On nonconvex caustics of convex billiards

Oliver Knill*

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Abstract

There are billiard tables of constant width, for which the billiard map has invariant curves in the phase space which belong to continuous but nowhere differentiable caustics. We apply this to construct ruled surfaces which have a nowhere differentiable lines of striction. We use it also to get Riemannian metrics on the sphere such that the caustic belonging at least one point on the sphere is nowhere differentiable. For three dimensional billiards, we find three dimensional billiard surfaces with nonconvex rough caustics.

1 Convex billiards

Describing the long time behavior of a path of a light ray or billiard ball in a convex domain is an interesting mathematical problem. Good starting points in the literature are [21, 31, 6, 33, 27]. In the case, when the domain is a strictly convex planar region, the return map to the boundary T defines an area preserving map ϕ of the annulus $A = \mathbf{T} \times [0,1]$, where the circle $\mathbf{T}$ parametrizes the table with arc length s: the impact point s at the boundary T and the angle of impact θ define the next impact point s_1 with angle θ_1. The map is given by $\phi : (s, \cos(\theta)) \mapsto (s_1, \cos(\theta_1))$. Noncontractible invariant curves Γ in A can define caustics γ in the interior of the table T. These are curves with the property that a ball, once tangent to γ stays tangent. The precise relation between the invariant curve Γ and the caustic γ is discussed in the next section.

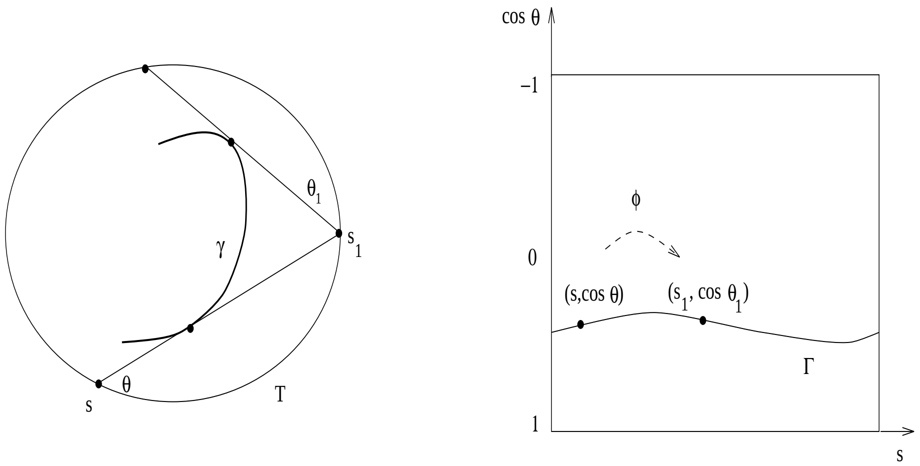

Figure 1) The billiard map $\phi : A \to A$ with an invariant curve Γ coming from a caustic γ.

*Division of Physics, Mathematics and Astronomy, Caltech, Pasadena, CA, 91125, Present address: Department of Mathematics, University of Arizona, Tucson, AZ, 85721, USA

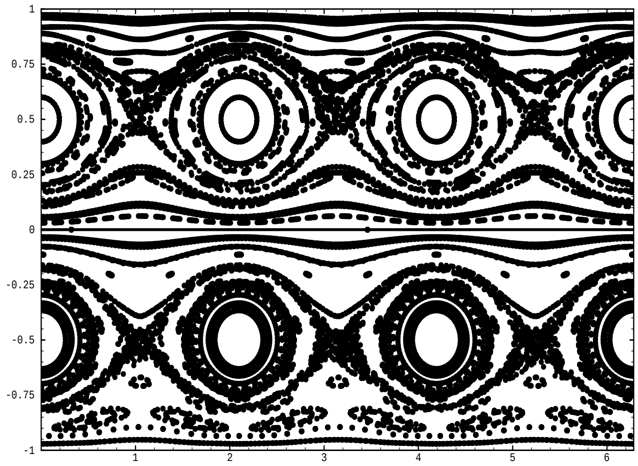

Figure 2) Some orbits of $\phi : A \to A$ where ϕ is a map obtained from a convex billiard table of equal thickness. Beside the invariant curve $\{\cos(\theta) = 0\}$, we see other invariant curves. The ones near the boundary illustrate Douady-Lazutkin's result.

Invariant curves Γ of ϕ are interesting for the dynamics because they divide the phase space A into invariant regions and prevent so ergodicity. They occur often because near the boundary of a strictly convex C^6 table, there are invariant curves belonging to convex caustics. They define "whisper galeries" (see [9]). Smoothness is crucial because a Baire generic convex billiard has no invariant curves [12]. A criterion of Mather [23] for the nonexistence of invariant curves is that the table T has somewhere a vanishing curvature. If the curvature of the table is unbounded at some point, then there are no invariant curves in a neighborhood of the boundary of A [20].

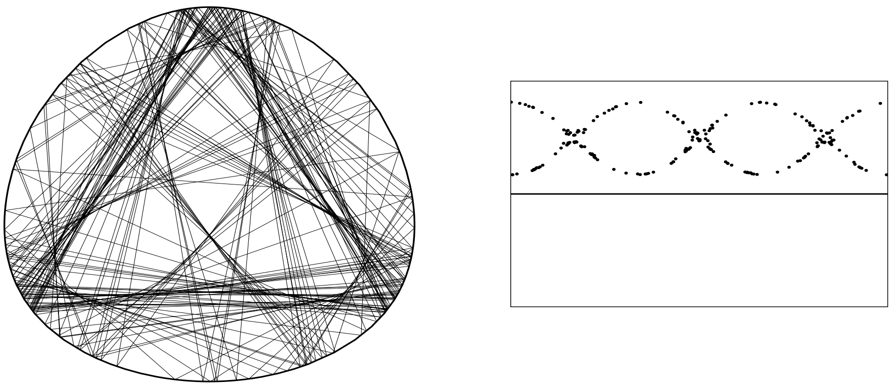

Figure 3a). A finite piece of a chaotic billiard trajectory. To the right, the same orbit is shown in the phase space A.

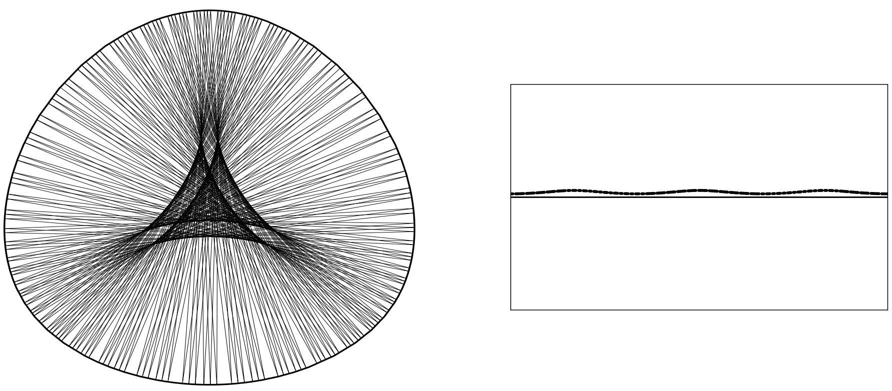

Figure 3b). A finite piece of an orbit with steep impact angle. The orbit in phase space A is near the invariant equator.

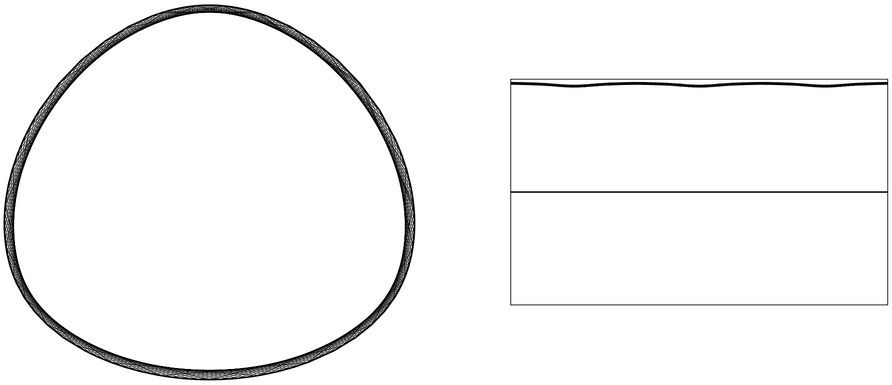

Figure 3c). A finite piece of a "whisper galery" orbit with angle of impact near π. The orbit in phase space A is near the invariant boundary.

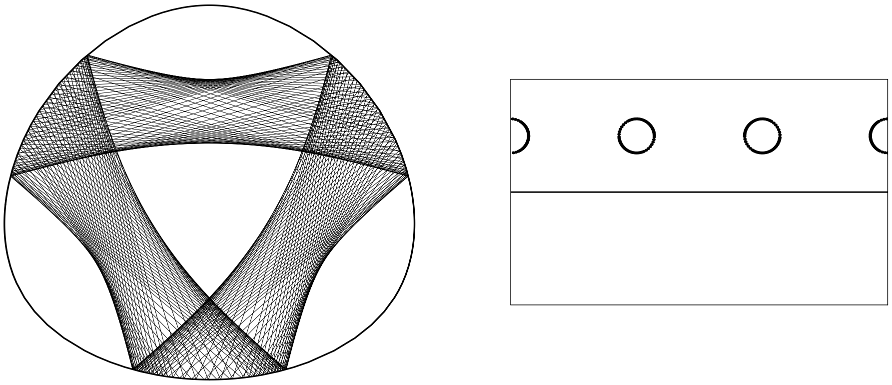

Figure 3d). A finite piece of a orbit near a linearly stable periodic orbit of period 3. The map ϕ^3 in phase space A leaves invariant a contractible curve. For conditions on the existence of such curves see [24].

We deal here with a class of billiard tables of constant width. In this case, invariant curves Γ exist a priori. We are interested in the regularity of the corresponding caustics γ. Part of the literature about caustics in billiards deals with convex caustics (e.g. [32, 1, 16]). Nonconvex caustics were drawn in [9] and further discussed in [15]. The aim of our article is to point out that there are caustics, which are nowhere differentiable and to illustrate the story with some pictures. Furthermore, we want to show the relation of billiard caustics with caustics in differential geometry. There, a caustics to a surface F in a manifold M is the place, where wave fronts radiating away from F stop being immersed manifolds. A caustic to a point $P \in M$ is the caustic of a sufficiently small geodesic ball around P. The geometry of such caustics in four dimensional space-time manifolds has astrophysical relevance because an astronomer sees a distorted picture of an object sitting in a point P for which the earth is located in the caustic [26, 17]. When is there a relation between billiard caustics and differential geometrical caustics? In the special situations considered here, the table can be embedded into a Riemannian sphere M such that the boundary T of the table is a wave front of a point $P \in M$. The caustic of P in the sphere is then the caustics of the billiard table. An other relation with differential geometry of surfaces is that on can define a two dimensional ruled surface from a billiard by considering the orbits in "space time". The line of friction of these surfaces project down to the caustic. We finally discuss caustics of three dimensional billiards in regions of equal thickness. Because in higher dimensions, only ellipsoids have convex caustics [5, 13], caustics of higher dimensional billiards must be nonconvex. They can additionally be quite rough.

Acknowledgments. I want to take the opportunity to thank Eugene Gutkin for discussions on billiards and caustics during a previous collaboration on billiards.

2 Caustics of convex billiards

Let T be a strictly convex billiard table determined by its radius of curvature $\rho \in L^2(\mathbf{T})$ which we assume to be positive everywhere. The function ρ has then an almost everywhere convergent Fourier expansion

$\rho(s) = \sum_{k\in\mathbf{Z}} a_k e^{iks}$. To formulate the results, it is convenient to take ρ in one of the Hilbert spaces

$$H^\mu(\mathbf{T}) = \{ f \in L^2(\mathbf{T}) \mid f(s) = \sum_{k\in\mathbf{Z}} a_k e^{iks}, ||f||_\mu = \sum_{k\in\mathbf{Z}} |k|^{2\mu} |a_k|^2 < \infty \}$$

which are all contained in the Hilbert space $L^2(\mathbf{T})$ of square integrable periodic functions. For $\mu > 1/2$, a function in $H^\mu(\mathbf{T})$ has some Hölder continuity and the Fourier series converge uniformly (e.g. [22]). More generally, by a Sobolev embedding theorem, if $\mu > 1/2 + r$, $r \in \mathbf{N}$, then every $f \in H^\mu(\mathbf{T})$ is r times continuously differentiable: $f \in C^r(\mathbf{T})$. We use here H^μ mainly for notational reason in order to control with a single parameter μ both nowhere-differentiable and smooth situations. Not much is lost for a reader who decides to ignore the spaces H^μ which are by the way useful in the calculus of variations (see [19]).

Douady remarked that a C^k invariant curve Γ of a C^{k+1} table defines a C^{k-1} caustic (see [9] Remark I.4.6. I'm grateful to J.M. Strelcyn for sending me a copy of [9]). This result holds also for intermediate degrees of smoothness and the formula allows to define the caustic for any C^2 table T.

Lemma 2.1 (R. Douady) *If the invariant curve Γ is in $H^\mu(\mathbf{T})$ and the table T is in $H^{\mu+1}(\mathbf{T},\mathbf{C})$, then it defines the caustic $\gamma \in H^{\mu-1}(\mathbf{T},\mathbf{C})$ by*

$$\boxed{\gamma(s) = T(s) + \frac{\sin(\theta)}{\kappa(s) + \Gamma'(s)} u(s)}$$

Here $u(s)$ is a unit vector in the direction of the trajectory and $\kappa(s) = \rho(s)^{-1}$ is the curvature at $T(s)$. This formula defines a "caustic" for any invariant curve of a billiard map belonging to a table T which has $\rho \in L^\infty(\mathbf{T})$, with $\rho \geq \epsilon > 0$.

Proof. Let $\delta(s) = \arg(T(s_1) - T(s))$ be the angle between the x-axis in $\mathbf{R}^2$ and the trajectory connecting $T(s), T(s_1)$. We have $\gamma(s) = T(s) + b(s)u(s)$, where $b(s) = \sin(\theta)(d\delta/ds)^{-1}$. If $\alpha(s)$ is the angle of the tangent in the point $T(s)$, then $\delta(s) = \theta(s) + \alpha(s)$ and $\delta' = \theta' + \alpha'(s) = \kappa(s) + \Gamma'(s)$, so that $b(s) = \sin(\theta)(\kappa(s) + \Gamma'(s))^{-1}$.

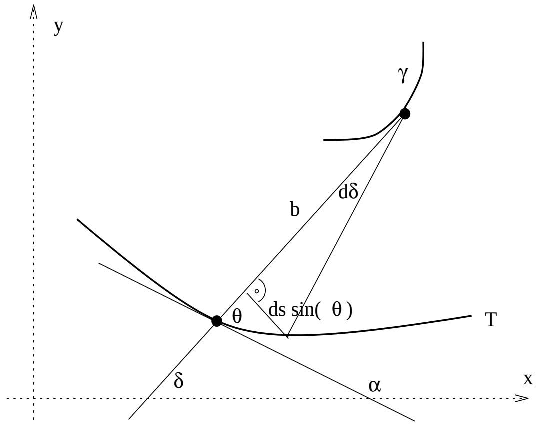

Figure 4). The relation $b(s) = \sin(\theta)(d\delta/ds)^{-1}$ in the proof of Lemma 2.1.

Both κ and Γ' are in $H^{\mu-1}$. Because the table is strictly convex, $\delta'(s) > 0$ for all s so that $\kappa + \Gamma' > 0$ and $\gamma \in H^{\mu-1}(\mathbf{T})$.

In [15], it was asked how to define in general a caustic of a convex C^2 table with only Lipschitz invariant curve Γ. The equation in Lemma 2.1 does this for a table with invariant circle Γ and bounded measurable curvature function κ. Because the derivative $\Gamma'(s)$ is a bounded measurable function by Birkhoff's theorem (see [21]), the image of the function $\gamma : \mathbf{T} \to \mathbf{C}$ is in general a measurable set in the complex plane. □

3 Caustics of tables of equal thickness

If $\Gamma(s) = \{\cos(\theta) = 0\}$ is an invariant curve of the billiard map, then T is a curve of equal thickness [29, 14]. Each orbit on Γ has period 2. A corresponding billiard trajectory tangent to the caustic both leaves and hits the table orthogonally. It is more convenient to parametrize such a table by the angle α of the tangent instead of the arc length s of the table. A bounded measurable function ρ satisfying $\rho(\alpha) + \rho(\alpha + \pi) = \text{const}$ and $\int_0^{2\pi} \rho(\alpha)e^{i\alpha}$ defines such a table in the complex plane by

$$T(\alpha) = T(0) + \int_0^\alpha \rho(\beta)e^{i\beta}\, d\beta \,, \tag{1}$$

if $\int_0^{2\pi} \rho(\alpha)e^{i\alpha}\, d\alpha = 0$. If $T \in H^{\mu+1}$, then $\rho \in H^\mu$. In the case, when the invariant curve Γ is the equator, $\Gamma'(s) = 0$ and the formula in Lemma 2.1 is

$$\gamma(\alpha) = T(\alpha) + \rho(\alpha)ie^{i\alpha} \,. \tag{2}$$

This is the formula for an evolute of the table T ([7]). If we plug in the Fourier series $\rho(\alpha) = \sum_{k\in\mathbf{Z}} a_k e^{ik\alpha}$, we get

$$\gamma(\alpha) = T(0) - i(\sum_{k\in\mathbf{Z}} \frac{a_k}{(k+1)}(e^{i(k+1)\alpha} - 1) + i\sum_{k\in\mathbf{Z}} a_k e^{i(k+1)\alpha} = \gamma(0) + i\sum_{k\in\mathbf{Z}} \frac{k}{k+1} a_k (e^{i(k+1)\alpha} - 1) \,.$$

We need $a_1 = 0$ in order that the table is closed. The condition $\rho(\alpha) + \rho(\alpha + \pi) = 2a_0$ holds if $a_{2k} = 0$ for $k \neq 0$. For ρ to be real, we have to require also $a_{-k} = \overline{a_k}$. We summarize this as follows:

Proposition 3.1 *Given a positive function $\rho(\alpha) = \sum_{k\in\mathbf{Z}} a_k e^{ik\alpha} \in H^\mu(\mathbf{T})$ with $a_{-k} = \overline{a_k}$ satisfying $a_1 = 0$ and $a_{2k} = 0$ for $k \neq 0$. It defines the $H^{\mu+1}$ table or equal thickness*

$$\boxed{T(\alpha) = T(0) - i\sum_{k\in\mathbf{Z}} \frac{a_k}{k+1}(e^{i(k+1)\alpha} - 1) \,.}$$

The corresponding billiard map ϕ has the equator $\Gamma = \{\cos(\theta) = 0\}$ as an invariant curve and Γ defines the H^μ caustic

$$\boxed{\gamma(\alpha) = \gamma(0) + i\sum_{k\in\mathbf{Z}} \frac{k}{k+1} a_k (e^{i(k+1)\alpha} - 1)}$$

where $\gamma(0) = T(0) + ia_0$. If $\mu > 1/2 + r$, then ϕ and the function γ are r times continuously differentiable.

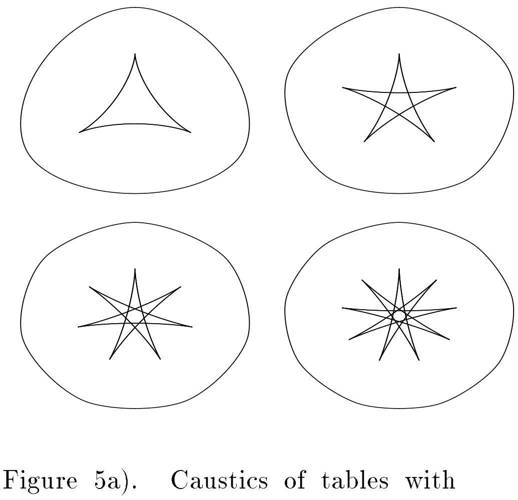

Figure 5a). Caustics of tables with equal thickness given by the curve determined through $\rho(\alpha) = a_0 + \cos((1 + 2k)\alpha)$ for $k = 1, 2, 3, 4$ with $a_0 = 4$.

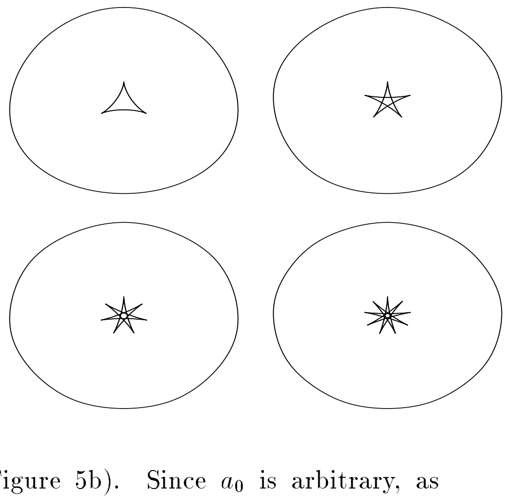

Figure 5b). Since a_0 is arbitrary, as long as $\rho > 0$, the tables are not unique. Here are the same caustics as in Figure 4b) but with tables using $a_0 = 10$.

From formulas (1), (2), we get $\gamma'(\alpha) = \rho'(\alpha) i e^{i\alpha}$. This shows that ρ has the same critical points as γ. We see explicitely the fact that the vertices of T correspond to cusps of γ.
The caustics in Figures 5),6) were drawn with Mathematica [35] using trigonometric polynomials $\rho(\alpha)$. While the functions γ were real analytic, the image of γ has in general singularities. Indeed, according to a result of Tabachnikov (see [30, 2]), the function ρ' has at least N zeros, if it is a trigonometric polynomial of degree N. If the zeros are simple, the curve γ has at least N cusps counted with multiplicity. For the caustics belonging to the equator, the map γ is 2 to 1. Therefore, if T has N nondegenerate vertices, then a caustic γ belonging to the equator has $[N/2]$ cusps.

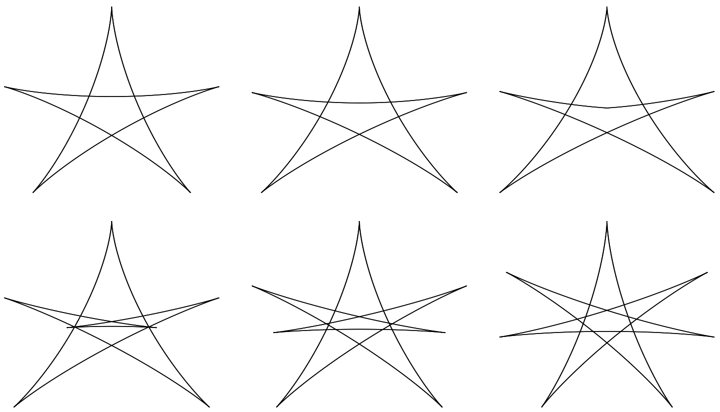

Figure 6a). Examples of caustics obtained from tables with $\rho(\alpha) = a_0 + g\cos(5k\alpha) + (1-g)\cos(7k\alpha)$ for $g = k/7$, $k = 1,2,3,4,5,6$. We see an interpolation between the caustics with 5-fold and 7-fold symmetry in Figure 5a).

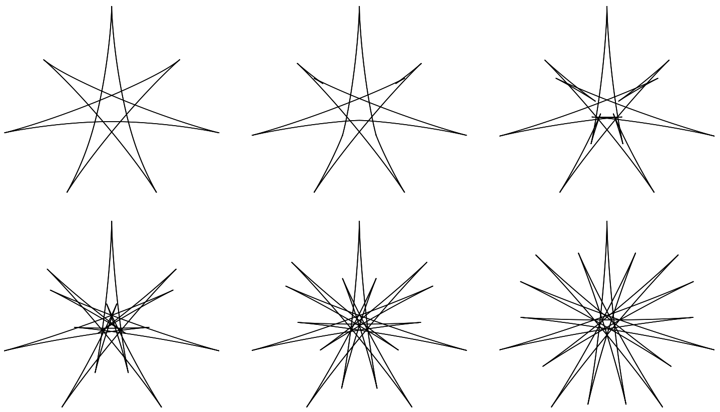

Figure 6b). Examples of caustics using $\rho(\alpha) = a_0 + g\cos(7k\alpha) + (1-g)\cos(17k\alpha)$ for $g = k/7$, $k = 1,2,3,4,5,6$. We see an interpolation between caustics with 7-fold and 17-fold symmetry.

4 Nowhere differentiable caustics

Corollary 4.1 *There exist continuous caustics γ of tables of equal thickness which have both coordinate functions of γ nowhere differentiable except maybe at the points $\alpha = k\pi/2, k = 0,\dots,3$.*

Proof. Let $1 > \mu > 1/2$. Take $a_k = k^{-\mu}$ for $k = 2^n$ and $a_k = 0$ else. This Weierstrass function ρ is in H^μ for $\mu' < \mu$ and so continuous. The corresponding caustic is also continuous because of Lemma 2.1. Up to a translation and a rotation in the plane, we have $\gamma(\alpha) = \sum_{k \in \mathbf{Z}} \frac{k}{k+1} a_k \exp(i(k+1)\alpha)$ so that

$$\begin{aligned}
\mathrm{Re}(\gamma)(\alpha) &= \sum_{k=2}^{\infty} \frac{k}{k+1} a_k \cos((k+1)\alpha) + \sum_{k=2}^{\infty} \frac{k}{k-1} a_k \cos((k-1)\alpha) \\
&= \sum_{k=2}^{\infty} a_k \left(\frac{k}{k+1}\cos(2\alpha) + \frac{k}{k-1}\right)\cos((k-1)\alpha) - \frac{k}{k+1}\sin(2\alpha)\sin((k-1)\alpha) \\
&= \sum_{k=2}^{\infty} a_k h_k(\alpha) .
\end{aligned}$$

Using $a_{2^l} h_{2^l}(\alpha_{m,\pm}) = \mp 2^{-l\mu} \frac{2^l}{2^l+1} \sin(2\alpha_{m,\pm})$ for $\alpha_{m,\pm} = \pm\frac{\pi}{2}2^{-l} + m 2\pi 2^{-l}, m \in \mathbf{Z}$ it is not difficult to check that $\mathrm{Re}(\gamma)$ is nowhere differentiable except maybe at the points $\alpha = 0, \pi$. Details and an analogue statement for $\mathrm{Im}(\gamma)$, (where we have to exclude the points $\alpha = \pm\pi/2$) are left to the reader. $\square$

Remarks.
1) The fact that we could have differentiability at some finite set of points is due to the symmetry of the curves. This symmetry is broken, if the a_k are no more real. We have no doubt that one could construct examples, where one has nowhere differentiability without exceptions.
2) Because by a theorem of Banach, nowhere differentiable functions are Baire generic among continuous functions [25], we expect also that a "general" curve of equal thickness produces nowhere differentiable caustics.
3) Because every convex not too long curve can be extended to a curve of equal thickness, there are many tables for which the caustic is in some interval nowhere differentiable.

5 Nowhere differentiable lines of striction of ruled surfaces

Place the billiard table in a plane $\{z = 0\}$ in Euclidean space $\mathbf{R}^3 \sim \mathbf{C} \times \mathbf{R}$. Let the billiard ball also move in the z direction by giving it a $z-$velocity $1/l$ if l is the length of the trajectory between successive impacts $\{(T(s_k), z_k)\}$ on the boundary of the two dimensional zylinder $T \times \mathbf{R}$. A ball starting at the boundary $(T, 0)$ hits the lifted table $(T, 1)$ in the plane $\{z = 1\}$. The union of all billiard ball trajectories which are in the plane tangent to the caustic γ form a ruled surface, a one parameter family of straight lines.

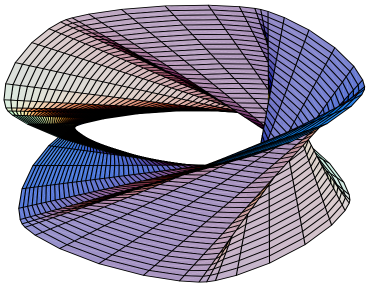

Figure 7a). A ruled surface constructed from a billiard table with $\rho(\alpha) = p + \cos(5\alpha)$. Such a table has exactly 5 flat invariant curves $\Gamma = \{\theta = \theta_i\}$, where θ_i solves $\tan(5\theta) = 5\tan(\theta)$ [14]. We see the ruled surface belonging to one of these invariant curves different from the equator.

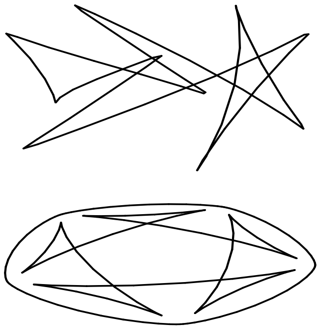

Figure 7b). The line of striction of the surface in Figure 7a) projects down to the caustic of the table. (For better visibility, the line of striction to the table has been lifted up a bit in this drawing).

We should mention, how one constructs the line of striction of a (not necessarily smooth) one parameter family of curves $\gamma(t), t \in [0, 1]$: consider for $n \in \mathbf{N}$ the curves $\gamma(k/n), k = 1, \dots, n$. Determine for each k the point on the curve $\gamma(k/n)$ which is nearest to $\gamma((k+1)/n)$. Connecting these $n-1$ points gives a polygon. Any accumulation point in $C([0, 1], \mathbf{R}^3)$ of such curves defines a generalized line of striction.
The line of striction of the ruled surface constructed from the table with a caustic projects along the z direction (see Figure 7a) to the caustic. However, unlike the caustic itself, the line of striction does not need to have self intersections. The example in Figure 7a) illustrates this.

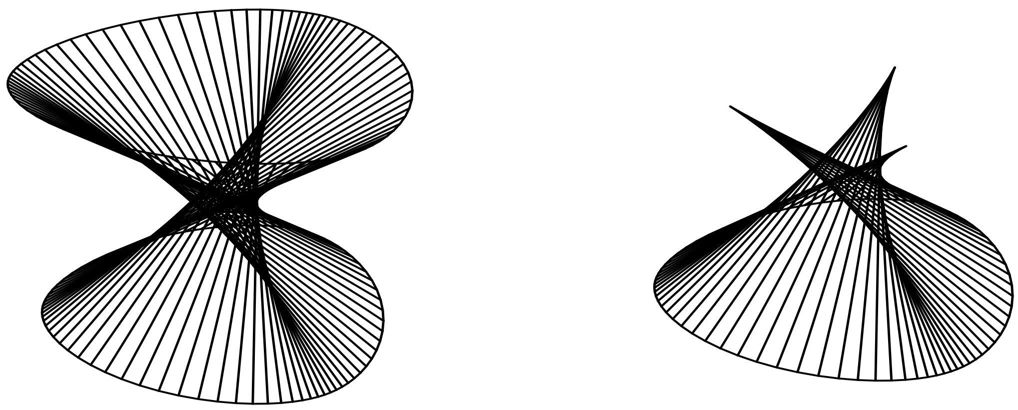

Figure 8). The ruled surface obtain from T given by $\rho(\alpha) = a_0 + \cos(3\alpha)$, where Γ is the equator. The projection of the line of striction is the caustic seen in Figure 5a). In the drawing to the right, the surface has been drawn only until the line of striction.

Corollary 5.1 *There exist differentiable ruled surfaces in $\mathbf{R}^3$, whose line of striction is a curve which has all coordinate functions nowhere differentiable (except maybe at four points).*

Proof. Take a nowhere differentiable caustic as constructed above. The two first coordinates of the line of striction are the two coordinates of the caustics and so nowhere differentiable functions. Because the third coordinate of the line of striction is $\rho(\alpha)$, which is a nowhere differentiable Weierstrass function, all coordinate functions are nowhere differentiable. $\square$

6 Nowhere differentiable caustics on compact manifolds

The caustic of a point on a Riemannian manifold is the set of intersection points of infinitesimally closed geodesics at this point. A motivating picture for studiing billiards is that the billiard flow can be considered as a limiting case of the geodesic flow [3]. One can now ask, whether caustics of billiards can be related to caustics of a family of geodesics starting at a point in a two dimensional Riemannian manifold M. Here, in the case of billiards in tables of equal thickness, an orbit tangent to the caustic can actually be related to an orbit of a geodesic flow on a sphere. It gives so manifolds, where the caustics of all orders of some point are known.

Corollary 6.1 *There exist metrics g on the two sphere S^2 and a point $P \in S^2$ such that the caustic of this point has nowhere differentiable coordinate functions (except maybe at four points). The metric can have the same smoothness as the table T.*

Proof. Take the north hemi-sphere with the usual metric and let P be the North pole. A geodesic trajectory starting at P hits the equator at a right angle. Deform the south-turning-circle below the equator into a curve T of equal thickness and fill the south cap with a flat metric. Put on the remaining annulus smooth coordinates $(\phi, u) \in \mathbf{T} \times [0, 1]$ such that $\{u = 0\}$ is the equator and $\{u = 1\} = T$. By taking a metric in this ring for which the coordinates are flat, a geodesic flow entering orthogonally the annulus at the equator leaves the annulus orthogonally to T and returns to P after having been tangent to the caustics γ. $\square$

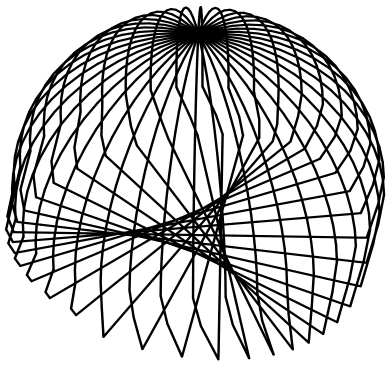

Figure 9). Geodesics starting on the north pole P of a deformed sphere M with a Riemannian metric g. The caustic of P on M is a caustic of a table T of equal thickness. Every geodesic starting in P is closed. All such geodesics have equal length. (This is a special case of a more general fact for closed geodesics through a point on a two dimensional surface [11]).

7 Are there examples of fractal caustics?

We have seen that the coordinate functions of caustics γ can be "rough". Can the image of γ become "fractal"? More precicely: do there exist convex tables T such that the image of γ has Hausdorff dimension μ for some $\mu > 1$? Corresponding questions for graphs of real functions can be difficult and often, one is only able to estimate the box-counting dimension as it is in the case of the Weierstrass function [10].

We could try to find the distortion of the map from the graph $G = \{(\alpha, \rho(\alpha))\} \subset \mathbf{R}^2$ of ρ to γ. Let $T(\alpha) = (f(\alpha), g(\alpha))$ be the table defined by a function $\rho \in H^\mu$. If ρ is a Weierstrass function then the graph G of ρ has box-counting dimension > 1 [10]. The transformation $\phi : \mathbf{R}^2 \to \mathbf{R}^2$: $\phi(\alpha, \beta) = (f(\alpha) - \sin(\alpha)\beta, g(\alpha) + \cos(\alpha)\beta)$ maps G onto the caustic γ. If it were invertible near $\gamma(\mathbf{T})$, the image ofγ would herite the fractal properties of G. Let's see: the Jacobian of ϕ is

$$D\phi(\alpha, \beta) = \begin{pmatrix} f'(\alpha) - \beta\cos(\alpha) & -\sin(\alpha) \\ g'(\alpha) - \beta\sin(\alpha) & \cos(\alpha) \end{pmatrix} .$$

Because $T'(\alpha) = \rho(\alpha)e^{i\alpha}$, one has $f'(\alpha) = \cos(\alpha)\rho(\alpha), g'(\alpha) = \sin(\alpha)\rho(\alpha)$ and unfortunately, $\det D\phi$ vanishes exactly on G. While this approach does not lead to caustics with fractional box counting dimension, we know why:

Proposition 7.1 *The caustic of a curve of equal thickness is the place, where the wave front formed by the ray of light starting orthogonally to the table of equal thickness fails to be an immersed submanifold.*

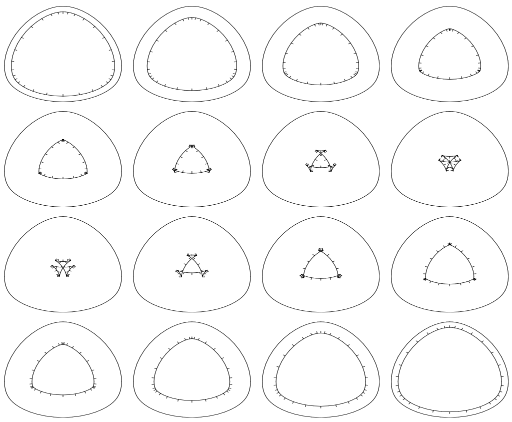

Figure 10a). Wave fronts from the table with $\rho(\alpha) = a_0 + \cos(3\alpha)$ radiating inside and reversing inside out during the time evolution. The wave fronts are the contour lines of the ruled surface of the caustic. γ.

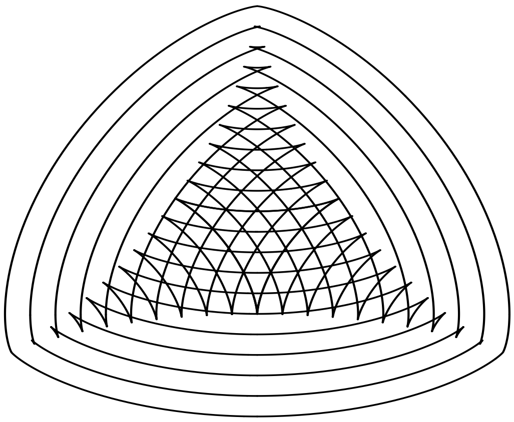

Figure 10b). A simultaneous view of the wave fronts indicates the location of the caustic. The family of fronts is called the standard Legendre collapse. It also illustrates a theorem of Arnold [2] stating that some of these wave fronts have should have at least four cusps.

When plotting caustics using larger and larger frequencies in ρ, the curves gets usually more and more cusps (see Figure 11 a). The pictures let expect that the box-counting dimension or even the Hausdorff dimension can be bigger than 1 for some caustics but we do not know.

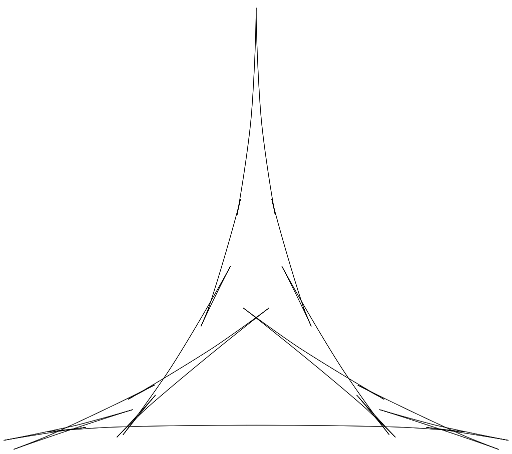

Figure 11a). A caustic $\alpha \mapsto \gamma(\alpha)$ of a table approximating the table given by the Weierstrass function ρ constructed in the proof of Corollary 4.1.

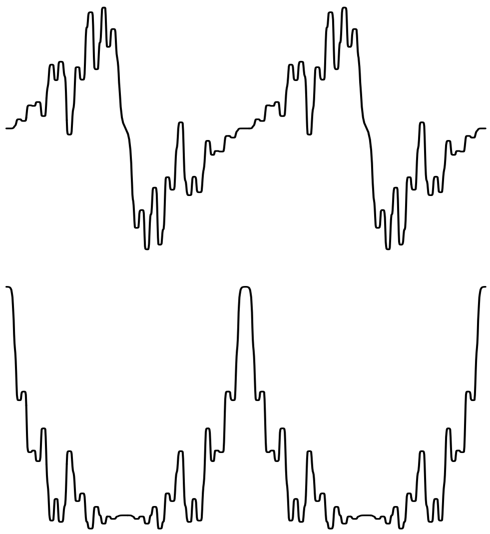

Figure 11b). The upper graph is the function $\alpha \mapsto \mathrm{Re}(\gamma)(\alpha)$, the lower graph is the function $\alpha \mapsto \mathrm{Im}(\gamma)(\alpha)$.

8 Caustics of three dimensional billiards

Not so much is known about higher dimensional billiards. Some few information can be found in [6, 31, 33]. A three dimensional version of a billiard is obtained by taking a C^2 convex surface R in $\mathbf{R}^3$ which is homeomorphic to the sphere. The return map to the boundary defines a map ϕ on a four dimensional manifold $R \times R$. This map ϕ leaves invariant a smooth measure. Generalising results from two to three dimensions are not straightforward. For example, a result of Wojtkowsky [34] shows that putting convex pieces of the boundary sufficiently far apart does not necessarily render the periodic orbits unstable as it does for two dimensional billiards. An other result is the nonexistence of convex caustics [13] which has the consequence that higher dimensional "whisper galeries" are not possible. We consider here caustics of three dimensional billiards. The title of our section certainly clashes with the title "only ellipsoids have caustics" in [13]! The reason is that the term caustic is often used for "convex caustic". That higher dimensional billiards can have nonconvex caustics is quite easy to see, even so we did not find this observation mentioned in the literature.

Lemma 8.1 *a) Take a curve T of equal thickness in the $x-z$ plane in $\mathbf{R}^3$. Assume that T is reflection symmetric with respect to the z-axes L. If we rotate T around L, we get a surface R of revolution which has the property of being a surface of equal thickness.*
b) A billiard trajectory orthogonal to one point P of R intersects the surface R a second time orthogonally. It corresponds to a 2-periodic orbit of the return map ϕ.

Proof. a) For any direction given by a line K through the origin, the thickness of the region in that direction is the thickness of a curve T which lies in a plane containing K. (If we place the region between two parallel planes, then the two points, which touch the planes belong to one of the rotated curves of equal thickness).
b) Consider now a line K orthogonal to the surface R in a point P. There is a plane Σ through this line which intersects R in a curve T of equal thickness. The line K intersects this curve T orthogonally in a point P_1. Because R is symmetric with respect to the reflections in Σ, the line K intersects also R orthogonally in P_1 and not only the curve T. □

Remark. We did not find mentioned nor drawn rotational symmetric surfaces of equal thickness in the litarature nor seen billiards inside them. Surfaces of equal thickness were drawn in [18] but they do not seem to be surfaces of revolution. For more information about regions of equal thickness, see [8].

By symmetry, it is clear that a two dimensional caustic of a three dimensional billiard is obtained by rotating the caustic γ of T around L.

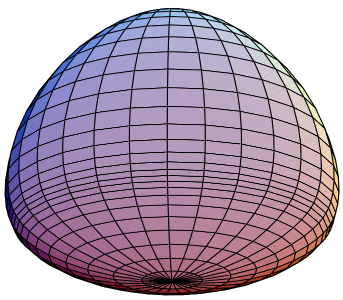

Figure 12a). A surface of equal thickness in space can be obtained by rotating a curve T of equal thickness around a symmetry axes L.

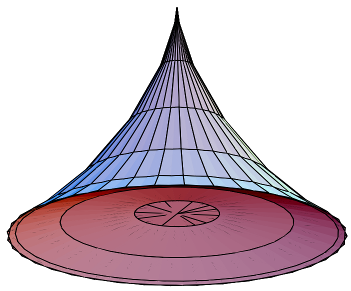

Figure 12b). The caustic to the surface of equal thickness is obtained by rotating the caustic γ of T around the symmetry axes L.

What does the existence of caustic say about the billiard dynamics? For two dimensional billiards, invariant curves prevented ergodicity. Here, the symmetry of the surface of revolution leads to a billiard map ϕ which has a Noether integral. This "angular momentum integral" exists in a similar way as the Clairaut integral exists for the geodesic flow on surfaces of revolution.

One can now ask about the dynamical properties of ϕ on a three dimensional leaf $I = const$. First of all, for each angular momentum, there are trajectories which lie in the plane orthogonal to L. They have as an intersection with R a circle of maximal radius. Depending on the angular momentum, such an orbit is periodic or quasiperiodic.

Consider now the situation, when the angular momentum is zero. An orbit moves in a plane through the rotation axis L. We observe that the three dimensional zero angular momentum leaf is foliated by two dimensional invariant manifolds on which the dynamics is the two dimensional billiard map (see Figure 13).

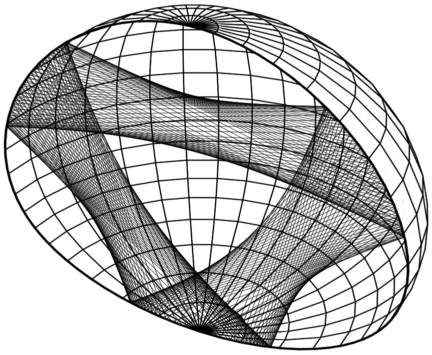

Figure 13). Trajectories with zero angular momentum in a three dimensional billiard of equal thickness move in two dimensional planes where they are obtained by iterating the two dimensional billiard map.

The fact that we can find the two dimensional billiard inside the three dimensional billiard has the consequence that we know the existence of many periodic orbits, more than have to be expected in general: a theorem of Babenko [4] assures only the existence of at least n geometrically distinct periodic orbits. Also, by the variational theorem in ergodic theory (see e.g. [21]), the topological entropy of the three dimensional billiard map is bigger or equal to the topological entropy of the two dimensional billiard because every invariant measure of the two dimensional billiard gives rise to an invariant measure of the three dimensional one. The two dimensional invariant surfaces in the phase space A which come from trajectories tangent to the caustic are all contained in the zero angular momentum surface. Hence, they do not directly influence the map on a leaf with positive angular momentum. Ergodicity is not a priori excluded on a three dimensional invariant leaf with nonzero angular momentum, even so one should expect nonergodicity because of quasiperiodic orbits on a set of positive measure.

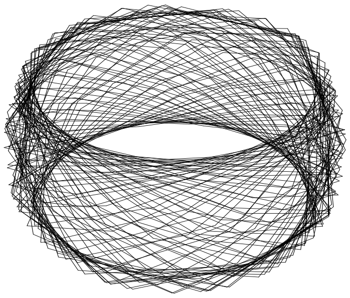

Figure 14a). Playing billiards in a surface of equal thickness. We see a finite piece of a trajectory in the surface of revolution determined by $\rho(\alpha) = 4 + \cos(3\alpha)$.

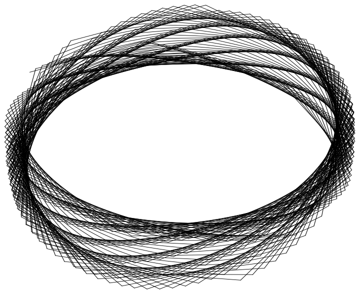

Figure 14b). An finite piece of an other orbit with a bit larger angular momentum. in the same surface as in Figure 14a).

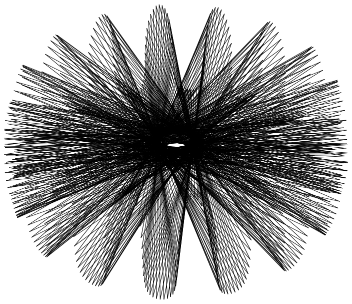

Figure 14c). This piece of orbit seems to ly on a two dimensional torus in the three dimensional leaf of constant angular momentum.

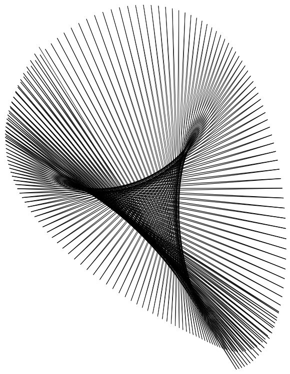

Figure 14d). A piece of an orbit with relatively small angular momementum.

It would be interesting to know more about the dynamics of ϕ on a nonzero angular momentum leaf. Unless the table is a sphere, trajectories do not all ly in planes [28]. The reader might want to discover, which parts of the discussed relations of caustics with differential geometrical caustics can be generalized to three (or higher dimensional) billiards in (hyper) regions in $\mathbf{R}^n$ of equal thickness.

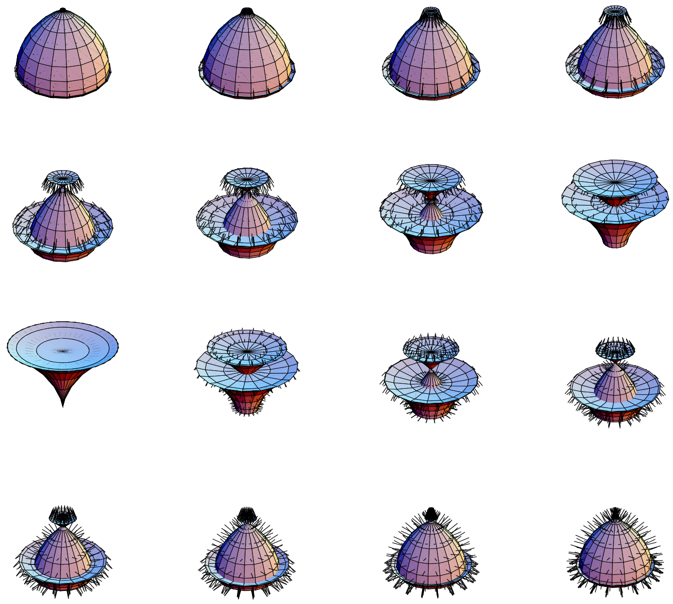

Figure 15). The standard Legendre collapse of a wave front moving from a surface of revolution of equal thickness. Two dimensional sections of this collapse are shown in Figure 10a). The surface turns inside out during this collapse as can be seen by the arrows which point inside the surface at the beginning and outside at the end Unlike in a famous movie, this "turning the sphere inside out" is not smooth: the smoothness gets lost at the points of the caustic.

Appendix: Is there a nonintegrable billiard with algebraic return map?

The numerical method, we used for computing orbits for the illustrations in Figure 2) and Figure 3) goes as follows: to construct the map $\phi : (\alpha, \theta) \mapsto (\alpha', \alpha' - \alpha - \theta)$, one has to find $\alpha' \neq \alpha$ so that $(T(\alpha) - T(\alpha'))e^{-(\alpha+\theta)}$ is real. This is equivalent to

$$\sum_{k \in \mathbf{Z}} \frac{a_k}{k+1} (\cos(k\alpha + \theta) - \cos((k+1)\alpha' - \alpha - \theta)) = 0 \, .$$

Numerically, the second root of this equation can be found very efficiently with a Newton method. (For three dimensional billiards, the situation is similar but slightly more complicated). In the simpest case $\rho(\alpha) = a_0 + \cos(3\alpha)$, we have the problem to solve

$$\cos(\alpha - 4x + \theta) - \cos(3\alpha - \theta) + 4a_0 \cos(\alpha - x + \theta) - 4a_0 \cos(\theta) + 2\cos(3\alpha + \theta) - 2\cos(\alpha + 2x + \theta) = 0$$

for $x = \alpha'$. An interesting question is, whether the second root $x \neq \alpha$ is an algebraic expression in α and θ. One has the problem to find roots of a specific class of octonic polynomials. More generally, one can ask, if there exists a table of equal thickness different from the circle, for which the return map is algebraic in some coordinates. We do not know of any smooth convex billiard table different from an ellipse, where the return map ϕ is an algebraic expression in some coordinates.

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