Study of a System of Coupled NLS Equations Under Periodic Boundary Conditions

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Abstract

The theory of a simple NLS equation under periodic boundary conditions has been recently studied by Y.Li et al, using the Floquet spectral analysis of its associated Zakharov–Shabat linear operator, with the purpose of describing homoclinic solutions of perturbations of the NLS equation. Here, we show that a system of three coupled NLS equations

$$iq_{1t} = q_{1xx} + 2|q_1|^2 q_1 + 4q_1|q_2|^2 + 2q_2^2 q_3^*$$

$$iq_{2t} = q_{2xx} + 2|q_1|^2 q_2 + 2|q_2|^2 q_2 + 2|q_3|^2 q_2 + 2q_1 q_2^* q_3$$

$$iq_{3t} = q_{3xx} + 2|q_3|^2 q_3 + 4q_3|q_2|^2 + 2q_2^2 q_1^*$$

with periodic boundary conditions $q_i(x+l) = q_i(x) \quad i = 1, 2, 3$ is integrable and specify the corresponding Zakharov–Shabat linear operator. We also attempt to describe its homoclinic orbits and give new results for its associated non–integrable perturbations through a Mel'nikov approach, using the spectral analysis of Lax 's operator and Floquet theory.

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1 Introduction

Homoclinic orbits have long been identified as a possible source of chaos in nonlinear finite dimensional dynamical systems, see [1], [2]. For example, when an integrable 3–dimensional system, containing a smooth connection between the stable and unstable invariant manifolds of a fixed point is perturbed, generally speaking, the connection is broken and the invariant manifolds intersect transversally. This yields isolated homoclinic orbits of the perturbed system, near which one expects lead to find shift map embeddings and *chaos*. Mel'nikov 's method allows one to measure the distance between the stable and unstable manifolds, under the perturbation and gives precise information regarding homoclinic (or heteroclinic) crossings.

In higher dimensions the situation is considerably more complicated, since phase space is no longer divided into separate regions by intersecting invariant manifolds. Still, for nonlinear dynamical systems of arbitrary but finite dimensions Mel'nikov type methods are available for detecting intersections (transversal or otherwise) between invariant manifolds [2], [3], [5]. In infinite dimensions, a similar approach to homaclinic solutions is often associated with the perturbations of integrable infinite dimensional equations of soliton theory.

In particular, it is found that small perturbations of the NLS and sine–Gordon [6], [7] equations may lead to chaotic behavior [6], [7]. Homoclinic orbits associated with the sine–Gordon and NLS equations have been obtained via *inverse spectral theory.* Having thus demonstrated the existence of a homoclinic structure, one can use inverse scattering for periodic problems, in order to explain how homoclinic crossings, in infinite dimensional phase space, can give rise to instabilities and chaos, as in the case of finite dimensional dynamical systems. An important ingredient for the successof this approach is the fact that the unperturbed system has a Hamiltonian formulation.

The theory of the NLS equation under periodic boundary conditions has been studied by Y.Li and D.McLaughlin [6]. Here, we consider *a Hamiltonian system* of three coupled NLS equations under periodic boundary conditions, together with the Floquet spectral theory of its associated Zakharov–Shabat linear operator [15], in order to study its dynamics under the in fluence of nonintegrable perturbations.

The complete integrability of our coupled NLS system is established through the determination of a Lax pair which we describe below. An alternative way to show this is by using the method of symmetry reduction, as applied e.g. to the SDYM (self–dual Yang–Mills) equations in the space $\mathbb{R}^{2,2}$ with the metric of the signature $(++--)$ [11], [12].

We consider the following system of coupled NLS equations:

$$\begin{aligned} iq_{1t} &= q_{1xx} + 2|q_1|^2 q_1 + 4q_1|q_2|^2 + 2q_2^2 q_3^* \\ iq_{2t} &= q_{2xx} + 2|q_1|^2 q_2 + 2|q_2|^2 q_2 + 2|q_3|^2 q_2 + 2q_1 q_2^* q_3 \\ iq_{3t} &= q_{3xx} + 2|q_3|^2 q_3 + 4q_3|q_2|^2 + 2q_2^2 q_1^* \end{aligned} \tag{1.1}$$

with periodic boundary conditions $q_i(x+l) = q_i(x) \quad i = 1, 2, 3.$

To view (1.1) as an infinite dimensional dynamical system, we first fix its phase space. We begin with the ambient space of $H^1_{per}([0,l], \mathbb{C}^6) \subset L^2([0,l], \mathbb{C}^6)$ of periodic complex valued 6–dimensional vector functions of x, which are square integrable over $[0,l]$ with square integrable first derivatives. It will be convenient to also define the following real Hilbert space:

$$\mathcal{A}^- = \{(\mathbf{q}, \mathbf{p}) \in H^1_{per} : p_i = -q_i^* \quad i = 1, 2, 3\}$$

For (1.1), the associated phase space is then the embedding of C^∞ in $\mathcal{A}^-$,

$$\mathcal{X} \equiv C^\infty \subset \mathcal{A}^-$$

in which, (1.1) is a Hamiltonian system:

$$\begin{pmatrix} i\mathbf{q}_t \\ i\mathbf{p}_t \end{pmatrix} = \mathbf{J} gradH, \qquad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \end{pmatrix} \tag{1.2}$$

with Hamiltonian $H : \mathcal{X} \longrightarrow \mathbb{R}$ given by:

$$H(\mathbf{q}, \mathbf{p}) = -i \int_0^l [\sum_{i=1}^{3} |q_i|^4 - |q_{ix}|^2] + 2(2|q_1|^2|q_2|^2 + 2|q_2|^2|q_3|^2 + q_1^* q_3^* q_2^2 + q_1 q_3 q_2^{*2})\, dx \tag{1.3}$$

2 Lax Pair of Coupled NLS equations

The system of NLS eqns, (1.1), can be obtained as isospectral deformation equations of a linear spectral problem that can be written in its general form:

$$\phi_x = L(\lambda, x, t)\phi \tag{2.2}$$

$$\phi_t = P(x, t, \lambda)\phi \tag{2.3}$$

where ϕ and L are complex–valued $N \times N$ matrix functions of x, t and rational in the spectral parameter λ. We consider as usual the case

$$L(\lambda, x, t) = \lambda A + Q(x, t) \tag{2.2a}$$

$$P(x, t, \lambda) = \sum_{j=0}^{n} P^{(j)} \lambda^j \tag{2.2b}$$

where A is a constant diagonal matrix,

$$A \equiv (ia_{jj})_{j=1}^N$$

and the *potential* matrix $Q(x, t)$ has zero diagonal elements $Q_{jj} = 0 \quad j = 1, \cdots, N$ We now concentrate on potentials $Q(x, t)$ which exponentially rapidly decrease as $|x| \longrightarrow \infty$ and introduce the right (left) matrix Jost solutions $F^{(+)}(F^{(-)})$ for the operator L which are determined by their asymptotic behavior

$$F^{(\pm)} \longrightarrow exp[\lambda A x] \quad as \quad x \longrightarrow \pm\infty$$

We consider a simple Lie algebra g. A homogeneous space of a Lie group G is any differentiable manifold M on which G acts transitively $(\forall p_1, p_2 \in M, \exists g \in G : g.p_1 = p_2)$. The subgroup G which leaves a given point $p_0 \in M$ fixed, is called the *isotropy* group at p_0 and is definrd by:

$$K \equiv K_{p0} = \{g \in G : g.p_0 = p_0\}$$

Let g and k be the Lie algebras of G and K respectively, and let m be the vector space complement of k in g. Then

$$g = k \oplus m, \qquad [k,k] \subset k$$

and m is identified with the tangent space $T_{p_0}M$ of $M = G/K$ at poinrt p_0.

We choose the element A of the Cartan subalgebra h of g: $A \in h \subset g$. If A is regular, $C_g(A) = h$. Let $k = C_g(A)$ and m be the complementary vector subspace of k in g: $g = k \oplus m$, m is identified with the tangent space $T_{p_0}(G/K)$ where K is the *isotropy* group of G. All elements of h are diagonalised.

Consider the second spectral equation (2.3) where $Q(x,t) \in m$, $P(x,t) \in g$, $P(x,t,\lambda) = P_k(x,t,\lambda) + P_m(x,t,\lambda)$ and $[A, P_k] = 0$ since $k = C_g(A)$.

The integrability condition of (2.3)–(2.2b) is:

$$Q_t = P_x - [Q, P] - \lambda[A, P_m] \tag{2.3}$$

where we have used $[A, P_k] = 0$, since $k = C_g(A)$, while (2.3) can be written

$$Q_t = P_{mx} - [Q, P_k] - \lambda[A, P_m] \tag{2.4a}$$

$$P_{kx} = [Q, P_m] \tag{2.4b}$$

we can immediately integrate the (2.4b)

$$P_k = \partial^{-1}[Q, P_m]$$

, so that the (2.4a) takes the form

$$Q_t = (\partial - adQ\partial^{-1}adQ - \lambda adA)P_m$$

where $(adX)Y \equiv [X, Y]$. Notice the recursion operator appears on the right:

$$(\partial - adQ\partial^{-1}adQ)P_m^{(j)} = (adA)P_m^{(j-1)} \quad j = 1, 2, \cdots, n+1$$

if $P(x,t,\lambda) = \sum_{j=0}^{n} P^{(j)}\lambda^j$.

Since adA is nonsingular on m, we can solve for $P_m^{(j-1)}$ at each step. At each step $P_k^{(j)}$ is obtained by integration and theprocess terminates with the N^{th} order flow:

$$Q_t = (\partial - adQ\partial^{-1}adQ)P_m^{(0)}$$

After some simple calculations for the NLS equations, (see: e.g. [10]) we have:

$$P_m^{(2)} = 0 \quad P_k^{(2)} = A \tag{2.5a}$$

$$P_m^{(1)} = Q \quad P_k^{(1)} = 0 \tag{2.5b}$$

$$P_m^{(0)} = \frac{1}{i} \sum_{\alpha=1}^{3} \partial_x q_\alpha e_\alpha - \partial_x p_\alpha e_{-\alpha} \tag{2.5c}$$

$$P_k^{(0)} = -\frac{1}{i} \sum_{\alpha,\beta=1}^{3} q_\alpha p_\beta [e_\alpha, e_\beta] \tag{2.5d}$$

where $e_\alpha, e_{-\alpha}$ satisfy:

$$Q = \sum_{\alpha=1}^{3} q_\alpha e_\alpha + p_\alpha e_{-\alpha} \tag{2.6}$$

where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_{-2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad e_{-3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

¿From (2.5a)–(2.5d) we finally have:

$$P = \begin{pmatrix} -i(|q_1|^2 + |q_2|^2) + \frac{i\lambda^2}{2} & -i(q_1 q_2^* + q_2 q_3^*) & -iq_{1x} + \lambda q_1 & -iq_{2x} + \lambda q_2 \\ -i(q_2 q_1^* + q_3 q_2^*) & -i(|q_2|^2 + |q_3|^2) + \frac{i\lambda^2}{2} & -iq_{2x} + \lambda q_2 & -iq_{3x} + \lambda q_3 \\ -iq_{1x}^* - \lambda q_1^* & -iq_{2x}^* - \lambda q_2^* & i(|q_2|^2 + |q_1|^2) - \frac{i\lambda^2}{2} & i(q_2 q_1^* + q_3 q_2^*) \\ -iq_{2x}^* - \lambda q_2^* & -iq_{3x}^* - \lambda q_3^* & i(q_1 q_2^* + q_2 q_3^*) & i(|q_2|^2 + |q_3|^2) - \frac{i\lambda^2}{2} \end{pmatrix}$$

or

$$P(x,t,\lambda) = P^{(0)} + Q\lambda + A\lambda^2 \tag{2.7}$$

Let us now consider the case of special plane–wave solutions, where q_1, q_2, q_3 are independent of x, i.e.

$$\hat{q}_j = a_j exp(in_j(t)), \quad j = 1,2,3 \tag{2.8}$$

Substituting (2.8) into (1.1) we have the following special solutions: $a_1 = a_2 = a_3 = a$, $n_2 = 0$ and $n_1 = -n_3$ where $n_1(t) = -8a^2t + \gamma_1$ Now, considering small perturbations of the form,

$$q_j(x,t) = \hat{q}_j(x,t)(1 + \epsilon_j(x,t)) \tag{2.9}$$

where $|\epsilon_j| \ll 1$, we substitute (2.9) into (1.1), keep linear terms in ϵ_j and obtain:

$$\begin{aligned}
\epsilon_{1t} &= -i\epsilon_{1xx} - 2i|a|^2(\epsilon_1 + \epsilon_1^*) - 6i|a|^2(\epsilon_2 + e_2^*) - 2i\epsilon_3^* \\
\epsilon_{2t} &= -i\epsilon_{2xx} - 2i|a|^2(\epsilon_1 + \epsilon_1^*) - 2i|a|^2(\epsilon_2 + \epsilon_2^*) - 2i|a|^2(\epsilon_3 + \epsilon_3^*) - 2i|a|^2(\epsilon_1 + \epsilon_2^* + \epsilon_3) \\
\epsilon_{3t} &= -i\epsilon_{3xx} - 2i|a|^2(\epsilon_3 + \epsilon_3^*) - 6i|a|^2(\epsilon_2 + e_2^*) - 2i\epsilon_1^*
\end{aligned} \tag{2.10}$$

Assuming

$$\epsilon_j(x,t) = \hat{\epsilon}_{j(-n)} exp(-i\mu_n x) + \hat{\epsilon}_{j(n)} exp(i\mu_n x) \tag{2.11}$$

where $\mu_n = 2\pi n/l$. Subistititing (2.11) into (2.10) and consider conjuagate equations (2.10) obtain the corresponding linearized matrix. We compute the corresponding growth rates and the plane–wave solutions (2.8) are unstable in x.

3 Inverse Spectral Transformation

Our system of coupled NLS equations can be integrated using a Zakharov–Shabat type linear system [15] (2.1). Compatibility of the equations (2.2) (2.3) ensures that the coefficients q_i satisfy the NLS system.

Focusing our attention upon the *spatial flow* (2.2), we let $Y^{(j)}, \quad j = 1,2,3,4$ be the fundamental solutions of ode (2.3), with the initial conditions:

$$Y^{(1)}(0) = (1,0,0,0)^{\perp}, Y^{(2)}(0) = (0,1,0,0)^{\perp}, Y^{(3)}(0) = (0,0,1,0)^{\perp}, Y^{(4)}(0) = (0,0,0,1)^{\perp}$$

With these, the differential equation may be written as an integral equation:

$$Y^{(j)} = e^{i\lambda x/2} Y^{(j)}(0) + \int_0^x exp[A\lambda(x-t)/2] Q Y^{(j)} \, dt \tag{3.1}$$

with

$$Y^{(j+2)} = \bar{Y}^{(j)} \quad j = 1,2$$

$\bar{Y}^{(j)}$ being the complex conjugate of $Y^{(j)}$. Now, equations (2.2), (2.3) become:

$$\Big[2i \begin{pmatrix} -\mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_2 \end{pmatrix} \frac{d}{dx} - \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R} & \mathbf{O} \end{pmatrix} \Big] \phi = \lambda \phi \tag{3.2}$$

$$\Big[2i \begin{pmatrix} -\mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_2 \end{pmatrix} \frac{d}{dt} - A^{-1} P \Big] \phi = \lambda^2 \phi \tag{3.3}$$

where

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2 \\ q_2 & q_3 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ r_2 & r_3 \end{pmatrix}$$

We define the non self–adjoint operator $\hat{L}$ as:

$$\hat{L} := 2i \begin{pmatrix} -\mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_2 \end{pmatrix} \frac{d}{dx} - \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R} & \mathbf{O} \end{pmatrix}$$

The spectrum $\sigma(\hat{L})$ is defined as the closure of the set of complex λ for which there exists a solution of

$$\hat{L}\phi = \lambda\phi$$

Now, Floquet theory is applied starting with the *fundamental matrix*

$$M = M(x, \lambda, \mathbf{q}) = columns\{Y^{(1)}, Y^{(2)}, Y^{(3)}, Y^{(4)}\}$$

which is defined as the 4×4 matrix solution of the linear problem (3.2) whose initial value at $x = 0$ is the identity matrix. One introduces the *transfer matrix*

$$T(\lambda; \mathbf{q}) \equiv M(l, \lambda, \mathbf{q})$$

defined by

$$\phi(x + l, x_0, \lambda) = T(\lambda; \mathbf{q})\phi(x, x_0, \lambda)$$

(where $\phi(x, x_0, \lambda)$ is a basis solutions of (3.2)), which transfers the basis $\phi(x, x_0, \lambda)$ forward by one period l of the potentials q_1, q_2, q_3.

We also define $\varrho(\lambda)$ as the eigenvalues of the transfer matrix T, which satisfy the characteristic equation:

$$\varrho^4 + I_1\varrho^3 + I_2\varrho^2 + I_3\varrho + 1 = 0 \tag{3.4}$$

where the scalar functions $I_j : \mathbb{C} \times H^1_{per} \longrightarrow \mathbb{C}, \quad j = 1, 2, 3$, are defined by

$$I_1 = \begin{vmatrix} \tau_{22} & \tau_{23} & \tau_{24} \\ \tau_{32} & \tau_{33} & \tau_{34} \\ \tau_{42} & \tau_{43} & \tau_{44} \end{vmatrix} + \begin{vmatrix} \tau_{21} & \tau_{23} & \tau_{24} \\ \tau_{31} & \tau_{33} & \tau_{34} \\ \tau_{41} & \tau_{43} & \tau_{44} \end{vmatrix} + \begin{vmatrix} \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \\ \tau_{41} & \tau_{42} & \tau_{43} \end{vmatrix} \tag{3.5a}$$

$$I_2 = \sum_{k=2}^{4} \begin{vmatrix} \tau_{31} & \tau_{3k} \\ \tau_{41} & \tau_{4k} \end{vmatrix} + \sum_{m=3}^{4} \begin{vmatrix} \tau_{32} & \tau_{3m} \\ \tau_{42} & \tau_{4m} \end{vmatrix} + \begin{vmatrix} \tau_{33} & \tau_{34} \\ \tau_{43} & \tau_{44} \end{vmatrix} \tag{3.5b}$$

$$I_3 = trace(T(\lambda)) \tag{3.5c}$$

We need to prove that the $I_j, \quad j = 1, 2, 3$ (evaluated on the critical points λ_k) are independent integrals of the system of coupled NLS, i.e

$$\{I_j, H\} = 0 \qquad j = 1, 2, 3 \tag{3.6}$$

where the Poisson bracket in (3.6) is defined by

$$\{F, G\} = \int_0^l \langle gradF, \mathbf{J} gradG \rangle \, dx \tag{3.7}$$

To compute the homoclinic orbits in the whiskers (infinite dimensional stable and unstable manifolds) we need the correspoding Gauge–Backlund transformation for the matrix of potentials, Q. Following the Gauge–Backlund theory (see [9]) we need the independent solutions of the Zakharov-Shabat linear system for a periodic solution $\mathbf{q}(x,t)$ of the coupled NLS equations which is quasiperiodic in t. We remark that two eigenfuctions for (3.2), (3.3) are

$$\phi(x,\lambda) = (\phi_1(x,\lambda), \phi_2(x,\lambda), \phi_3(x,\lambda), \phi_4(x,\lambda))^{\perp}$$

$$\bar{\phi}(x,\lambda^*) = (\phi_3^*(x,\lambda^*), \phi_4^*(x,\lambda^*), -\phi_1^*(x,\lambda^*), -\phi_2^*(x,\lambda^*))^{\perp}$$

and define a transformation matrix $\mathcal{G}$ as [9]:

$$\mathcal{G} = \mathcal{G}(\lambda, \nu, \phi) \equiv NLN^{-1} \tag{3.8}$$

where N is the fundamental matrix of solutions and $L = diag[\lambda - \nu, \lambda - \nu, \lambda - \nu^*, \lambda - \nu^*]$. We define U and Ψ by

$$U = \mathbf{Q} + [\mathcal{G}, \mathbf{K}] \tag{3.9}$$

and

$$\Psi(x,t,\lambda) \equiv \mathcal{G}(\lambda, \nu; \phi)\phi(x,t;\lambda) \tag{3.10}$$

where

$$\mathbf{K} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_2 \end{pmatrix}$$

Formulas (3.9) and (3.10) are the Backlund transformations for the potential matrix and eigenfunctions, respectively.

Equations (1.1) describe an infinite dimensional Hamiltonian system with an infinite number of commuting constants of the motion. The common level sets of these constants of motion are generically infinite dimensional tori of maximal dimension. Typically, these constants of motion have linearly independent gradients at certain critical tori. These critical tori have dimensions which are lower than maximal, but otherwise arbitrary and can either be stable or unstable.(In the unstable case, the constants of motion have a saddle structure in a neighborhood of the critical torus). An unstable critical torus has unstable manifold of dimension $N + M$, where M denotes the dimension of the torus itself and N will be shown to be finite. A phase point on this unstable (stable) manifold approaches the critical torus as $t \longrightarrow \pm\infty$. Such a phase point is said to lie on an orbit which is *homoclinic to the critical torus.*

4 A Perturbed System of NLS Equations

In this section, we consider a perturbation of the system of NLS equation generated by a perturbed vector field $\epsilon\mathbf{f}(\mathbf{q})$. The perturbed system has the following form:

$$\begin{pmatrix} i\mathbf{q}_t \\ i\mathbf{p}_t \end{pmatrix} = \mathbf{J}gradH + \epsilon\mathbf{f}(\mathbf{q},\mathbf{p}) \tag{4.1}$$

We can prove the existence of invariants manifolds of (4.1) and define the vector of distance between the stable and unstable whiskers.

In this setting the *Mel'nikov Vector* can be defined using the three independent integrals of the motion, I_1, I_2, I_3 as follows:

$$M_j = \int_{-\infty}^{\infty} \langle gradI_j, \mathbf{f}\rangle(U_h)\, dt \quad j = 1,2,3 \tag{4.2}$$

where the integrand is evaluated along the unperturbed homoclinic orbit $\mathbf{q} = U_h$ (See [2], [6]).

We concentiate on a neighborhood of plane–wave solutions of the pde's of the form (2.8), with $(\alpha,\gamma) \in U \times \mathbb{R}^3$. By varying the constant amplitudes $(\alpha_1, \alpha_2, \alpha_3)$ and the phases γ, the solutions fill out an invariant plane in the infinite dimensional phase space. Within this hyperplane, we consider the set

$$\mathcal{A} = \{\alpha \in U : U = (\alpha_{11}, \alpha_{12}) \times (\alpha_{21}, \alpha_{22}) \times (\alpha_{31}, \alpha_{32})\}$$

corresponding to the existence of double points in the spectral configuration of the solution (see [4]). In the unperturbed phase space, $\mathcal{A}$ is a six–dimensional hyperbolic invariant set with stable and unstable manifolds denoted by $W^s_0(\mathcal{A}) = W^u_0(\mathcal{A})$. In general (from spectral analysis) $\mathbb{R}^{2N} \times V \times T^3$, $V \subset \mathbb{R}^3$ denotes the family of plane wave solutions with N complex double points.

In the perturbed system (4.1) $q_j(t)$ persists as a periodic orbit of saddle type. For ϵ sufficiently small it possesses stable and unstable manifolds $W^{s,u}_\epsilon$ which are ϵ–close to $W^{s,u}_0$ respectively. To determine whether W^s_ϵ, W^u_ϵ intersect transversally, we need to compute the Mel'nikov vector which provides an $\mathcal{O}(\epsilon)$–measure of the distance between these perturbed manifolds.

Now, after a proper choice of plane wave solutions q_j, the coefficients of the characteristic equation of the transfer matrix T are such that the following condition is satisfied:

$$I_1 = I_3 = trace(T(\lambda))$$

Thus, equation (3.4) now take the form

$$(\varrho^2 + \Delta_1\varrho + 1)(\varrho^2 + \Delta_2\varrho + 1) = 0 \tag{4.3}$$

where

$$\Delta_i : \mathbb{C} \times H^1_{per} \longrightarrow \mathbb{C} \quad i = 1,2$$

$$\Delta_1 = \frac{1}{2}(trT \pm \sqrt{(trT)^2 - 4(I_2 - 2)}) \tag{4.4a}$$

$$\Delta_2 = \frac{1}{2}(trT \mp \sqrt{(trT)^2 - 4(I_2 - 2)}) \tag{4.4b}$$

and $\varrho^{(1)}$, $\varrho^{(2)}$ denote the Floquet multipliers which are related to Δ_1, Δ_2 by

$$\varrho^{(i)}(\lambda) = \frac{1}{2}(\Delta_i(\lambda) \pm \sqrt{\Delta_i^2 - 4})$$

$\varrho^{(i)}$ and Bloch eigenfunctions ψ of $\hat{L}$ are well defined functions on the Riemann surfaces for $(\lambda^{(i)}, \sqrt{\Delta_i^2(\lambda^{(i)}) - 4})$, $i = 1, 2$

We define the critical point as the value of λ such that

$$\frac{d\Delta_i}{d\lambda} = 0$$

and multiple points

$$\{\lambda \in \mathbb{C} : \lambda \quad are \quad critical \quad points \quad and \quad \Delta_i(\lambda) = \pm 2\}$$

The multiplicity of a multiple point is defined to be the order of vanishing of $\Delta_i(\lambda) \mp 2$.

We need to prove that the quantities $\Delta_i(\lambda)$ generates an infinity family of NLS constants of motion for each critical value of λ.

$$\{\Delta_i, H\} = 0 \quad i = 1, 2$$

and for this case to determine whether stable and unstable manifolds intersect transverally. We compute a Mel'nikov vector, which provides an an $\mathcal{O}(\epsilon)$–measure of the distance between these perturbed manifolds.

$$M_j = \int_{-\infty}^{\infty} \langle gradF_{kj}, \mathbf{f}(\mathbf{q}, \mathbf{p})\rangle(U_h)\, dt \quad j = 1, 2 \tag{4.5}$$

where $F_{kj} = \Delta_j(\lambda_{kj}^c(\mathbf{q}), \mathbf{q})$, λ_{k1}^c, $\quad \lambda_{k2}^c$ are the k–critical points of Δ_1, Δ_2

the integrand is evaluated along the unperturbed homoclinic orbit $\mathbf{q} = U_h$, which correspond to the critical point from the Riemann surface.

Acknowledgements

Vassilios M. Rothos thanks the State Scholarships Foundations of Greece (S.S.F) for support. This research was partially supported by a P.EN.ED grant from the Greek Secretariat of Research and Technology of the Greek Ministry of Industry, Energy and Technology. I would also like to thank Prof. T. Bountis, for many useful discussions.

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