}[\,\{0 \to t\}\,] = \langle \sigma(0)\sigma(t) \rangle_{\mathbb{L}} \geq \mathbb{M}[\{\lambda \subset \mathbb{L}\} \cap \{0 \to t\}]\,, \tag{6.52}$$

since for $\lambda \subset \mathbb{L}$

$$q_{\mathbb{L}}(\lambda) \geq q(\lambda)\,. \tag{6.53}$$

Therefore

$$\begin{aligned} \langle \sigma(0)\sigma(t) \rangle_{\mathbb{L}} &\geq \mathbb{M}[\{\lambda \subset \mathbb{L}\} \cap \{0 \to t\}] \\ &= \mathbb{M}[\{\lambda \subset \mathbb{L}\}\,|\,\{0 \to t\}] \langle \sigma(0)\sigma(t) \rangle\,. \end{aligned} \tag{6.54}$$

□

The next proposition is the equivalent of Proposition 6.1 for the lattice $\mathbb{L}$. In its proof we use a result of subsection 7.2.

Proposition 6.2 *Let $\beta < \beta_c$, $h \geq 0$ and $J(e)$ be defined by (4.5). Let $t \in \Sigma_0$ with $0 < t(1)$, and $a \in \mathbb{N}$ with $2a < t(1)$. Let B_t be the square box*

$$B_t := \{s \in \mathbb{L} : 0 \leq s(1) \leq t(1)\,;\, 0 \leq s(2) \leq t(1)\}\,. \tag{6.55}$$

Then

$$\sum_{\substack{\lambda:\delta\lambda=\{0,t\} \\ \lambda \text{ inside } B_t}} q_{\mathbb{L}}(\lambda) \geq \langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}\left[1 - O(|t(1)|^{5/2}\exp\{-2a\tau_{\mathrm{bd}}((1,0))\})\right]\exp\{-O(a)\}\,. \tag{6.56}$$

λ inside B_t *means that $\lambda \subset B_t$ and no edge of λ, except the first and the last one, is adjacent to a site $t' \in B_t$ with*

$$t'(1) = 0 \text{ or } t'(1) = t(1) \text{ or } t'(2) = t(1)\,. \tag{6.57}$$

Proof. We define $u_a := (a, 0)$ and $v_a := (t(1) - a, 0)$. The first part of the proof is identical with the one of Proposition 6.1. Thus we have

$$\sum_{\substack{\lambda:\delta\lambda=\{0,t\} \\ \lambda \text{ inside } B_t}} q_{\mathbb{L}}(\lambda) \geq \exp\{-O(a)\} \sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\} \\ \lambda \text{ inside } B_t}} q_{\mathbb{L}}(\lambda)\,. \tag{6.58}$$

We adapt the proof of Lemma 6.3. Consider λ such that $\delta\lambda = \{u_a, v_a\}$ with initial point u_a. Assume that λ touches the boundary of the box B_t at t_*. Let $\lambda(s_*) := t_*$. There are two cases.

1. $t_*(2) = t(1)$. Then there is a last time s_1 such that $s_1 < s_*$ with $\lambda(s_1) \in \Sigma_0$ and a first time $s_2 > s_*$ such that $\lambda(s_2) \in \Sigma_0$. Lemma 5.1 implies that

$$\sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\} \\ t_* \in \lambda}} q_{\mathbb{L}}(\lambda) \leq O(\exp\{-2\tau(t)\})\,. \tag{6.59}$$

We write the right–hand side of (6.59) as

$$O(\exp\{-2\tau(t)\}) = \frac{O(\exp\{-2\tau(t)\})}{\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}}\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}\,. \tag{6.60}$$

The lower bound on the boundary two–point function of section 7.2 and Proposition 4.2 imply that

$$\frac{O(\exp\{-2\tau(t)\})}{\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}} \leq O\Big(|t(1)|^{3/2}\exp\{-\tau(t)\}\Big)\,. \tag{6.61}$$

2. $t_*(1) = 0$ or $t_*(1) = t(1)$. Then from Lemma 5.4 we have

$$\begin{aligned}
\sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\} \\ t_* \in \lambda}} q_{\mathbb{L}}(\lambda) &\leq \langle\sigma(u_a)\sigma(t_*)\rangle_{\mathbb{L}}\langle\sigma(v_a)\sigma(t_*)\rangle_{\mathbb{L}} && (6.62)\\
&\leq \langle\sigma(-u_a)\sigma(t_*)\rangle_{\mathbb{L}}\langle\sigma(v_a)\sigma(t_*)\rangle_{\mathbb{L}} \\
&\leq \langle\sigma(-u_a)\sigma(v_a)\rangle_{\mathbb{L}} \\
&\leq \langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}} O\Big(|t(1)|^{3/2}\exp\{-2\tau_{\mathrm{bd}}(u_a)\}\Big)\,. && (6.63)
\end{aligned}$$

□

7 Correction to the exponential decay

7.1 Lower bound for the two–point function

There are two cases for which it is easy to derive a lower bound for the two–point function $\langle\sigma(0)\sigma(t)\rangle$, namely when $t(2)=0$ or $t(1)=t(2)$ and $t_2(1)=t_2(2)$.

Proposition 7.1 *Let $J(e)=\beta$ for all edges e of $\mathbb{Z}^2$.*

1. *Let $t=(t(1),0)$. Then*

$$\langle\sigma(0)\sigma(t)\rangle \geq \frac{1}{4(2|t(1)|+1)}\exp\{-\tau(t)\}\,. \tag{7.1}$$

2. *Let $t=(s,s)$. Then*

$$\langle\sigma(0)\sigma(t)\rangle \geq \frac{1}{4(2|s|+1)}\exp\{-\tau(t)\}\,. \tag{7.2}$$

Proof. We prove 1. Consider the set B

$$B:=\{t'\in\mathbb{Z}^2:\,\max\{|t'(1)|,|t'(2)|\}=|t(1)|\}\,, \tag{7.3}$$

which separates 0 and $t=(nt(1),0)$ for any positive $n\in\mathbb{N}$. By Proposition 2.2 point 1. we have

$$\langle\sigma(0)\sigma(t)\rangle \leq \sum_{t'\in B}\langle\sigma(0)\sigma(t')\rangle\langle\sigma(t')\sigma(t)\rangle\,. \tag{7.4}$$

Using the symmetry properties of the two–point function and Proposition 2.2 point 2. we have for any $t'\in B$

$$\langle\sigma(0)\sigma(t')\rangle \leq \langle\sigma(0)\sigma(t'')\rangle \quad\text{and}\quad \langle\sigma(t')\sigma(t)\rangle \leq \langle\sigma(t'')\sigma(t)\rangle\,, \tag{7.5}$$

where $t''=(t(1),0)$. Therefore

$$\langle\sigma(0)\sigma(t)\rangle \leq 4(2|t(1)|+1)\langle\sigma(0)\sigma(t'')\rangle\langle\sigma(t'')\sigma(t)\rangle\,. \tag{7.6}$$

By iteration we get

$$\langle\sigma(0)\sigma(t)\rangle \leq [4(2|t(1)|+1)]^n[\langle\sigma(0)\sigma(t'')\rangle]^n\,. \tag{7.7}$$

Taking the logarithm, dividing by n and taking the limit $n\to\infty$ we prove the statement. □

Remark: The asymptotic behaviour of the two–point function is known explicitly [MW] chapters XI and XII. We do not need the exact expression. For our purpose it is sufficient to know that there exists a constant C such that

$$\langle\sigma(0)\sigma(t)\rangle \geq \frac{C}{\sqrt{|t|}}\exp\{-\tau(t)\}\,. \tag{7.8}$$

The prefactor of the exponential can be understood heuristically for $\langle\sigma(0)\sigma(t)\rangle$ with $t=(t(1),0)$. Indeed, by Lemma 6.1

$$\begin{aligned}\langle\sigma(0)\sigma(t)\rangle &= \chi(t(1),0)\cdot \mathbb{M}[\,\{t'(2)=0\}\mid \cup_{t':t'(1)=t(1)}\{0\to t'\}] \\ &\geq \exp\{-\tau(t)\}\cdot \mathbb{M}[\,\{t'(2)=0\}\mid \cup_{t':t'(1)=t(1)}\{0\to t'\}]\,.\end{aligned} \tag{7.9}$$

The second factor, which can be interpreted as a conditional probability, is of the order $O(1/\sqrt{|t(1)|})$ (Central Limit Theorem). It can be analyzed by perturbative methods [G], [BLP2], [BF]. The same is true for general t [DKS].

7.2 Lower bound for the boundary two–point function

We study the boundary two–point function on $\mathbb{L}$; the coupling constants are given by (4.5).

Proposition 7.2 *Let $J(e)$ be given by (4.5), $\beta<\beta_c$ and $t\in\Sigma_0$.*

1. *For all $h\geq 0$*
$$\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,h}\geq(\tanh J)^2\cdot\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,1}\,. \tag{7.10}$$

2. *Let $h>h_c(\beta)$ (see Proposition 4.2). Then there exists a constant $C=C(h,\beta)$ such that*
$$\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,h}\geq C\exp\{-\tau_{\rm bd}(t;\beta,h)\}\,. \tag{7.11}$$

Remark: The asymptotic behaviour of the two–point function is known explicitly [MW] chapter VII and [P]. We do not need the exact expression. For our purpose it is sufficient to know that for the case $h=1$ there exists a constant C such that for $t\in\Sigma_0$,

$$\langle\sigma(0)\sigma(t)\rangle\geq\frac{C}{|t|^{3/2}}\exp\{-\tau_{\rm bd}(t)\}\,. \tag{7.12}$$

The prefactor of the exponential can be understood heuristically. Indeed, by Lemma 6.4

$$\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}\geq\mathbb{M}[\,\{\lambda\subset\mathbb{L}\}\,|\,\{0\to t\}\,]\langle\sigma(0)\sigma(t)\rangle\,. \tag{7.13}$$

The term $\mathbb{M}[\,\{\lambda\subset\mathbb{L}\}\,|\,\{0\to t\}\,]$, which can be interpreted as a conditional probability, is $O(1/|t|)$; (use the reflection principle of random walks). Together with the factor $O(1/\sqrt{|t|})$ of $\langle\sigma(0)\sigma(t)\rangle$ we get the $O(1/|t|^{3/2})$ factor.

Proof. By GKS inequalities the boundary two–point function decreases if we set $h=0$ and $J(e)=0$ for all edges e adjacent to a site $t'\in\Sigma_0$, except the vertical edges adjacent to $t'=0$ and $t'=t$. It is now immediate to sum over the variables $\sigma(0)$ and $\sigma(t)$ explicitly.

We prove 2. assuming Lemma 7.1 proven below. Let $t=(t(1),0)$ with $t(1)>0$. We set $t_k:=kt$ for $k=1,2,\ldots,n$. Let $a\in\mathbb{N}$, $t(1)>a$, and I be the interval

$$I:=\{t\in\mathbb{L}:\,t_1(1)-a\leq t(1)\leq t_1(1)\,,\,t(2)=0\}\,. \tag{7.14}$$

We have

$$\begin{aligned}\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} &= \mathbb{M}_{\mathbb{L}}[\{0\to t_n\}] \\ &= \mathbb{M}_{\mathbb{L}}[E_I^c\,|\,\{0\to t_n\}]\mathbb{M}_{\mathbb{L}}[\{0\to t_n\}]+\mathbb{M}_{\mathbb{L}}[E_I\cap\{0\to t_n\}]\,,\end{aligned}\tag{7.15}$$

where E_I is the event $\{\lambda\cap I\neq\emptyset\}$ and E_I^c the complementary event. We choose a so that

$$\mathbb{M}[E_I^c\,|\,\{0\to t_n\}]\leq 1/2\,,\tag{7.16}$$

which is possible according to Lemma 7.1 if $t(1)$ is large enough. Thus we have

$$\begin{aligned}\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} &\leq 2\mathbb{M}_{\mathbb{L}}[E_I\cap\{0\to t_n\}] \\ &\leq 2\sum_{u\in I}\sum_{\substack{\lambda:\delta\lambda=\{0,t_n\}\\ u\in\lambda}} q_{\mathbb{L}}(\lambda) \\ &\leq 2a\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}\langle\sigma(t_1)\sigma(t_n)\rangle_{\mathbb{L}}\,.\end{aligned}\tag{7.17}$$

We have used Lemma 5.1 and the monotonicity property of the boundary two-point function, which is proven in the same way as the corresponding property for the two–point function on $\mathbb{Z}^2$. By GKS inequalities and translation–invariance

$$\begin{aligned}\frac{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}}{\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}} &\leq \frac{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}}{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}\langle\sigma(t_1-(a,0))\sigma(t_1)\rangle_{\mathbb{L}}} \\ &= \frac{1}{\langle\sigma(0)\sigma((a,0))\rangle_{\mathbb{L}}}\,.\end{aligned}\tag{7.18}$$

If we set

$$C_*:=\frac{\langle\sigma(0)\sigma((a,0))\rangle_{\mathbb{L}}}{2a}\,,\tag{7.19}$$

then

$$\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}}\leq C_*^{-1}\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}\langle\sigma(t_1)\sigma(t_n)\rangle_{\mathbb{L}}\,.\tag{7.20}$$

We can iterate this result,

$$\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}}\leq C_*^{-n}\Big(\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}\Big)^n\,.\tag{7.21}$$

Therefore, if $t(1)$ is large enough, then

$$-\tau_{\mathrm{bd}}(t;\beta,h)=\lim_{n\to\infty}\frac{1}{n}\ln\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}}\leq-\ln C_*+\ln\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}\,.\tag{7.22}$$

□

Lemma 7.1 *Let $\beta<\beta_c$ and $h>h_c(\beta)$. Let $I=[x_1,x_2]$ be a finite non-empty interval in Σ_0. Let $0<x_1$ and $x_2<t\in\Sigma_0$. Then there exist ε positive, n_ε and C_1 such that for all x_1,x_2 with $|x_2-x_1|\geq n_\varepsilon$*

$$\mathbb{M}_{\mathbb{L}}[\{\lambda\cap I=\emptyset\}\,|\,\{0\to t\}]\leq C_1\exp\{-\varepsilon|x_2-x_1|\}\,.\tag{7.23}$$

Proof. We have

$$\mathbb{M}_{\mathbb{L}}[\,\{0 \to t\}\,] = \langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}\,. \tag{7.24}$$

Let λ be a random line such that $\delta\lambda = \{0,t\}$ and $\lambda \cap I = \emptyset$. Let s_1 be the last time when λ touches Σ_0 at the left hand side of I, and let s_2 be the first time that λ touches Σ_0 at the right hand side of I. We set $u := \lambda(s_1)$ and $v := \lambda(s_2)$. We necessarily have $u < x_1 < x_2 < v$. From Lemmas 5.4 and 5.1 we get

$$\mathbb{M}_{\mathbb{L}}[\,\{\lambda \cap I = \emptyset\} \cap \{0 \to t\}\,] \le \sum_{u,v} \exp\{-\tau(v-u)\}\langle \sigma(0)\sigma(u)\rangle_{\mathbb{L}}\langle \sigma(v)\sigma(t)\rangle_{\mathbb{L}}\,. \tag{7.25}$$

By GKS inequalities

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}} \ge \langle \sigma(0)\sigma(u)\rangle_{\mathbb{L}}\langle \sigma(u)\sigma(v)\rangle_{\mathbb{L}}\langle \sigma(v)\sigma(t)\rangle_{\mathbb{L}}\,, \tag{7.26}$$

so that

$$\mathbb{M}_{\mathbb{L}}[\,\{\lambda \cap I = \emptyset\}\,|\,\{0 \to t\}\,] \le \sum_{u,v} \frac{\exp\{-\tau(v-u)\}}{\langle \sigma(u)\sigma(v)\rangle_{\mathbb{L}}}\,. \tag{7.27}$$

We know that

$$\lim_{n\to\infty} -\frac{1}{n}\ln\langle \sigma(0)\sigma(nt_1)\rangle_{\mathbb{L}} = \tau^*_{\text{bd}}\,, \tag{7.28}$$

where

$$\tau^*_{\text{bd}} = \tau_{\text{bd}}(t_1) \quad , \quad t_1 = (1,0)\,. \tag{7.29}$$

Let $0 < 2\varepsilon < \tau^* - \tau^*_{\text{bd}}$; $\tau^* = \tau(t_1)$. We can find n_ε so that for all $n \ge n_\varepsilon$,

$$\langle \sigma(0)\sigma(nt_1)\rangle_{\mathbb{L}} \ge \exp\{-n(\tau^*_{\text{bd}} + \varepsilon)\}\,. \tag{7.30}$$

From this inequality and $\tau(u-v) = |u-v|\cdot\tau^*$ the lemma follows. □

Remark: Using Proposition 7.2 we can improve Lemma 7.1. There exists a constant C such that for any interval $I = [x_1, x_2]$ we have

$$\mathbb{M}_{\mathbb{L}}[\,\{\lambda \cap I = \emptyset\}\,|\,\{0 \to t\}\,] \le C\exp\{-(\tau^* - \tau^*_{\text{bd}})\cdot|x_2 - x_1|\}\,. \tag{7.31}$$

Part II

Ising model at low temperature

We study the large deviations of the magnetization of the Ising model for $\beta > \beta_c$ (i.e. below the critical temperatures). We analyze in particular boundary effects. Some estimates of part I are essential. The results are valid only for the two–dimensional case.

8 Low–temperature representation

There is a representation of the Gibbs measure in Λ with $+$ boundary condition in terms of contours, which is similar to the one of (2.4). To each configuration ω, which satisfies the Λ^+–boundary condition, we associate a family $\underline{\gamma} = \underline{\gamma}(\omega)$ of compatible contours on the dual lattice $(\mathbb{Z}^2)^*$: let $\mathcal{E}^*(\omega)$ be the set of edges of $(\mathbb{Z}^2)^*$,

$$\mathcal{E}^*(\omega) := \{e^* : [\sigma(t)(\omega)\sigma(t')(\omega) - 1] = -2,\ \langle t, t' \rangle = (e^*)^*\}\,; \tag{8.1}$$

then decompose the set $\mathcal{E}^*(\omega)$ into compatible contours $\underline{\gamma}$. In (8.1) $(e^*)^*$ is the dual edge of e^*. Two important remarks: All contours γ of $\underline{\gamma}(\omega)$ are closed, i.e. $\delta\gamma = \emptyset$; *not* all families of compatible contours are allowed as it is the case for the high–temperature representation. This motivates the

Definition 8.1 *A family of compatible contours $\underline{\gamma}$ in Λ^* is* Λ^+–**compatible** *if and only if there exists a configuration ω satisfying the Λ^+–boundary condition, such that $\underline{\gamma}$ is the family of contours of ω.*

Let $\Lambda \subset \mathbb{Z}^2$. The set $\Lambda^* \subset (\mathbb{Z}^2)^*$ is by definition the set

$$\Lambda^* := \{t^* \in (\mathbb{Z}^2)^* : t^* \text{ is a corner of a plaquette } p(t),\, t \in \Lambda\}\,. \tag{8.2}$$

Any family of contours of a configuration ω satisfying the Λ^+–boundary condition is in Λ^*. Given a closed contour γ on the dual lattice $(\mathbb{Z}^2)^*$ there exists a unique configuration ω_γ having γ as single contour and such that $\omega(t) = 1$ for all t, except for a finite number. The **interior** of γ, $\mathrm{int}\gamma$ is

$$\mathrm{int}\gamma := \{t \in \mathbb{Z}^2 : \omega_\gamma(t) = -1\}\,. \tag{8.3}$$

The **exterior** of γ is $\mathrm{ext}\gamma := \mathbb{Z}^2 \backslash \mathrm{int}\gamma$. The **volume** of γ is

$$\mathrm{vol}\gamma := |\mathrm{int}\gamma|\,. \tag{8.4}$$

A contour γ of a configuration ω is **external** if there is no other contour γ' of the configuration such that $\mathrm{int}\gamma \subset \mathrm{int}\gamma'$. Let γ be a contour of a configuration ω satisfying the Λ^+–boundary condition. The **closure of the interior of γ in Λ**, $\overline{\mathrm{int}}\gamma$, is the union of $\mathrm{int}\gamma$ and the set of all $t \in \Lambda \backslash \mathrm{int}\gamma$, such that $\omega'(t) = 1$ for *any* configuration ω' with the properties: 1) ω' satisfies the Λ^+–boundary condition; 2) γ is an external contour in ω'. If Λ is large enough $\overline{\mathrm{int}}\gamma$ is independent of Λ. The **closure of the exterior of γ in Λ**, $\overline{\mathrm{ext}}\gamma$, is the union of ext$\gamma$ and the set of all $t \in \Lambda\backslash\mathrm{ext}\gamma$, such that $\omega'(t) = -1$ for *any* configuration ω' with the properties: 1) ω' satisfies the Λ^+–boundary condition; 2) γ is an external contour in ω'.

Let $J^*(e^*)$ be the dual coupling to $J(e)$. The $*$–**weight** of a contour is

$$w^*(\gamma) := \prod_{e^*\in\gamma} \tanh J^*(e^*) \equiv \prod_{e^*\in\gamma} \exp\{-2J(e)\}\,. \tag{8.5}$$

The normalization constant $\Xi(\Lambda)^+$ appearing in the definition of μ_Λ^+ can be written as

$$\begin{aligned}\Xi(\Lambda)^+ &= \sum_{\omega} \prod_{\substack{e=\langle t,t'\rangle:\\ e\cap\Lambda\neq\emptyset}} \exp\{J(e)[\sigma(t)(\omega)\sigma(t')(\omega)-1]\} \\ &= \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset\\ \Lambda^+-\mathrm{comp.}}} \prod_{\gamma\in\underline{\gamma}} \prod_{e^*\in\gamma} \exp\{-2J(e)\} \\ &= \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset\\ \Lambda^+-\mathrm{comp.}}} \prod_{\gamma\in\underline{\gamma}} w^*(\gamma)\,.\end{aligned} \tag{8.6}$$

Let $\Lambda \subset \mathbb{Z}^2$ and $\underline{\gamma}'$ be a family of Λ^*–compatible closed contours. We set

$$Z^+(\Lambda|\underline{\gamma}';J) := \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset\\ \underline{\gamma}\cup\underline{\gamma}'\ \Lambda^+-\mathrm{comp.}}} w^*(\underline{\gamma})\,. \tag{8.7}$$

If $\underline{\gamma}' = \emptyset$, then

$$Z^+(\Lambda|\emptyset;J) = Z^+(\Lambda;J)\,. \tag{8.8}$$

Lemma 8.1 *Let Λ be a finite subset of $\mathbb{Z}^2$; $J(e)$ be non–negative coupling constants. Then*

$$\Xi(\Lambda;J)^+ = Z^+(\Lambda;J)\,. \tag{8.9}$$

Let $\underline{\gamma}'$ be a family of Λ^*–compatible closed contours. Then the probability $P_\Lambda^+[\underline{\gamma}']$, computed with respect to the measure μ_Λ^+, is given by

$$P_\Lambda^+[\underline{\gamma}'] = w^*(\underline{\gamma})\frac{Z^+(\Lambda|\underline{\gamma}';J)}{Z^+(\Lambda;J)}\,. \tag{8.10}$$

If Λ is simply connected, then any family of Λ^*–compatible contours is Λ^+–compatible; furthermore

$$Z^+(\Lambda;J) = Z(\Lambda^*;J^*) \tag{8.11}$$

and

$$P_\Lambda^+[\underline{\gamma}'] = w^*(\underline{\gamma})\frac{Z(\Lambda^*|\underline{\gamma}';J^*)}{Z(\Lambda^*;J^*)}\,. \tag{8.12}$$

Proof. The first part of the lemma follows from the comparison of the high–temperature and low–temperature representations. If Λ is simply connected then we construct explicitely the configuration ω starting from the boundary for any family of Λ^+–compatible contours. $\square$

9 Phase of small contours

A basic idea in [DKS] is the introduction of an intermediate length–scale in the analysis of the large deviations of the magnetization. One distinguishes between small and large contours. We study here the large deviations for the magnetization under the condition that all contours of a configuration ω are small. Our main result is Proposition 9.1. It is inspired by the appendix of Schonmann and Shlosman in [SS]; see also Ioffe [I2] and Pisztora [Pi] for related and former results of this kind.

9.1 Definition of the phase of small contours

Let l be some positive integer; we set

$$B(0;l) := \{t = (t(1), t(2)) \in (\mathbb{Z}^2)^* : \, -l \leq t(i) < l \, , \, i = 1,2\} \, . \tag{9.1}$$

Definition 9.1 *Let $s \in \mathbb{N}$. A contour γ is s–***small***, or* **small***, if there is a translate of $B(0;s)$ which contains γ.*

Let Λ be a finite subset of $\mathbb{Z}^2$. The phase of small contours is described by the conditionned measure

$$P_\Lambda^{+,s}[\,\cdot\,] := P_\Lambda^+[\,\cdot\,|\{\text{all contours } s\text{–small}\}] \, . \tag{9.2}$$

The expectation value is denoted by $\langle\,\cdot\,\rangle_\Lambda^{+,s}$, or $P_\Lambda^{+,s}[\,\cdot\,]$. It is convenient to use

$$I_\Lambda^s(\omega) := \begin{cases} 1 & \text{if } w \text{ satisfies the } \Lambda^+\text{–boundary condition} \\ & \text{and each contour of } \omega \text{ is } s\text{–small,} \\ 0 & \text{otherwise.} \end{cases} \tag{9.3}$$

The function $\omega \mapsto I_\Lambda^s(\omega)$ is increasing. Furthermore, if Λ is the union of two disjoint connected components Λ_1 and Λ_2, then

$$I_\Lambda^s(\omega) = I_{\Lambda_1}^s(\omega) \cdot I_{\Lambda_2}^s(\omega) \, . \tag{9.4}$$

Lemma 9.1 *Let $J(e) \geq 0$ for all edges e. Let $l, s \in \mathbb{N}$,*

$$\Lambda_1 : \; = \; B(0;l) \, , \tag{9.5}$$

$$\Lambda_2 : \; = \; B(0;l+s+1) \, . \tag{9.6}$$

Suppose that $\Lambda \supset \Lambda_2$.

1. *Let f be Λ_1–local and g is $\Lambda \backslash \Lambda_2$–local. Then*

$$|\, \langle f\, g\rangle_\Lambda^{+,s} - \langle f\rangle_\Lambda^{+,s} \cdot \langle g\rangle_\Lambda^{+,s} \,| \leq \max_{\Lambda_1 \subset \Lambda' \subset \Lambda_2} |\, \langle f\rangle_\Lambda^{+,s} - \langle f\rangle_{\Lambda'}^{+,s} \,| \cdot \langle g\rangle_\Lambda^{+,s} \, . \tag{9.7}$$

2. If furthermore f is increasing, then

$$\langle f\, g\rangle_\Lambda^{+,s} \le \frac{1}{\langle I^s_{\Lambda_2}\rangle^+_{\Lambda_2}}\, \langle f\rangle^{+,s}_{\Lambda_1}\cdot \langle g\rangle^{+,s}_\Lambda\,, \tag{9.8}$$

$$\langle f\, g\rangle_\Lambda^{+,s} \ge \langle I^s_{\Lambda_2}\rangle^+_{\Lambda_2}\, \langle f\rangle^{+,s}_{\Lambda_2}\cdot \langle g\rangle^{+,s}_\Lambda\,, \tag{9.9}$$

and

$$\langle f\, g\rangle_\Lambda^{+,s} \ge \langle f\rangle^{+}_{\Lambda_2}\cdot \langle g\rangle^{+,s}_\Lambda\,. \tag{9.10}$$

Proof. By definition

$$\langle f\, g\rangle_\Lambda^{+,s} = \frac{\langle f\, g\, I^s_\Lambda\rangle^+_\Lambda}{\langle I^s_\Lambda\rangle^+_\Lambda}\,. \tag{9.11}$$

Suppose that γ is an external contour in a configuration ω and that $I^s_\Lambda(\omega) = 1$. Then

$$\Lambda_2 \not\subset \mathrm{int}\gamma\,. \tag{9.12}$$

Moreover, if

$$\mathrm{int}\gamma \cap (\Lambda\backslash\Lambda_2) \neq \emptyset\,; \tag{9.13}$$

then

$$\overline{\mathrm{int}}\gamma \cap \Lambda_1 = \emptyset\,. \tag{9.14}$$

Let $\gamma_1(\omega),\ldots,\gamma_n(\omega)$ be all external contours of ω such that

$$\mathrm{int}\gamma_i(\omega) \cap (\Lambda\backslash\Lambda_2) \neq \emptyset\,,\ i=1,\ldots,n\,; \tag{9.15}$$

we define the random set

$$\Lambda(\omega) := (\Lambda\backslash\Lambda_2) \bigcup_{i=1,\ldots,n} \overline{\mathrm{int}}\gamma_i(\omega)\,. \tag{9.16}$$

We have

$$\langle f\, g\rangle_\Lambda^{+,s} = \sum_{\Lambda''\subset\Lambda} \langle f\, g\,|\,\{\Lambda(\,\cdot\,)=\Lambda''\}\rangle^{+,s}_\Lambda \cdot P^{+,s}_\Lambda[\{\Lambda(\,\cdot\,)=\Lambda''\}]\,. \tag{9.17}$$

The Markov property and (9.4) imply

$$\langle f\, g\,|\,\{\Lambda(\,\cdot\,)=\Lambda''\}\rangle^{+,s}_\Lambda = \langle f\rangle^{+,s}_{\Lambda\backslash\Lambda''}\cdot \langle g\,|\,\{\Lambda(\,\cdot\,)=\Lambda''\}\rangle^{+,s}_\Lambda\,. \tag{9.18}$$

If $P^{+,s}_\Lambda[\{\Lambda(\,\cdot\,)=\Lambda''\}]\neq 0$, then $\Lambda_1\subset\Lambda\backslash\Lambda''\subset\Lambda_2$. This gives (9.7). If f is increasing, then FKG–inequalities and $\Lambda_1\subset\Lambda\backslash\Lambda''\subset\Lambda_2$ imply that

$$\langle f\, I^s_{\Lambda_2}\rangle^+_{\Lambda_2} \le \langle f\, I^s_{\Lambda\backslash\Lambda''}\rangle^+_{\Lambda\backslash\Lambda''} \le \langle f\, I^s_{\Lambda_1}\rangle^+_{\Lambda_1} \tag{9.19}$$

and

$$\langle f\, I^s_{\Lambda\backslash\Lambda''}\rangle^+_{\Lambda\backslash\Lambda''} \ge \langle f\rangle^+_{\Lambda\backslash\Lambda''}\cdot \langle I^s_{\Lambda\backslash\Lambda''}\rangle^+_{\Lambda\backslash\Lambda''} \ge \langle f\rangle^+_{\Lambda_2}\cdot \langle I^s_{\Lambda\backslash\Lambda''}\rangle^+_{\Lambda\backslash\Lambda''}\,. \tag{9.20}$$

From (9.18) and (9.20) follows (9.10). Since

$$\frac{\langle I^s_{\Lambda_1}\rangle^+_{\Lambda_1}}{\langle I^s_{\Lambda\backslash\Lambda''}\rangle^+_{\Lambda\backslash\Lambda''}} \le \frac{1}{\langle I^s_{\Lambda\backslash\Lambda''}\rangle^+_{\Lambda\backslash\Lambda''}} \le \frac{1}{\langle I^s_{\Lambda_2}\rangle^+_{\Lambda_2}} \ \text{ and } \ \frac{\langle I^s_{\Lambda_2}\rangle^+_{\Lambda_2}}{\langle I^s_{\Lambda\backslash\Lambda''}\rangle^+_{\Lambda\backslash\Lambda''}} \ge \langle I^s_{\Lambda_2}\rangle^+_{\Lambda_2}\,, \tag{9.21}$$

(9.18) and inequalities (9.19), (9.21) give (9.8) and (9.9). $\square$

Remark: There is a version of (9.10) for f decreasing,

$$\langle f\, g\rangle_\Lambda^{+,s} \leq \langle f\rangle_{\Lambda_2}^{+} \cdot \langle g\rangle_\Lambda^{+,s} \,. \tag{9.22}$$

In the proof of Lemma 9.1 we only use (9.4), the Markov property and FKG–inequalities. Lemma 9.1 is therefore also true if we replace the measure μ_Λ^+ by another measure, which has the Markov property, as long as FKG–inequalities remain valid; for example we may consider the Ising model with arbitrary external field.

We derive some consequences of Lemma 9.1 for a model with the coupling constants

$$J(e) := \begin{cases} \beta \geq 0 & \text{if } e = \langle t, t'\rangle,\ t(2) \geq 0 \text{ and } t'(2) \geq 0, \\ h\beta \geq 0 & \text{if } e = \langle t, t'\rangle,\ t(2) \leq -1 \text{ or } t'(2) \leq -1. \end{cases} \tag{9.23}$$

We recall a result of [BLP2].

Lemma 9.2 *For any $\beta > \beta_c$ there exists $\overline{a}(\beta) > 0$ and K such that*

$$|\,\langle \sigma_A\rangle_{V_1}^{+} - \langle \sigma_A\rangle_{V_2}^{+}\,| \leq K \sum_{t\in A}\ \sum_{t'\in V_1\Delta V_2} \exp\{-\overline{a}(\beta)|t-t'|\}\,, \tag{9.24}$$

where $A \subset V_1 \cap V_2$ and $V_1 \Delta V_2 = (V_1\backslash V_2) \cup (V_2\backslash V_1)$.

Lemma 9.3 *Let $J(e)$ be the coupling constants given by (9.23) with $\beta > \beta_c$. Let $s \in \mathbb{N}$ and $t \in \mathbb{L}$ with $t(2) > 2s+1$. Let $\Lambda \subset \mathbb{L}$ such that*

$$\min\{|t(i) - u(i)| :\ u \notin \Lambda,\ i = 1,2\} > 2s+1\,. \tag{9.25}$$

Then there exists a positive constant $\kappa = \kappa(\beta)$ (see (9.34)) such that

$$\langle \sigma(t)\rangle^{+,\beta} \leq \langle \sigma(t)\rangle_\Lambda^{+,s,J} \leq \langle \sigma(t)\rangle^{+,\beta} + O(s^4)\exp\{-\kappa\cdot s\})\,. \tag{9.26}$$

Suppose furthermore that $t' \in \Lambda$ and

$$\min\{|t'(i) - t(i)| :\ i = 1,2\} > 2s+1\,. \tag{9.27}$$

Then

$$|\,\langle \sigma(t)\,\sigma(t')\rangle_\Lambda^{+,s} - \langle \sigma(t)\rangle_\Lambda^{+,s}\cdot\langle \sigma(t')\rangle_\Lambda^{+,s}\,| \leq O(s^4)\exp\{-\kappa\cdot s\}\cdot\langle \sigma(t')\rangle_\Lambda^{+,s}\,. \tag{9.28}$$

Proof. Let Λ_1 be a translate of the box $B(0;s)$ with t in its "center". Let Λ_2 be the translate (same translation) of the box Λ_2 of Lemma 9.1 with $l = s$. The first inequality follows from (9.10), with $g \equiv 1$, and from FKG–inequalities,

$$\langle \sigma(t)\rangle^{+,\beta} \leq \langle \sigma(t)\rangle_{\Lambda_2}^{+,J} \leq \langle \sigma(t)\rangle_\Lambda^{+,s,J}\,. \tag{9.29}$$

Lemma 9.1 implies

$$\langle \sigma(t)\rangle_\Lambda^{+,s,J} \leq \frac{1}{\langle I_{\Lambda_2}^s\rangle_{\Lambda_2}^{+}}\cdot\langle \sigma(t)\rangle_{\Lambda_1}^{+,s,J}\,. \tag{9.30}$$

In Λ_1 all contours are s–small, so that we have

$$\langle \sigma(t) \rangle_{\Lambda}^{+,s,J} \leq \frac{1}{\langle I^s_{\Lambda_2} \rangle^+_{\Lambda_2}} \cdot \langle \sigma(t) \rangle_{\Lambda_1}^{+,J} . \tag{9.31}$$

If the diameter $d(\gamma)$ of γ is smaller than s, then γ is s–small. Lemma 5.6 and Lemma 8.1 give

$$\frac{1}{\langle I^s_{\Lambda_2} \rangle^+_{\Lambda_2}} \leq 1 + O(s^4) \exp\{-s\alpha(\beta^*)\} , \tag{9.32}$$

with $\alpha(\beta^*)$ of Lemma 5.6. Lemma 9.2 gives

$$| \, \langle \sigma(t) \rangle_{\Lambda_1}^{+,J} - \langle \sigma(t) \rangle^{+,\beta} \, | \leq O(s) \exp\{-s\overline{a}(\beta)\} . \tag{9.33}$$

Define $\kappa(\beta)$ so that

$$\max \left[\exp\{-s\alpha(\beta^*)\} , \exp\{-s\overline{a}(\beta)\} \right] \leq \exp\{-\kappa(\beta) \cdot s\} . \tag{9.34}$$

The second affirmation is a consequence of (9.7) and (9.26). $\square$

9.2 Large deviations in the phase of small contours

We start with the elementary

Lemma 9.4 *Let $Y_1, \ldots, Y_n$ be independent random variables such that*

$$\mathbb{E} \left[\exp(aY_i) \right] \tag{9.35}$$

exists for all $a \in \mathbb{R}$ and all $i = 1, \ldots, n$. Let

$$m_n := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[Y_i \right] , \tag{9.36}$$

$$f_n(a) := \frac{1}{n} \sum_{i=1}^{n} \ln \mathbb{E} \left[\exp(aY_i) \right] , \tag{9.37}$$

and

$$\mathrm{var}_n^{\#} := \sup_{a \in \mathbb{R}} \frac{d^2}{da^2} f_n(a) . \tag{9.38}$$

Then, for any $x > 0$,

$$\mathrm{Prob}\left[\, \Big| \sum_{i=1}^{n} (Y_i - \mathbb{E} \left[Y_i \right]) \Big| \geq nx \, \right] \leq \exp\Big(-n \frac{x^2}{2\mathrm{var}_n^{\#}}\Big) . \tag{9.39}$$

Proof. Suppose that

$$\sum_{i=1}^{n} (Y_i - \mathbb{E} \left[Y_i \right]) \geq nx . \tag{9.40}$$

Then for any $a \geq 0$, we have

$$\begin{aligned}
\mathrm{Prob}[\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y_i]) \geq nx] &\leq \exp(-nxa - nm_n a + nf_n(a)) \qquad (9.41)\\
&\leq \exp(-nxa + n\frac{1}{2}a^2 \mathrm{var}_n^{\#})\\
&\leq \exp\Big(-n\frac{x^2}{2\mathrm{var}_n^{\#}}\Big).
\end{aligned}$$

The proof for the case $\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y_i]) \leq -nx$ is similar. □

Proposition 9.1 *Let $J(e) \geq 0$ for all edges e.*

1. Let

$$\mathrm{var}_\Lambda^+ := \frac{1}{|\Lambda|}\sum_{t,t' \in \Lambda}\Big(\langle\sigma(t')\sigma(t)\rangle_\Lambda^+ - \langle\sigma(t')\rangle_\Lambda^+\,\langle\sigma(t)\rangle_\Lambda^+\Big). \qquad (9.42)$$

For any $x \geq 0$,

$$P_\Lambda^+\Big[\Big\{\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle_\Lambda^+) \geq x|\Lambda|\Big\}\Big] \leq \exp\Big(-|\Lambda|\frac{x^2}{2\mathrm{var}_\Lambda^+}\Big). \qquad (9.43)$$

2. Let $l, s \in \mathbb{N}$,

$$\begin{aligned}
\Lambda_1 : &= B(0; l), \qquad (9.44)\\
\Lambda_2 : &= B(0; l+s+1). \qquad (9.45)
\end{aligned}$$

Suppose that $\Lambda' \subset \mathbb{Z}^2$ is the union of n' disjoint translates $\mathcal{B}_i$ of the box Λ_2 and that Λ'' is the union of n'' disjoint translates $\mathcal{B}_j$ of the box Λ_2 such that $\Lambda' \cap \Lambda'' = \emptyset$. Let $\Lambda := \Lambda' \cup \Lambda''$, $N := n' + n''$ and

$$P_\Lambda^s := P_{\Lambda'}^{+,s} \otimes P_{\Lambda''}^{-,s}. \qquad (9.46)$$

Let

$$\Delta_{\Lambda'}^{+,s} := \max_{\Lambda_1 \subset \Lambda_3 \subset \Lambda_2}\Big|\frac{1}{|\Lambda'|}\sum_{t\in\Lambda'}\langle\sigma(t)\rangle_{\Lambda'}^{+,s} - \frac{1}{|\Lambda_1|}\sum_{t\in\Lambda_1}\langle\sigma(t)\rangle_{\Lambda_3}^{+,s}\Big|, \qquad (9.47)$$

and

$$\Delta_{\Lambda''}^{-,s} := \max_{\Lambda_1 \subset \Lambda_3 \subset \Lambda_2}\Big|\frac{1}{|\Lambda''|}\sum_{t\in\Lambda''}\langle\sigma(t)\rangle_{\Lambda''}^{-,s} - \frac{1}{|\Lambda_1|}\sum_{t\in\Lambda_1}\langle\sigma(t)\rangle_{\Lambda_3}^{-,s}\Big|. \qquad (9.48)$$

If

$$x \equiv y + 2\frac{4(s+1)(2l+s+1)}{|\Lambda_2|} + \frac{n'\Delta_{\Lambda'}^{+,s} + n''\Delta_{\Lambda''}^{-,s}}{N}, \qquad (9.49)$$

and $y \geq 0$, then

$$P_\Lambda^s\Big[\Big\{|\sum_{t\in\Lambda}(\sigma(t) - \langle\sigma(t)\rangle_\Lambda^s)| \geq x|\Lambda|\Big\}\Big] \leq \exp\Big(-N\frac{y^2}{2}\Big). \qquad (9.50)$$

Remarks: 1. A variant of 1. is: for any $x \geq 0$,

$$P_\Lambda^{+,s}\Big[\Big\{\sum_{t\in\Lambda}(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^+)\geq x|\Lambda|\Big\}\Big]\leq \tag{9.51}$$

$$\Big(1-P_\Lambda^+[\{\exists\,\gamma \text{ not small}\}]\Big)^{-1}\exp\Big(-|\Lambda|\frac{x^2}{2\mathrm{var}_\Lambda^+}\Big)\,.$$

2. We have a similar proposition if we consider Λ^-–boundary condition. In particular 1. becomes in this case

For any $x \geq 0$,

$$P_\Lambda^-\Big[\Big\{\sum_{t\in\Lambda}(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^-)\leq -x|\Lambda|\Big\}\Big]\leq\exp\Big(-|\Lambda|\frac{x^2}{2\mathrm{var}_\Lambda^-}\Big)\,. \tag{9.52}$$

By symmetry

$$\mathrm{var}_\Lambda^- = \mathrm{var}_\Lambda^+\,. \tag{9.53}$$

Proof.

The proof of 1. is the same as the proof of Lemma 9.4. Let

$$f_\Lambda(a) := \frac{1}{|\Lambda|}\ln\langle\exp[a\sum_{t\in\Lambda}\sigma(t)]\rangle_\Lambda^+\,. \tag{9.54}$$

It is sufficient to notice that GHS–inequalities give

$$\sup_{a\geq 0}\frac{d^2}{da^2}f_\Lambda(a)\leq\mathrm{var}_\Lambda^+\,. \tag{9.55}$$

We prove 2. The proof is similar to the proof of Lemma 9.1. We define for each box $\mathcal{B}_i$ a random variable Y_i. Each box $\mathcal{B}_i$ is a translate of Λ_2; denote by $\mathcal{B}_i'$ the translate of Λ_1 by the same translation and set

$$Y_i := \frac{1}{|\Lambda_1|}\sum_{t\in\mathcal{B}_i'}\sigma(t)\,. \tag{9.56}$$

Then

$$P_\Lambda^s\Big[\Big\{\,|\sum_{t\in\Lambda}(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^s)\,|\geq x|\Lambda|\Big\}\Big]\leq \tag{9.57}$$

$$P_\Lambda^s\Big[\Big\{\,|\sum_{j=1}^N Y_j-\frac{1}{|\Lambda_2|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle_\Lambda^s\,\Big|\geq N\Big(x-2\frac{4(s+1)(2l+s+1)}{|\Lambda_2|}\Big)\Big\}\Big]\,.$$

We define a random set $\Lambda(\omega)$. Let $\gamma_1(\omega),\ldots,\gamma_n(\omega)$ be all external contours of ω such that $\mathrm{int}\gamma_k$ has a non–empty intersection with at least two different boxes $\mathcal{B}_k$; we set

$$\Lambda(\omega) := \bigcup_{i=1,\ldots,n}\overline{\mathrm{int}}\gamma_i\,. \tag{9.58}$$

By construction,
$$\Lambda(\omega) \cap \mathcal{B}'_k = \emptyset\,, \tag{9.59}$$
for all ω such that all contours are s–small. If $\widehat{x} = x - 8(s+1)(2l+s+1)/|\Lambda_2|$, then
$$P^s_\Lambda\Big[\Big\{\,|\sum_{j=1}^N Y_j - \frac{1}{|\Lambda_2|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle^s_\Lambda)\,| \geq N\widehat{x}\Big\}\Big] = \tag{9.60}$$
$$\sum_{\Lambda'\subset\Lambda} P^s_\Lambda\Big[\Big\{\,|\sum_{j=1}^N Y_j - \frac{1}{|\Lambda_2|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle^s_\Lambda)\,| \geq N\widehat{x}\Big\}\,\Big|\,\{\Lambda(\,\cdot\,)=\Lambda'\}\Big]\cdot P^s_\Lambda[\{\Lambda(\,\cdot\,)=\Lambda'\}]\,.$$
Let Λ' be such that $P^s_\Lambda[\{\Lambda(\,\cdot\,)=\Lambda'\}] \neq 0$, so that the variables Y_i, $i=1,\ldots,N$, are independent with respect to the probability measure $P^s_\Lambda[\,\cdot\,|\,\{\Lambda(\,\cdot\,)=\Lambda'\}]$. Using (9.49) we get
$$P^s_\Lambda\Big[\Big\{\,|\sum_{j=1}^N Y_j - \frac{1}{|\Lambda_2|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle^s_\Lambda)\,| \geq N\widehat{x}\Big\}\,\Big|\,\{\Lambda(\,\cdot\,)=\Lambda'\}\Big] \leq \tag{9.61}$$
$$P^s_\Lambda\Big[\Big\{\,|\sum_{j=1}^N\Big(Y_j - \langle Y_j\,|\,\{\Lambda(\,\cdot\,)=\Lambda'\}\rangle^s_\Lambda\Big)\,| \geq Ny\Big\}\,\Big|\,\{\Lambda(\,\cdot\,)=\Lambda'\}\Big]\,.$$
Since the random variables Y_j are bounded by one, their variances are also bounded by one, so that we conclude by using Lemma 9.4. □

Comments: 1. In the proof of (9.50) we used a trivial bound on the variances of the variables Y_j. Since there is a constraint on the size of the contours we expect that the variance goes to zero when $|\Lambda_L| \to \infty$; this would improve the end–result (9.50).

2. To apply this proposition we need to control the quantities $\langle\sigma(t)\rangle^+_\Lambda$, var^+_Λ, $\langle\sigma(t)\rangle^{+,s}_\Lambda$ and (9.47), (9.48). We make some remarks concerning that point.

2a. If $J(e)=\beta$ for all edges e and $\beta > \beta_c$ then FKG–inequalities give
$$\langle\sigma(t)\rangle^+_\Lambda \geq m^* \quad \text{and} \quad \langle\sigma(t)\rangle^-_\Lambda \leq -m^*\,, \tag{9.62}$$
and we can use Lemma 9.2 to estimate
$$|\langle\sigma(t)\rangle^+_\Lambda - m^*| \quad \text{or} \quad |\langle\sigma(t)\rangle^-_\Lambda + m^*|\,. \tag{9.63}$$
Moreover, GHS–inequalities give
$$\begin{aligned}\mathrm{var}^-_\Lambda = \mathrm{var}^+_\Lambda \;&=\; \frac{1}{|\Lambda|}\sum_{t,t'\in\Lambda}\{\,\langle\sigma(t')\sigma(t)\rangle^+_\Lambda - \langle\sigma(t')\rangle^+_\Lambda\langle\sigma(t)\rangle^+_\Lambda\,\} \\ &\leq\; \frac{1}{|\Lambda|}\sum_{t,t'\in\Lambda}\{\,\langle\sigma(t')\sigma(t)\rangle^+ - \langle\sigma(t')\rangle^+\langle\sigma(t)\rangle^+\,\} \\ &\leq\; \sum_{t\in\mathbb{Z}^2}\{\,\langle\sigma(0)\sigma(t)\rangle^{+,\beta} - m^*(\beta)^2\,\}\,.\end{aligned} \tag{9.64}$$

The quantity

$$\chi := \sum_{t\in\mathbb{Z}^2} \{ \langle\sigma(0)\sigma(t)\rangle^{+,\beta} - m^*(\beta)^2 \} \tag{9.65}$$

is called susceptibility. It coincides with the one defined by (6.2) when $\beta \leq \beta_c$. Indeed, in that latter case $m^* = 0$ and $\langle\sigma(t)\sigma(0)\rangle^{+,\beta} = \langle\sigma(t)\sigma(0)\rangle^{\beta}$. It is finite for $\beta > \beta_c$ in the 2D Ising model.

2b. Let the coupling constants $J(e)$ be given by (9.23) with $\beta > \beta_c$ and $h \geq 1$. Let $\Lambda \subset \mathbb{L}$; GHS–inequalities imply

$$\mathrm{var}_\Lambda^{-,\beta,h} = \mathrm{var}_\Lambda^{+,\beta,h} \leq \mathrm{var}_\Lambda^{+,\beta,1} \leq \chi\,. \tag{9.66}$$

Moreover, for all $t \in \mathbb{L}$, $t(2) \geq 1$ we have by FKG–inequalities (see proof of (5.27))

$$\langle\sigma(t)\rangle_{\Lambda\backslash\Sigma_0}^{-,\beta,1} \leq \langle\sigma(t)\rangle_\Lambda^{-,\beta,h} \leq \langle\sigma(t)\rangle_\Lambda^{-,\beta,1}\,, \tag{9.67}$$

so that we can use Lemma 9.2 to compare $\langle\sigma(t)\rangle_\Lambda^{-,\beta,h}$ with $-m^*$.

2c. Let the coupling constants $J(e)$ be given by (9.23) with $\beta > \beta_c$ and $0 < h \leq 1$. In that case we use

Lemma 9.5 *Let the coupling constants $J(e)$ be given by (9.23) with $\beta > \beta_c$. Let $t \in \Lambda$ such that*

$$\min\{|t(i) - t'(i)| : t' \notin \Lambda\,,\ i = 1,2\} \geq 2s > 0\,. \tag{9.68}$$

There exists a positive constant $\overline{a}(\beta)$ (see Lemma 9.2) such that

$$\begin{aligned}\langle\sigma(t)\rangle^{+,\beta} - 2P_\Lambda^+[\{\exists\,\gamma \text{ not s} - \text{small}\}] \leq \langle\sigma(t)\rangle_\Lambda^{+,\beta,h} \leq \qquad\qquad & (9.69)\\ \langle\sigma(t)\rangle^{+,\beta} + 2P_\Lambda^+[\{\exists\,\gamma \text{ not s} - \text{small}\}] + O(s)\exp\{-s\overline{a}(\beta)\}\,. &\end{aligned}$$

Proof. Let $\mathcal{E}$ be the event: all external contours γ in ω, which have at least one edge on the boundary of Λ^*, are s–small. We have

$$|\,\langle\sigma(t)\rangle_\Lambda^{+,\beta,h} - \langle\sigma(t)\,|\,\mathcal{E}\rangle_\Lambda^{+,\beta,h}\cdot P_\Lambda^{+,\beta,h}[\,\mathcal{E}\,]\,| \leq P_\Lambda^{+,\beta,h}[\,\{\exists\gamma \text{ not } s\text{–small}\}\,]\,. \tag{9.70}$$

By FKG–inequalities

$$\langle\sigma(t)\,|\,\mathcal{E}\rangle_\Lambda^{+,\beta,h} \geq \langle\sigma(t)\rangle^{+,\beta}\,. \tag{9.71}$$

The lower bound follows from (9.70) and (9.71). The upper bound follows by using Lemma 9.2 to show that $\langle\sigma(t)\rangle^{+,\beta} \leq \langle\sigma(t)\rangle^{+,\beta} + O(s)\exp(-s\overline{a}(\beta))$. $\square$

There is of course a similar result with $-$ boundary condition instead of $+$ boundary condition. In case Λ is simply connected we can use Lemmas 8.1 and 5.6 to estimate

$$P_\Lambda^{+,\beta,h}[\,\{\exists\,\gamma \text{ not } s\text{–small}\}\,]\,. \tag{9.72}$$

To get an upper bound on $\mathrm{var}_\Lambda^{+,\beta,h}$ we use GKS inequalities,

$$\langle\sigma(t)\sigma(t')\rangle_\Lambda^{+,\beta,h} \leq \langle\sigma(t)\sigma(t')\rangle_\Lambda^{+,\beta,1}\,, \tag{9.73}$$

and estimate by Lemmas 9.5 and 9.2

$$\langle \sigma(t) \rangle_\Lambda^{+,\beta,h} \langle \sigma(t') \rangle_\Lambda^{+,\beta,h} - \langle \sigma(t) \rangle_\Lambda^{+,\beta,1} \langle \sigma(t') \rangle_\Lambda^{+,\beta,1} . \tag{9.74}$$

3. Using the above method, it is possible to improve the results on the phase of small contours given in [I2] and [SS]. In these papers, the probability which was considered was $P_\Lambda^{+,s}[\frac{1}{\Lambda}\sum_{t\in\Lambda}\sigma(t) - m^* < -\epsilon]$. In such a case, we can apply the preceding method with $l = C(\epsilon)s$, with C sufficiently large. We then use the fact that there exist $\mu(\epsilon) > 0$ and $\epsilon'(\epsilon) > 0$ such that at least μN boxes have a magnetization at most $m^* - \epsilon'$. Using the fact that $\{\frac{1}{\Lambda}\sum_{t\in\Lambda}\sigma(t) - m^* < -\epsilon\}$ is decreasing we can first remove the constraint on the size of contours and then replace Λ' by Λ_2. The event so obtained can be estimated using the results of [S]. This gives ($s = L^b$)

$$P_\Lambda^{+,s}[\frac{1}{\Lambda}\sum_{t\in\Lambda}\sigma(t) - m^* < -\epsilon] \leq (\exp(-O(L^b))^{O(L^{2-2b})} = \exp(-O(L^{2-b})) \tag{9.75}$$

This exponent can be shown to be optimal using the method of proof of lemma 12.3.

10 Large deviations: lower bound

We derive a lower bound for large deviations of the magnetization of an Ising model in a finite box. It is given by the solution of the isoperimetric problem discussed in the introduction, hence it depends on the choice of the boundary condition and the shape of the box. In the next section we show that the lower bound is optimal.

Let $r_1, r_2 \in \mathbb{N}$; we define the box $\Lambda_L = \Lambda_L(r_1, r_2)$

$$\Lambda_L := \{t \in \mathbb{Z}^2 : -r_1 L \le t(1) < r_1 L \,;\; 0 \le t(2) < 2r_2 L\}\,. \tag{10.1}$$

We choose the coupling constants as in (9.23),

$$J(e) := \begin{cases} \beta \ge 0 & \text{if } e = \langle t, t'\rangle,\ t(2) \ge 0 \text{ and } t'(2) \ge 0, \\ h\beta \ge 0 & \text{if } e = \langle t, t'\rangle,\ t(2) \le -1 \text{ or } t'(2) \le -1. \end{cases} \tag{10.2}$$

With these coupling constants the Gibbs measure μ_L^{h}, $h \ge 0$, is equal to the Gibbs measure in Λ_L with Λ_L^+–boundary condition, $\mu_L^{\mathrm{h}} = \mu_L^+ \equiv \mu_{\Lambda_L}^+$. The case $h < 0$ is equivalent to a $\Lambda_L^{\pm}$–boundary condition, with the same nonnegative coupling constants. By definition a configuration ω satisfies the $\Lambda_L^{\pm}$–boundary condition if

$$\omega(t) := \begin{cases} 1 & \text{if } t \notin \Lambda_L,\ t(2) \ge 0, \\ -1 & \text{if } t \notin \Lambda_L,\ t(2) < 0. \end{cases} \tag{10.3}$$

The Gibbs measure $\mu_L^{\pm}$ is defined by

$$\mu_L^{\pm}(\omega) := \begin{cases} \Xi^{\pm}(\Lambda_L)^{-1} \exp(-H_{\Lambda_L}(\omega)) & \text{if } \omega(t) \text{ satisfies the } \Lambda_L^{\pm}\text{–bd. cond.,} \\ 0 & \text{otherwise.} \end{cases} \tag{10.4}$$

It is technically convenient to consider separately the cases $h \ge 0$ and $h < 0$. Below, when $h \ge 0$ we write probabilities with respect to μ_L^{h} by $P_L^+[\,\cdot\,]$ and when $h < 0$ by $P_L^{\pm}[\,\cdot\,]$. The functional $\mathtt{W}$ is denoted by $\mathtt{W}_+$, resp. $\mathtt{W}_-$, when $h \ge 0$, resp. $h < 0$.

Let $\beta > \beta_c$ and $m^* = m^*(\beta) > 0$ be the spontaneous magnetization. We choose m and c such that $-m^* < m < m^*$ and $0 < c < 1/2$. We define the event

$$A(m;c) := \{\omega : \,|\sum_{t \in \Lambda_L} \omega(t) - m|\Lambda_L|\,| \le |\Lambda_L| \cdot L^{-c}\}\,. \tag{10.5}$$

The main results of this section, Theorems 10.1 and 10.2, are lower bounds on

$$P_L^+[A(m;c)] \quad \text{and} \quad P_L^{\pm}[A(m;c)]\,. \tag{10.6}$$

10.1 Positive boundary magnetic field

We consider the case of Λ_L^+–boundary condition. We prove the

Theorem 10.1 *Assume that*

1. *The coupling constants are defined by (10.2) with* $\beta > \beta_c$. $-m^* < m < m^*$ *and* $c := 1/2 - \delta$, $\delta > 0$.
2. $\mathtt{W}_+$ *is defined by (1.9) with*
$$\widehat{\tau}(x) := \tau(x;\beta^*)\,, \tag{10.7}$$
the decay–rate of the two–point function (see Proposition 4.1 5.), and
$$\widehat{\tau}_{\mathrm{bd}} := \tau_{\mathrm{bd}}((1,0);\beta^*,h^*)\,, \tag{10.8}$$
the decay–rate of the boundary two–point function (see Definition 4.2). The parameter h^* *is defined by the relation*
$$\exp\{-2\beta h\} = \tanh\beta^* h^*\,. \tag{10.9}$$
3. $\mathcal{C}$ *is a simple closed rectifiable curve, which is the boundary of a convex body of volume* $4r_1r_2\frac{m^*-m}{2m^*}$ *in the rectangle* Q.

If $0 < \varepsilon < \delta$, *then there exists* $L(\varepsilon) = L(\varepsilon,\beta,h)$ *such that for all* $L \geq L(\varepsilon)$,
$$P_L^+[A(m;c)] \geq \exp\left\{-L\cdot\mathtt{W}_+(\mathcal{C}) - |\mathcal{C}|O(L^{3/4+\varepsilon/2})\right\}\,. \tag{10.10}$$

Remark: It is possible to get a better estimate if we use the fact that $(1,0)$ is a minimum of $\tau(x;\beta)$ on the unit sphere. We can prove, by modifying the definition of the set A_φ (10.20), that
$$P_L^+[A(m;c)] \geq \exp\left\{-L\cdot\mathtt{W}_+(\mathcal{C}) - |\mathcal{C}|O(L^{1/2+\varepsilon})\right\}\,. \tag{10.11}$$

Proof. The basic strategy of the proof is taken from section 7 in [Pf2]. Given the boundary of a convex body, we define a polygonal appromixation of it. Then, by summing over all large contours passing through the vertices of the polygonal approximation we can estimate the probability of the event $A(m;c)$ in terms of the functional $\mathtt{W}_+$ using Propositions 6.1 and 6.2. We divide the proof into five steps.

Step 1. Definition of a polygonal approximation of $\mathcal{C}$.

Let $[0,|\mathcal{C}|] \to \mathbb{R}^2$, $a \mapsto \mathcal{C}(a)$ be a unit–speed parametrization of $\mathcal{C}$ so that the orientation of $\mathcal{C}$ is counter–clockwise. We define
$$\dot{\mathcal{C}}_+(a) := \lim_{b\downarrow 0}\frac{\mathcal{C}(a+b)-\mathcal{C}(a)}{b}\quad,\quad \dot{\mathcal{C}}_-(a) := \lim_{b\downarrow 0}\frac{\mathcal{C}(a-b)-\mathcal{C}(a)}{b}\,. \tag{10.12}$$

For any unit vector $x \in \mathbb{R}^2$ we set

$$\theta(x) := \text{ interior angle between } (1,0) \text{ and } x\,. \tag{10.13}$$

We define two maps, ϕ_+ and ϕ_-: $[0, |\mathcal{C}|] \to \mathbb{R}/2\pi$,

$$\phi_+(a) := \theta(\dot{\mathcal{C}}_+(a)) \quad , \quad \phi_-(a) := \theta(\dot{\mathcal{C}}_-(a))\,. \tag{10.14}$$

Let $L \in \mathbb{N}$ and set

$$\delta_L := L^{-1/2+\varepsilon}\,. \tag{10.15}$$

We partition the interval $[0, |\mathcal{C}|]$ into subintervals $I_i := [\,a_{i-1}, a_i\,]$, $i = 1, \ldots N_L$. We set $a_0 := 0$. If $\mathcal{C} \cap w_Q$ is a non–empty interval, then we assume that this interval is $[\mathcal{C}(a_0), \mathcal{C}(a_1)]$. Otherwise $[a_0, a_1]$ is the maximal interval such that

$$|\phi_-(a) - \phi_+(a_0)| \leq \delta_L \quad , \quad \forall a \in [a_0, a_1]\,; \tag{10.16}$$

$[a_1, a_2]$ is the maximal interval such that

$$|\phi_-(a) - \phi_+(a_1)| \leq \delta_L \quad , \quad \forall a \in [a_1, a_2]\,, \tag{10.17}$$

and so on, with $a_{N_L} = |\mathcal{C}|$. Since the curve $\mathcal{C}$ describes the boundary of a convex body the number of subintervals N_L is bounded by

$$N_L \leq \frac{2\pi}{\delta_L} = 2\pi L^{1/2-\varepsilon}\,. \tag{10.18}$$

We set

$$t_i := \mathcal{C}(a_i) \quad , \quad i = 0, \ldots N_L - 1\,, \tag{10.19}$$

and construct a polygon $\mathcal{P}_L$. Let $\varphi = 0, \pi/2, \pi, 3\pi/2$ and set

$$A_\varphi := \{a \in [0, |\mathcal{C}|] : \, |\phi_+(a) - \varphi| \leq \delta_L^{1/2}\,, \, |\phi_-(a) - \varphi| \leq \delta_L^{1/2}\}\,, \tag{10.20}$$

and

$$\overline{A}_\varphi := \bigcup_{j:\, I_j \cap A_\varphi \neq \emptyset} I_j\,. \tag{10.21}$$

1. $\mathcal{P}_L^0$ is the polygon with vertices $t_0, t_1, \ldots, t_{N_L} = t_0$.

2. If there exists $a \in \overline{A}_0$, with $\mathcal{C}(a)(2) \leq \delta_L$, then we define a new polygon $\mathcal{P}_L^1$ as follows. Let t_l be the vertex of $\mathcal{P}_L^0$ with $t_l = \mathcal{C}(a_l)$, $a_l \in \overline{A}_0$ and $t_l(1)$ minimal; let t_r be the vertex of $\mathcal{P}_L^0$ with $t_r = \mathcal{C}(a_r)$, $a_r \in \overline{A}_0$ and $t_r(1)$ maximal. We replace the polygonal line with vertices $t_j = \mathcal{C}(a_j)$, $a_j \in \overline{A}_0$, by the polygonal line with vertices $t_l, u_l := (t_l(1), 0), u_r := (t_r(1), 0), t_r$. (We may have $t_l = u_l$ or $t_r = u_r$.)
   Otherwise we set $\mathcal{P}_L^1 := \mathcal{P}_L^0$.

3. We do a similar construction with $\varphi = \pi/2, \pi, 3\pi/2$, defining successively polygons $\mathcal{P}_L^2, \mathcal{P}_L^3, \mathcal{P}_L^4$.

4. If a side of $\mathcal{P}_L^4$ has a length larger than $2\delta_L$, then we subdivide the side so that at the end all sides obtained by that subdivision have a length between δ_L and $2\delta_L$. The resulting polygon is $\mathcal{P}_L$.

We give a lower bound for $\mathtt{W}_+(\mathcal{C})$ in terms of $\mathtt{W}_+(\mathcal{P}_L)$. Conditions (10.16), (10.17) and (4.2) of Proposition 4.1 imply that

$$|\,\mathtt{W}_+(\mathcal{C}) - \mathtt{W}_+(\mathcal{P}_L^0)\,| \leq |\mathcal{C}| O(L^{-1/2+\varepsilon})\,. \tag{10.22}$$

Condition (10.20) and (4.2) imply that

$$|\mathtt{W}_+(\mathcal{P}_L^0) - \mathtt{W}_+(\mathcal{P}_L^4)| \leq |\mathcal{C}| O(\delta_L^{1/2})\,, \tag{10.23}$$

when $\widehat{\tau}_{\mathrm{bd}} = \widehat{\tau}((1,0))$. In the general case, $\widehat{\tau}_{\mathrm{bd}} \leq \widehat{\tau}((1,0))$,

$$\mathtt{W}_+(\mathcal{C}) \geq \mathtt{W}_+(\mathcal{P}_L^4) - |\mathcal{C}| O(\delta_L^{1/2})\,. \tag{10.24}$$

4. does not modify this inequality, so that

$$\mathtt{W}_+(\mathcal{C}) \geq \mathtt{W}_+(\mathcal{P}_L) - |\mathcal{C}| O(L^{-1/4+\varepsilon/2})\,. \tag{10.25}$$

Step 2. Scaling and definition of a set of closed contours $\mathcal{G}_L$.

Let $L\mathcal{P}_L$ be the polygon obtained by scaling $\mathcal{P}_L$ by a factor L and shifting it by $(0,-1/2)$. In the following we suppose that we have possibly slightly modified $L\mathcal{P}_L$ so that its vertices are in Λ_L^*. A vertex of the scaled polygon is denoted by Lt_j. We do the following construction.

1. For each side of the scaled polygon, which are along the boundary of Λ_L^*, we construct a box as in Proposition 6.2.
2. For the other sides we construct a box as in Proposition 6.1.

We define a set of closed contours $\mathcal{G}_L = \{\Gamma\}$.

1. Each $\Gamma \in \mathcal{G}_L$ is closed and passes through all vertices of $L\mathcal{P}_L$.
2. Let γ_k be the part of Γ between two vertices, say Lt_k and Lt_{k+1}. If for the side $[Lt_k, Lt_{k+1}]$ there is a box B_k such that
   - 2a. it is at distance at least $L^{1/2+\varepsilon}$ from the boundary of Λ_L^*,
   - 2b. it does not intersect another box,
   - 2c. its side has a length larger than $L^{\varepsilon/2}$,

   then γ_k is any contour inside B_k with $\delta\gamma_k = \{Lt_k, Lt_{k+1}\}$.
3. In all other cases γ_k coincides with a fixed contour of minimal length between Lt_k and Lt_{k+1}.

The total length of the fixed contours inside boxes, whose sides have a length smaller than $L^{\varepsilon/2}$, is bounded by

$$O(L^{\varepsilon/2}) N_L \leq O(L^{1/2-\varepsilon/2})\,, \tag{10.26}$$

because these sides have not been subdivided in the construction of $\mathcal{P}_L$. The total length of the fixed contours inside boxes, which do not verify 2a or 2b, is bounded by

$$O(L^{3/4+\varepsilon/2})\,. \tag{10.27}$$

Indeed, the number of boxes which are not disjoint is finite; if 2a does not hold, we have

$$|\phi_\pm(a)-\varphi| \geq \delta_L^{1/2} \quad , \quad \varphi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\,, \tag{10.28}$$

so that the length of these fixed contours cannot exceed $O(L^{3/4+\varepsilon/2})$ since

$$\delta_L^{1/2} \cdot O(L^{3/4+\varepsilon/2}) = O(L^{1/2+\varepsilon})\,. \tag{10.29}$$

Step 3. Estimation of $P_L^+[A(m;c)\,|\,\{\Gamma\text{external; all }\gamma\neq\Gamma\text{ s-small}\}]$.

Let $\Gamma \in \mathcal{G}_L$. We estimate

$$P_L^+[A(m;c)\,|\,\{\Gamma\text{external; all }\gamma\neq\Gamma\text{ s-small}\}] = \tag{10.30}$$

$$1 - P_L^+[\,|\{\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L|\,| > |\Lambda_L|\cdot L^{-c}\}\,|\,\{\Gamma\text{external; all }\gamma\neq\Gamma\text{ s-small}\}]\,.$$

We use Proposition 9.1. We define the small contours by the condition

$$s := [L^{\delta/2}]\,, \tag{10.31}$$

and take

$$l := [L^{1/2}]\,. \tag{10.32}$$

In our case $x = O(L^{-c})$; the three terms (9.47), (9.48) and $4(s+1)(2l+s+1)/|\Lambda_2|$ are of the same order $O(L^{-1/2+\delta/2}) \ll O(L^{-c})$ (use Lemma 9.3). We introduce a grid in Λ_L, with an elementary cell congruent to Λ_2. The volume of Λ_2 is $O(L)$. Therefore the total number of cells of the grid in Λ_L is $O(L)$. The total number of cells of the grid, which have a non–empty intersection with one given box constructed in step 2 is at most

$$O(L^{1/2+\varepsilon})^2/O(L) = O(L^{2\varepsilon})\,. \tag{10.33}$$

There are at most $O(L^{1/2-\varepsilon})$ boxes in the construction of step 2: the number of sides of $\mathcal{P}_L^4$ is at most $O(L^{1/2-\varepsilon})$; there are at most $O(L)/O(L^{1/2+\varepsilon}) = O(L^{1/2-\varepsilon})$ boxes of sides $O(L^{1/2+\varepsilon})$. Therefore the total number of cells which intersect the boxes constructed in step 2. is at most $O(L^{1/2-\varepsilon})\cdot O(L^{2\varepsilon}) = O(L^{1/2+\varepsilon})$. Consequently N in Proposition 9.1 is $O(L)$. Notice also that the total volume of the boxes constructed in step 2 is bounded by $O(L^{1/2-\varepsilon})\cdot O(L^{1/2+\varepsilon})^2 = O(L^{3/2+\varepsilon})$, which is smaller than $|\Lambda_L|\cdot L^{-c}$ for L large enough. Thus (see e.g. section 7 in [Pf2] for details)

$$P_L^+[\{|\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L|\,| > |\Lambda_L|\cdot L^{-c}\}\,|\,\{\Gamma\text{external; all }\gamma\neq\Gamma\text{ s-small}\}] \tag{10.34}$$

$$\leq O(\exp\{-O(L^{2\delta})\})\,.$$

Step 4. Estimation of $P_L^+[\,\{\Gamma\text{external; all } \gamma\neq\Gamma \text{ s-small}\}\,]$.

Define
$$\Lambda_L(\mathrm{ext}\Gamma) := \Lambda_L\backslash\overline{\mathrm{int}\Gamma}\,, \tag{10.35}$$
and
$$\Lambda_L(\mathrm{int}\Gamma) := \Lambda_L\backslash\overline{\mathrm{ext}\Gamma}\,. \tag{10.36}$$
We have
$$P_L^+[\,\{\Gamma\text{external; all } \gamma\neq\Gamma \text{ s-small}\}\,] = w^*(\Gamma)\frac{Z^{+,s}(\Lambda_L(\mathrm{ext}\Gamma))Z^{+,s}(\Lambda_L(\mathrm{int}\Gamma))}{Z^+(\Lambda)}\,, \tag{10.37}$$
where $Z^{+,s}(\Lambda')$ is defined as $Z^+(\Lambda')$ in (8.7), but by summing only over s–small contours. Although $\Lambda_L(\mathrm{ext}\Gamma)$ is not simply connected, any $\Lambda_L(\mathrm{ext}\Gamma)^*$–compatible family of s–small closed contours is $\Lambda_L(\mathrm{ext}\Gamma)^+$–compatible. Therefore using Lemma 8.1 we have
$$P_L^+[\{\Gamma\text{external; all } \gamma\neq\Gamma \text{ s-small}] = \tag{10.38}$$
$$w^*(\Gamma)\frac{Z(\Lambda_L^*|\Gamma)}{Z(\Lambda_L^*)}\cdot\langle\{\text{all }\gamma\ s\text{–small}\}\rangle_{\Lambda_L(\mathrm{ext}\Gamma)^*}\cdot\langle\{\text{all }\gamma\ s\text{–small}\}\rangle_{\Lambda_L(\mathrm{int}\Gamma)^*} =$$
$$q_{\Lambda_L^*}(\Gamma;\beta^*,h^*)\cdot\langle\{\text{all }\gamma\ s\text{–small}\}\rangle_{\Lambda_L(\mathrm{ext}\Gamma)^*}\cdot\langle\{\text{all }\gamma\ s\text{–small}\}\rangle_{\Lambda_L(\mathrm{int}\Gamma)^*}\,.$$
Lemma 5.6 implies (if diameter $d(\gamma)\leq s$, then γ is s–small)
$$\langle\{\text{all }\gamma\ s\text{–small}\}\rangle_{\Lambda_L(\mathrm{ext}\Gamma)^*}\geq 1-O(L^{2+2\delta}\exp\{-\alpha L^{\delta/2}\})\,. \tag{10.39}$$
A similar estimate holds for $\langle\{\text{all }\gamma\ s\text{–small}\}\rangle_{\Lambda_L(\mathrm{int}\Gamma)^*}$.

Before doing the last step it is useful to summarize all above estimates. We have
$$P_L^+[A(m;c)]\geq \tag{10.40}$$
$$\sum_{\Gamma\in\mathcal{G}_L}P_L^+[A(m;c)\,|\,\{\Gamma\text{ext.; all }\gamma\neq\Gamma\text{ s-small}\}]\cdot P_L^+[\{\Gamma\text{ext.; all }\gamma\neq\Gamma\text{ s-small}\}]\geq$$
$$\Big(1-O(\exp\{-O(L^{2\delta})\})\Big)\cdot\sum_{\Gamma\in\mathcal{G}_L}P_L^+[\{\Gamma\text{ext.; all }\gamma\neq\Gamma\text{ s-small}\}]\geq$$
$$\Big(1-O(\exp\{-O(L^{2\delta})\})\Big)\cdot\Big(1-O(L^{2+2\delta}\exp\{-\alpha L^{\delta/2}\})\Big)^2\sum_{\Gamma\in\mathcal{G}_L}q_{\Lambda_L^*}(\Gamma;\beta^*,h^*)\,.$$

Step 5. Estimation of $P_L^+[A(m;c)]$ in terms of the functional $\mathtt{W}_+$.

It remains to control the last sum. Lemmas 5.4 and 5.3 give
$$\begin{aligned}\sum_{\Gamma\in\mathcal{G}_L}q_{\Lambda_L^*}(\Gamma) &\geq \sum_{\Gamma=\{\gamma_i\}\in\mathcal{G}_L}\prod_i q_{\Lambda_L^*}(\gamma_i)\\ &\geq \sum_{\Gamma=\{\gamma_i\}\in\mathcal{G}_L}\prod_i q_{\mathbb{L}}(\gamma_i)\,.\end{aligned} \tag{10.41}$$

Any box B_i, in which γ_i is not fixed, is at a distance at least $O(L^{1/2+\varepsilon})$ from Σ_0; hence Lemma 5.3 implies that
$$\sum_{\Gamma=\{\gamma_i\}\in\mathcal{G}_L}\prod_i q_{\mathbb{L}}(\gamma_i)\geq \tag{10.42}$$

$$\exp\left\{-O(|B_i|)\mathrm{e}^{-\tau^*(\beta^*)L^{1/2+\varepsilon}}\right\}\sum_{\Gamma=\{\gamma_i\}\in\mathcal{G}_L}\prod_i q(\gamma_i)\,.$$

Our choice of $\mathcal{G}_L$ implies that the sums over all γ_i are independent, so that we can estimate them using Propositions 6.1 and 6.2. For example, in the case of Proposition 6.1 when $[Lt_i, Lt_{i+1}]$ is a side of $L\mathcal{P}_L$,

$$\sum_{\substack{\gamma_i:\delta\gamma_i=\{Lt_i,Lt_{i+1}\}\\ \gamma_i\ \text{inside}\, B_i}} q(\gamma_i) \geq \langle\sigma(Lt_i)\sigma(Lt_{i+1})\rangle\Big[1-O(L^{1/2+\varepsilon})\exp\{-O(a)\}\Big]\exp\{-O(a)\}\,. \tag{10.43}$$

We choose $a := c_1 \ln L$, c_1 large enough, and use (7.8), (7.12) and Proposition 7.2 to prove the existence of two positive constants c_2 and c_3 such that

$$\sum_{\substack{\gamma_i:\delta\gamma_i=\{Lt_i,Lt_{i+1}\}\\ \gamma_i\ \text{inside}\, B_i}} q(\gamma_i) \geq \Big(1-O(L^{-c_2})\Big)\frac{\exp\{-\mathtt{W}_+([Lt_i, Lt_{i+1}])\}}{|Lt_{i+1}-Lt_i|^{c_3}}\,. \tag{10.44}$$

The length of the fixed γ_i does not exceed $O(L^{3/4+\varepsilon/2})$ and

$$\mathtt{W}_+(\mathcal{C}) \geq \mathtt{W}_+(\mathcal{P}_L) - |\mathcal{C}|O(L^{-1/4+\varepsilon/2})\,, \tag{10.45}$$

so that the above estimates imply the theorem. $\square$

Remark: We allow fluctuations of the magnetization in

$$A(m;c) = \{\omega : \,|\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L|\,| \leq |\Lambda_L|\cdot L^{-c}\}\,, \tag{10.46}$$

with $0 < c < 1/2$, because our estimates of Propositions 6.1 and 6.2 are done using square boxes (see step 2.). The total volume of the boxes is $O(L^{3/2+\varepsilon})$; it must be small compared to $|\Lambda_L|\cdot L^{-c} = O(L^{3/2+\delta})$.

10.2 Negative boundary magnetic field

We consider the case of $\Lambda_L^{\pm}$–condition. We prove the

Theorem 10.2 *Assume that*

1. *The coupling constants are defined by (10.2) with $\beta > \beta_c$.*
   $-m^* < m < m^*$ and $c := 1/2 - \delta$, $\delta > 0$.
2. *$\mathtt{W}_-$ is defined by (1.9) with*

$$\widehat{\tau}(x) := \tau(x;\beta^*)\,, \tag{10.47}$$

   the decay–rate of the two–point function (see Proposition 4.1 5.), and

$$\widehat{\tau}_{\mathrm{bd}} := -\tau_{\mathrm{bd}}((1,0);\beta^*,h^*)\,, \tag{10.48}$$

   τ_{bd} being the decay–rate of the boundary two–point function.
   The parameter h^* is defined by the relation

$$\exp\{-2\beta|h|\} = \tanh\beta^* h^*\,. \tag{10.49}$$

3. *$\mathcal{C}$ is a simple closed rectifiable curve, which is the boundary of a convex body of volume $4r_1r_2\frac{m^*-m}{2m^*}$ in the rectangle Q.*

If $0 < \varepsilon < \delta$, then there exists $L(\varepsilon) = L(\varepsilon,\beta,h)$ such that for all $L \geq L(\varepsilon)$,

$$P_L^{\pm}[A(m;c)] \geq \exp\left\{-L\cdot \mathtt{W}_-(\mathcal{C}) - |\mathcal{C}|O(L^{3/4+\varepsilon/2})\right\}\,. \tag{10.50}$$

The proof of Theorem 10.2 is similar to that of Theorem 10.1. There is one important difference, which we discuss now. If ω satisfies a $\Lambda_L^{\pm}$–boundary condition then there is *always* an open contour with a fixed left–hand end–point t_1^* and a fixed right–hand end–point t_2^*. We denote this particular contour by Γ^*.

Definition 10.1 *A family of compatible contours $\underline{\gamma}$ in Λ_L^* is $\Lambda_L^{\pm}$–**compatible** if and only if there exists a configuration ω satisfying the $\Lambda_L^{\pm}$–boundary condition, such that $\underline{\gamma}$ is the family of contours of ω.*

The normalization constant $\Xi(\Lambda_L)^{\pm}$ appearing in the definition of $\mu_L^{\pm}$ can be written as

$$\begin{aligned}\Xi(\Lambda_L)^{\pm} &= \sum_{\omega}\prod_{\substack{e=\langle t,t'\rangle:\\ e\cap\Lambda_L\neq\emptyset}}\exp\{J(e)[\sigma(t)(\omega)\sigma(t')(\omega)-1]\} \\ &= \sum_{\substack{\underline{\gamma}:\\ \Lambda_L^{\pm}-\mathrm{comp.}}}\prod_{\gamma\in\underline{\gamma}}\prod_{e^*\in\gamma}\exp\{-2J(e)\} \\ &= \sum_{\substack{\underline{\gamma}:\\ \Lambda_L^{\pm}-\mathrm{comp.}}}\prod_{\gamma\in\underline{\gamma}} w^*(\gamma)\,.\end{aligned} \tag{10.51}$$

We set

$$Z^{\pm}(\Lambda_L|\underline{\gamma}';J) := \sum_{\substack{\underline{\gamma}:\,\underline{\gamma}\cup\underline{\gamma}' \\ \Lambda_L^{\pm}-\mathrm{comp}}} w^*(\underline{\gamma})\,. \tag{10.52}$$

Since Λ_L is simply connected we have the important identity

$$\frac{Z^{\pm}(\Lambda_L;J)}{Z^{+}(\Lambda_L;J)} = \sum_{\Gamma^*} w^*(\Gamma^*)\frac{Z^{\pm}(\Lambda_L|\Gamma^*;J)}{Z^{+}(\Lambda_L;J)} = \langle\sigma(t_1^*)\sigma(t_2^*)\rangle^{J^*}_{\Lambda_L^*}\,. \tag{10.53}$$

This quantity can be controlled by Propositions 4.2 and 7.2.

Proof of Theorem 10.2: If $\mathcal{C}\cap\{x\in Q\,:\,x(2)=0\}=\emptyset$, then $\mathcal{C}$ is at a positive distance from $\{x\in Q\,:\,x(2)=0\}$. We can repeat the proof of Theorem 10.1. The open contour Γ^* is constraint to remain in an horizontal strip of width δ_L. If L is large enough the construction performed in the proof of Theorem 10.1 does not interfere with these Γ^*. We consider the case when $\mathcal{C}$ is the boundary of a convex body whose intersection with $\{x\in Q\,:\,x(2)=0\}$ is a non–empty interval $I:=[a,b]$.

Let $\mathcal{C}\backslash I$ be the part of $\mathcal{C}$ which does not intersect $\{x\in Q\,:\,x(2)=0\}$, and define $\mathcal{C}'$ as a unit–speed parametrized curve going from the bottom left corner of Q along $\{x\in Q\,:\,x(2)=0\}$ up to a, then following $\mathcal{C}\backslash I$ up to b and finally going from b to the bottom right corner of Q $\{x\in Q\,:\,x(2)=0\}$. We proceed as in the proof of Theorem 10.1 and define a polygonal interpolation of $\mathcal{C}'$. The proof is the same, up to minor, straightforward modifications. The set $\mathcal{G}_L$ is formed by contours Γ^*. Dividing and multiplying by $Z^+(\Lambda_L;J)$ and using identity (10.53) we can conclude. Since $\langle\sigma(t_1^*)\sigma(t_2^*)\rangle^{J^*}_{\Lambda_L^*}$ of (10.53) appears in the denominator the relevant functional is now $\mathtt{W}_-$. $\square$

11 Large deviations: upper bound

By Theorems 10.1 and 10.2, if $0 < \varepsilon < 1/2 - c$, then

$$P_L^+[A(m;c)] \geq \exp\left\{-L \cdot \mathtt{W}^*(m) - O(L^{3/4+\varepsilon/2})\right\}. \tag{11.1}$$

for L large enough, where

$$\mathtt{W}^*(m) := \inf\Big\{\mathtt{W}(\mathcal{C}) : \mathcal{C} \subset Q\,,\ \mathrm{vol}\,\mathcal{C} = 4r_1 r_2 \frac{m^* - m}{2m^*}\Big\}, \tag{11.2}$$

and

$$\mathtt{W} = \begin{cases} \mathtt{W}_+ & + \text{ boundary condition,} \\ \mathtt{W}_- & \pm \text{ boundary condition.} \end{cases} \tag{11.3}$$

We show that these lower bounds are optimal. To do this we analyze the measures in terms of large contours. The basic idea is to make a coarse–grained description of the large contours. We consider separately the cases of positive and negative boundary fields.

The basic estimates come from Lemmas 5.4, 5.5 and Proposition 9.1. As pointed out in the first comment following the proof of that proposition, (9.50) is probably not a sharp bound. For that reason we prove optimality only for

$$A(m;c) = \{\omega : |\sum_{t\in\Lambda_L} \omega(t) - m|\Lambda_L|| \leq |\Lambda_L| \cdot L^{-c}\}, \tag{11.4}$$

with $c = 1/4 - \delta$, $\delta > 0$ instead of $c = 1/2 - \delta$, $\delta > 0$. Notice that the choice $c = 1/4 - \delta$ does not lead to better lower bounds in section 10.

11.1 Positive boundary magnetic field

For $h \geq 0$ and $\mathcal{C} \subset Q$ we have

$$\mathtt{W}(\mathcal{C}) = \int_{\mathcal{C}} \widehat{\tau}(n(s))\,ds + \left[\widehat{\tau}_{\mathrm{bd}} - \widehat{\tau}((1,0))\right]|\mathcal{C} \cap w_Q| \geq \int_{\mathcal{C}_+} \widehat{\tau}(n(s))\,ds\,, \tag{11.5}$$

where $\mathcal{C}_+ := \mathcal{C} \setminus w_Q$.

Let $r_1, r_2 \in \mathbb{N}$ and $\Lambda_L = \Lambda_L(r_1, r_2)$ be the box

$$\Lambda_L = \{t \in \mathbb{Z}^2 : -r_1 L \leq t(1) < r_1 L\,;\ 0 \leq t(2) < 2r_2 L\}. \tag{11.6}$$

We treat the case of the Λ_L^+–boundary condition. The constant c is fixed, $c = 1/4 - \delta$, $\delta > 0$. The cut–off between small and large contours is chosen as

$$s := [L^{\delta'}]. \tag{11.7}$$

with $\delta > \delta' > 0$.

In each configuration ω with Λ_L^+–boundary condition we denote the large contours by $\Gamma_1, \Gamma_2, \ldots$. We choose a total order on $(\mathbb{Z}^2)^*$:

$$t < t' \Longleftrightarrow t(2) < t'(2) \quad \text{or} \quad t(2) = t'(2) \text{ and } t(1) < t'(1). \tag{11.8}$$

The origin of Γ_i is the first point of Γ_i. The parametrization of Γ_i, $s \mapsto \Gamma_i(s)$, is chosen so that it is counterclockwise and $\Gamma_i(0)$ is the origin of Γ_i. The coarse-grained description of Γ_i consists of defining a sequence of points $S_i = (t_{i0}, t_{i1}, \ldots, t_{in_i})$. The procedure is similar to the one used in the proof of Lemma 5.6, but here we must treat the points of Γ_i on the line $\{t \in (\mathbb{Z}^2)^* : t(2) = -1/2\}$ with special care.

1. $t_{i0} := \Gamma_i(0)$. If $t_{i0}(2) = -1/2$ and there exists s such that $\Gamma_i(s)(2) = -1/2$, then go to 5. Otherwise go to 2.
2. Let s_1 be the first integer time such that Γ_i is outside the square $B(t_{i0}; [L^{\delta'}])$. We set $t_{i1} := \Gamma_i(s_1)$.
3. Let s_2 be the first integer time greater than s_1 such that Γ_i is outside the square $B(t_{i1}; [L^{\delta'}])$. We set $t_{i2} := \Gamma_i(s_2)$.
4. The procedure is iterated until it stops.
5. If $t_{i0}(2) = -1/2$ and there exists $s \in \mathbb{N}$ such that $\Gamma_i(s)(2) = -1/2$, then $t_{i1} := \Gamma_i(s_1)$ with s_1 the largest integer time such that
$$\Gamma_i(s_1)(2) = -1/2 \quad \text{and} \quad \Gamma_i(s)(2) \leq [L^{\delta'}] \quad \forall s \in [0, s_1]\,. \tag{11.9}$$
6. If for all $s > s_1$ $\Gamma_i(s)(2) \neq -1/2$, then apply the procedure 2. to 4. to the part of Γ_i, $\{\Gamma_i(s) : s \geq s_1\}$. Otherwise go to 7.
7. Let s_2 be the first integer time greater than s_1, such that $\Gamma_i(s_2)(2) > [L^{\delta'}]$. We set $t_{i2} := \Gamma_i(s_2)$. Let $s*$ be the first integer time greater than s_2 such that $\Gamma_i(s^*)(2) = -1/2$. Apply the procedure 2. to 4. to the part of Γ_i, $\{\Gamma_i(s) : s_2 \leq s \leq s^*\}$. Then apply the procedure starting at 1. to the part of Γ_i, $\{\Gamma_i(s) : s \geq s^*\}$.

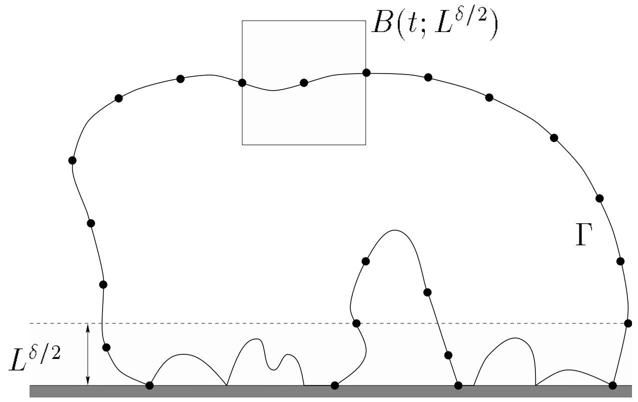


Figure 1: Coarse-graining of a large contour Γ touching the lower wall; the dots represent the sequence of points $S_i = \{t_{i0}, \ldots, t_{in_i}\}$.

Let $S := (t_1, \ldots, t_n)$ be an ordered sequence of points and $\mathcal{P}(S)$ be the corresponding closed polygonal line with vertices $(t_1, \ldots, t_n)$. To each Γ_i we associate a closed polygonal line $\mathcal{P}(\Gamma_i)$:
$$\mathcal{P}(\Gamma_i) := \mathcal{P}(S_i)\,, \tag{11.10}$$

where $S_i = (t_{i0}, t_{i1}, \dots, t_{in_i})$ is the ordered sequence of points defined by the above procedure. We set

$$B(S_i) := \left\{t \in \Lambda_L :\ t(2) \leq [L^{\delta'}]\right\} \bigcup_{t_{ij} \in S_i} \left(B(t_{ij}; [L^{\delta'}]) \cap \Lambda_L\right). \tag{11.11}$$

$\mathtt{W}_+$ is defined as in Theorem 10.1 and we set

$$\mathtt{W}_+(S_1, \dots, S_k) := \sum_{j=1}^{k} \mathtt{W}_+(\mathcal{P}(S_j)). \tag{11.12}$$

If Γ is a large contour with $\mathcal{P}(\Gamma) = \mathcal{P}(S_i)$, then Γ is inside $B(S_i)$ and

$$\{\Gamma(s) :\ s_j < s < s_{j+1}\} \cap \{t \in (\mathbb{Z}^2)^* :\ t(2) = -1/2\} = \emptyset, \tag{11.13}$$

whenever $\Gamma(s_j)(2) = t_j(2) \neq -1/2$ or $\Gamma(s_{j+1})(2) = t_{j+1}(2) \neq -1/2$. Using this property we can estimate $P_L^+[\{S_1, \dots, S_k\}]$ in terms of the functional $\mathtt{W}_+$ (see Lemma 5.5),

$$P_L^+[\{S_1, \dots, S_k\}] \leq \exp\{-\mathtt{W}_+(S_1, \dots, S_k)\}. \tag{11.14}$$

Let ω_{Γ_i} be the unique configuration satisfying the Λ_L^+–boundary condition having Γ_i as single contour. The **interior** of $\mathcal{P}(S_i)$ is $\mathrm{Int}\mathcal{P}(S_i) := \{t \in \Lambda_L : \omega_{\Gamma_i}(t) = -1\} \backslash B(S_i)$, where Γ_i is any contour such that $\mathcal{P}(\Gamma_i) = \mathcal{P}(S_i)$. The **volume** of $\mathcal{P}(S_i)$ is $\mathrm{Vol}\mathcal{P}(S_i) := |\mathrm{Int}\mathcal{P}(S_i)|$. The **closure** of $\mathrm{Int}(\mathcal{P}(S_i))$ is $\overline{\mathrm{Int}}\mathcal{P}(S_i) := |\mathrm{Int}\mathcal{P}(S_i)| \cup B(S_i)$.

Let $\omega_{\underline{\Gamma}}$ be the unique configuration satisfying the Λ_L^+–boundary condition having $\underline{\Gamma} := (\Gamma_1, \Gamma_2, \dots, \Gamma_k)$ as the complete set of its contours. The **interior** of $\underline{S} := (S_1, \dots, S_k)$ is $\mathrm{Int}\underline{S} := \{t \in \Lambda_L : \omega_{\underline{\Gamma}}(t) = -1\} \backslash \bigcup_i B(S_i)$, where $\underline{\Gamma}$ is any set of contours such that $\mathcal{P}(\Gamma_i) = \mathcal{P}(S_i)$, for all $\Gamma_i \in \underline{\Gamma}$. The **phase volume** of $\underline{S}$ is $\alpha(\underline{S})|\Lambda_L| := |\mathrm{Int}\underline{S}|$.

Lemma 11.1 *We assume that the coupling constants are defined by (10.2), $\beta > \beta_c$, and that $\mathtt{W}_{\mathtt{t}+}$ is defined as in Theorem 10.1. Then for any $\eta < \delta'$ and $T > 0$*

$$P_L^+[\{\sum_{j \geq 1} \mathtt{W}_+(\mathcal{P}(S_j)) \geq T\}] \leq \exp\left\{-T[1 - O(L^{\eta - \delta'})]\right\}. \tag{11.15}$$

Proof. We have

$$P_L^+[\{\sum_{j \geq 1} \mathtt{W}_+(\mathcal{P}(S_j)) \geq T\}] = \tag{11.16}$$

$$\sum_{k \geq 1} \sum_{S_1, \dots S_k} P_L^+[\{\mathtt{W}_+(S_1, \dots, S_k) \geq T\}].$$

Let $q(x,k)$ be the number of integer solutions of $1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq x$, $\sum_{i=1}^k \alpha_i = x$, k fixed, and $q(x)$ the number of integer solutions of $1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq x$ and $\sum_{i=1}^k \alpha_i = x$, k arbitrary. For large x

$$q(x) \sim \frac{1}{4\sqrt{3}x} \exp\left(2\pi\sqrt{x/6}\right). \tag{11.17}$$

Let us consider k polygonal lines $\mathcal{P}(S_1),\ldots,\mathcal{P}(S_k)$, where $S_i=(t_{i0},t_{i1},\ldots,t_{in_i})$. $L^{O(N)}$ is a rough estimate of the number of families of k polygonal lines with $n_1+\cdots+n_k=N$. Therefore the number of families of polygonal lines with $n_1+\cdots+n_k=N$, k arbitrary is bounded by

$$\sum_k q(N,k)L^{O(N)}=\exp\{NO(\ln L)\}\,. \tag{11.18}$$

Suppose that

$$\mathtt{W}_+(S_1,\ldots,S_k)=\sum_{i=1}^k \mathtt{W}_+(\mathcal{P}(S_i))=T'\,. \tag{11.19}$$

Using (11.5) we have

$$N\equiv n_1+\cdots+n_k\leq K_1T'L^{-\delta'}\,, \tag{11.20}$$

with $K_1\leq 2(\min_{\|n\|=1}\widehat{\tau}(n))^{-1}$.

$$\begin{aligned} P_L^+[\{S_1,\ldots,S_k\}] &\leq \exp\{-\mathtt{W}_+(S_1,\ldots,S_k)\} \\ &= \exp\{-\mathtt{W}_+(S_1,\ldots,S_k)+NL^\eta\}\exp\{-NL^\eta\} \\ &\leq \exp\left\{-\mathtt{W}_+(S_1,\ldots,S_k)(1-O(L^{-\delta'+\eta}))\right\}\exp\{-NL^\eta\}\,. \end{aligned} \tag{11.21}$$

Therefore

$$\begin{gathered} P_L^+[\{\sum_{j\geq 1}\mathtt{W}_+(\mathcal{P}(S_j))\geq T\}]\leq \\ \exp\left\{-T[1-O(L^{\eta-\delta'})]\right\}\sum_{N\geq 1}\exp\{NO(\ln L)-NL^\eta\}\leq \\ \exp\left\{-T[1-O(L^{\eta-\delta'})]\right\}\,. \end{gathered} \tag{11.22}$$

□

Lemma 11.2 *We assume that the coupling constants are defined by (10.2), $\beta>\beta_c$. Let $c=1/4-\delta$, $\delta>0$ and $-m^*<m<m^*$. For any $\eta>0$*

$$P_L^+\left[\left\{\left|\alpha(\underline{S})-\frac{m^*-m}{2m^*}\right|\geq\frac{1+\eta}{2m^*L^c}\right\}\,\middle|\,A(m;c)\right]\leq\exp\{-O(L)\}\,, \tag{11.23}$$

provided L is large enough.

Proof. We set

$$E_1(m;c):=\left\{\left|\alpha(\underline{S})-\frac{m^*-m}{2m^*}\right|\geq\frac{1+\eta}{2m^*L^c}\right\}\,. \tag{11.24}$$

We partition $E_1(m;c)$ into sets indexed by the set of large contours. Let

$$[\underline{\Gamma}]:=\{\omega:\underline{\Gamma}\text{ is the family of large contours of }\omega\}. \tag{11.25}$$

We write

$$P_L^+[E_1(m;c)\,|\,A(m;c)]=\sum_{\substack{\underline{\Gamma}:\\ [\underline{\Gamma}]\subset E_1(m;c)}} P_L^+[A(m;c)\,|\,[\underline{\Gamma}]]\cdot\frac{P_L^+[[\underline{\Gamma}]]}{P_L^+[A(m;c)]}\,. \tag{11.26}$$

Since (11.1) and Lemma 11.1 hold, we can find a constant K such that

$$P_L^+[\,E_1(m;c)\cap\{\sum_i \mathtt{W}_+(\mathcal{P}(S_i))\geq KL\}\,|\,A(m;c)\,]\leq \tag{11.27}$$

$$P_L^+[\,\{\sum_i \mathtt{W}_+(\mathcal{P}(S_i))\geq KL\}\,|\,A(m;c)\,]\leq \exp\{-O(L)\}\,.$$

It is sufficient to control in (11.26) the terms with $\underline{\Gamma}$ such that

$$\sum_i \mathtt{W}_+(\mathcal{P}(S_i))\leq KL\,. \tag{11.28}$$

Implicitely we therefore suppose that this condition is satisfied in the rest of the proof. In particular the total length of the polygonal lines is at most $O(L)$. Suppose that $\underline{\Gamma}=\{\Gamma_1,\ldots,\Gamma_k\}$ and that $\mathcal{P}(\Gamma_j)=\mathcal{P}(S_j)$, $j=1,\ldots,k$. Each Γ_j is inside some set $B(S_j)$. Under condition (11.28) inequality (11.20) reads

$$N=n_1+\cdots+n_k\leq O(L^{1-\delta'})\,, \tag{11.29}$$

which implies that

$$|\bigcup_i B(S_i)\,|\leq O(L^{1+\delta'}) \tag{11.30}$$

for any $\underline{\Gamma}$ such that (11.28) holds.

We set

$$\alpha(\underline{\Gamma})|\Lambda_L|:=|\{t\in\Lambda_L\,:\,\omega_{\underline{\Gamma}}(t)=-1\}|\,. \tag{11.31}$$

Notice that

$$P_L^+[\,A(m;c)\,|\,[\underline{\Gamma}]\,]=P^{\star,s}_{\Lambda(\underline{\Gamma})}[\,A(m;c)\,]\,, \tag{11.32}$$

where we introduced

$$\Lambda(\underline{\Gamma})=\bigcap_{\Gamma\in\underline{\Gamma}}(\overline{\mathrm{Int}}\Gamma\cup\overline{\mathrm{Ext}}\Gamma) \tag{11.33}$$

and the boundary condition $\star$ is given by any $\omega\in[\underline{\Gamma}]$.

We write

$$\sum_{t\in\Lambda_L}\sigma(t)(\omega)-m|\Lambda_L|=\Big(\sum_{t\in\Lambda_L}\sigma(t)(\omega)-\langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\Big)+\Big(\langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}-m|\Lambda_L|\Big)\,. \tag{11.34}$$

On the other hand, using lemma 9.3 and (11.30), we have

$$\begin{aligned}\langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} &= m^*|\Lambda_L|(1-2\alpha(\underline{\Gamma}))\pm O(L^{1+\delta'}) &(11.35)\\ &= m^*|\Lambda_L|(1-2\alpha(\underline{S}))\pm O(L^{1+\delta'})\,, &(11.36)\end{aligned}$$

so that, for every $\omega\in A(m;c)$,

$$\begin{aligned}\Big|\sum_{t\in\Lambda_L}\sigma(t)(\omega)-\langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\Big| &\geq \Big|\langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}-m|\Lambda_L|\Big|-\Big|\sum_{t\in\Lambda_L}\sigma(t)(\omega)-m|\Lambda_L|\Big|\\ &\geq \frac{1+\eta}{L^c}|\Lambda_L|-O(L^{1+\delta'})-L^{-c}|\Lambda_L|\\ &\geq \frac{\eta}{2}|\Lambda_L|L^{-c} &(11.37)\end{aligned}$$

Thus we have

$$P_L^+[\,A(m;c)\,|\,[\underline{\Gamma}]\,] \leq P^{\star,s}_{\Lambda(\underline{\Gamma})}\Big[\Big|\sum_{t\in\Lambda_L}\sigma(t)(\omega) - \langle\sum_{t\in\Lambda_L}\sigma(t)\rangle^{\omega,s}_{\Lambda(\underline{\Gamma})}\Big| \geq \frac{\eta}{2}|\Lambda_L|L^{-c}\Big]\,. \quad (11.38)$$

We estimate (11.38) by Proposition 9.1. We introduce a grid composed of squares whose sides have length $[L^{1/4}]$. There are $O(L^{3/2})$ squares of the grid in Λ_L. Since (11.29) holds, there are at most $O(L^{1-\delta'})$ squares of the grid, which have a non–empty intersection with $\cup_j B(S_j)$. The squares of the grid not intersecting $\cup_j B(S_j)$ play the role of the boxes $\mathcal{B}_i$. The term $4(s+1)(2l+s+1)/|\Lambda_2|$ is $O(L^{-1/4+\delta'}) \ll O(L^{-c})$. The same is true for (9.47) and (9.48) as a consequence of Lemma 9.3 and of the upper bound $O(L)$ on the total length of the polygonal lines. Proposition 9.1 implies that

$$P_L^+[\,A(m;c)\,|\,[\underline{\Gamma}]\,] \leq \exp\{-O(L^{1+2\delta})\}\,, \quad (11.39)$$

provided L is large enough. □

Theorem 11.1 *Assume that*

1. *The coupling constants are defined by (10.2) with $\beta > \beta_c$.*
   $-m^* < m < m^*$ and $c := 1/4 - \delta$, $\delta > 0$.
2. *$\mathtt{W}_+$ is defined as in Theorem 10.1.*
3. *$\mathtt{W}_+^*(m)$ is defined by*

$$\mathtt{W}_+^*(m) := \inf\{\mathtt{W}_+(\mathcal{C}) : \mathcal{C}\subset Q\,,\ \mathrm{vol}\,\mathcal{C} = 4r_1r_2\frac{m^*-m}{2m^*}\}\,. \quad (11.40)$$

Let $\eta < \delta' < \delta$ and

$$\begin{aligned} A(m;c): &= \{\omega : |\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L|\,| \leq |\Lambda_L|\cdot L^{-c}\}\,; & (11.41)\\ E_1(m;c): &= \left\{\left|\alpha(\underline{S}) - \frac{m^*-m}{2m^*}\right| < \frac{1+\eta}{2m^*L^c}\right\}\,; & (11.42)\\ E_2(m;c): &= \left\{\sum_i \mathtt{W}_+(S_i) \leq L\cdot \mathtt{W}_+^*(m)\big[1+O(L^{\eta-\delta'})\big]\right\}\,. & (11.43)\end{aligned}$$

Then, for L large enough,

$$P_L^+[\,E_1(m;c)\cap E_2(m;c)\,|\,A(m;c)\,] \geq 1 - \exp\Big\{-O(L^{1+\eta-\delta'})\Big\} \quad (11.44)$$

and

$$|\frac{1}{L}\ln P_L^+[\,A(m;c)\,] + \mathtt{W}_+^*(m)| \leq O(L^{\eta-\delta'})\,. \quad (11.45)$$

Proof. The first affirmation follows from Theorem 10.1, Lemma 11.1 and Lemma 11.2. We prove the second affirmation. Theorem 10.1 implies that, for L large enough,

$$-\frac{1}{L}\ln P_L^+[\,A(m;c)\,] \leq \mathtt{W}_+^*(m) + O(L^{-1/4+\varepsilon/2})\,, \quad (11.46)$$

with $0<\varepsilon<\delta$. Let $\widetilde{E}_1(m;c)$ be the complementary event of $E_1(m;c)$. We have

$$\begin{aligned} P_L^+[\,A(m;c)\,] &= P_L^+[\,A(m;c)\cap E_1\,]+P_L^+[\,A(m;c)\cap \tilde{E}_1\,] \\ &= P_L^+[\,A(m;c)\cap E_1\,]+P_L^+[\,\tilde{E}_1\,|\,A(m;c)\,]\cdot P_L^+[\,A(m;c)\,]\,. \end{aligned} \tag{11.47}$$

Therefore, setting $A=A(m;c)$,

$$(1-P_L^+[\,\tilde{E}_1\,|\,A\,])\cdot P_L^+[\,A\,]\le P_L^+[\,E_1\,]\,. \tag{11.48}$$

The inequality

$$\sum_i \mathrm{Vol}\mathcal{P}(S_i)\ge \alpha(\underline{S})|\Lambda_L|\ge \Big(\frac{m^*-m}{2m^*}-\frac{1+\eta}{2m^*L^c}\Big)|\Lambda_L| \tag{11.49}$$

implies that

$$\sum_i \mathtt{W}_+(\mathcal{P}(S_i))\ge \mathtt{W}_+^*(m+\frac{1+\eta}{L^c})L\,. \tag{11.50}$$

Suppose that in Q we put besides $\mathcal{C}(m)$, a convex body realizing the minimum $\mathtt{W}_+^*(m)$, another convex body $\mathcal{C}((1+\eta)/L^c)$ realizing the minimum for $\mathtt{W}_+^*((1+\eta)/L^c)$, then

$$\mathtt{W}_+^*(m+\frac{1+\eta}{L^c})+\mathtt{W}_+^*((1+\eta)/L^c)\ge \mathtt{W}_+^*(m)\,. \tag{11.51}$$

Therefore

$$\begin{aligned} (1-P_L^+[\,\tilde{E}_1\,|\,A\,])\cdot P_L^+[\,A\,] &\le P_L^+\Big[\Big\{\sum_i \mathtt{W}_+(\mathcal{P}(S_i))\ge \mathtt{W}_+^*(m+\frac{1+\eta}{L^c})L\Big\}\Big] \\ &\le P_L^+\Big[\Big\{\sum_i \mathtt{W}_+(\mathcal{P}(S_i))\ge \mathtt{W}_+^*(m)L-\mathtt{W}_+^*((1+\eta)/L^c)L\Big\}\Big]\,. \end{aligned} \tag{11.52}$$

Using Lemma 11.1, for L large enough,

$$-\frac{1}{L}\ln P_L^+[\,A(m;c)\,]\ge \mathtt{W}_+^*(m)-O(L^{\eta-\delta'})\,. \tag{11.53}$$

$\square$

11.2 Negative boundary magnetic field

We consider the $\Lambda_L^{\pm}$–boundary condition. The remarks of subsection 10.2 apply. By definition the open contour Γ^* is a large contour. We associate to Γ^* a sequence of points $S^* := (t_{*0}, \ldots, t_{*N})$ using the same procedure as for the other contours. $\mathcal{P}(S^*)$ is the open polygonal line with vertices S^*. We thus obtain a family $(\mathcal{P}(S_1), \ldots, \mathcal{P}(S_q), \mathcal{P}(S^*), \mathcal{P}(S_1'), \ldots, \mathcal{P}(S_p'))$ of polygonal lines. We have distinguished between the polygonal lines with no edge belonging to the line $\{t \in \mathbb{R}^2 : t(2) = -1/2\}$, which are denoted by $(\mathcal{P}(S_1), \ldots, \mathcal{P}(S_q))$, and the other ones denoted by $(\mathcal{P}(S_1'), \ldots, \mathcal{P}(S_p'))$. We will now associate to the set of polygonal lines $(\mathcal{P}(S^*), \mathcal{P}(S_1'), \ldots, \mathcal{P}(S_p'))$ a new set of *closed* polygonal lines $(\mathcal{P}(S_{q+1}), \ldots, \mathcal{P}(S_k))$. This is done in the following way:

1. Consider the family of polygonal lines $(\mathcal{P}(S^*), \mathcal{P}(S_1'), \ldots, \mathcal{P}(S_p'))$; remove all edges of these polygonal lines belonging to the line $\{t \in \mathbb{R}^2 : t(2) = -1/2\} \cap \Lambda^*$; call this set of edges $\mathcal{E}^*$.

2. Close these new polygonal lines by adding the set $(\{t \in \mathbb{R}^2 : t(2) = -1/2\} \cap \Lambda^*) \setminus \mathcal{E}^*$.

This defines a set of closed polygonal lines, which are denoted $(\mathcal{P}(S_{q+1}), \ldots, \mathcal{P}(S_k))$.

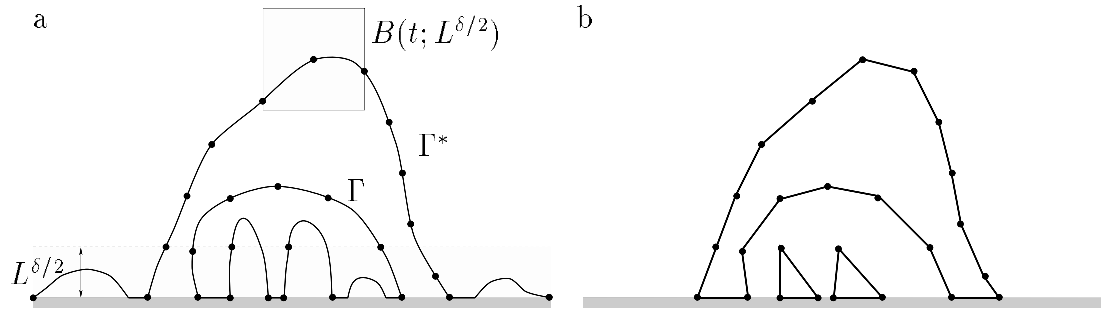


Figure 2: a) Coarse-graining of a large contour Γ touching the lower wall and of the open contour Γ^*; the dots represent the sequence of points obtained by the procedure described. b) The *three* resulting closed polygonal lines.

Remark: We do not modify the large contours. The relation between the family $(S_1, \ldots, S_k)$ and the large contours of the configuration is that these contours must be compatible with the original family $(S_1, \ldots, S_q, S^*, S_1', \ldots, S_p')$.

Notice that the above construction is such that we have the identity

$$\mathtt{W}_-(S_1, \ldots, S_k) = \mathtt{W}_+(S_1, \ldots, S_q, S^*, S_1', \ldots, S_p') - \tau_{\mathrm{bd}}(2r_1 + 1)L \tag{11.54}$$

where

$$\mathtt{W}_-(S_1, \ldots, S_k) := \sum_{i=1}^{k} \mathtt{W}_-(\mathcal{P}(S_i))\,. \tag{11.55}$$

Lemma 11.3 *In the setting described above, there exists a constant K_2 such that*

$$P_L^{\pm}[\{S_1, \ldots, S_k\}] \leq K_2\, L^{3/2} \exp\{-\mathtt{W}_-(S_1, \ldots, S_k)\}\ . \tag{11.56}$$

Proof. We write $P_L^{\pm}[\,\{S_1,\ldots,S_k\}\,]$ as a quotient

$$P_L^{\pm}[\,\{S_1,\ldots,S_k\}\,] =: \frac{Z^{\pm}(\Lambda_L|S_1,\ldots,S_k)}{Z^{\pm}(\Lambda_L)}\,. \tag{11.57}$$

Dividing and multiplying by $Z^{+}(\Lambda_L)$ we must consider the quotients

$$\frac{Z^{\pm}(\Lambda_L|S_1,\ldots,S_k)}{Z^{+}(\Lambda_L)} \quad , \quad \frac{Z^{\pm}(\Lambda_L)}{Z^{+}(\Lambda_L)}\,. \tag{11.58}$$

The first quotient is estimated using Lemmas 5.4, 5.5 and the above remark,

$$\frac{Z^{\pm}(\Lambda_L|S_1,\ldots,S_k)}{Z^{+}(\Lambda_L)} \leq \exp\{-\mathtt{W}_{+}(S_1,\ldots,S_q,S^*,S'_1,\ldots,S'_p)\} \tag{11.59}$$

The second quotient is estimated as in subsection 10.2, using Proposition 7.2 and the remark following it,

$$\frac{Z^{\pm}(\Lambda_L)}{Z^{+}(\Lambda_L)} = \langle\sigma(t_1^*)\sigma(t_2^*)\rangle_{\Lambda_L^*} \geq C\,(2r_1L)^{-3/2}\exp\{-\tau_{\mathrm{bd}}(t_2^*-t_1^*)\}\,. \tag{11.60}$$

These inequalities give, using (11.54),

$$P_L^{\pm}[\,\{S_1,\ldots,S_k\}\,] \leq C^{-1}(2r_1)^{3/2}\,L^{3/2}\exp\{-\mathtt{W}_{-}(S_1,\ldots,S_k)\}\,. \tag{11.61}$$

□

Lemma 11.4 *We assume that the coupling constants are defined by (10.2), $\beta>\beta_c$, and that $\mathtt{W}_{-}$ is defined as in Theorem 10.2. Then for any $\eta<\delta'<\delta$ and $T>0$*

$$P_L^{\pm}[\,\{\sum_{j\geq 1}\mathtt{W}_{-}(\mathcal{P}(S_j))\geq T\}\,] \leq \exp\Big\{-T[1-O(L^{\eta-\delta'})]+O(L^{1+\eta-\delta'})\Big\}\,. \tag{11.62}$$

Proof. The proof is essentially the same as that of Lemma 11.1. We have

$$P_L^{\pm}[\,\{\sum_{j\geq 1}\mathtt{W}_{-}(\mathcal{P}(S_j))\geq T\}\,] = \tag{11.63}$$
$$\sum_{k\geq 1}\sum_{S_1,\ldots S_k} P_L^{\pm}[\,\{\mathtt{W}_{-}(S_1,\ldots,S_k)\geq T\}\,]\,.$$

Suppose that

$$\mathtt{W}_{-}(S_1,\ldots,S_k) = T' \equiv T'_{+} - T'_{-}\,, \tag{11.64}$$

with T'_{+}, resp. T'_{-}, the positive, resp. negative, part of the functional $\mathtt{W}_{-}$. The total number N of vertices of the polygonal lines $\mathcal{P}(S_i)$, $i=1,\ldots,k$, is bounded by (see (11.20))

$$N \leq T'_{+}K_1L^{-\delta'}\,. \tag{11.65}$$

Therefore

$$\begin{aligned} P_L^{\pm}[\,\{S_1,\ldots,S_k\}\,] &\leq \exp\{-\mathtt{W}_{-}(S_1,\ldots,S_k)\}\,K_2L^{3/2} \\ &= \exp\Big\{-T'_{+}+T'_{-}+NL^{\eta}\Big\}\,K_2L^{3/2}\exp\{-NL^{\eta}\} \\ &\leq \exp\Big\{-T'_{+}(1-O(L^{-\delta'+\eta}))+T'_{-}\Big\}\,K_2L^{3/2}\exp\{-NL^{\eta}\} \\ &\leq \exp\Big\{-\mathtt{W}_{-}(1-O(L^{-\delta'+\eta}))+O(L^{1-\delta'+\eta})\Big\}\,\cdot \\ &\qquad K_2L^{3/2}\exp\{-NL^{\eta}\}\,. \end{aligned} \tag{11.66}$$

The end of the proof is the same as that of Lemma 11.1.

$$P_L^{\pm}[\{\sum_{j\geq 1}\mathtt{W}_-(\mathcal{P}(S_j))\geq T\}]\leq \tag{11.67}$$

$$\exp\left\{-T[1-O(L^{\eta-\delta'})]+O(L^{1-\delta'+\eta})\right\}\sum_{N\geq 1}K_2L^{3/2}\exp\{NO(\ln L)-NL^{\eta}\}\leq$$

$$\exp\left\{-T[1-O(L^{\eta-\delta'})]+O(L^{1-\delta'+\eta})\right\}.$$

□

Defining $\alpha(S_1,\ldots S_k):=\alpha(S_1,\ldots,S_q,S^*,S'_1,\ldots,S'_p)$, the next lemma is proven like lemma 11.2.

Lemma 11.5 *We assume that the coupling constants are defined by (10.2), $\beta>\beta_c$. Let $c=1/4-\delta$, $\delta>0$ and $-m^*<m<m^*$. For any $\eta>0$*

$$P_L^{\pm}\left[\left\{\left|\alpha(\underline{S})-\frac{m^*-m}{2m^*}\right|\geq\frac{1+\eta}{2m^*L^c}\right\}\,\middle|\,A(m;c)\right]\leq\exp\{-O(L)\}\,, \tag{11.68}$$

provided L is large enough.

We can now prove Theorem 11.2 as we proved Theorem 11.1.

Theorem 11.2 *Assume that*

1. *The coupling constants are defined by (10.2) with $\beta>\beta_c$.* *$-m^*<m<m^*$ and $c:=1/4-\delta$, $\delta>0$.*
2. *$\mathtt{W}_-$ is as in Theorem 10.2.*
3. *$\mathtt{W}_-^*(m)$ is defined by*

$$\mathtt{W}_-^*(m):=\inf\{\mathtt{W}_-(\mathcal{C}):\mathcal{C}\subset Q\,,\,\mathrm{vol}\,\mathcal{C}=4r_1r_2\frac{m^*-m}{2m^*}\}\,. \tag{11.69}$$

Let $\eta<\delta'<\delta$ and

$$A(m;c):\;=\;\{\omega:|\sum_{t\in\Lambda_L}\omega(t)-m|\Lambda_L||\leq|\Lambda_L|\cdot L^{-c}\}\,; \tag{11.70}$$

$$E_1(m;c):\;=\;\left\{\left|\alpha(\underline{S})-\frac{m^*-m}{2m^*}\right|<\frac{1+\eta}{2m^*L^c}\right\}\,; \tag{11.71}$$

$$E_2(m;c):\;=\;\left\{\sum_i\mathtt{W}_-(S_i)\leq L\cdot\mathtt{W}_-^*(m)\Big[1+O(L^{\eta-\delta'})\Big]\right\}\,. \tag{11.72}$$

Then, for L large enough,

$$P_L^{\pm}[\,E_1(m;c)\cap E_2(m;c)\,|\,A(m;c)\,]\geq 1-\exp\left\{-O(L^{1+\eta-\delta'})\right\} \tag{11.73}$$

and

$$|\frac{1}{L}\ln P_L^{\pm}[\,A(m;c)\,]+\mathtt{W}_-^*(m)|\leq O(L^{\eta-\delta'})\,. \tag{11.74}$$

12 Macroscopic droplet

In this last section we consider the limit of the lattice spacing going to zero. We suppose that $\beta > \beta_c$, $h \in \mathbb{R}$, $-m^* < m < m^*$ and $c = 1/4 - \delta > 0$ fixed. We define the canonical Gibbs measure $\langle \cdot | m \rangle_L(\beta, h)$ by

$$\langle \cdot | m \rangle_L(\beta, h) := \begin{cases} \langle \cdot | A(m; c) \rangle_L^+(\beta, h) & \text{if } h \geq 0, \\ \langle \cdot | A(m; c) \rangle_L^{\pm}(\beta, h) & h < 0. \end{cases} \tag{12.1}$$

In that section we treat both cases $h \geq 0$ and $h < 0$ simultaneously. We set

$$\langle \cdot \rangle_L(\beta, h) := \begin{cases} \langle \cdot \rangle_L^+(\beta, h) & \text{if } h \geq 0, \\ \langle \cdot \rangle_L^{\pm}(\beta, |h|) & h < 0. \end{cases} \tag{12.2}$$

As in the preceding section a contour is small if and only if it can be put inside a translate of the box $B(0; [L^{\delta'}])$, $0 < \delta' < \delta$. The specific choice of δ' is made later on; δ' is small. We do the analysis in the box $\Lambda_L(r_1, r_2)$ and at the end we scale everything by $1/L$ and take the limit of the lattice spacing going to zero.

Let $C \subset \mathbb{Z}^2$; **the empirical magnetization in** C is

$$m_C(\omega) := \frac{1}{|C|} \sum_{t \in C} \sigma(t)(\omega) \,. \tag{12.3}$$

Let $0 < a < 1$; we introduce a grid $\mathcal{L}(a)$ in Λ_L made of cells which are translates of the square box $B(0; [L^a])$. The specific choice of a is made later on; a is close to 1. In most of the cells the empirical magnetization is close to m^* or $-m^*$ with high probability (see Theorem 12.1).

Let $\mu > 0$ so that $a + \mu < 1$; we say that a **polygonal line is small** if $\overline{\mathrm{Int}}\mathcal{P}(S_i)$ can be put inside a translate of the box $B(0; [L^{a+\mu}])$; otherwise the polygonal line is **large**.
We partition the cells of $\mathcal{L}(a)$ into four sets.
A cell C is **polluted** if

$$\Big| C \cap \Big(\bigcup_{\mathcal{P}(S)\,\mathrm{small}} \overline{\mathrm{Int}}\mathcal{P}(S) \Big) \Big| \geq L^{2a - \eta''} \,, \tag{12.4}$$

with η'' a small positive number to be chosen later on.
A cell of $\mathcal{L}(a)$ is an **interface–cell** if it is not polluted and it has a non–empty intersection with a large polygonal line.
A cell of $\mathcal{L}(a)$ is called a **phase–cell** if it is neither polluted nor an interface–cell and it is entirely contained inside Λ_L.
The remaining cells are called **boundary cells**.

Lemma 12.1 *Let $\omega \in E_1(m; c) \cap E_2(m; c)$ and suppose $\delta' < a + \mu < 1 - \eta''$. Then, uniformly in ω,*

$$\begin{aligned} &\#\{C \textit{ cell of } \mathcal{L}(a) : C \textit{ is polluted}\} \leq O(L^{1-a+\mu+\eta''}) \\ &\#\{C \textit{ cell of } \mathcal{L}(a) : C \textit{ is an interface–cell}\} \leq O(L^{1-a}) \\ &\#\{C \textit{ cell of } \mathcal{L}(a) : C \textit{ is a boundary–cell}\} \leq O(L^{1-a}) \end{aligned}$$

Proof. We estimate the total volume of the region containing small polygonal lines. We partition the small polygonal lines into families. The first family contains all small polygonal lines $\mathcal{P}(S)$ with $\mathrm{Int}\mathcal{P}(S) = \emptyset$. We then partition the remaining polygonal lines into families so that for each family

$$[L^{a+\mu}]^2 \leq \Big|\bigcup_{\mathcal{P}(S)} \mathrm{Int}\mathcal{P}(S)\Big| \leq 10[L^{a+\mu}]^2 \tag{12.5}$$

(except possibly for the last family which may not satisfy the lower bound). The total length of the members of a family satisfying the latter inequalities is at least $K_3 L^{a+\mu}$ (isoperimetric inequality). Since the total length of the polygonal lines is at most $K'L$, we have at most $O(L^{1-a-\mu})$ families. Consequently, the total volume of these small polygonal lines is bounded by $O(L^{1+a+\mu})$. The volume of $B(\underline{S})$ is bounded by $O(L^{1+\delta'})$ (see (11.20) and (11.65)). Hence the total volume of the closure of the interior of these small polygonal lines is at most $O(L^{1+a+\mu})$.
The number of polluted cells is therefore at most $O(L^{1+a+\mu})/L^{2a-\eta''} = O(L^{1-a+\mu+\eta''})$.
To count the number of interface–cells we estimate the number of points we need in order to make a coarse–grained description of large polygonal lines using a reference box $B(0;[L^a])$ according to the method of the previous sections. Since the total length of the polygonal lines is at most $K'L$, the total number of interface–cells is at most $2K'L^{1-a}$.
The number of boundary cells is bounded by $O(L^{1-a})$. □

Let $\varepsilon(L)$ be a positive decreasing function such that $\lim_{L\to\infty}\varepsilon(L) = 0$ (see Lemma 12.2). Notice that a phase–cell cannot be surrounded by a small polygonal line; otherwise it would be polluted. We define the event E_3: in any phase–cell C the empirical magnetization satisfies

$$|\, m_C(\omega) - m^* \,| \leq \varepsilon(L)\,, \tag{12.6}$$

if the phase–cell is outside all external large contours or inside an even number of large contours, otherwise

$$|\, m_C(\omega) + m^* \,| \leq \varepsilon(L)\,. \tag{12.7}$$

Theorem 12.1 *Let $\beta > \beta_c$, $h \in \mathbb{R}$, $-m^* < m < m^*$ and $c = 1/4 - \delta > 0$. Let $\langle\,\cdot\,|m\rangle_L(\beta,h)$ be the canonical Gibbs state. Let E_1 and E_2 be the events defined in Theorems 11.1 or 11.2. Let $\eta' > 0$ be such that $2a - \delta' - 2\eta' > 1$. Then there exists a positive constant κ (see (12.40)) such that for L large enough*

$$\mathrm{Prob}[\, E_3 \,|\, E_1 \cap E_2 \,] \geq 1 - \exp\{-O(L^\kappa)\} \tag{12.8}$$

and

$$\mathrm{Prob}[\, E_3 \cap E_1 \cap E_2 \,] \geq 1 - \exp\{-O(L^\kappa)\}\,. \tag{12.9}$$

Proof. Let $A \equiv A(m;c)$, E_3^c complementary event to E_3, and $E_{1,2} := E_2 \cap E_1$. By definition

$$\mathrm{Prob}[\, E_3^c \,|\, E_{1,2} \,] \;=\; \frac{\langle\, E_3^c \cap E_{1,2}\,|m\rangle_L}{\langle\, E_{1,2}\,|m\rangle_L} \tag{12.10}$$

$$= \frac{\langle\, E_3^c \cap E_{1,2} \cap A \,\rangle_L}{\langle\, E_{1,2} \cap A \,\rangle_L}$$
$$= \langle\, A \,|\, E_3^c \cap E_{1,2} \rangle_L \, \frac{\langle\, E_3^c \,|\, E_{1,2} \,\rangle_L}{\langle\, A \,|\, E_{1,2} \,\rangle_L}$$
$$\leq \frac{\langle\, E_3^c \,|\, E_{1,2} \,\rangle_L}{\langle\, A \,|\, E_{1,2} \,\rangle_L}\,.$$

The numerator and denominator are estimated in the following lemmas.

Lemma 12.2 *Let*

$$\lim_{L\to\infty} \frac{\max(\, L^{-\eta'}, L^{-\eta''}\,)}{\varepsilon(L)} = 0\,. \tag{12.11}$$

1. If the phase–cell C is outside all external large contours or inside an even number of large contours, then for L large enough

$$\langle\, \{|m_C(\omega) - m^*| \geq \varepsilon(L)\} \,|\, E_{1,2})\,\rangle_L \leq \exp\{-O(L^{2a-2\delta'-2\eta'})\varepsilon(L)^2\}\,. \tag{12.12}$$

2. If the phase–cell C is inside an odd number of large contours, then for L large enough

$$\langle\, \{|m_C(\omega) + m^*| \geq \varepsilon(L)\} \,|\, E_{1,2})\,\rangle_L \leq \exp\{-O(L^{2a-2\delta'-2\eta'})\varepsilon(L)^2\}\,. \tag{12.13}$$

Proof. We prove 1. Let $\underline{S}$ such that $E(\underline{S}) \subset E_{1,2}$; let $\underline{\Gamma}$ be a family of large contours defining $\underline{S}$. We denote by $E(\underline{\Gamma})$ the set of configurations with $\underline{\Gamma}$ as family of large contours. It is sufficient to prove that

$$\langle\, \{|m_C(\omega) - m^*| \geq \varepsilon(L)\} \,|\, E(\underline{\Gamma})\,\rangle_L \leq \exp\{-O(L^{2a-2\delta'-2\eta'})\varepsilon(L)^2\}\,, \tag{12.14}$$

with $O(L^{2a-2\delta'-2\eta'})$ uniform in $\underline{\Gamma}$ and $E(\underline{\Gamma}) \subset E(\underline{S}) \subset E_{1,2}$. Let

$$C^* := C \cap \Big(\bigcup_{\mathcal{P}(S)\,\mathrm{small}} \overline{\mathrm{Int}}\mathcal{P}(S) \Big)\,. \tag{12.15}$$

For L large enough (use $\varepsilon(L) > L^{-\eta''}$)

$$\begin{aligned} \langle\, \{|m_C(\omega) - m^*| \geq \varepsilon(L)\} \,|\, E(\underline{\Gamma})\,\rangle_L \leq \\ \langle\, \{|m_{C\backslash C^*}(\omega) - m^*| \geq 2\varepsilon(L)/3\} \,|\, E(\underline{\Gamma})\,\rangle_L\,. \end{aligned} \tag{12.16}$$

We have $C\backslash C^* \subset \Lambda(\underline{\Gamma})$ (see (11.33)) and

$$\begin{aligned} \langle\, \{|m_{C\backslash C^*}(\omega) - m^*| \geq 2\varepsilon(L)/3\} \,|\, E(\underline{\Gamma})\,\rangle_L = \\ \langle\, \{|m_{C\backslash C^*}(\omega) - m^*| \geq 2\varepsilon(L)/3\}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\,, \end{aligned} \tag{12.17}$$

$\langle\,\cdot\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}$ being the Gibbs measure in $\Lambda(\underline{\Gamma})$ with $\star$ boundary condition (see section 11.1), conditionned on the fact that there are only small contours. Using Lemmas 9.2, 9.3 or 9.5 we get

$$|\,\langle\, m_{C\backslash C^*}(\omega)\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m^*\,| \leq O(\exp\{-L^{\delta'}\})\,. \tag{12.18}$$

We apply Proposition 9.1 with $l = L^{\delta'+\eta'}$ and use $\varepsilon(L) > L^{-\eta'}$. The number of cells of $\mathcal{L}(\delta'+\eta')$, which have a non–empty intersection with $B(S_i)$, $S_i \in \underline{S}$, is bounded by $O(L^{1-\delta'})$; indeed, the union of all boxes $B(t;[L^{\delta'+\eta'}])$ centered at all vertices of the polygonal lines $\mathcal{P}(\underline{S})$ contains $B(S_i)$ for any $S_i \in \underline{S}$, and there are at most $K_1 L^{1-\delta'}$ vertices for the polygonal lines $\mathcal{P}(\underline{S})$. Since $2a - 2\delta' - 2\eta' > 1 - \delta'$, the number of cells of $\mathcal{L}(\delta' + \eta')$ which do not intersect any $B(S_i)$ and are inside the cell C is $O(L^{2a-2\delta'-2\eta'})$; we have for L large enough (Proposition 9.1)

$$\langle \{ | m_{C\setminus C^*}(\omega) - m^* | \geq 2\varepsilon(L)/3 \} \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \leq \tag{12.19}$$

$$\langle \{ | m_{C\setminus C^*}(\omega) - \langle m_{C\setminus C^*}(\omega) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} | \geq \varepsilon(L)/2 \} \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \leq$$
$$\exp\{-O(L^{2a-2\delta'-2\eta'})\varepsilon(L)^2\} .$$

□

Lemma 12.3 *For L large enough*

$$\langle A(m;c) \,|\, E_{1,2} \rangle_L \geq \exp\{-O(L^{2-c-\delta'})\} . \tag{12.20}$$

Proof. It is sufficient to prove, uniformily in $\underline{\Gamma}$ with $E(\underline{\Gamma}) \subset E(\underline{S}) \subset E_{1,2}$, that

$$\langle A(m;c) \,|\, E(\underline{\Gamma}) \rangle_L \geq \exp\{-O(L^{2-c-\delta'})\} . \tag{12.21}$$

Let $\underline{\Gamma}$ be given. All contours $\gamma \notin \underline{\Gamma}$ in $\omega \in E(\underline{\Gamma})$ are s–small, $s = [L^{\delta'}]$. Since $E(\underline{\Gamma}) \subset E_{1,2}$ the phase volume $\alpha(\underline{S})$ satisfies

$$\Big| \alpha(\underline{S}) - \frac{m^* - m}{2m^*} \Big| < \frac{1+\eta}{2m^* L^c} , \tag{12.22}$$

with η some fixed positive number smaller than δ'. We have $|\Lambda_L \setminus \Lambda(\underline{\Gamma})| \leq 2K' L^{1+\delta'}$; hence

$$\Big| \langle \sum_{t\in\Lambda_L} \sigma(t) \,|\, E(\underline{\Gamma}) \rangle_L - \langle \sum_{t\in\Lambda(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \Big| \leq O(L^{1+\delta'}) . \tag{12.23}$$

We have

$$\langle \sum_{t\in\Lambda_L} \sigma(t) \,|\, E(\underline{\Gamma}) \rangle_L = m^* |\Lambda_L| (1 - 2\alpha(\underline{S})) \pm O(L^{1+\delta'}) . \tag{12.24}$$

Therefore

$$\Big| \langle \sum_{t\in\Lambda(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m|\Lambda_L| \Big| \leq \frac{1+2\eta}{L^c} |\Lambda_L| , \tag{12.25}$$

for L large enough. If

$$\Big| \langle \sum_{t\in\Lambda(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m|\Lambda_L| \Big| \leq \frac{1-\eta}{L^c} |\Lambda_L| , \tag{12.26}$$

then $\langle A(m;c) \,|\, E(\underline{\Gamma}) \rangle_L$ is very close to one (use Proposition 9.1). Suppose that

$$\Big| \langle \sum_{t\in\Lambda(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m|\Lambda_L| \Big| > \frac{1-\eta}{L^c} |\Lambda_L| . \tag{12.27}$$

To be specific we consider the case ($\epsilon \leq 3\eta$)

$$\langle \sum_{t\in\Lambda(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = m|\Lambda_L| + \frac{1-\eta+\varepsilon}{L^c}|\Lambda_L|\,. \tag{12.28}$$

In this case, the mean magnetization is too high in $\Lambda(\underline{\Gamma})$. Let Λ^+ be the component of $\Lambda(\underline{\Gamma})$ where the $\star$ boundary condition corresponds to $+$ boundary condition. We construct a region $\Delta \subset \Lambda^+$ of suitable volume and we impose zero magnetization inside Δ in order to reduce the total magnetization. First let us compute the volume of Δ. It is specified by the condition

$$\langle \sum_{t\in\Lambda(\underline{\Gamma})\backslash\Delta} \sigma(t) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = m|\Lambda_L|\,, \tag{12.29}$$

that is,

$$\begin{aligned}\langle \sum_{t\in\Lambda(\underline{\Gamma})} \sigma(t) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} &- \langle \sum_{t\in\Delta} \sigma(t) \rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \\ &= m|\Lambda_L| + \frac{1-\eta+\varepsilon}{L^c}|\Lambda| - |\Delta|m^* \\ &= m|\Lambda_L|\,,\end{aligned} \tag{12.30}$$

which implies that

$$|\Delta| = \frac{1-\eta+\varepsilon}{m^* L^c}|\Lambda|\,. \tag{12.31}$$

We construct Δ as a union of cubes which are translate of $B(0,[L^{\delta'}])$ so that all contours inside these boxes are small.
We introduce the grid $\mathcal{L}(\delta')$. The number of cells of $\mathcal{L}(\delta')$ which intersects some $B(S_i)$ is bounded by $O(L^{1-\delta'})$. The total number of cells of $\mathcal{L}(\delta')$ is $O(L^{2-2\delta'})$ so that it is always possible to find $O(L^{2-c-2\delta'})$ cells not intersecting any $B(S_i)$, for L large enough.
Let $0 < \delta'' < \delta'$. Inside each selected cells $\mathcal{B}_j$ there is in the center a translate $\mathcal{B}'_j$ of the box $B(0,[L^{\delta'} - 2L^{\delta''}])$. We define the event $\widetilde{A}$:

1. all contours which have a non–empty intersection with $\Lambda(\underline{\Gamma})\backslash\Delta$ or with at least two $\mathcal{B}_j$ are $L^{\delta''}$–small;

2. $$\Big| \sum_{t\in\Lambda(\underline{\Gamma})\backslash\Delta} \sigma(t) - m|\Lambda_L| \Big| \leq |\Lambda_L|/2L^c\,; \tag{12.32}$$

3. for each box $\mathcal{B}'_j$ we have

$$\Big| \sum_{t\in\mathcal{B}'_j} \sigma(t) \Big| \leq |\mathcal{B}'_j|/L^{c\delta'}\,. \tag{12.33}$$

By definition $\widetilde{A} \subset A(m;c)$. Therefore

$$\langle\, A(m;c)\,|\, E(\underline{\Gamma})\,\rangle_L \geq \langle\, \widetilde{A}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\,. \tag{12.34}$$

Let $\widetilde{A}_{1,2}$ be the event defined by conditions 1. and 2. only. Then

$$\langle\, \widetilde{A}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = \langle\, \widetilde{A} \,|\, \widetilde{A}_{1,2}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\,\langle\, \widetilde{A}_{1,2}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\,. \tag{12.35}$$

The first term $\langle\, \widetilde{A} \,|\, \widetilde{A}_{1,2}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}$ is estimated using Theorems 10.1 and 10.2. Denote by $\underline{\gamma}(\omega)$ all external contours in ω which have a non–empty intersection with $\Lambda(\underline{\Gamma})\backslash\Delta$ or with at least two $\mathcal{B}_j$, and by $\widetilde{A}_{1,2}(\underline{\gamma}')$ the set of $\omega \in \widetilde{A}_{1,2}$ such that $\underline{\gamma}(\omega) = \underline{\gamma}'$. Then

$$\langle\, \widetilde{A} \,|\, \widetilde{A}_{1,2}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = \sum_{\underline{\gamma}'} \langle\, \widetilde{A} \,|\, \widetilde{A}_{1,2}(\underline{\gamma}')\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\,\frac{\langle\, \widetilde{A}_{1,2}(\underline{\gamma}')\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}}{\langle\, \widetilde{A}_{1,2}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}}\,. \tag{12.36}$$

Under the condition $\widetilde{A}_{1,2}(\underline{\gamma}')$ the events inside each box $\mathcal{B}'_j$ become independent. Since the boxes $\mathcal{B}'_j$ are isometric to $B(0, L^{\delta'} - 2L^{\delta''})$ there is no condition on the contours inside these boxes. In each box we have a large deviation as in Theorems 10.1 and 10.2 with $m = 0$ and $\widetilde{L} = L^{\delta'} - 2L^{\delta''}$ instead of L. Therefore, applying these theorems with $\mathcal{C}$ a Wulff shape in the center of each $\mathcal{B}'_j$,

$$\begin{aligned}\langle\, \widetilde{A} \,|\, \widetilde{A}_{1,2}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} &\geq \exp\{-O(L^{\delta'})\cdot O(L^{2-c-2\delta'})\} \\ &\geq \exp\{-O(L^{2-c-\delta'})\}\,.\end{aligned} \tag{12.37}$$

Proposition 9.1 and Lemma 5.6 imply that $\lim_{L\to\infty} \langle\, \widetilde{A}_{1,2}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = 1$. Indeed, let $\chi(\delta')$ be the event that all contours are $L^{\delta'}$–small and $\chi(\delta'')$ the event that all contours are $L^{\delta''}$–small. Then

$$\begin{aligned}\langle\, \widetilde{A}_{1,2}\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} &\geq \langle\, \widetilde{A}_{1,2}\,\chi(\delta'')\,\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \\ &= \langle\, \widetilde{A}_{1,2} \,|\, \chi(\delta'')\,\rangle^{\star}_{\Lambda(\underline{\Gamma})}\,\frac{\langle\, \chi(\delta'')\,\rangle^{\star}_{\Lambda(\underline{\Gamma})}}{\langle\, \chi(\delta')\,\rangle^{\star}_{\Lambda(\underline{\Gamma})}}\,.\end{aligned} \tag{12.38}$$

Lemma 5.6 implies that the numerator and denominator of the quotient tend to 1 as $L \to \infty$; Proposition 9.1 implies that $\langle\, \widetilde{A}_{1,2} \,|\, \chi(\delta'')\,\rangle^{\star}_{\Lambda(\underline{\Gamma})}$ tends to 1 as $L \to \infty$. $\square$

We now conclude the proof of Theorem 12.1.

Recall that $\varepsilon(L) > L^{-\eta'}$ and $c = 1/4 - \delta > 0$; from Lemmas 12.2 and 12.3

$$\mathrm{Prob}[\, E_3 \,|\, E_1 \cap E_2\,] \geq 1 - \exp\{-O(L^{\kappa})\} \tag{12.39}$$

follows if we can find a such that $1 > a > 0$, δ' such that $0 < \delta' < \delta$ and $0 < \eta'$ so that the hypothesis of Theorem 12.1 is satisfied and

$$\kappa := 2a - \delta' - 4\eta' - 2 + c > 0\,, \tag{12.40}$$

that is

$$a > 1 - \frac{c}{2} + \frac{\delta'}{2} + 2\eta'\,, \tag{12.41}$$

which is always possible. The last affirmation

$$\mathrm{Prob}[\, E_3 \cap E_1 \cap E_2\,] \geq 1 - \exp\{-O(L^{\kappa})\} \tag{12.42}$$

is a consequence of (12.39) and Theorems 11.1 and 11.2. $\square$

12.1 Continuum limit

We consider the model in the box Λ_L and scale everything by $1/L$, so that after scaling the box is the rectangle Q. We define the set of macroscopic droplets at equilibrium as

$$\mathcal{D}(m) := \{\, \mathcal{V} \subset Q : |\mathcal{V}| = \frac{m^* - m}{2m^*}|Q| \,,\, \mathtt{W_t}(\partial\mathcal{V}) = \mathtt{W}^*_{\mathtt{t}}(m) \,\}\,. \tag{12.43}$$

For each $\mathcal{V} \in \mathcal{D}(m)$ we have a magnetization profile,

$$\rho_{\mathcal{V}}(x) := \begin{cases} m^* & \text{if } x \in Q\backslash\mathcal{V}, \\ -m^* & x \in \mathcal{V}. \end{cases} \tag{12.44}$$

Let f be a real–valued function on Q; we set

$$d_1(f, \mathcal{D}(m)) := \inf_{\mathcal{V}\in\mathcal{D}(m)} \int_Q dx\, |\, f(x) - \rho_{\mathcal{V}}(x)\,|\,. \tag{12.45}$$

For each ω we define a magnetization profile $\rho_L(x;\omega)$ on Q. We subdivide the box Λ_L by the cells of the grid $\mathcal{L}(a)$. In each cell C we define the empirical magnetization $m_C(\omega)$. Then

$$\rho_L(x;\omega) := m_C(\omega) \text{ if } Lx \in C \tag{12.46}$$

where Lx is the point $x \in Q$ scaled by L.

Let $\omega \in E_{1,2,3} := E_1 \cap E_2 \cap E_3$ and let $\mathcal{P}(\underline{S}) = \{\mathcal{P}(S_i)(\omega) : i = 1, \ldots, k\}$ be the polygonal lines defined by the configuration ω. Using these polygonal lines scaled by $1/L$ we define a set $\mathcal{V}(\underline{S}) \subset Q$ with the following properties (see Theorems 11.1 and 11.2)

1. The set $\mathcal{V}(\underline{S}) \supset \mathrm{Int}\underline{S}$ and its volume is such that

$$\Big|\, |\mathcal{V}(\underline{S})| - \frac{m^* - m}{2m^*}|Q| \,\Big| \leq \frac{1+\eta}{2m^* L^c}|Q|\,; \tag{12.47}$$

2. The boundary $\partial\mathcal{V}(\underline{S})$ of $\mathcal{V}(\underline{S})$ is such that $\partial\mathcal{V}(\underline{S}) \subset \cup_i \mathcal{P}(S_i)$ and

$$\mathtt{W_t}(\partial\mathcal{V}(\underline{S})) \leq \mathtt{W}^*_{\mathtt{t}}(m) + O(L^{\eta-\delta})\,. \tag{12.48}$$

The boundary of the set $\mathcal{V}(\underline{S})$ may have several connected components. We define an auxiliary set $\widehat{\mathcal{V}}(\underline{S})$ by translating some of these components so that $\widehat{\mathcal{V}}(\underline{S})$ has the same volume as $\mathcal{V}(\underline{S})$ and $\mathtt{W_t}(\partial\mathcal{V}(\underline{S})) = \mathtt{W_t}(\partial\widehat{\mathcal{V}}(\underline{S}))$. The boundary of $\widehat{\mathcal{V}}(\underline{S})$ can be parametrized by a single Lipschitz map $t \mapsto (u(t), v(t))$. We compare the set $\widehat{\mathcal{V}}(\underline{S})$ with the droplets in $\mathcal{D}(m)$. Given two sets $F \subset Q$ and $G \subset Q$ their distance is

$$d(F,G) := \max\{\sup_{s\in F} \inf_{t\in G} \|s-t\|\,,\, \sup_{t\in G} \inf_{s\in F} \|s-t\|\}\,, \tag{12.49}$$

with $\|s-t\|$ the Euclidean distance of $s,t \in Q$. The following lemma, inspired by Corollary 3.2 in [DP], shows that one component of $\widehat{\mathcal{V}}(\underline{S})$ is close to a droplet of $\mathcal{D}(m)$ and that the total volume of the other components is small.

Lemma 12.4 *Let $\varepsilon > 0$. There exists a function $\delta(\varepsilon)$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ such that if $V \subset Q$ has the properties*

1. *the boundary of V is parametrized by a unit–speed Lipschitz parametrization $t \mapsto (u(t), v(t))$,*
2. *the volume of V is larger than $|Q|(m^* - m)/2m^* - \varepsilon$,*
3. $\mathtt{W_t}(\partial V) \leq \mathtt{W}^*_\mathtt{t}(m) + \varepsilon$,

then

$$\inf_{\mathcal{V}\in\mathcal{D}(m)} d(\mathcal{V}, V) \leq \delta(\varepsilon)\,. \tag{12.50}$$

Proof. Suppose that there exists $\delta' > 0$ and V_n, $n \in \mathbb{N}$, such that $\varepsilon = 1/n$ and

$$\inf_{\mathcal{V}\in\mathcal{D}(m)} d(\mathcal{V}, V_n) \geq \delta'\,. \tag{12.51}$$

Let $t \mapsto (u_n(t), v_n(t))$ be the unit–speed Lipschitz parametrization of the boundary of V_n. We choose the parametrization in such way that

$$|V_n| = \frac{1}{2}\int_{\partial V_n} (v'_n\, u_n - u'_n\, v_n)\,. \tag{12.52}$$

By our hypothesis the length of the boundary ∂V_n is uniformly bounded, so that we can parametrize all boundaries by maps defined on a single interval $I \subset \mathbb{R}$ (we still denote the parametrizations by $(u_n(t), v_n(t))$). Since the parametrizations are Lipschitz with a Lipschitz constant bounded by one, the maps $t \mapsto (u_n(t), v_n(t))$ are equicontinuous. By Ascoli's Theorem we can extract a uniformly convergent subsequence so that $(u^*(t), v^*(t)) = \lim_k (u_{n_k}(t), v_{n_k}(t))$ is the boundary of a set V^* with volume

$$|V^*| = \lim_{k\to\infty} |V_{n_k}| \geq |Q|(m^* - m)/2m^*\,. \tag{12.53}$$

By the uniform convergence of the sequence we have

$$\liminf_{k\to\infty} \Big[\widehat{\tau}_{\mathrm{bd}} - \widehat{\tau}((1,0))\Big]\,|\partial V_{n_k} \cap w_Q| \geq \Big[\widehat{\tau}_{\mathrm{bd}} - \widehat{\tau}((1,0))\Big]\,|\partial V^* \cap w_Q|\,, \tag{12.54}$$

since $\Big[\widehat{\tau}_{\mathrm{bd}} - \widehat{\tau}((1,0))\Big] \leq 0$. A classical theorem (see e.g. [Da] chapter 3) gives

$$\liminf_{k\to\infty} \int_{\partial V_{n_k}} \widehat{\tau}(n(s))ds \geq \int_{\partial V^*} \widehat{\tau}(n(s))ds\,, \tag{12.55}$$

since $\widehat{\tau}$ is convex. Therefore

$$\mathtt{W_t}(\partial V^*) \leq \lim_{k\to\infty} \mathtt{W_t}(\partial V_{n_k}) \leq \mathtt{W}^*_\mathtt{t}(m)\,, \tag{12.56}$$

and $V^* \in \mathcal{D}(m)$, which contradicts the existence of δ'. □

Corollary 12.1 *Under the hypothesis of Lemma 12.4, if ε is small enough, then one connected component of V is at distance at most $\delta(\varepsilon)$ from a droplet of $\mathcal{D}(m)$ and the total volume of the remaining components is at most $O(\delta(\varepsilon))$.*

Theorem 12.2 *Let $\beta > \beta_c$, $h \in \mathbb{R}$, $-m^* < m < m^*$ and $c = 1/4 - \delta > 0$. Let $\langle \cdot \,|\, m \rangle_L(\beta, h)$ be the canonical Gibbs state. Then there exists a positive function $\overline{\varepsilon}(L)$, $\lim_{L\to\infty} \overline{\varepsilon}(L) = 0$, and $\kappa > 0$ (see 9.34) such that for L large enough*

$$\mathrm{Prob}[\, \{\, d_1(\rho_L(\,\cdot\,;\omega), \mathcal{D}(m)) \le \overline{\varepsilon}(L) \,\}\,] \ge 1 - \exp\{-O(L^\kappa)\}\,. \tag{12.57}$$

Proof. Let $\omega \in E_{1,2,3}$ and let $\mathcal{P}(\underline{S}) = \{\mathcal{P}(S_i)(\omega) : i = 1, \ldots, k\}$ be the polygonal lines defined by the configuration ω. We define $\mathcal{V}(\underline{S}) \subset Q$ with properties (12.47) and (12.48) as above and set

$$\rho_L(x; \underline{S}) := \begin{cases} m^* & \text{if } x \in Q \backslash \mathcal{V}(\underline{S}), \\ -m^* & x \in \mathcal{V}(\underline{S}). \end{cases} \tag{12.58}$$

There exist two positive numbers μ and η'' (see Lemma 12.1),

$$\mu + \eta'' < 1 - a\,, \tag{12.59}$$

such that, if $\omega \in E_{1,2,3}$ and $\mathcal{P}(\underline{S})(\omega) = \mathcal{P}(\underline{S})$, then uniformly in $\omega \in E_{1,2,3}$

$$\int_Q dx\, |\, \rho_L(x;\omega) - \rho_L(x;\underline{S})\, | \le O(L^{a-1+\mu+\eta''}) + \varepsilon(L)|Q| + O(L^{a-1})\,. \tag{12.60}$$

The first term on the right hand side is the contribution coming from the polluted cells, the second term from the phase–cells and the last one from the interface–cells and boundary–cells. We define

$$d_1(L) := \sup_{\omega \in E_{1,2,3}} d_1(\rho_L(\,\cdot\,; \underline{S}(\omega)), \mathcal{D}(m))\,. \tag{12.61}$$

Then Lemma 12.4 and Corollary 12.1 imply that $\lim_{L\to\infty} d_1(L) = 0$. Theorem 12.2 follows by choosing

$$\overline{\varepsilon}(L) := O(L^{a-1+\mu+\eta''}) + \varepsilon(L)|Q| + O(L^{a-1}) + d_1(L)\,. \tag{12.62}$$

$\square$

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