Ergodic Properties of Infinite Harmonic Crystals: an Analytic Approach

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Abstract

We prove that the quantum dynamics of a class of infinite harmonic crystals becomes ergodic and mixing in the following sense: if H_m is the m-particle Schrödinger operator, $\omega_{\beta,m}(A) = \mathrm{Tr}\,(A \exp -\beta H_m)/\mathrm{Tr}\,(\exp -\beta H_m)$ the corresponding quantum Gibbs distribution over the observables A, B, $\psi_{m,\lambda}$ the coherent states in the m-th particle Hilbert space, $g_{m,\lambda} = (\exp -\beta H_m)\psi_{m,\lambda}$ then

$$\lim_{t\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{T}\int_0^T \langle e^{iH_nt}Ae^{-iH_nt}\psi_{m,\lambda},\psi_{m,\lambda}\rangle\,dt = \lim_{m\to\infty}\omega_{\beta,m}(A)$$

if the classical infinite dynamics is ergodic, and

$$\lim_{t\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\omega_{\beta,m}(e^{iH_nt}Ae^{-iH_nt}B) = \lim_{m\to\infty}\omega_{\beta,m}(A)\cdot\lim_{m\to\infty}\omega_{\beta,m}(B)$$

if it is in addition mixing. The classical ergodicity and mixing properties are recovered as $\hbar \to 0$, and $\lim_{m\to\infty}\omega_{\beta,m}(A)$ turns out to be the average over a classical Gibbs measure of the symbol generating A under Weyl quantization.

1 Introduction

This paper deals with the ergodic theory of a class of infinite quantum systems, the harmonic crystals. In this introduction we review the relevance of the infinitely many particle limit in detecting chaotic behaviour of quantum systems, state the results and motivate why to our opinion is convenient to examine the problem via pseudodifferential operators.

Let H be the quantization of a Hamiltonian generating a flow S_t on a constant energy manifold $M_E \subset R^m$, $A, B \in \mathcal{L}(\mathcal{H})$ any suitable quantum observable in $\mathcal{H} = L^2(\mathbb{R}^m)$, and let $\sigma(H)$ be discrete and simple, with projections P_n on the eigenvectors $\{u_n : n = 0, 1, \ldots\}$. If quantum chaotic behaviour (if any) is to be characterized in terms of ergodicity and mixing, we have to consider the quantum microcanonical ensemble at energy E, i.e. the application $\omega_{\Delta,E}$ mapping any $A \in \mathcal{L}(\mathcal{H})$ into:

$$\omega_{\Delta,E}(A) = \frac{\mathrm{Tr}\, A \sum_{n:E-\Delta<E_n<E} P_n}{\mathrm{Tr}\, \sum_{n:E-\Delta<E_n<E} P_n} \equiv \frac{\mathrm{Tr}\, A\delta(H-E)}{\mathrm{Tr}\, \delta(H-E)} \tag{1.1}$$

(see [Ru] §1.3; $\Delta > 0$ is arbitrarily small). The quantum evolution $A_H(t) = e^{iHt}Ae^{-iHt}$ of A leaves $\omega_{\Delta,E}(A)$ invariant. Hence the consequent definition of mixing is (see Appendix 2 for details)

$$\lim_{t\to\infty} \omega_{\Delta,E}(A_H(t)B) = \omega_{\Delta,E}(A) \cdot \omega_{\Delta,E}(B) \tag{1.2}$$

We can always find in $\mathcal{H}$ (see Appendix 2 below for the easy verification) a family of normalized vectors $(\psi_\lambda)_{\lambda\in\Lambda}, \Lambda = \mathbb{R}^{2m}$ complete for $\omega_{\Delta,E}$, namely

$$\omega_{\Delta,E}(A) = \int_\Lambda \langle A\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}}\, d\nu_{\Delta,E}(\lambda), \qquad \forall\, A \in \mathcal{L}(\mathcal{H}) \tag{1.3}$$

for a well determined probability measure $\nu_{\Delta,E}(\lambda)$ on Λ.
Then (1.2) becomes

$$\int_\Lambda \langle A_H(t)B\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}} d\nu_{\Delta,E}(\lambda) \to \int_\Lambda \langle A\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}} d\nu_{\Delta,E}(\lambda) \int_\Lambda \langle B\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}} d\nu_{\Delta,E}(\lambda) \tag{1.4}$$

as $|t| \to \infty$. This entails the following representation of the quantum ergodicity notion (see again Appendix 2): for any $A \in \mathcal{L}(\mathcal{H})$ and for $d\nu$-almost all $\lambda \in \mathbb{R}^{2m}$,

$$\frac{1}{T}\int_0^T \langle A_H(t)\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}} dt \to \int_\Lambda \langle A\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}} d\nu_{\Delta,E}(\lambda) = \omega_{\Delta,E}(A) \quad \text{as} \quad |T| \to \infty. \tag{1.5}$$

On the other hand it is well known (and easy to verify) that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle \psi, A_H(t)\psi\rangle\, dt = \sum_{n=0}^\infty |\lambda_n|^2 \langle u_n, Au_n\rangle \tag{1.6}$$

Here $\psi = \sum_{n=0}^{\infty} \lambda_n u_n$ is any normalized quantum state expanded on the eigenvector basis (u_n). (1.6) is the Von Neumannn definition of quantum ergodicity[VN] on the microcanonical ensemble. Now the verification of (1.5) requires H to have continuous spectrum ([Ru], §1.3), and (1.6) shows that the time average cannot eliminate the dependence on the initial datum $\psi \equiv \{\lambda_n\}_{n=0}^{\infty}$.

This *a priori* lacking of ergodicity, and *a fortiori* of mixing, looks as a manifestation of the so called "quantum suppression of classical chaos", which however can disappear when the number of particles tends to infinity. This has been remarked in different contexts and within different approaches in [Ch], [JLPC], [JL], [Be]. Hence the quantum counterparts of chaotic systems with infinitely many degrees of freedom (for a recent review see [Be]) are the best candidates to look for chaotic behaviour. The simplest one is the infinite linear harmonic system

$$\ddot{q}_i = -2 \sum_{i,j \in \mathbb{Z}} V_{ij} q_j \tag{1.7}$$

We prove that, when the couplings V_{ij} generate an infinite dimensional dynamics ϕ_t ergodic *with respect to the (infinite dimensional) Gibbs measure* $d\mu_G(\beta)$ [LL, Ti, VH], the quantum evolution is ergodic, and mixing if ϕ_t is in addition mixing. The averages are now to be computed on the quantum canonical ensemble (Gibbs state at inverse temperature β), i.e., the application ω_β mapping any $A \in \mathcal{L}(\mathcal{H})$ into $\omega_\beta(A) = \dfrac{\operatorname{Tr} JAe^{-\beta H}}{\operatorname{Tr} e^{-\beta H}}$. More precisely, denote:

$$q_m(x,\xi) = \frac{1}{2}|\xi|^2 + \langle V_m x, x\rangle, \qquad V_m = (V_{i,j})_{\substack{|i|\le m \\ |j|\le m}} \tag{1.8}$$

the $(2m+1)$ dimensional Hamiltonian defined on $\Lambda_m = (\mathbb{R}^{2m+1})^2$; $H_m = Op^W(q_m)$ the operator on $L^2_m \equiv L^2(\mathbb{R}^{2m+1})$ defined by its Weyl quantization, $A = Op^W(a)$ the operator on L^2_m quantizing $a \circ \Pi_{m_1}(x,\xi)$ (m_1 fixed) where a is any smooth classical observable on Λ_{m_1} and $\Pi_{m_1}(x,\xi) \equiv (x,\xi)_{|i|\le m_1}$.

Then the present results are (see Theorems 2.2, 2.3 and Proposition 5.1 for a sharper version): $\forall \beta > 0$

$$\lim_{T\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} \frac{1}{T}\int_0^T \langle A_n(t)\psi_{\lambda,m}, \psi_{\lambda,m}\rangle_{L^2_m}\, dt = \lim_{m\to\infty}\int_{\Lambda_\infty} \langle A\psi_{\lambda,m}, \psi_{\lambda,m}\rangle_{L^2_m} d\nu_m(\lambda) \tag{1.9}$$

for ν-almost any λ, and

$$\lim_{t\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} \omega_{\beta,m}(A_n(t)B) = \lim_{m\to\infty}\omega_{\beta,m}(A)\cdot \lim_{m\to\infty}\omega_{\beta,m}(B) \tag{1.10}$$

Here:

$$\omega_{\beta,m}(A) = \frac{\operatorname{Tr} Ae^{-\beta H_m}}{\operatorname{Tr} e^{-\beta H_m}}; \quad \omega_{\beta,m}(A_n(t)B) = \frac{\operatorname{Tr} A_n(t)Be^{-\beta H_m}}{\operatorname{Tr} e^{-\beta H_m}}; \tag{1.11}$$

$A_n(t)$ is the Heisenberg observable corresponding to A under the quantum evolution of H_n;

$$\psi_{\lambda,m} = \frac{\exp(-\beta H_m/2) f_{\lambda,m}}{\| \exp(-\beta H_m/2) f_{\lambda,m}\|} \quad \nu_m(\lambda) = \frac{\| \exp(-\beta H_m/2) f_{\lambda,m}\|^2 \, d\lambda}{\int_{\Lambda_m} \| \exp(-\beta H_m/2) f_{\lambda,m}\|^2 \, d\lambda}$$

$f_{\lambda,m}$ being the Bargmann coherent states (a set of vectors in $L^2(\mathbb{R}^{2m+1})$ indexed by $\lambda \in \Lambda_m$ whose definition is recalled in Appendix 2); $\nu(\lambda) = \lim_{m\to\infty} \nu_m(\lambda)$.

Remark 1. The mixing property with respect to the KMS states in the CCR algebra of the infinite harmonic crystal (which has the same W^* closure of the pseudifferential algebra we use) is proved in [Be], Example 4.46, through the asymptotic abelianess of the Weyl algebra automorphism generated by the dynamics of the infinite crystal, when $\sigma(V)$ is purely absolutely continuous so that classical mixing holds [LL]. The asymptotic abelianess may however fail if $\sigma(V)$ is only continuous and the classical system is only ergodic. Hence the ergodicity result (1.9) requires in general an independent proof.

Remark 2. The main reason why, to our opinion, an "analytic" proof, based on pseudodifferential calculus, is in any case useful is that the notion (1.4) is proved to have the expected classical limit (Appendix 2).

Additional reasons are the following:

1. One finds the r.h.sides of (1.9),(1.10) to be the relevant classical averages:

$$\lim_{m\to\infty} \int_{\Lambda_\infty} \langle A_{m_1} \psi_{\lambda,m}, \psi_{\lambda,m}\rangle \, d\nu_m(\lambda) = \int_{\Lambda_\infty} a \circ \Pi_{m_1} \, d\hat{\mu}_\beta \tag{1.12}$$

$$\lim_{m\to\infty} \frac{\mathrm{Tr}\, A_{m_1} e^{-\beta H_m}}{\mathrm{Tr}\, e^{-\beta H_m}} \cdot \frac{\mathrm{Tr}\, B_{m_1} e^{-\beta H_m}}{\mathrm{Tr}\, e^{-\beta H_m}} = \int_{\Lambda_\infty} a \circ \Pi_{m_1} \, d\hat{\mu}_\beta \cdot \int_{\Lambda_\infty} b \circ \Pi_{m_1} \, d\hat{\mu}_\beta \tag{1.13}$$

   Here $\hat{\mu}_\beta = \lim_{m\to\infty} \hat{\mu}_{\beta,m}$, where $\hat{\mu}_{\beta,m}$ is the (explicitly constructed) Gibbs measure on Λ whose Weyl quantization yields $e^{-\beta H_m}$. It turns out that $\hat{\mu}_\beta$ depends on $\hbar$ and reduces to $\mu_G(\beta)$ as $\hbar \to 0$, because $e^{-\beta q_m}$ is just the principal symbol of $e^{-\beta H_m}$ realized as a pseudodifferential operator.

2. If the initial states $\psi_{m,\lambda}$ belong to an explicitly constructed set (the image under $e^{-\beta H_m}$ of "almost all" coherent states on Λ_m), the $m \to \infty$ limit can actually eliminate the dependence of the r.h.s. (1.9) on the particular state in the set.

3. Unlike the algebraic proof, the analytic one can be in principle extended to systems quantizing non-linear classical equations. Work in this direction is in progress: it can be proved [GJLM] that in some non linear cases the above results are still true in the sense of the formal power series in $\hbar$.

We conclude this introduction with the remark that the dynamical mechanism generating chaotic behaviour, in the classical case and in the quantum one as well, is but free propagation of the chaotic initial condition: the infinite harmonic crystal goes indeed over (when the spacing goes to zero, and for special choices of V) to the free wave equation (equivalently, there exist coordinates in which the particle motions are free) and the chaotic initial condition is selected by the invariant Gibbs measure. This situation is referred to as kinematic chaos [JLP].

The paper is organized as follows: in the next Section we state assumptions and results, after a brief recall of the infinite dimensional classical harmonic dynamics; in §3 and in §4 we prove the quantum ergodicity and the quantum mixing, respectively, in the most general formulation. In §5 and §6 we prove a sharper formulation of the above results when $\exp -\beta H_m$ is replaced by $Op^W(\exp -\beta q_m)$ and the family of vectors in L^2_m is specialized to the coherent states. Appendix 1 contains the proof of some technical lemmas, and Appendix 2 contains the discussion of our results in the light of the existing notions of quantum ergodicity and mixing, together with the verification that they have the expected classical limit.

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2 Assumptions and Statement of the Results

In the notation of [LL], to which we refer the reader for any further detail on the system of infinitely many oscillators, let $V = (V_{i,j})_{i,j\in\mathbb{Z}}$ be an infinite real-symmetric matrix; q_m and V_m are as in (1.8) and $\Lambda_m = (\mathbb{R}^{2m+1})^2$.

We write $S_m(1)$ for the set of C^∞ functions on Λ_m which are bounded together with all their derivatives, and for $a \in S_m(1)$ we denote $Op^W(a)$ the Weyl quantization (with $\hbar = 1$) of the symbol (equivalently, classical observable) a, explicitly given by the oscillatory integral :

$$Op^W(a)u(x) = (2\pi)^{-(2m+1)} \int_{\Lambda_m} e^{i\langle (x-y),\xi\rangle} a(\frac{x+y}{2}, \xi) u(y)\, dy\, d\xi \tag{2.1}$$

for all $u \in \mathcal{S}(\mathbb{R}^{2m+1})$. In particular the Schrödinger operator H_m on $L^2(\mathbb{R}^{2m+1})$

$$H_m := Op^W(q_m) = \frac{1}{2}(\sum_{j=1}^{2m+1} D^2_{x_j}) + \langle V_m x, x\rangle, \qquad D_{x_j} = -i\frac{\partial}{\partial x_j} \tag{2.2}$$

quantizes the Hamiltonian q_m describing m oscillators coupled through V_m.

We assume from now on

(H1) $$|V_{ij}| = \mathcal{O}(|i-j|^{-\infty}), \qquad |i-j| \to +\infty$$

and $\exists\, 0 < \varepsilon < M < \infty$ such that $\forall m \geq 0$, $\sigma(V_m) \subset [\varepsilon, M]$.

In particular, $V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is bounded and strictly positive, with $\sigma(V) \subset [\varepsilon, M]$.

(H2) The operator V acting on $\ell^2(\mathbb{Z})$ has no point spectrum.

Denote

$$\Lambda_\infty := \bigcup_{k\in\mathbb{N}} \Big\{(x_j,\xi_j)\}; \qquad |x_j| + |\xi_j| = \mathcal{O}(|j|^k),\ |j| \to \infty\Big\} := \bigcup_{k\in\mathbb{N}} \mathcal{H}_k \tag{2.3}$$

It is proved in [LL] that under condition (H1), Λ_∞ is invariant under the classical evolution of infinitely many degrees of freedom defined as follows

$$\phi_t(x,\xi) \equiv \phi(t,x,\xi) = e^{tB}(x,\xi), \qquad \forall (x,\xi) \in \Lambda_\infty, \forall t \in \mathbb{R} \tag{2.4}$$

where $B(x,\xi)$ is the infinite-dimensional Hamiltonian vector field generated by q_m when $m \to \infty$

$$B(x,\xi) = \left(\xi_j, -2\sum_{k\in\mathbb{Z}} V_{jk} x_k\right)_{j\in\mathbb{Z}} \tag{2.5}$$

Moreover, if $\Pi_m : \Lambda_\infty \to \Lambda_m$ denotes the projection

$$\Pi_m(x,\xi) = (x_j,\xi_j)_{|j|\leq m} \tag{2.6}$$

for any $(x,\xi) \in \mathcal{H}_k$ one has

$$\phi(t,x,\xi) = \lim_{m\to\infty} \phi_{m,t}(\Pi_m(x,\xi)) \in \mathcal{H}_k \tag{2.7}$$

where $\phi_{m,t} = \exp tH_{q_m}$, $H_{q_m} = \left(\dfrac{\partial q_m}{\partial \xi}, -\dfrac{\partial q_m}{\partial x}\right)$ is the vector field generated by q_m, and the limit is taken with respect to the natural Banach space topology of $\mathcal{H}_k$.

Now by (H1) the operator $V^{-\frac{1}{2}}$ exists and is continuous on $\ell^2(\mathbb{Z})$. This assumption and (H2) allow Lanford and Lebowitz[LL] to prove the existence of the infinite dimensional, ergodic Gibbs measure $d\mu_G(\beta)$ on Λ_∞, namely

1.
$$\int_{\Lambda_\infty} \varphi \circ \Pi_{m_1}\, d\mu_G(\beta) = \lim_{m\to\infty} \int_{\Lambda_m} \varphi \circ \Pi_{m_1}(x,\xi) e^{-\beta q_m(x,\xi)} \frac{dx\, d\xi}{Z_m}, \quad \forall \varphi \in C_b^0(\mathbb{R}^{2m_1+1}) \tag{2.8}$$

where

$$Z_m(\beta) = \int_{\Lambda_m} e^{-\beta q_m(x,\xi)}\, dx\, d\xi \tag{2.9}$$

is the m-particle partition function;

2. The Gibbs measure is invariant and ergodic with respect to the flow $\phi(t; x, \xi)$, namely the continuous dynamical system $(\Lambda_\infty, \phi_t, d\mu_G(\beta))$ is ergodic.

An example of an infinite matrix satisfying (H1)-(H2) is given by $V = W$ where

$$W_{ij} = 0, \qquad |i-j| \geq 2, \qquad W_{ii} = 1, \qquad W_{i,i+1} = W_{i,i-1} = \alpha \tag{2.10}$$

with $|\alpha| < \dfrac{1}{2}$, $\alpha \in \mathbb{R}$. The properties (H1), (H2) are proved e.g. in [Sj].

To state our result we need to establish some further notation. For $f \in L^2(\mathbb{R}^{2m+1})$ and $(x, \xi) \in \Lambda_m$, we introduce the Wigner function of f

$$w_f(x, \xi) = \int_{\mathbb{R}^{2m+1}} e^{iu\xi} f(x - \frac{u}{2}) \overline{f(x + \frac{u}{2})}\, du \tag{2.11}$$

and we restrict our attention to a random set of states f in the following sense : for all $m \in \mathbb{N}$, we consider a measure space (X_m, θ_m) with positive measure θ_m, and a family $(f_\lambda)_{\lambda \in X_m}$ of functions in $L^2(\mathbb{R}^{2m+1})$ such that :

(H3) For $dxd\xi$-almost all $(x, \xi) \in \Lambda_m$, the application $X_m \ni \lambda \mapsto w_{f_\lambda}(x, \xi)$ is in $L^1(X_m, d\theta_m)$ with non negative values, and the quantity $\int_{X_m} w_{f_\lambda}(x, \xi) d\theta_m(\lambda)$ is ($dxd\xi$-almost everywhere) constant with respect to (x, ξ).

Here we can notice that, at least formally, (H3) is implied by the property (to be compared with (8.10)):

$$\mathrm{Tr}(A) = \int \langle A f_\lambda, f_\lambda \rangle d\theta_m(\lambda)$$

for any trace-class operator A. Indeed we have $w_{f_\lambda}(x, \xi) = \langle A_{x,\xi} f_\lambda, f_\lambda \rangle$ with $A_{x,\xi} f(y) = e^{2i(y-x)\xi} f(2x - y)$, which actually is not trace-class, but whose distributional kernel $K_{x,\xi}(y, y') = e^{2i(y-x)\xi} \delta(y' + y = 2x)$ formally satisfies: $\int K_{x,\xi}(y, y) dy = 1$.

In the last section we develop an example (the so-called coherent states) where (H3) is satisfied. Note that in any case, $w_f(x, \xi)$ is real and satisfies : $\int w_f(x, \xi) dx d\xi = (2\pi)^{2m+1} \|f\|^2$.

As we shall see, (H3) implies among other things that

$$\int_{X_m} \|e^{-\beta H_m/2} f_\lambda\|^2 d\theta_m(\lambda) < +\infty \tag{2.12}$$

so that we can consider the following probability measure on X_m:

$$d\nu_m(\lambda) = \frac{\|e^{-\beta H_m/2} f_\lambda\|^2 d\theta_m(\lambda)}{\int_{X_m} \|e^{-\beta H_m/2} f_\lambda\|^2 d\theta_m(\lambda)}. \tag{2.13}$$

Now let

$$W_\beta = \sqrt{2}V^{-\frac{1}{2}}\tanh\frac{\beta V^{\frac{1}{2}}}{\sqrt{2}} \tag{2.14}$$

(which is well defined on $\ell^2(\mathbb{Z})$), and for $\beta > 0$, denote $\hat{\mu}_\beta$ the Gaussian probability measure on Λ_∞ with mean zero and covariance given by:

$$\begin{aligned} E[x_i x_j] &= \langle (2VW_\beta)^{-1} e_i, e_j \rangle_{\ell^2(\mathbb{Z})} \\ E[\xi_i \xi_j] &= \langle W_\beta^{-1} e_i, e_j \rangle_{\ell^2(\mathbb{Z})} \\ E[x_i \xi_j] &= 0 \end{aligned} \tag{2.15}$$

where $e_i = (\delta_{ij})_{j\in\mathbb{Z}}$. Then our first main result is :

Theorem 2.1 *Assume* (H1)-(H3). *Then :*

(i) For any $\beta > 0$, the dynamical system $(\Lambda_\infty, \phi_t, \hat{\mu}_\beta)$ is ergodic ;

(ii) For $m_1 \in \mathbb{N}$ fixed and $a \in S_{m_1}(1)$, denote

$$g_{m,\beta,\lambda} = \frac{e^{-\frac{1}{2}\beta H_m} f_\lambda}{\| e^{-\frac{1}{2}\beta H_m} f_\lambda \|}$$

and

$$A(m,n,T,\lambda) = \frac{1}{T}\int_0^T \left\langle e^{itH_n} Op^W(a\circ \Pi_{m_1}) e^{-itH_n} g_{m,\beta,\lambda}\, ,\, g_{m,\beta,\lambda} \right\rangle_{L^2(\mathbb{R}^{2m+1})} dt.$$

Then one has

$$\lim_{T\to\infty} \limsup_{n\to\infty} \limsup_{m\to\infty} \int_{X_m} \left| A(m,n,T,\lambda) - \int_{\Lambda_\infty} a\circ\Pi_{m_1}\, d\hat{\mu}_\beta \right| d\nu_m(\lambda) = 0.$$

Remarks.

1. $A(m,n,T,\lambda)$ can be made arbitrarily close to $\int a\circ\Pi_{m_1}\, d\hat{\mu}_\beta$ in $L^1(X_m, d\nu_m(\lambda))$ by first choosing T, then $n = n(T)$, and finally $m = m(n,T)$ large enough. The *pointwise* convergence of $A(m,n,T,\lambda)$ is proved in Proposition 5.1 below, choosing for f_λ a particular set of coherent states.

2. Note that $A(m,n,T,\lambda)$ is well defined since the action of $e^{itH_n}Op^W(a\circ\Pi_{m_1})e^{-itH_n}$ on $g_{m,\beta,\lambda}$ which is a C^∞ function on $\mathbb{R}^{2m+1}$ is well defined for $n \le m$.

3. For β small, we have $W_\beta = \beta I + \mathcal{O}(\beta^3)$ and therefore the covariance of $\hat{\mu}_\beta$ coincides with the one of the usual Gibbs measure $\mu_G(\beta)$ up to a $\mathcal{O}(\beta^3)$-error term. In this sense, we can say that $\hat{\mu}_\beta$ and $\mu_G(\beta)$ are asymptotically equal for small β's (that is for large temperatures).

4. The measure $\hat{\mu}_\beta$ can be seen as the limit when $m \to +\infty$ of the probability measure on Λ_m obtained by normalizing $e^{-q_{\beta,m}(x,\xi)}dxd\xi$, where

$$q_{\beta,m}(x,\xi) = q_m(W_{\beta,m}^{1/2}x, W_{\beta,m}^{1/2}\xi) \tag{2.16}$$

and

$$W_{\beta,m} = \sqrt{2}V_m^{-\frac{1}{2}}\tanh\frac{\beta V_m^{\frac{1}{2}}}{\sqrt{2}}. \tag{2.17}$$

(In fact, one can prove that (H1) and the spectral theorem imply that for any continuous function f on $\mathbb{R}$, $\langle f(V_m)e_i, e_j\rangle$ tends to $\langle f(V)e_i, e_j\rangle$ as $m \to \infty$.)

This relation between $\hat{\mu}_\beta$ and the usual Gibbs measure reflects the relation between the usual quantum Gibbs measure $e^{-\beta H_m}$ and the Weyl quantization of the classical Gibbs measure $e^{-\beta q_m}$, namely (see Lemma 3.1 below):

$$e^{-\beta H_m} = C_{\beta,m}Op^W(e^{-q_{\beta,m}})$$

where $C_{\beta,m}$ is a constant. In particular, if we denote $\#$ the Weyl composition of symbols, we get (with some other constant $C'_{\beta,m}$):

$$e^{-q_{\beta,m}}\#e^{-q_{\beta,m}} = C'_{\beta,m}e^{-q_{2\beta,m}} \tag{2.18}$$

which also explains the fact that $\hat{\mu}_\beta$ appears in the result given the above choice of $g_{m,\beta,\lambda}$, dictated by the standard requirement $\operatorname{Tr} e^{-\beta H_m} < +\infty$.

To state the mixing property we need two additional assumptions

(H4) The spectrum of V on $\ell^2(\mathbb{Z})$ is absolutely continuous.

(H5) The matrix $W_{\beta,m} = \sqrt{2}V_m^{-\frac{1}{2}}\tanh\frac{\beta V_m^{\frac{1}{2}}}{\sqrt{2}}$ satisfies:

$$(W_{\beta,m})_{i,j} = \mathcal{O}(|i-j|^{-\infty})$$

uniformly with respect to m, i and j.

Note that (H5) is satisfied e.g. for V of the form $V = I + \alpha J$ where J admits only a finite number of non-zero diagonals and $\alpha \in \mathbb{R}$ is chosen small enough. In particular, the example given in (2.10) satisfies (H1) and (H4)-(H5) if $|\alpha|$ is small enough. Note also that the absolute continuity of $\sigma(V)$ implies ([LL]) that the continuous dynamical system $(\Lambda_\infty, \phi_t, \mu_G)$ enjoys the mixing property. For $m_1 \in \mathbb{N}$ and $a \in S_{m_1}(1)$, we denote $\hat{a} \in \mathcal{S}'(\Lambda_{m_1})$ the usual Fourier transform of a formally given by the integral:

$$\hat{a}(x^*, \xi^*) = \int_{\Lambda_{m_1}} e^{-i\langle (x,\xi),(x^*,\xi^*)\rangle}a(x,\xi)dxd\xi. \tag{2.19}$$

Then the result is:

Theorem 2.2 *Assume* (H1) *and* (H4)-(H5). *For $m_1 \in \mathbb{N}$ fixed, $a, b \in S_{m_1}(1)$, and $n \geq m_1$ denote*

$$A = Op^W(a \circ \Pi_{m_1})$$
$$A_n(t) = e^{itH_n} A e^{-itH_n} = Op^W(a \circ \Pi_{m_1} \circ \phi_{n,t}) := Op^W(a_{n,t})$$
$$B = Op^W(b \circ \Pi_{m_1}).$$

Then we have

$$\lim_{m\to\infty} \frac{\mathrm{Tr}(Ae^{-\beta H_m})}{\mathrm{Tr}(e^{-\beta H_m})} = \int_{\Lambda_\infty} a \circ \Pi_{m_1}\, d\hat{\mu}_\beta := \omega_\beta(A) \tag{2.20}$$

and if moreover $\hat{a}$ and $\hat{b}$ are bounded measures on Λ_{m_1}, one has:

$$\lim_{t\to\infty} \lim_{n\to\infty} \omega_\beta(A_n(t)B) = \omega_\beta(A) \cdot \omega_\beta(B) \tag{2.21}$$

Remark. Although this corresponds to the notion of quantum mixing already existing in the framework of W^* dynamical systems, our procedure permits us to completely avoid to realize any algebra of operators on an infinite dimensional space.

Under an additional assumption on V the results of Theorems 2.1 and 2.2 admit a less cumbersome formulation which eliminates the necessity of the double limit with respect to m and n. The further assumption is :

(H6) For all $m \geq 0$ there exists a $(2m+1) \times (2m+1)$ real-symmetric matrix $\tilde{V}_m$ satisfying the same assumption (H1) as V_m, and such that :
(i) $\forall i, j \in \mathbb{Z}$, $\langle \tilde{V}_m^{-1} e_i, e_j \rangle$ tends to $\langle V^{-1} e_i, e_j \rangle_{\ell^2}$ as $m \to +\infty$;
(ii) $\forall m, n \in \mathbb{Z}$, the operator $\Pi_m \tilde{V}_n^{\frac{1}{2}} \Pi_n$ becomes independent of n for n sufficiently large ;
(iii) The operator $\tilde{V}_m^{-\frac{1}{2}} \Pi_m \tilde{V}_n^{\frac{1}{2}} \Pi_n$ tends strongly to the identity on each $\mathcal{H}_k$ ($k \in \mathbb{N}$) as $m \to +\infty$.

It is not very difficult to verify that an example of such V satisfying (H4) (in addition to (H1)-(H2)) is given by $V = W^2$ where W is as in the example (2.10) : in this case one can take $\tilde{V}_m = (W_m)^2$ where W_m is extracted from W as in (1.8).

Under assumption (H6), we define for $0 < \beta < (2M)^{-1/2}$:

$$\tilde{q}_{\beta,m}(x,\xi) = \langle F_{\beta,m}\xi, \xi \rangle + \langle G_{\beta,m} x, x \rangle \tag{2.22}$$

where (denoting I_m the identity on $\mathbb{R}^{2m+1}$):

$$F_{\beta,m} = \frac{1}{2\beta} \tilde{V}_m^{-1} \left(I_m - (I_m - 2\beta^2 \tilde{V}_m)^{\frac{1}{2}} \right) \tag{2.23}$$

$$G_{\beta,m} = 2\tilde{V}_m F_{\beta,m}. \tag{2.24}$$

We also consider on X_m the probability measure:

$$d\tilde{\nu}_m(\lambda) = \frac{\|Op^W(e^{-\tilde{q}_{\beta,m}})f_\lambda\|^2 d\theta_m(\lambda)}{\int_{X_m} \|Op^W(e^{-\tilde{q}_{\beta,m}})f_\lambda\|^2 d\theta_m(\lambda)}. \tag{2.25}$$

Then the result is:

Theorem 2.3 *Assume* (H1)-(H3) *and* (H6), *and, for* $m_1 \in \mathbb{N}$ *fixed,* $a \in S_{m_1}(1)$ *and* $0 < \beta < (2M)^{-1/2}$, *denote*

$$\tilde{g}_{m,\beta,\lambda} = \frac{Op^W(e^{-\tilde{q}_{\beta/2,m}})f_\lambda}{\|Op^W(e^{-\tilde{q}_{\beta/2,m}})f_\lambda\|}$$

and

$$\tilde{A}(m,T,\lambda) = \frac{1}{T}\int_0^T \left\langle e^{itH_m} Op^W(a\circ\Pi_{m_1})e^{-itH_m}\tilde{g}_{m,\beta,\lambda}\,,\,\tilde{g}_{m,\beta,\lambda}\right\rangle_{L^2(\mathbb{R}^{2m+1})} dt.$$

Then one has

(i)

$$\lim_{T\to\infty}\limsup_{m\to\infty}\int_{X_m}\left|\tilde{A}(m,T,\lambda) - \int_{\Lambda_\infty} a\circ\Pi_{m_1}\, d\mu_G(\beta)\right| d\tilde{\nu}_m(\lambda) = 0.$$

(iii) Assume furthermore (H4). Then (2.20) becomes

$$\lim_{m\to\infty}\frac{\mathrm{Tr} A\, Op^W(e^{-\beta q_m})}{\mathrm{Tr} Op^W(e^{-\beta q_m})} = \int_{\Lambda_\infty} a\circ\Pi_{m_1}\, d\mu_G\beta := \tilde{\omega}_\beta(A) \tag{2.26}$$

and if moreover $\hat{a}$ *and* $\hat{b}$ *are bounded measures on* Λ_{m_1}, *(2.21) becomes:*

$$\lim_{t\to\infty}\lim_{m\to\infty}\tilde{\omega}_\beta(A_m(t)B) = \tilde{\omega}_\beta(A)\cdot\tilde{\omega}_\beta(B) \tag{2.27}$$

Remarks.

1. Here, the choice of the quadratic form $\tilde{q}_{\beta,m}$ is dictated from the fact that we have (see Lemma 5.2 below):

$$e^{-\tilde{q}_{\beta,m}}\# e^{-\tilde{q}_{\beta,m}} = C''_{\beta,m} e^{-2\beta\tilde{q}_m} \tag{2.28}$$

where

$$\tilde{q}_m(x,\xi) = \frac{1}{2}|\xi|^2 + \langle\tilde{V}_m x, x\rangle.$$

In view of assumption (H6)(i), this explains why we get the usual Gibbs measure in the limit $m\to+\infty$. Note that we also have $F_{\beta,m} = \frac{\beta}{2}I_m + \mathcal{O}(\beta^3)$ et $G_{\beta,m} = \beta V_m + \mathcal{O}(\beta^3)$, so that $\tilde{q}_{\beta,m}$ is asymptotically equal to $\beta\tilde{q}_m$ as $\beta\to 0_+$.

2. For β small it is possible to compare $Op^W(e^{-\tilde{q}_{\beta,m}})$ with $e^{-\beta\tilde{H}_m}$, where $\tilde{H}_m = Op^W(\tilde{q}_m) = \frac{1}{2}(\sum_{j=1}^{2m+1} D^2_{x_j}) + \langle \tilde{V}_m x, x\rangle$. Actually, denoting $\tilde{\lambda}_1, \dots, \lambda_{2m+1}$ the eigenvalues of $\tilde{V}_m$, we get by standard formulas (see (3.3) below) that $Op^W(e^{-\tilde{q}_{\beta,m}})$ is unitarily equivalent to

$$\bigotimes_{j=1}^{2m+1} \frac{1}{\sqrt{1-\beta^2\tilde{\lambda}_j/2}} \exp\left[-\frac{1}{2}\left(\ln\frac{1+\beta\sqrt{\tilde{\lambda}_j/2}}{1-\beta\sqrt{\tilde{\lambda}_j/2}}\right)(D^2_{x_j}+x_j^2)\right] \tag{2.29}$$

while under the same unitary transformation $e^{-\beta\tilde{H}_m}$ becomes

$$\bigotimes_{j=1}^{2m+1} e^{-\beta\sqrt{\tilde{\lambda}_j/2}(D^2_{x_j}+x_j^2)}. \tag{2.30}$$

Since for small β we have:

$$\frac{1}{2}\ln\frac{1+\beta\sqrt{\tilde{\lambda}_j/2}}{1-\beta\sqrt{\tilde{\lambda}_j/2}} = \beta\sqrt{\frac{\tilde{\lambda}_j}{2}}\big(1+\mathcal{O}(\beta)\big)$$

we get from (2.29)-(2.30):

$$Op^W(e^{-\tilde{q}_{\beta,m}}) = e^{-\beta\tilde{H}_m(1+\beta R_{\beta,m})} \tag{2.31}$$

where $[R_{\beta,m}, \tilde{H}_m] = 0$ and $\frac{1}{2m+1}R_{\beta,m}$ is uniformly bounded in (β, m).

3. The previous remark proves that for β small, $[Op^W(e^{-\tilde{q}_{\beta,m}}), \tilde{H}_m] = 0$. This is true for all positive β, by formula (3.4) below, and the fact that

$$e^{-\tilde{q}_{\beta,m}} \circ \exp tH_{\tilde{q}_m}(x,\xi) = e^{-\tilde{q}_{\beta,m}} \circ \exp tH_{\tilde{q}_{\beta,m}}((2F_{\beta,m})^{-\frac{1}{2}}x, (2F_{\beta,m})^{-\frac{1}{2}}\xi)$$

is constant with respect to $t \in \mathbb{R}$.

4. Since the choice of the lattice $\mathbb{Z}$ does not play any role at all in our proofs, it can be replaced without modification by any lattice $\Gamma \subset \mathbb{R}^d$ $(d \geq 1)$ of the type considered in [LL], in which case the model describes an infinite harmonic crystal in $\mathbb{R}^d$. Also the choice of Λ_m is unessential, in the sense that any other choice $\Lambda'_m \to \Lambda_\infty$ in a reasonable way leads to the same results.

3 Proof of Theorem 2.1

Now we turn to the proof of Theorem 2.1. The fact that $(\Lambda_\infty, \phi_t, \hat{\mu}_\beta)$ is ergodic essentially follows from the arguments of [LL], but, for the sake of completeness, we give a sketch of the proof. First, the invariance of $\hat{\mu}_\beta$ under ϕ_t is a consequence of the commutativity between the operator B defined in (2.5) and the operator $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \ni (x, \xi) \mapsto (W_\beta^{\frac{1}{2}} x, W_\beta^{\frac{1}{2}} \xi)$. Now, denote $\hat{h}_1$ the closure in $L^2(\Lambda_\infty, d\hat{\mu}_\beta)$ of the set of all finite sums of $(a_j x_j + b_j \xi_j)$'s, $(a_j, b_j \in \mathbb{C} \; ; \; j \in \mathbb{Z})$. Then, denoting $d(\mathbb{Z})$ the elements of $\ell^2(\mathbb{Z})$ with finite support, the application

$$
\begin{array}{rcl}
\Theta \; : \; d(\mathbb{Z}) \oplus d(\mathbb{Z}) & \rightarrow & \hat{h}_1 \\
a \oplus b & \mapsto & \sum_j (a_j x_j + b_j \xi_j)
\end{array}
$$

can be extended into an isomorphism from $D((2VW_\beta)^{-1/2}) \oplus D(W_\beta^{-1/2})$ to $\hat{h}_1$ (where $D(A)$ denotes the domain of the operator A). Moreover, identifying $D((2VW_\beta)^{-1/2}) \oplus D(W_\beta^{-1/2})$ with $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ in an obvious way, we see that the action of ϕ_t on $\hat{h}_1$ is represented on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ (via the two previous identifications) by its infinitesimal generator $U = \begin{pmatrix} 0 & -V^{-1/2} \\ V^{1/2} & 0 \end{pmatrix}$. Since, by assumption (H2), U has no point spectrum the result (i) follows by an abstract argument (see [LL]Prop.4.2).

To prove (ii) we first show :

Lemma 3.1 *There exists a constant $C_{\beta,m}$ such that :*

$$
e^{-\beta H_m} = C_{\beta,m} Op^W(e^{-q_{\beta,m}})
$$

Proof Let $\lambda_1, ..., \lambda_{2m+1}$ be the eigenvalues of V_m, and denote $y = (y_1, ..., y_{2m+1})$ the coordinates in $\mathbb{R}^{2m+1}$ corresponding to an orthonormal basis of eigenvectors of V_m. Then H_m becomes:

$$
H'_m = -\frac{1}{2}\Delta_y + \sum_j \lambda_j y_j^2
$$

while the operator $K_m = Op^W(e^{-q_{\beta,m}})$ is transformed into :

$$
K'_m = \bigotimes_{j=1}^{2m+1} Op^W \left(\exp - \left(\frac{1}{\sqrt{2\lambda_j}} \tanh(\beta\sqrt{\frac{\lambda_j}{2}})\eta_j^2 + \sqrt{2\lambda_j}\tanh(\beta\sqrt{\frac{\lambda_j}{2}}) y_j^2 \right) \right).
$$

(Here η is the dual variable of y, and $\xi = ({}^tM)^{-1}\eta = M\eta$ since $x = My$ with M orthogonal.) Then the change of variables

$$
y_j \mapsto z_j = (2\lambda_j)^{1/4} y_j
$$

transforms H'_m into:

$$H''_m = \sum_j \sqrt{\frac{\lambda_j}{2}}(D^2_{z_j} + z_j^2) \tag{3.1}$$

and K'_m into :

$$K''_m = \bigotimes_{j=1}^{2m+1} Op^W \left(e^{-\tanh\left(\beta\sqrt{\frac{\lambda_j}{2}}\right)\left(\zeta_j^2+z_j^2\right)} \right). \tag{3.2}$$

Next, consider the well known one-dimensional identity valid for $0 < a < 1$

$$Op^W(e^{-a(x^2+\xi^2)}) = \frac{1}{\sqrt{1-a^2}} e^{-\frac{a}{2}x^2} e^{-\frac{a}{1-a^2}D_x^2} e^{-\frac{a}{2}x^2}$$

which can be for instance verified by explicit computation of the Weyl symbol of the r.h.s.. Then, using the formula (see e.g.[He])

$$e^{-\frac{x^2}{2}} e^{-tD_x^2} e^{-\frac{x^2}{2}} = \exp\left[-\frac{\ln(z+\sqrt{z^2-1})}{4k\sqrt{z^2-1}} (D_x^2 + 4k^2(z^2-1)x^2) \right]$$

with $k = \dfrac{1}{4t}$, $z = 2t+1$ $(t>0)$, we get in particular for $0 < a < 1$:

$$Op^W(e^{-a(x^2+\xi^2)}) = \frac{1}{\sqrt{1-a^2}} \exp\left[-\frac{1}{2}\left(\ln\frac{1+a}{1-a}\right)(D_x^2 + x^2) \right] \tag{3.3}$$

Taking $a = \tanh(\beta\sqrt{\frac{\lambda_j}{2}})$, the Lemma follows from (3.1)-(3.3). □

Now, since the flow generated by q_n defines a linear canonical transformation on Λ_n, we have:

$$e^{itH_n} Op^W(a) e^{-itH_n} = Op^W(a \circ \exp tH_{q_n}), \qquad \forall a \in S_n(1) \tag{3.4}$$

This relation (an “exact Egorov theorem”, going back at least to Van Hove (see e.g.[Fo])) holds only in the Weyl quantization [BS]§5.2.

For $n \le m$ and $(x,\xi) \in \Lambda_m$, denote

$$\rho(m,\lambda) = \| e^{-\frac{1}{2}\beta H_m} f_\lambda \|^2 \tag{3.5}$$

$$a_{n,T}(x,\xi) = \frac{1}{T}\int_0^T (a \circ \Pi_{m_1} \circ \exp tH_{q_n} \circ \Pi_n)(x,\xi))dt. \tag{3.6}$$

Using Lemma 3.1, (3.4) and (2.11), we get:

$$A(m,n,T,\lambda) = \frac{C^2_{\beta,m}}{\rho(m,\lambda)} \int_{\Lambda_m} \left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}} \right)(x,\xi) w_{f_\lambda}(x,\xi) dx d\xi \tag{3.7}$$

where # is the Weyl composition of symbols on Λ_m:

$$(a\#b)(x,\xi) = \pi^{-2(2m+1)} \int_{\Lambda_m^2} a(x+y,\xi+\eta) b(x+z,\xi+\zeta) e^{2i(\zeta y - z\eta)} dy d\eta dz d\zeta. \tag{3.8}$$

Taking advantage of assumption (H3), we get from (3.7)

$$\int_{X_m} A(m,n,T,\lambda)\rho(m,\lambda)d\theta_m(\lambda) = C_0 \int_{\Lambda_m} \left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi)dxd\xi \quad (3.9)$$

$$\int_{X_m} |A(m,n,T,\lambda)|\rho(m,\lambda)d\theta_m(\lambda) \leq C_0 \int_{\Lambda_m} |\left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi)|dxd\xi \quad (3.10)$$

where $C_0 = C_0(\beta,m)$ is a constant, which can be computed by taking $a \equiv 1$ in (3.9):

$$C_0 = \left(\int e^{-q_{\beta,m}} \# e^{-q_{\beta,m}} dxd\xi\right)^{-1} \int_{X_m} \rho(m,\lambda)d\theta_m(\lambda). \quad (3.11)$$

This also proves (2.12) so that, using the notation (2.13) we can rewrite (3.9)-(3.10) as:

$$\int_{X_m} A(m,n,T,\lambda)d\nu_m(\lambda) = C_1 \int_{\Lambda_m} \left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi)dxd\xi \quad (3.12)$$

$$\int_{X_m} |A(m,n,T,\lambda)|d\nu_m(\lambda) \leq C_1 \int_{\Lambda_m} |\left(e^{-q_{\beta,m}} \# a_{n,T} \# e^{-q_{\beta,m}}\right)(x,\xi)|dxd\xi \quad (3.13)$$

with

$$C_1 = \left(\int e^{-q_{\beta,m}} \# e^{-q_{\beta,m}} dxd\xi\right)^{-1}.$$

Now we make use of the two following properties of the operation $\#$ (valid e.g. for any a, b, c in $\mathcal{S}(\Lambda_m)$):

$$\int_{\Lambda_m} (a\#b\#c)(x,\xi)dxd\xi = \int_{\Lambda_m} (b\#c\#a)(x,\xi)dxd\xi \quad (3.14)$$

$$\int_{\Lambda_m} (a\#b)(x,\xi)dxd\xi = \int_{\Lambda_m} a(x,\xi)b(x,\xi)dxd\xi. \quad (3.15)$$

The property (3.14) is just a consequence of the cyclicity of the trace of operators, and (3.15) comes from a direct computation using (3.8).

Here our symbol a is not supposed to be in $\mathcal{S}(\Lambda_m)$, but an easy argument of density allows us to deduce from (3.12) and (3.14)-(3.15) (using also (2.18)):

$$\int_{X_m} A(m,n,T,\lambda)d\nu_m(\lambda) = \int_{\Lambda_m} a_{n,T}(x,\xi) \left[e^{-q_{\beta,m}(x,\xi)} dxd\xi\right]_N \quad (3.16)$$

where we have used the notation:

$$\int_E f\,[d\mu]_N = \frac{1}{\mu(E)} \int_E f\,d\mu \quad (3.17)$$

for any finite positive measure μ on a set E.

Now the problem is to rewrite also (3.13) in this way, despite the appearance of the modulus. The argument to do this is based upon the following:

Lemma 3.2 *There exists a positive definite quadratic form $Q_{\beta,m}(x,\xi,y,\eta)$ on Λ_m^2 such that for all $a\in S_m(1)$:*

$$e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}}=C'_{\beta,m}\tilde{a}e^{-q_{2\beta,m}}$$

where $C'_{\beta,m}$ is the constant appearing in (2.18), and

$$\tilde{a}(x,\xi)=\int_{\Lambda_m}a(y,\eta)\left[e^{-Q_{\beta,m}(x,\xi,y,\eta)}dyd\eta\right]_N.$$

Proof - See Appendix 1.

We deduce in particular from Lemma 3.2 the existence of a positive C^∞ function $\gamma(x,\xi)$ on Λ_m such that for all $a\in S_m(1)$:

$$\int_{\Lambda_m}(e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}})dxd\xi=\int_{\Lambda_m}a(x,\xi)\gamma(x,\xi)dxd\xi \tag{3.18}$$

$$\int_{\Lambda_m}|e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}}|dxd\xi\leq\int_{\Lambda_m}|a(x,\xi)|\gamma(x,\xi)dxd\xi. \tag{3.19}$$

By (3.12), (3.16) and (3.18) we get that γ equals a constant times $e^{-q_{2\beta,m}}$, which by (3.13) and (3.19) allows us to conclude that :

$$\int_{X_m}|A(m,n,T,\lambda)|\,d\nu_m(\lambda)\leq\int_{\Lambda_m}|a_{n,T}(x,\xi)|\left[e^{-q_{\beta,m}(x,\xi)}dxd\xi\right]_N. \tag{3.20}$$

Without loss of generality, we can assume from now on that $\int_{\Lambda_\infty}(a\circ\Pi_{m_1})d\hat{\mu}_\beta=0$, and then it remains to estimate the r.h.s. of (3.20). Since $a_{n,T}(x,\xi)$ depends only on $\Pi_n(x,\xi)$, we can let m go to $+\infty$ in (3.20) and we get:

$$\limsup_{m\to\infty}\int_{X_m}|A(m,n,T,\lambda)|\,d\nu_m(\lambda)\leq\int_{\Lambda_\infty}|a_{n,T}(x,\xi)|d\hat{\mu}_\beta. \tag{3.21}$$

Then we use (2.7) to let n go to $+\infty$ in (3.22). By the dominated convergence theorem, we then obtain:

$$\limsup_{n\to\infty}\limsup_{m\to\infty}\int_{X_m}|A(m,n,T,\lambda)|\,d\nu_m(\lambda)\leq\int_{\Lambda_\infty}|\frac{1}{T}\int_0^T(a\circ\Pi_{m_1}\circ\phi_t)(x,\xi)dt|d\hat{\mu}_\beta. \tag{3.22}$$

Finally, we let T go to $+\infty$. By the ergodicity property, we have that

$$\frac{1}{T}\int_0^T(a\circ\Pi_{m_1}\circ\phi_t)(x,\xi)dt\to\int_{\Lambda_\infty}(a\circ\Pi_{m_1})d\hat{\mu}_\beta=0$$

for $\hat{\mu}_\beta$-almost all (x,ξ) in Λ_∞. Therefore, applying again the dominated convergence theorem, we get from (3.22):

$$\lim_{T\to+\infty}\limsup_{n\to\infty}\limsup_{m\to\infty}\int_{X_m}|A(m,n,T,\lambda)|\,d\nu_m(\lambda)\leq 0 \tag{3.23}$$

and this completes the proof of Theorem 2.1. $\square$

4 Proof of Theorem 2.2

Let us now proceed to the proof of Theorem 2.2. Denote

$$\omega_{\beta,m}(A) = \frac{\mathrm{Tr}(Ae^{-\beta H_m})}{\mathrm{Tr}(e^{-\beta H_m})}. \tag{4.1}$$

Using Lemma 3.1 and (3.15) we see that

$$\omega_{\beta,m}(A) = \int_{\Lambda_m} a \circ \Pi_{m_1}(x,\xi) \left[e^{-q_{\beta,m}(x,\xi)}\, dx\, d\xi\right]_N \tag{4.2}$$

so that the first assertion (2.20) of the theorem is obvious.

For $m \geq n \geq m_1$ we also have:

$$\omega_{\beta,m}(A_n(t)B) = \int_{\Lambda_m} a_{n,t}\#(b \circ \Pi_{m_1})(x,\xi) \left[e^{-q_{\beta,m}(x,\xi)}\, dx\, d\xi\right]_N. \tag{4.3}$$

For $X = (x,\xi)$ and $Y = (y,\eta) \in \Lambda_m$, we denote

$$\sigma(X,Y) = \xi y - x\eta \tag{4.4}$$

the canonical symplectic form on Λ_m. Then by (3.8) we have:

$$a_{n,t}\#(b \circ \Pi_{m_1})(X) = \pi^{-2(2m+1)} \int_{\Lambda_n^2} (a_{n,t} \circ \Pi_n)(Y)(b \circ \Pi_{m_1})(Z) e^{2i[\sigma(Y,X)+\sigma(Z,Y-X)]} dY\, dZ \tag{4.5}$$

By the Fourier inversion formula and the assumption on a, b, we can write for any $Y_1 \in \Lambda_{m_1}$:

$$a(Y_1) = (2\pi)^{-2(2m_1+1)} \int_{\Lambda_{m_1}} e^{\langle Y_1, Y^*\rangle} \hat{a}(Y^*) dY^* \tag{4.6}$$

and a similar formula for b. Here we have used an abuse of notation by writing $\hat{a}(Y^*)dY^*$ for the (non necessarily Lebesgue absolutely continuous) measure defined by the Fourier transform of a.

In particular, taking $Y_1 = \Pi_{m_1}\phi_{n,t}\Pi_n(Y)$ in (4.6) and substituing in (4.5), we get:

$$a_{n,t}\#(b \circ \Pi_{m_1})(X) = \pi^{-2(2m+1)}(2\pi)^{-4(2m_1+1)} \times$$
$$\int e^{2i[\sigma(Y,X)+\sigma(Z,Y-X)]+i\langle \phi_{n,t}\Pi_n(Y), Y^*\rangle + i\langle Z, Z^*\rangle} \hat{a}(Y^*)dY^*\, \hat{b}(Z^*)dZ^*\, dY\, dZ$$

where the integration runs over $(Y^*, Z^*, Y, Z) \in \Lambda_{m_1} \times \Lambda_{m_1} \times \Lambda_m \times \Lambda_m$, and Λ_{m_1} has been identified in an obvious way with a subspace of Λ_n and of Λ_m.

Interpreting the integration over (Y, Z) as an oscillatory one, we can first integrate with respect to Z, and we obtain (using the well-known identity $\int_{\mathbb{R}^d} e^{2i(x-y)\xi} d\xi = \pi^d \delta(y = x)$):

$$a_{n,t}\#(b \circ \Pi_{m_1})(X) = (2\pi)^{-4(2m_1+1)} \int e^{i\langle Z^*, X\rangle + i\langle \phi_{n,t}\Pi_n(X + \tilde{Z}^*/2), Y^*\rangle} \hat{a}(Y^*)dY^*\, \hat{b}(Z^*)dZ^* \tag{4.7}$$

where we have denoted $\tilde{Z}^* = (-\zeta^*, z^*)$ if $Z^* = (z^*, \zeta^*)$.

Now, inserting (4.7) into (4.3), and making the change of variables $X \mapsto X - \tilde{Z}^*/2$, this gives:

$$\omega_{\beta,m}(A_n(t)B) = (2\pi)^{-4(2m_1+1)} \times$$
$$\int_{\Lambda_m \times \Lambda_{m_1}^2} e^{i\langle Z^*, X\rangle + i\langle \phi_{n,t}\Pi_n(X), Y^*\rangle} \left[e^{-q_{\beta,m}(X-\tilde{Z}^*/2)}dX\right]_N \hat{a}(Y^*)dY^*\,\hat{b}(Z^*)dZ^*.$$

and therefore, writing $q_{\beta,m}(X) = \langle Q_{\beta,m}X, X\rangle$ with $Q_{\beta,m}(x,\xi) = (V_m W_{\beta,m}x\,,\, W_{\beta,m}\xi)$:

$$\omega_{\beta,m}(A_n(t)B) = (2\pi)^{-4(2m_1+1)} \int_{\Lambda_{m_1}^2} \Gamma_{m,n,t}(Y^*, Z^*) e^{-q_{\beta,m}(\tilde{Z}^*)/4} \hat{a}(Y^*)dY^*\,\hat{b}(Z^*)dZ^*. \tag{4.8}$$

where

$$\Gamma_{m,n,t}(Y^*, Z^*) = \int_{\Lambda_m} e^{i\langle Z^*, X\rangle + \langle \tilde{Z}^*, Q_{\beta,m}X\rangle + i\langle \phi_{n,t}\Pi_n(X), Y^*\rangle} \left[e^{-q_{\beta,m}(X)}dX\right]_N \tag{4.9}$$

is of the form:

$$\Gamma_{m,n,t}(Y^*, Z^*) = \int_{\Lambda_m} F_{n,t,Y^*,\tilde{Z}^*}(\Pi_{m_1}Q_{\beta,m}X, \Pi_n X))\, \left[e^{-q_{\beta,m}(X)}dX\right]_N \tag{4.10}$$

with F_{n,t,Y^*,Z^*} smooth and uniformly bounded together with all its derivatives on $\Lambda_{m_1} \times \Lambda_n$. To let m tend to infinity in (4.10), we use the following lemma (which is the point where (H5) is used):

Lemma 4.1 *Let $F \in C^\infty(\Lambda_{m_1} \times \Lambda_n)$ be uniformly bounded together with all its derivatives. Then*

$$\int_{\Lambda_m} F(\Pi_{m_1}Q_{\beta,m}X, \Pi_n X))\, \left[e^{-q_{\beta,m}(X)}dX\right]_N \to \int_{\Lambda_\infty} F(\Pi_{m_1}Q_\beta X, \Pi_n X))\, d\hat{\mu}_\beta \quad (m \to \infty)$$

where $\hat{\mu}_\beta$ is defined in (2.15), *and Q_β is defined on Λ_∞ by:*

$$Q_\beta(x,\xi) = (VW_\beta x, W_\beta \xi), \qquad W_\beta \sqrt{2}V^{-\frac{1}{2}}\tanh\frac{\beta V^{\frac{1}{2}}}{\sqrt{2}}.$$

Proof- See Appendix 1.

Now, for any fixed (n, t, Y^*, Z^*), we see on (4.9)-(4.10) that, as $m \to \infty$, $\Gamma_{m,n,t}(Y^*, Z^*)$ tends to:

$$\Gamma_{n,t}(Y^*, Z^*) = \int_{\Lambda_\infty} e^{i\langle Z^*, X\rangle + \langle \tilde{Z}^*, Q_\beta X\rangle + i\langle \phi_{n,t}\Pi_n(X), Y^*\rangle} d\hat{\mu}_\beta \tag{4.11}$$

which in turns is of the form:

$$\Gamma_{n,t}(Y^*, Z^*) = \int_{\Lambda_\infty} f_{Z^*}(X) g_{Y^*}(\Pi_{m_1}\phi_{n,t}\Pi_n X) d\hat{\mu}_\beta \tag{4.12}$$

with f_{Z^*} and g_{Y^*} uniformly bounded, and g_{Y^*} continuous on Λ_{m_1}. Then, using (2.7) and the dominated convergence theorem, we see on (4.12) that, as $n \to \infty$, $\Gamma_{n,t}(Y^*, Z^*)$ tends to:

$$\Gamma_t(Y^*, Z^*) = \int_{\Lambda_\infty} f_{Z^*}(X) g_{Y^*}(\Pi_{m_1} \phi_t X) d\hat{\mu}_\beta. \tag{4.13}$$

Now, the same arguments used in the proof of Theorem 2.1 (i) (see the begining of section 3) lead to the fact that under (H4) the classical dynamical system $(\Lambda_\infty, \phi_t, \hat{\mu}_\beta)$ is mixing. As a consequence, we get from (4.13):

$$\Gamma_t(Y^*, Z^*) \to \int f_{Z^*}(X) d\hat{\mu}_\beta \cdot \int g_{Y^*}(\Pi_{m_1} X) d\hat{\mu}_\beta \quad \text{as} \quad t \to \infty. \tag{4.14}$$

Summing up (4.9)-(4.14), we have proved that for any fixed $(Y^*, Z^*) \in \Lambda^2_{m_1}$, we have:

$$\lim_{t\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \Gamma_{m,n,t}(Y^*, Z^*) = \int_{\Lambda_\infty} e^{i\langle Z^*, X\rangle + \langle \tilde{Z}^*, Q_\beta X\rangle} d\hat{\mu}_\beta(X) \cdot \int_{\Lambda_\infty} e^{i\langle X, Y^*\rangle} d\hat{\mu}_\beta(X) \tag{4.15}$$

and because of the translation invariance of the Lebesgue measure on Λ_m, and the fact that $\langle Z^*, \tilde{Z}^*\rangle = 0$, it is also easy to verify that

$$\int_{\Lambda_\infty} e^{i\langle Z^*, X\rangle + \langle \tilde{Z}^*, Q_\beta X\rangle} d\hat{\mu}_\beta(X) = e^{\langle Q_\beta \tilde{Z}^*, \tilde{Z}^*\rangle/4} \int_{\Lambda_\infty} e^{i\langle Z^*, X\rangle} d\hat{\mu}_\beta(X) \tag{4.16}$$

Since by assumption $\hat{a}(Y^*)dY^*$ and $\hat{b}(Z^*)dZ^*$ are bounded measures on Λ_{m_1}, and $|\Gamma_{m,n,t}(Y^*, Z^*) e^{-q_{\beta,m}(\tilde{Z}^*)/4}| = 1$, we can use the dominated convergence theorem in (4.8) and conclude from (4.15)-(4.16) (using also the obvious fact that $q_{\beta,m}(\tilde{Z}^*)$ tends to $\langle Q_\beta \tilde{Z}^*, \tilde{Z}^*\rangle$ as $m \to \infty$) that:

$$\begin{aligned}
&\lim_{t\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \omega_{\beta,m}(A_n(t)B) = (2\pi)^{-4(2m_1+1)} \times \\
&\int_{\Lambda_{m_1}\times\Lambda_\infty} e^{i\langle X, Y^*\rangle} \hat{a}(Y^*) dY^* \, d\hat{\mu}_\beta(X) \cdot \int_{\Lambda_{m_1}\times\Lambda_\infty} e^{i\langle Z^*, X\rangle} \hat{b}(Z^*) dZ^* \, d\hat{\mu}_\beta(X) \\
&= \int_{\Lambda_\infty} a \circ \Pi_{m_1} d\hat{\mu}_\beta \cdot \int_{\Lambda_\infty} b \circ \Pi_{m_1} d\hat{\mu}_\beta
\end{aligned}$$

where the last equality comes again from the Fourier-inverse formula. $\square$.

5 Proof of Theorem 2.3

Lemma 5.2 *For any pair of positive definite real-symmetric matrices F and G on Λ_m, there exists a constant $C = C(F, G, m)$ such that:*

$$e^{-(\langle F\xi,\xi\rangle + \langle Gx,x\rangle)} \# e^{-(\langle F\xi,\xi\rangle + \langle Gx,x\rangle)} = Ce^{-2\left(\langle F(F+G^{-1})^{-1}G^{-1}\xi,\xi\rangle + \langle (F+G^{-1})^{-1}x,x\rangle\right)}.$$

Proof - See Appendix 1.

In particular, taking $F = F_{\beta,m}$ and $G = G_{\beta,m}$ defined in (2.23)-(2.24) we get easily (2.28) from Lemma 5.2. Then computations analogous to those of the previous section ((3.7) through (3.20)) lead to:

$$\int_{X_m} \left|\tilde{A}(m,T,\lambda)\right| d\tilde{\nu}_m(\lambda) \leq \int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-\beta\tilde{q}_m(x,\xi)}dxd\xi\right]_N. \tag{5.17}$$

Now in the r.h.s. (5.17) we make the change of variables :

$$x = \tilde{V}_m^{-\frac{1}{2}} y \qquad \xi = \eta$$

which gives :

$$\int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-\beta\tilde{q}_m(x,\xi)}dxd\xi\right]_N = \int_{\Lambda_m} |a_{m,T}(\tilde{V}_m^{-\frac{1}{2}}y,\eta)| \left[e^{-\beta(\eta^2+y^2)}dyd\eta\right]_N. \tag{5.18}$$

Since the quadratic form in the exponent is now diagonal, we can integrate over $n \geq m$ variables so that

$$\int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-\beta\tilde{q}_m(x,\xi)}dxd\xi\right]_N = \int_{\Lambda_n} |a_{m,T}(\tilde{V}_m^{-\frac{1}{2}}\Pi_m y,\Pi_m\eta)| \left[e^{-\beta(\eta^2+y^2)}dyd\eta\right]_N \tag{5.19}$$

for any $n \geq m$. Coming back to the old variables on Λ_n, this gives:

$$\int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-\beta\tilde{q}_m(x,\xi)}dxd\xi\right]_N = \int_{\Lambda_n} |a_{m,T}(\tilde{V}_m^{-\frac{1}{2}}\Pi_m\tilde{V}_n^{\frac{1}{2}}x,\Pi_m\xi)| \left[e^{-\beta\tilde{q}_n(x,\xi)}dxd\xi\right]_N. \tag{5.20}$$

Now, by assumption (H4)(ii), for n large enough, the function $a_{m,T}(\tilde{V}_m^{-\frac{1}{2}}\Pi_m V_n^{\frac{1}{2}}x,\Pi_m\xi)$ depends only on a fixed number of variables independent of n. Then, by assumption (H4)(i) and standard results on the Gaussian measures (see e.g. [LL]), letting n tend to $+\infty$ in (5.20), we get:

$$\int_{\Lambda_m} |a_{m,T}(x,\xi)| \left[e^{-\beta\tilde{q}_m(x,\xi)}dxd\xi\right]_N = \int_{\Lambda_\infty} |a_{m,T}(\tilde{V}_m^{-\frac{1}{2}}\Pi_m\tilde{V}_n^{\frac{1}{2}}\Pi_n x,\Pi_m\xi)| d\mu_G(\beta). \tag{5.21}$$

Finally, using assumption (H4)(iii), (2.7), and the uniform (with respect to $m \geq 0$) continuity of $\exp tH_{q_m} \circ \Pi_m$ on each $\mathcal{H}_k$, we see that for all $(x,\xi) \in \Lambda_\infty$ and $t \in \mathbb{R}$:

$$(a\circ\Pi_{m_1}\circ\exp tH_{q_m})(\tilde{V}_m^{-\frac{1}{2}}\Pi_m\tilde{V}_n^{\frac{1}{2}}\Pi_n x,\Pi_m\xi) \to (a\circ\Pi_{m_1}\circ\phi_t)(x,\xi) \quad \text{as} \quad m \to +\infty. \tag{5.22}$$

It follows from (5.22), (5.21), (5.17) and the dominated convergence theorem that:

$$\limsup_{m\to+\infty} \int_{X_m} \left|\tilde{A}(m,T,\lambda)\right| d\tilde{\nu}_m(\lambda) \leq \int_{\Lambda_\infty} \left|\frac{1}{T}\int_0^T (a\circ\Pi_{m_1}\circ\phi_t)(x,\xi)dt\right| d\mu_G(\beta). \tag{5.23}$$

Letting T tend to infinity in (5.23), the ergodicity of the system $(\Lambda_\infty, \phi_t, \mu_G(\beta))$ and the fact that we can restrict to the case $\int a\circ\Pi_{m_1}d\mu_G(2\beta) = 0$ yield the assertion.

□

6 Coherent States: Sharpening the Ergodicity Result

Now we take $X_m = \Lambda_m$, $d\theta_m(\lambda) = d\lambda$, and for $\lambda = (\lambda_x, \lambda_\xi) \in \Lambda_m$ the coherent states defined by:

$$f_\lambda(x,\xi) = e^{ix\lambda_\xi - (x-\lambda_x)^2/2}. \tag{6.1}$$

Then a direct computation gives:

$$w_{f_\lambda}(x,\xi) = 2^{2m+1}\pi^{m+\frac{1}{2}} e^{-(\xi-\lambda_\xi)^2-(x-\lambda_x)^2} \tag{6.2}$$

so that (H3) is obviously satisfied. Therefore the results of Theorems 2.1 and 2.2 hold for V satisfying (H1)-(H2) (respectively (H1), (H2) and (H6)). The new fact which appears in this situation is :

Lemma 6.1 *Under (6.1), the two measures $d\nu_m(\lambda)$ and $d\tilde{\nu}_m(\lambda)$, defined respectively by (2.13) and (2.25), are Gaussian probability measures on $X_m = \Lambda_m$.*

Proof - In each case, the measure is of the form $C\|Op^W(e^{-q})f_\lambda\|^2 d\lambda$ where C is a constant and q is a positive definite quadratic form on Λ_m. Moreover, by computations analogous e.g. to those for (3.7) we have :

$$\|Op^W(e^{-q})f_\lambda\|^2 = C' \int_{\Lambda_m} (e^{-q}\#e^{-q})(x,\xi) w_{f_\lambda}(x,\xi) dx d\xi \tag{6.3}$$

where C' is another constant. Then the result follows immediately by (6.2) and the fact that $e^{-q}\#e^{-q} = C''e^{-q'}$ where q' is a positive definite quadratic form on Λ_m and C'' is a constant. □

Now denote L_m and $\tilde{L}_m$ the two real-symmetric positive definite $(4m+2)\times(4m+2)$-matrices defined by :

$$d\nu_m(\lambda) = \left[e^{-\langle L_m\lambda,\lambda\rangle} d\lambda\right]_N \tag{6.4}$$

$$d\tilde{\nu}_m(\lambda) = \left[e^{-\langle \tilde{L}_m\lambda,\lambda\rangle} d\lambda\right]_N. \tag{6.5}$$

Then for any bounded function $A(\lambda)$ we have :

$$\int_{\Lambda_m} A(\lambda) d\nu_m(\lambda) = \int_{\Lambda_m} A(L_m^{-\frac{1}{2}}\lambda) \left[e^{-|\lambda|^2} d\lambda\right]_N \tag{6.6}$$

and therefore, by an argument similar to the one leading to (5.21) :

$$\int_{\Lambda_m} A(\lambda) d\nu_m(\lambda) = \int_{\Lambda_\infty} A(L_m^{-\frac{1}{2}}\Pi_m\lambda) d\nu_G(\lambda) \tag{6.7}$$

where $d\nu_G(\lambda)$ is the infinite dimensional Gibbs measure obtained by taking the limit of $\left[e^{-|\lambda|^2}d\lambda\right]_N$ on Λ_n as $n\to+\infty$. A formula analogous to (6.7) is also true for $d\tilde{\nu}_m(\lambda)$, and therefore it follows from Theorems 2.1 and 2.2 that we have in this situation :

$$\lim_{T\to\infty}\limsup_{n\to\infty}\limsup_{m\to\infty}\int_{\Lambda_\infty}\left|A(m,n,T,L_m^{-\frac{1}{2}}\Pi_m\lambda)-\int_{\Lambda_\infty}a\circ\Pi_{m_1}\,d\hat{\mu}_\beta\right|d\nu_G(\lambda)=0 \quad (6.8)$$

$$\lim_{T\to\infty}\limsup_{m\to\infty}\int_{\Lambda_\infty}\left|\tilde{A}(m,T,\tilde{L}_m^{-\frac{1}{2}}\Pi_m\lambda)-\int_{\Lambda_\infty}a\circ\Pi_{m_1}\,d\mu_G(\beta)\right|d\nu_G(\lambda)=0. \quad (6.9)$$

Finally, using a very standard argument of measure theory, we easily deduce from (6.8) and (6.9) the following:

Proposition 6.1 *Assume (H1)-(H2) and choose the set of coherent states (6.1). Then there exist sequences $(T_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$, $(n_k)_{k\in\mathbb{N}}$ simultaneously tending to $+\infty$ such that for ν_G-almost all $\lambda\in\Lambda_\infty$:*

$$\lim_{k\to+\infty}A(m_k,n_k,T_k,L_{m_k}^{-\frac{1}{2}}\Pi_{m_k}\lambda)=\int_{\Lambda_\infty}a\circ\Pi_{m_1}\,d\hat{\mu}_\beta.$$

If moreover (H3) is satisfied, there exist sequences $(T_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ both tending to $+\infty$ such that for ν_G-almost all $\lambda\in\Lambda_\infty$:

$$\lim_{k\to+\infty}\tilde{A}(m_k,T_k,\tilde{L}_{m_k}^{-\frac{1}{2}}\Pi_{m_k}\lambda)=\int_{\Lambda_\infty}a\circ\Pi_{m_1}\,d\mu_G(\beta).$$

Remarks.

1. An analogous result holds if (6.1) is replaced by the more general case $f_\lambda(x,\xi)=e^{ix\lambda_\xi-\langle F_m(x-\lambda_x),x-\lambda_x\rangle}$, F_m being any positive definite symmetric matrix.

2. Actually, one can replace L_m by any other symmetric matrix L'_m such that $K_m=(L'_m)^{-1/2}L_m(L'_m)^{-1/2}$ is a diagonal matrix and the measure $\left[e^{-\langle K_m\lambda,\lambda\rangle}d\lambda\right]_N$ on Λ_m admits a limit $d\nu_\infty$ as $m\to+\infty$. In this case the "ν_G-almost all λ" of the Proposition must be replaced by "ν_∞-almost all λ".

7 Appendix 1

1. Proof of Lemma 3.2

Using (3.8), we see that $e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}}$ can be put under the form:

$$(e^{-q_{\beta,m}}\#a\#e^{-q_{\beta,m}})(x,\xi)=C_1\int_{\Lambda_m^4}a(Y_1)e^{-q_1(x,\xi,Y_1,Y_2,Y_3,Y_4)}dY_1dY_2dY_3dY_4 \quad (7.1)$$

where all along this proof C_j $(j = 1, 2...)$ will denote complex constants, q_1 is a complex quadratic form on Λ_m^5, and the integral (7.1) is oscillatory. Moreover, a direct computation gives:

$$\int_{\Lambda_m^3} e^{-q_1(x,\xi,Y_1,Y_2,Y_3,Y_4)} dY_2 dY_3 dY_4 = C_2 e^{-Q(x,\xi,Y_1)} \tag{7.2}$$

where $C_2 \in \mathbb{R}$ and Q is a positive definite quadratic form. Actually, this can also be seen without computation in the following way: the existence of the complex constant C_2 and the complex quadratic form Q such that (7.2) holds is clear, and if a is real then $Op^W(e^{-q_{\beta,m}} \# a \# e^{-q_{\beta,m}}) = Op^W(e^{-q_{\beta,m}}) Op^W(a) Op^W(e^{-q_{\beta,m}})$ is a symmetric operator. As a consequence $e^{-q_{\beta,m}} \# a \# e^{-q_{\beta,m}}$ must be real for a real, which implies that Ce^{-Q} is real (and hence both C and Q are). Moreover, one can show easily that the application $S_m(1) \ni a \mapsto e^{-q_{\beta,m}} \# a \# e^{-q_{\beta,m}}$ maps continuously $S_m(1)$ into $\mathcal{S}(\Lambda_m)$, so that Q is necessarily positive definite.

When $a \equiv 1$, we get from (2.18), (7.1), (7.2) :

$$C_1 C_2 \int_{\Lambda_m} e^{-Q(x,\xi,Y_1)} dY_1 = e^{-\tilde{q}_{2\beta,m}}. \tag{7.3}$$

Then the result follows from (7.1), (7.2), (7.3) by writing :

$$(e^{-q_{\beta,m}} \# a \# e^{-q_{\beta,m}})(x,\xi) = \left(C_1 C_2 \int_{\Lambda_m} e^{-Q(x,\xi,Y_1)} dY_1 \right) \int_{\Lambda_m} a(Y_1) \left[e^{-Q(x,\xi,Y_1)} dY_1 \right]_N .$$

□

2. Proof of Lemma 4.1

Denote $\Delta(m)$ the difference between the two expressions. For any $p, q \in \mathbb{N}$, we write:

$$\Delta(m) = \Delta_1(m,p,q) + \Delta_2(m,p,q) + \Delta_3(m,p) + \Delta_4(p,q) \tag{7.4}$$

with

$$\Delta_1(m,p,q) = \int_{\Lambda_m} F(\Pi_{m_1} Q_{\beta,q} \Pi_p X, \Pi_n X) \left[e^{-q_{\beta,m}(X)} dX \right]_N - \int_{\Lambda_\infty} F(\Pi_{m_1} Q_{\beta,q} \Pi_p X, \Pi_n X)\, d\hat{\mu}_\beta$$

$$\Delta_2(m,p,q) = \int_{\Lambda_m} \left(F(\Pi_{m_1} Q_{\beta,m} \Pi_p X, \Pi_n X) - F(\Pi_{m_1} Q_{\beta,q} \Pi_p X, \Pi_n X) \right) \left[e^{-q_{\beta,m}(X)} dX \right]_N$$

$$\Delta_3(m,p) = \int_{\Lambda_m} \left(F(\Pi_{m_1} Q_{\beta,m} X, \Pi_n X) - F(\Pi_{m_1} Q_{\beta,m} \Pi_p X, \Pi_n X) \right) \left[e^{-q_{\beta,m}(X)} dX \right]_N$$

$$\Delta_4(p,q) = \int_{\Lambda_\infty} \left(F(\Pi_{m_1} Q_{\beta,q} \Pi_p X, \Pi_n X) - F(\Pi_{m_1} Q_\beta X, \Pi_n X) \right) d\hat{\mu}_\beta.$$

Now, by assumption on F, there exists a positive constant C such that for any m and p:

$$|\Delta_3(m,p)| \le C \int_{\Lambda_m} \|\Pi_{m_1} Q_{\beta,m}(X - \Pi_p X)\|_{\ell^\infty} \left[e^{-q_{\beta,m}(X)} dX\right]_N \tag{7.5}$$

and we have (with obvious notations):

$$\begin{aligned} \|\Pi_{m_1} Q_{\beta,m}(X - \Pi_p X)\|_{\ell^\infty} \quad &\le \sup_{|i|\le m_1} \sum_{|j|>p} |(Q_{\beta,m})_{i,j} X_j| \\ &\le \sup_{|i|\le m_1} \left(\sum_{|j|>p} j^2 |(Q_{\beta,m})_{i,j}|^2\right)^{1/2} \left(1 + \sum_{j\neq 0} \frac{X_j^2}{j^2}\right)^{1/2} \end{aligned}$$

and therefore, using (H5):

$$\|\Pi_{m_1} Q_{\beta,m}(X - \Pi_p X)\|_{\ell^\infty} \le \frac{C'}{p} \left(1 + \sum_{j\neq 0} \frac{X_j^2}{j^2}\right). \tag{7.6}$$

with a constant C' independant of m and p. Since also

$$\int_{\Lambda_m} X_j^2 \left[e^{-q_{\beta,m}(X)} dX\right]_N = (Q_{\beta,m}^{-1})_{j,j} \le \|Q_{\beta,m}^{-1}\|_{\mathcal{L}(\ell^2)} = \mathcal{O}(1) \tag{7.7}$$

uniformly with respect to m, we deduce from (7.5), (7.6):

$$|\Delta_3(m,p)| = \mathcal{O}(p^{-1}) \tag{7.8}$$

uniformly with respect to m and p.

In a similar way, using the fact that for fixed p, both finite dimensional matrices $\Pi_{m_1} Q_{\beta,m} \Pi_p$ and $\Pi_{m_1} Q_{\beta,q} \Pi_p$ tend to $\Pi_{m_1} Q_\beta \Pi_p$ as m and q tend to infinity, one can prove that:

$$\Delta_2(m,p,q) \to 0 \quad \text{as} \quad m \text{ and } q \to \infty. \tag{7.9}$$

Moreover, for any fixed (p,q) we see that

$$\Delta_1(m,p,q) \to 0 \quad \text{as} \quad m \to \infty. \tag{7.10}$$

The same arguments also give, subtituing Q_β to $Q_{\beta,m}$, that

$$\Delta_4(p,q) \to 0 \quad \text{as} \quad p \text{ and } q \to \infty. \tag{7.11}$$

Then, choosing $\varepsilon > 0$ arbitrarily small, one can first fix p large enough so that $|\Delta_3(m,p)| \le \varepsilon$ for all (m,q) and $|\Delta_4(p,q)| \le \varepsilon$ for all q sufficiently large, then fix q large enough so that $|\Delta_2(m,p,q)| \le \varepsilon$ for all sufficiently large m, and finally get $|\Delta_1(m,p,q)| \le \varepsilon$, and thus $|\Delta(m)| \le 4\varepsilon$, by taking m large enough. □

3. Proof of Lemma 5.2

From (3.8) we get easily:

$$\begin{aligned} e^{-(\langle F\xi,\xi\rangle+\langle Gx,x\rangle)}\# e^{-(\langle F\xi,\xi\rangle+\langle Gx,x\rangle)} &= \pi^{-2(2m+1)}\left|\int_{\Lambda_m} e^{2i\zeta y-\langle G(x+y),x+y\rangle-\langle F(\xi+\zeta),\xi+\zeta\rangle}dyd\zeta\right|^2 \\ &= \pi^{-2(2m+1)}|I|^2 \end{aligned} \tag{7.12}$$

where

$$I = e^{-\langle Gx,x\rangle-\langle F\xi,\xi\rangle}\int e^{iy(2\zeta+2iGx)-\langle Gy,y\rangle-2\langle F\xi,\zeta\rangle-\langle F\zeta,\zeta\rangle}dyd\zeta. \tag{7.13}$$

Making the change of variables $y'=\sqrt{2}G^{\frac{1}{2}}y$ and integrating first with respect to y' we get:

$$I = Ce^{-\langle Gx,x\rangle-\langle F\xi,\xi\rangle}\int e^{-\langle G^{-1}(\zeta+iGx),\zeta+iGx\rangle-2\langle F\xi,\zeta\rangle-\langle F\zeta,\zeta\rangle}d\zeta \tag{7.14}$$

where C is a constant, and therefore, setting $\zeta'=\sqrt{2}(F+G^{-1})^{\frac{1}{2}}\zeta$, this gives:

$$I = C'e^{-\langle F\xi,\xi\rangle-\langle (F+G^{-1})^{-1}(x-iF\xi),x-iF\xi\rangle} \tag{7.15}$$

where C' is a constant. Then (7.12) and (7.15) yield the result. □

8 Appendix 2: Remarks on Quantum Mixing and Ergodicity

The elementary remarks collected here, useful to clarify the subsequent statements, are presumably known but we were unable to locate a precise reference.

We first formulate into an abstract setting the definitions recalled in the introduction. Let H be a positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$ such that $\sigma(H)$ is discrete and simple and $e^{-\beta H}$ is trace-class for any positive β. Given $A\in\mathcal{L}(\mathcal{H})$ let $\omega(A)$ be the quantum microcanonical measure defined as in (1.1) or the corresponding quantum Gibbs measure at inverse temperature β

$$\omega(A)=\frac{\mathrm{Tr}\,(Ae^{-\beta H})}{\mathrm{Tr}\,(e^{-\beta H})} \tag{8.1}$$

indifferently. Let also $\mathcal{A}$ be a weakly closed sub-algebra of $\mathcal{L}(\mathcal{H})$ invariant under the action of e^{itH}. In this general context we assume (and verify below in our specific case) the existence of a family of normalized states $(\psi_\lambda)_{\lambda\in\Lambda}$ complete for ω on $\mathcal{A}$, in the sense that there is a probability measure $d\nu(\lambda)$ on the set Λ such that

$$\omega(A)=\int_\Lambda \langle A\psi_\lambda,\psi_\lambda\rangle_{\mathcal{H}}\,d\nu(\lambda),\qquad \forall\; A\in\mathcal{A}. \tag{8.2}$$

Then the quantum mixing property on $\mathcal{A}$, *defined* as

$$\omega(A_H(t)B) \to \omega(A)\omega(B) \quad \text{as} \quad |t| \to \infty \tag{8.3}$$

for any operators $A, B \in \mathcal{A}$ can be rewritten as

$$\int_\Lambda \langle A_H(t)B\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}} d\nu(\lambda) \to \int_\Lambda \langle A\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}} d\nu(\lambda) \int_\Lambda \langle B\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}} d\nu(\lambda). \tag{8.4}$$

Remark that (1.1, 8.1, 8.2) imply the invariance property

$$\int_\Lambda \langle A_H(t)\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}}\, d\nu(\lambda) = \int_\Lambda \langle A\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}}\, d\nu(\lambda) \tag{8.5}$$

and by analogy with the classical dynamical systems, a natural possible definition of quantum ergodicity is that for any $A \in \mathcal{A}$:

$$\frac{1}{T}\int_0^T \langle A_H(t)\psi_\lambda, \psi_\lambda\rangle_{\mathcal{H}}\, dt \to \omega(A) \quad \nu(\lambda) - \text{a.e.} \quad \text{as} \quad T \to \infty. \tag{8.6}$$

This definition is also motivated from the fact that in most situations the quantum mixing property (8.4) implies (8.6). Indeed, still assuming that $\sigma(H)$ is discrete and simple, we see in (1.6) that for any $A \in \mathcal{A}$ the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle A_H(t)\varphi, \psi\rangle_{\mathcal{H}}\, dt := \langle \bar{A}\varphi, \psi\rangle_{\mathcal{H}} \tag{8.7}$$

exists for all $\varphi, \psi \in \mathcal{H}$, and the operator $\bar{A} \in \mathcal{A}$ is invariant under the action of e^{itH}. As a consequence, the definition (8.6) of quantum ergodicity is equivalent to the fact that for any $A \in \mathcal{A}$ we have the identity

$$\langle \bar{A}\psi_\lambda, \psi_\lambda\rangle = \omega(A) \tag{8.8}$$

for ν-almost every λ. Now it is easy to see that the quantum mixing property implies that for any $A, B \in \mathcal{A}$, $\omega(\bar{A}B) = \omega(A)\omega(B)$. Therefore, if we assume moreover (which is obviously true in the concrete example discussed below) that for any $A \in \mathcal{A}$, the family $(\omega(AB))_{B\in\mathcal{A}}$ determines $\langle A\psi_\lambda, \psi_\lambda\rangle$ for ν-almost every λ, we deduce immediately that the quantum mixing implies (8.8), and hence quantum ergodicity.

Concerning this definition of quantum ergodicity, we remark that it is trivially included in the notion of ergodicity of the W^* dynamical systems [Be, BR] with respect to the triple $(\mathcal{A}, \Theta, \phi)$ where Θ is the automorphism of $\mathcal{A}$ generated by the unitary group e^{iHt} and ϕ is the state defined by the microcanonical or canonical measure.

Let us now turn to an explicit construction, in the particular case $\mathcal{H} = L^2(\mathbb{R}^m)$ ($m < +\infty$ fixed) mentioned in the introduction, of the measures $d\nu(\lambda)$, both in the microcanonical case and in the canonical one as well, through some natural choice of the set $\{\psi_\lambda : \lambda \in \Lambda\}$. This will also enable us to recover the classical definitions of mixing and ergodicity out of (8.3) and (8.6) at the classical limit $h \to 0$.

More precisely, for $\lambda = (\lambda_x, \lambda_\xi) \in \mathbb{R}^{2m}$ consider the Bargmann coherent states defined on $\mathbb{R}^{2m}$:

$$f_\lambda(x) = (\pi h)^{-m/4} e^{ix\lambda_\xi/h - (x-\lambda_x)^2/2h}. \tag{8.9}$$

Then it is well known (see e.g.[BS] Chapt.5) that for any trace class operator A on $L^2(\mathbb{R}^m)$, one has:

$$\int_{\mathbb{R}^{2m}} \langle Af_\lambda, f_\lambda\rangle_{L^2(\mathbb{R}^m)} d\lambda = \mathrm{Tr}\,(A)\ . \tag{8.10}$$

In particular, since $e^{-\beta H}$ is trace class on $L^2(\mathbb{R}^m)$, then

$$\int \|e^{-\beta H/2} f_\lambda\|^2 d\lambda = \int \langle e^{-\beta H} f_\lambda, f_\lambda\rangle d\lambda = \mathrm{Tr}\,(e^{-\beta H}) < +\infty$$

Hence we can consider the following probability measures on $\mathbb{R}^{2m}$:

$$d\nu_m(\lambda) = \frac{\|e^{-\beta H/2} f_\lambda\|^2 d\lambda}{\int \|e^{-\beta H/2} f_\lambda\|^2 d\lambda}, \quad d\nu_{\Delta,E}(\lambda) = \frac{\|\delta(H-E) f_\lambda\|^2 d\lambda}{\int \|\delta(H-E) f_\lambda\|^2 d\lambda}. \tag{8.11}$$

where as in §1, $\delta(H-E) = \sum_{n:E-\Delta<E_n<E} P_n$ with $\Delta > 0$ fixed. If we also set

$$\psi^c_\lambda = \frac{e^{-\beta H/2} f_\lambda}{\|e^{-\beta H/2} f_\lambda\|}, \quad \psi^{mc}_\lambda = \frac{\delta(H-E) f_\lambda}{\|\delta(H-E) f_\lambda\|}. \tag{8.12}$$

then we have the following result (to be compared with (8.2)):

Lemma 8.1 *For any bounded operator A on $L^2(\mathbb{R}^m)$, the following identities hold:*

$$\frac{\mathrm{Tr}\,\left(Ae^{-\beta H}\right)}{\mathrm{Tr}\,(e^{-\beta H})} = \int \langle A\psi^c_\lambda, \psi^c_\lambda\rangle d\nu_m(\lambda), \quad \frac{\mathrm{Tr}\,(A\delta(H-E))}{\mathrm{Tr}\,(\delta(H-E))} = \int \langle A\psi^{mc}_\lambda, \psi^{mc}_\lambda\rangle d\nu_{\Delta,E}(\lambda).$$

Proof- Just write:

$$\begin{aligned} &\mathrm{Tr}\,\left(Ae^{-\beta H}\right) = \mathrm{Tr}\,\left(e^{-\beta H/2} A e^{-\beta H/2}\right), \\ &\mathrm{Tr}\,(A\delta(H-E)) = \mathrm{Tr}(\delta(H-E)A\delta(H-E)) \end{aligned}$$

and use (8.10). □.

Consider now the particular case where A is the h-Weyl quantization of a classical observable $a = a(x,\xi) \in \mathcal{S}(\mathbb{R}^{2m})$, namely the operator $Op^W_h(a)$ defined by the oscillatory integral:

$$Op^W_h(a)u(x) = (2\pi h)^{-m} \int e^{i(x-y)\xi/h} a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi. \tag{8.13}$$

A well known direct application of the stationary phase method yields

$$\lim_{h\to 0}\langle Op_h^W(a)f_\lambda, f_\lambda\rangle = a(\lambda).$$

As a consequence, if we also have

$$H = Op_h^W(q)$$

for some symbol $q \in C^\infty(\mathbb{R}^{2m})$, then the semiclassical symbolic and functional calculus of pseudodifferential operators (see [Ro]) immediately implies:

Lemma 8.2 *For any $\lambda \in \mathbb{R}^{2m}$,*

$$\lim_{h\to 0}\langle Op_h^W(a)\psi_\lambda, \psi_\lambda\rangle = a(\lambda).$$

Moreover,

$$\lim_{h\to 0}\int_{\mathbb{R}^{2m}}\langle Op_h^W(a)\psi_\lambda^c, \psi_\lambda^c\rangle d\nu_m(\lambda) = \int a(\lambda)d\mu^c(\lambda)$$
$$d\mu^c(\lambda) = \frac{e^{-\beta q}d\lambda}{\int_{\mathbb{R}^{2m}} e^{-\beta q}d\lambda}$$

and

$$\lim_{h\to 0}\int_{\mathbb{R}^{2m}}\langle Op_h^W(a)\psi_\lambda^{mc}, \psi_\lambda^{mc}\rangle d\nu_{\Delta,E}(\lambda) = \int_{\mathbb{R}^{2m}} a(\lambda)d\mu^{mc}(\lambda)$$
$$d\mu^{mc}(\lambda) = \frac{\delta(q-E)d\lambda}{\int_{\mathbb{R}^{2m}}\delta(q-E)d\lambda}$$

Since, by the semiclassical Egorov theorem ([Ro], §5.4) the principal symbol of $e^{itH/h}Op_h^W(a)e^{-itH/h}$ is given by

$$a_t(x,\xi) = a(\phi_t(x,\xi))$$

where ϕ_t is the Hamiltonian flow generated by q, it follows that for any $a, b \in \mathcal{S}(\mathbb{R}^{2m})$:

$$\lim_{h\to 0}\langle e^{itH/h}Op_h^W(a)e^{-itH/h}Op_h^W(b)\psi_\lambda, \psi_\lambda\rangle = a(\phi_t(\lambda))b(\lambda). \tag{8.14}$$

It has now become clear out of Lemma 8.2 and (8.14) that the quantum notions of mixing and ergodicity given by (8.3) and (8.6) formally yield the corresponding classical notions as $h \to 0$.

As a final remark let us mention that if A is a pseudodifferential operator also the Von Neumann definition (1.6) reproduces the classical one at the classical limit if $\langle u_n, Au_n\rangle$ tends to the phase average of the symbol of A, as verified in many instances (see e.g.[Sc, CdV, HMR, Ze1, Ze2, DEGI]), in which H is the quantization of a Hamiltonian generating an ergodic flow. Some authors ([Sa, Ze2] assume this limiting property as the very definition of quantum ergodicity.

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