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Differential equation for temperature-dependent correlation function of Delta-interacting Bose gas

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Abstract

We consider the quantum nonlinear Schrödinger equation in one space and one time dimension. It is equivalent to Bose gas with delta interaction. We are interested in the non-free-fermionic case. We calculate temperature-dependent correlation functions. The determinant representation for correlation functions simplifies in the small mass limit of the Bose particle. In this limit we describe the correlation function in terms of the Maxwell-Bloch differential equation. We solve this equation at large distance and evaluate asymptotics of correlation functions.

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1 Introduction

In this paper we consider correlation functions of exactly solvable models. Our approach is based on the determinant representation of quantum correlation functions [1]. We consider the thermodynamics of Bose gas with delta-interaction at finite temperature $T > 0$. The one dimensional Bose gas with delta interaction is described by the canonical Bose fields $\psi(x)$ and $\psi^+(x)$ with the commutation relations :

$$[\psi(x), \psi^+(y)] = \delta(x-y), \quad [\psi(x), \psi(y)] = [\psi^+(x), \psi^+(y)] = 0. \tag{1.1}$$

The Hamiltonian of the model is

$$H = \int dx \left(\frac{1}{2m} \frac{\partial}{\partial x} \psi^+(x) \frac{\partial}{\partial x} \psi(x) + g\psi^+(x)\psi^+(x)\psi(x)\psi(x) - h\psi^+(x)\psi(x) \right), \tag{1.2}$$

where $m > 0$ is the mass, $g > 0$ is the coupling constant and $h > 0$ is the chemical potential. The Hamiltonian H acts on the Fock space with the vacuum vector $|vac\rangle$. The vacuum vector $|vac\rangle$ is characterized by the relation :

$$\psi(x)|vac\rangle = 0. \tag{1.3}$$

The dual vacuum vector $\langle vac|$ is characterized by the relations :

$$\langle vac|\psi^+(x) = 0, \quad \langle vac \mid vac\rangle = 1. \tag{1.4}$$

The corresponding equation of motion,

$$i\frac{\partial}{\partial t}\psi = [\psi, H] = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\psi + 2g\psi^+\psi\psi - h\psi, \tag{1.5}$$

is called the quantum nonlinear Schrödinger equation in one space and one time dimension. The quantum field theory problem is reduced to a quantum mechanics problem. It is well known that in the N particle sector the eigenvalue problem $H|\psi_N\rangle = E_N|\psi_N\rangle$, is equivalent to the one described by the quantum mechanics N body Hamiltonian :

$$H_N = -\frac{1}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial z_j^2} + 2g \sum_{1\le j<k\le N} \delta(z_k - z_j) - Nh. \tag{1.6}$$

The meaning of the eigenvalue problem $H_N\psi_N = E_N\psi_N$, is to be understood as the free equation,

$$-\frac{1}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial z_j^2}\psi_N = (E_N + Nh)\psi_N \quad \text{for} \quad z_j \neq z_k, \tag{1.7}$$

supplemented by the boundary conditions :

$$\frac{1}{2m}\left\{\left(\frac{\partial\psi_N}{\partial z_i}-\frac{\partial\psi_N}{\partial z_j}\right)\Big|_{z_i=z_j+0}-\left(\frac{\partial\psi_N}{\partial z_i}-\frac{\partial\psi_N}{\partial z_j}\right)\Big|_{z_i=z_j-0}\right\}=2g\psi_N|_{z_i=z_j\pm 0}\,,\ 1\le i\ne j\le N. \quad (1.8)$$

E.H. Lieb and W. Linger [2] solved the eigenvalue problem $H_N\psi_N=E_N\psi_N$. They constructed the eigen-functions $\psi_N=\psi_N(z_1,\cdots,z_N|\lambda_1,\cdots,\lambda_N)$ by means of the Bethe Ansatz. The eigen-function $\psi_N=\psi_N(z_1,\cdots,z_N|\lambda_1,\cdots,\lambda_N)$ depends on the spectral parameter $\lambda_1<\cdots<\lambda_N$. The spectral parameters $\lambda_1<\cdots<\lambda_N$ are determined by the periodic boundary conditions :

$$\psi_N(z_1,\cdots,z_j+L,\cdots,z_N|\lambda_1,\cdots,\lambda_N)=\psi_N(z_1,\cdots,z_j,\cdots,z_N|\lambda_1,\cdots,\lambda_N), \quad (1.9)$$

which amounts to the Bethe Ansatz equations :

$$e^{i\lambda_j L}=-\prod_{k=1}^{N}\frac{\lambda_j-\lambda_k+2img}{\lambda_j-\lambda_k-2img},\quad j=1,\cdots,N. \quad (1.10)$$

Here $L>0$ is the size of the box. The eigenvalue of the Hamiltonian H_N is given by

$$E_N=\sum_{j=1}^{N}\left(\frac{1}{2m}\lambda_j^2-h\right). \quad (1.11)$$

E.H. Lieb and W. Linger [2], [3] discussed the zero temperature thermodynamic limit. The ground state and its excitations are described by linear integral equations. C.N. Yang and C.P. Yang [4] discussed the finite temperature thermodynamic limit. The state of thermodynamic equilibrium is described by non-linear integral equations. The density of particles $\rho_p(\lambda)$ and the density of holes $\rho_h(\lambda)$ are described by the following non-linear integral equations :

$$2\pi\rho_t(\lambda) = 1+\int_{-\infty}^{\infty}K(\lambda,\mu)\rho_p(\mu)d\mu, \quad (1.12)$$

$$D = \frac{N}{L}=\int_{-\infty}^{\infty}\rho_p(\mu)d\mu, \quad (1.13)$$

$$\varepsilon(\lambda) = \frac{\lambda^2}{2m}-h-\frac{T}{2\pi}\int_{-\infty}^{\infty}K(\lambda,\mu)\ln\left(1+e^{-\frac{\varepsilon(\mu)}{T}}\right)d\mu, \quad (1.14)$$

where $T>0$ is temperature and $D=\frac{N}{L}$ is the density of particles. Here the functions $\varepsilon(\lambda)$ and $\rho_t(\lambda)$ are defined by

$$\frac{\rho_h(\lambda)}{\rho_p(\lambda)}=e^{\frac{\varepsilon(\lambda)}{T}},\quad \rho_t(\lambda)=\rho_p(\lambda)+\rho_h(\lambda). \quad (1.15)$$

The integral kernel $K(\lambda,\mu)$ is defined by

$$K(\lambda,\mu)=\frac{4mg}{(\lambda-\mu)^2+(2mg)^2}. \quad (1.16)$$

Consider the local density operator $j(x) = \psi^+(x)\psi(x)$. In this paper we consider the mean value of the operator :

$$\exp\left(\alpha Q(x)\right). \tag{1.17}$$

Here α is an arbitrary complex parameter and $Q(x)$ is the operator of the number of particles on the interval $[0, x]$:

$$Q(x) = \int_0^x \psi^+(y)\psi(y)dy. \tag{1.18}$$

We are interested in the generating function of temperature-dependent correlation function defined by

$$\langle \exp\left(\alpha Q(x)\right)\rangle_T = \frac{\mathrm{tr}\left(\exp\left(-\frac{H}{T}\right)\exp\left(\alpha Q(x)\right)\right)}{\mathrm{tr}\left(\exp\left(-\frac{H}{T}\right)\right)}. \tag{1.19}$$

The expectation value $\langle \exp\left(\alpha Q(x)\right)\rangle_T$ is a remarkable quantity, because a lot of interesting correlation function can be extracted from $\langle \exp\left(\alpha Q(x)\right)\rangle_T$. For example the density correlation function,

$$\langle j(x)j(0)\rangle_T = \frac{\mathrm{tr}\left(\exp\left(-\frac{H}{T}\right)j(x)j(0)\right)}{\mathrm{tr}\left(\exp\left(-\frac{H}{T}\right)\right)} \tag{1.20}$$

can be derived by

$$\langle j(x)j(0)\rangle_T = \frac{1}{2}\frac{\partial^2}{\partial x^2}\langle Q(x)^2\rangle_T = \frac{1}{2}\frac{\partial^2}{\partial x^2}\,\frac{\partial^2}{\partial \alpha^2}\langle \exp\left(\alpha Q(x)\right)\rangle_T\Big|_{\alpha=0}. \tag{1.21}$$

In this paper we are interested in the small mass limit of the Bose particle :

$$m \to 0,\ g \to \infty \ \text{ such that the product } \ c = 2mg \ \text{ is fixed.} \tag{1.22}$$

We want to emphasize that the small mass limit is not a free-fermionic limit. The scattering matrix of the particles λ_p and λ_h is equal to

$$S(\lambda_p, \lambda_h) = \exp\left(-i\delta(\lambda_p, \lambda_h)\right), \quad \lambda_p > \lambda_h, \tag{1.23}$$

where the scattering phase δ satisfying the following integral equation :

$$\delta(\lambda_p, \lambda_h) - \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\lambda_p, \mu)\vartheta(\mu)\delta(\mu, \lambda_h) = i\ln\left(\frac{ic + \lambda_p - \lambda_h}{ic - \lambda_p + \lambda_h}\right). \tag{1.24}$$

Here we used

$$\vartheta(\lambda) = \frac{1}{1 + e^{\frac{\varepsilon(\lambda)}{T}}} = \frac{\rho_p(\lambda)}{\rho_t(\lambda)}. \tag{1.25}$$

Therefore the small mass limit is not a free-fermionic limit. In the small mass limit we will show that the expectation value $\langle \exp(\alpha Q(x))\rangle_T$ is described in terms of a solution of the Maxwell-Bloch equation [5]. The plan of this paper is as follows. In section 2 we summarize known results of determinant representations for correlation functions. In section 3 we consider the small mass limit of temperature correlation functions. In section 4 we write down differential equations for correlation functions in the small mass limit. Temperature correlation functions are described in terms of a τ-function of the Maxwell Bloch equation in the small mass limit. In section 5 we evaluate asymptotics of the correlation functions.

2 Determinant representation with dual fields

The purpose of this section is to summarize the known results of determinants representation for temperature correlation functions [1]. First, we introduce the dual fields $\phi_j(\lambda)$, $(j = 1, \cdots, 4)$ defined by

$$\phi_j(\lambda) = p_j(\lambda) + q_j(\lambda), \quad (j = 1, \cdots, 4). \tag{2.1}$$

Here the fields $p_j(\lambda)$ and $q_j(\lambda)$ are defined by the commutation relations :

$$[p_j(\lambda), p_k(\mu)] = [q_j(\lambda), q_k(\mu)] = 0, \quad [p_j(\lambda), q_k(\mu)] = H_{j,k}(\lambda, \mu), \quad (j, k = 1, \cdots, 4). \tag{2.2}$$

Here we used

$$H_{j,k}(\lambda,\mu) = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}_{j,k} \ln(h(\lambda,\mu)) + \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix}_{j,k} \ln(h(\mu,\lambda)), \tag{2.3}$$

where

$$h(\lambda,\mu) = \frac{1}{ic}(\lambda - \mu + ic). \tag{2.4}$$

The dual fields $\phi_j(\lambda)$ commute.

$$[\phi_j(\lambda), \phi_k(\mu)] = 0, \quad (j, k = 1, \cdots, 4). \tag{2.5}$$

We introduce the auxiliary Fock space with the auxiliary vacuum vector $|0\rangle$. The auxiliary vacuum vector $|0\rangle$ is characterized by

$$p_j(\lambda)|0\rangle = 0, \quad (j = 1, \cdots, 4). \tag{2.6}$$

The auxiliary dual vacuum $\langle 0|$ is characterized by

$$\langle 0|q_j(\lambda) = 0, \quad (j = 1, \cdots, 4), \quad \langle 0 \mid 0 \rangle = 1. \tag{2.7}$$

We want to emphasize that the dual fields $\phi_j(\lambda)$ $(j = 1, \cdots, 4)$ and the auxiliary Fock space can be written in terms of the four standard Bose fields $\psi_j(\lambda),\ \psi_j^+(\mu),\ (j = 1, \cdots, 4)$ and the standard Fock vacuum $|0\rangle$ and the dual Fock vacuum $\langle 0|$:

$$[\psi_j(\lambda), \psi_k^+(\mu)] = \delta_{j,k}\delta(\lambda - \mu), \quad [\psi_j(\lambda), \psi_k(\mu)] = [\psi_j^+(\lambda), \psi_k^+(\mu)] = 0, \quad (j, k = 1, \cdots, 4). \tag{2.8}$$

$$\psi_j(\lambda)|0\rangle = 0, \qquad \langle 0|\psi_j^+(\lambda) = 0. \tag{2.9}$$

Actually, the dual fields can be realized by

$$p_j(\lambda) = \psi_j(\lambda), \quad q_k(\mu) = \sum_{l=1}^{4} \int_{-\infty}^{\infty} H_{l,k}(\nu, \mu)\psi_l^+(\nu)d\nu, \quad (j, k = 1, \cdots, 4). \tag{2.10}$$

Next we prepare two integral operators $\hat{V}_T$ and $\hat{K}_T$. The integral operator $\hat{V}_T$ is defined by

$$\left(\hat{V}_T f\right)(\lambda) = \int_{-\infty}^{\infty} V_T(\lambda, \mu) f(\mu) d\mu. \tag{2.11}$$

The integral kernel $V_T(\lambda, \mu)$ is defined by product $V_T(\lambda, \mu) = V(\lambda, \mu)\vartheta(\mu)$. The first factor $V(\lambda, \mu)$ is defined by

$$\begin{aligned} V(\lambda, \mu) &= \frac{1}{c}\left\{t(\lambda, \mu) + t(\mu, \lambda)\exp\left(-ix(\lambda - \mu) + \phi_1(\mu) - \phi_1(\lambda)\right)\right. \\ &+ \left.\exp\left(\alpha + \phi_3(\lambda) + \phi_4(\mu)\right)\left(t(\mu, \lambda) + t(\lambda, \mu)\exp\left(-ix(\lambda - \mu) + \phi_2(\lambda) - \phi_2(\mu)\right)\right)\right\}. \end{aligned} \tag{2.12}$$

where

$$t(\lambda, \mu) = \frac{(ic)^2}{(\lambda - \mu)(\lambda - \mu + ic)}. \tag{2.13}$$

We call the second-factor $\vartheta(\lambda)$ the Fermi weight :

$$\vartheta(\lambda) = \frac{1}{1 + e^{\frac{\varepsilon(\lambda)}{T}}} = \frac{\rho_p(\lambda)}{\rho_t(\lambda)}. \tag{2.14}$$

Because the dual fields $\phi_j(\lambda)$ commute with each other, we can define the quantity $\det\left(1 + \frac{1}{2\pi}\hat{V}_T\right)$. The integral operator $\hat{K}_T$ is defined by

$$\left(\hat{K}_T f\right)(\lambda) = \int_{-\infty}^{\infty} K_T(\lambda, \mu) f(\mu) d\mu. \tag{2.15}$$

The integral kernel $K_T(\lambda, \mu)$ is defined by $K_T(\lambda, \mu) = K(\lambda, \mu)\vartheta(\mu)$. $K(\lambda, \mu)$ is defined in (1.16). Now we state the results which we will use in the following sections.

Theorem 2.1 *[6] Its, Izergin, Korepin and Slavnov In terms of the dual fields $\phi_j(\lambda)$ $(j = 1, \cdots, 4)$, we can express the expectation value $\langle \exp(\alpha Q(x)) \rangle_T$ by the Fredholm determinant :*

$$\langle \exp(\alpha Q(x)) \rangle_T = \frac{\langle 0| \det\left(1 + \frac{1}{2\pi}\hat{V}_T\right)|0\rangle}{\det\left(1 - \frac{1}{2\pi}\hat{K}_T\right)}. \tag{2.16}$$

Here the symbol $\det\left(1 + \frac{1}{2\pi}\hat{V}_T\right)$ *represents the Fredholm determinant corresponding to the following Fredholm integral equation of the second kind:*

$$\left(\left(1 + \frac{1}{2\pi}\hat{V}_T\right) f\right)(\lambda) = g(\lambda), \quad \textit{for } \lambda \in (-\infty, \infty). \tag{2.17}$$

The denominator $\det\left(1 - \frac{1}{2\pi}\hat{K}_T\right)$ *represents the Fredholm determinant corresponding to the following Fredholm integral equation of the second kind :*

$$\left(\left(1 - \frac{1}{2\pi}\hat{K}_T\right) f\right)(\lambda) = g(\lambda), \quad \textit{for } \lambda \in (-\infty, \infty). \tag{2.18}$$

3 The small mass limit of the Bose particle

In this section we will show that at the small mass limit : $m \to 0$, $g \to \infty$, such that $c = 2mg$ is fixed, a simplification occurs. As explained in the Introduction, the scattering matrix depends on the product $c = 2mg$, not just on g. Therefore the limit of small mass is not a free-fermion limit. We want to emphasize this point. In the sequel we consider the limit of small mass. First we evaluate the solution of the Yang-Yang equation.

$$\varepsilon(\lambda) \quad = \quad \frac{\lambda^2}{2m} - h - \frac{T}{2\pi}\int_{-\infty}^{\infty} K(\lambda, \mu) \ln\left(1 + e^{-\frac{\varepsilon(\mu)}{T}}\right) d\mu. \tag{3.1}$$

This is done following [4].

Lemma 3.1 *In the small mass limit of the Bose particle, a solution of the Yang-Yang equation (3.1) is evaluated as :*

$$\varepsilon(\lambda) = \frac{\lambda^2}{2m} - h + O(\sqrt{m}). \tag{3.2}$$

Proof In [4] C.N. Yang and C.P. Yang derived the following inequalities.

$$\frac{\lambda^2}{2m} - h \geq \varepsilon(\lambda) \geq \frac{\lambda^2}{2m} + x_0, \tag{3.3}$$

where x_0 is defined by the integral equation :

$$x_0 = -h - \frac{T}{2\pi}\int_{-\infty}^{\infty} K(0,\mu)\ln\left(1+e^{-\frac{1}{T}\left(\frac{\mu^2}{2m}+x_0\right)}\right)du. \tag{3.4}$$

The existence of x_0 is proved in [4]. Let us change the integration variable to $\nu = \frac{\mu}{\sqrt{2m}}$. In the limit of small mass, $\sqrt{cg}$ tends to ∞. Therefore we obtain :

$$\begin{aligned} x_0 &= -h - \frac{T}{\pi}\int_{-\infty}^{\infty}\frac{\sqrt{cg}}{(\sqrt{cg})^2+\nu^2}\ln\left(1+e^{-\frac{1}{T}(\nu^2+x_0)}\right)d\nu & (3.5)\\ &= -h + x_0 - \frac{T}{\pi}\int_{-\infty}^{\infty}\frac{\sqrt{cg}}{(\sqrt{cg})^2+\nu^2}\ln\left(e^{\frac{1}{T}x_0}+e^{-\frac{1}{T}\nu^2}\right)d\nu & (3.6)\\ &= -h - \frac{T}{\pi}\frac{1}{\sqrt{cg}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ln\left(1+e^{-\frac{1}{T}(\nu^2+x_0)}\right)d\nu + O(m). & (3.7)\end{aligned}$$

When we assume $|x_0| \to \infty$, this contradicts to (3.6). Therefore we can assume that $|x_0|$ is bounded. Therefore, from the equation (3.7), we can deduce $x_0 = -h + O(\sqrt{m})$.

□

¿From lemma 3.1, we can evaluate the Fermi weight $\vartheta(\lambda)$. The Fermi weight $\vartheta(\lambda)$ has a very sharp maximum at $\lambda = 0$, from which it decreases to 0 very fast. Therefore a simplification occurs. First we consider the dual fields. In the sequel we consider the case that the spectral parameters are restricted to $\lambda, \mu \approx O(\sqrt{m})$. We observe the simplification of the commutation relations :

$$[p_j(\lambda), q_k(\mu)] = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}_{j,k} \frac{i}{c}(\mu-\lambda) + O(m). \tag{3.8}$$

Therefore we can identify pairs of fields :

$$p_1(\lambda) = -p_2(\lambda), \qquad p_3(\lambda) = p_4(\lambda), \tag{3.9}$$

$$q_1(\lambda) = -q_2(\lambda), \qquad q_3(\lambda) = q_4(\lambda), \tag{3.10}$$

$$\phi_1(\lambda) = -\phi_2(\lambda), \qquad \phi_3(\lambda) = \phi_4(\lambda). \tag{3.11}$$

Furthermore, because the first term of the commutation relation (3.8) is a linear function of the spectral parameters, we can choose a representation of fields such that $\phi_j(\lambda)$ are linear functions of the spectral parameter λ :

$$\phi_j(\lambda) = \phi_j(0) + \phi_j'(0)\lambda, \quad \phi_j(0) = p_j(0) + q_j(0), \ \phi_j'(0) = p_j'(0) + q_j'(0), \quad (j = 1,3). \tag{3.12}$$

Here the commutation relations are :

$$[p_j(0), q_k(0)] = 0 = [p'_j(0), q'_k(0)], \quad (j,k = 1,3), \tag{3.13}$$

$$[p'_1(0), q_3(0)] = \frac{i}{c} = -[p'_3(0), q_1(0)], \quad [p_3(0), q'_1(0)] = \frac{i}{c} = -[p_1(0), q'_3(0)]. \tag{3.14}$$

The actions on the auxiliary vacuum are :

$$p_1(0)|0\rangle = p_3(0)|0\rangle = p'_1(0)|0\rangle = p'_3(0)|0\rangle = 0, \tag{3.15}$$

$$\langle 0|q_1(0) = \langle 0|q_3(0) = \langle 0|q'_1(0) = \langle 0|q'_3(0) = 0 \tag{3.16}$$

Furthermore we arrive at the following formula.

Theorem 3.2 *In the small mass limit of the Bose particle, the expectation value of the Fredholm determinant simplifies as follows:*

$$\langle 0| \det \left(1 + \frac{1}{2\pi}\hat{V}_T\right) |0\rangle \longmapsto \langle 0| \det \left(1 + \hat{V}_{0,T}\right) |0\rangle. \tag{3.17}$$

Here the symbol $\hat{V}_{0,T}$ is the integral operator defined by

$$\left(\hat{V}_{0,T} f\right)(\lambda) = \int_{-\infty}^{\infty} V_{0,T}(\lambda,\mu) f(\mu) d\mu, \tag{3.18}$$

where the integral kernel is defined by product

$$V_{0,T}(\lambda,\mu) = \left(\frac{e^{\hat{\alpha}} - 1}{\pi}\right) \frac{\sin\frac{\hat{x}}{2}(\lambda - \mu)}{\lambda - \mu} \vartheta_0\left(\frac{\mu}{\sqrt{2mT}}, \frac{h}{T}\right), \tag{3.19}$$

where

$$\vartheta_0(\mu,\beta) = \frac{1}{1 + e^{\mu^2 - \beta}}. \tag{3.20}$$

Here we used the abbreviations:

$$\hat{\alpha} = \alpha + \hat{\alpha}_p + \hat{\alpha}_q, \ \hat{x} = x + \hat{x}_p + \hat{x}_q, \tag{3.21}$$

$$\hat{\alpha}_p = 2p_3(0), \ \hat{\alpha}_q = 2q_3(0), \ \hat{x}_p = -ip'_1(0), \ \hat{x}_q = -iq'_1(0). \tag{3.22}$$

The commutation relations and the actions on the auxiliary vacuum become:

$$[\hat{x}_p, \hat{\alpha}_q] = \frac{2}{c}, \qquad [\hat{\alpha}_p, \hat{x}_q] = \frac{2}{c}, \tag{3.23}$$

$$\hat{x}_p|0\rangle = 0 = \hat{\alpha}_p|0\rangle, \qquad \langle 0|\hat{x}_q = 0 = \langle 0|\hat{\alpha}_q. \tag{3.24}$$

The dual fields $\hat{\alpha}$ and $\hat{x}$ commute with each other, $[\hat{\alpha}, \hat{x}] = 0$.

Proof. From lemma 3.1, the Fermi weight $\vartheta(\lambda)$ has a very sharp maximum at $\lambda = 0$ and decrease to 0 very fast. When we consider the integral operator $\hat{V}_T$, we can restrict our consideration to the case of the spectral parameters $\lambda, \mu \approx O(\sqrt{m})$. Therefore we can use the above dual fields simplification. We can identify four dual fields to two dual fields, which are linear in the spectral parameters λ, μ. Furthermore, since the relations $[p_3'(0), \phi_1(\lambda) - \phi_1(\mu)] = 0,\ [q_3'(0), \phi_1(\lambda) - \phi_1(\mu)] = 0$ and $\langle 0|q_3'(0) = 0,\ p_3'(0)|0\rangle = 0$ hold, we can drop $p_3'(0),\ q_3'(0)$ in the expectation value $\langle 0| \det\left(1 + \frac{1}{2\pi}\hat{V}_T\right)|0\rangle$. Next we perform a similarity transformation $\exp\left(\frac{i}{2}\lambda(x - i\phi_1'(0))\right)$ which leaves the Fredholm determinant invariant. Finally we substitute the Fermi weight $\vartheta(\mu)$ by the modified Fermi weight $\vartheta_0\left(\frac{\mu}{\sqrt{2mT}}, \frac{h}{T}\right)$. We get the desired formula. □

The denominator of the expectation value (2.16) becomes the following :

$$\det\left(1 - \frac{1}{2\pi}\hat{K}_T\right) = 1 - \frac{\sqrt{2T}}{\pi c}\ d\left(\frac{h}{T}\right)\sqrt{m} + O(m), \tag{3.25}$$

where we used

$$d(\beta) = \int_{-\infty}^{\infty} \vartheta_0(\mu, \beta) d\mu. \tag{3.26}$$

The density D can be written as :

$$D = \frac{N}{L} = \int_{-\infty}^{\infty} \rho_p(\mu) d\mu = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1 + e^{\frac{1}{T}\left(\frac{\mu^2}{2m} - h\right)}} d\mu + O(m). \tag{3.27}$$

Therefore we can write

$$\det\left(1 - \frac{1}{2\pi}\hat{K}_T\right) = 1 - \frac{2}{c}D + O(m). \tag{3.28}$$

Therefore we arrive at the simplified formula for correlation functions.

Corollary 3.3 *In the small mass limit of the Bose particle, the temperature correlation function simplifies as follows :*

$$\langle \exp\left(\alpha Q(x)\right)\rangle_T \longmapsto \langle 0| \det\left(1 + \hat{V}_{0,T}\right)|0\rangle \left(1 + \frac{2}{c}D\right). \tag{3.29}$$

Here $D = \frac{N}{L}$ is the density of the thermodynamic limit.

4 Maxwell-Bloch differential equation

In this section we consider the differential equation for the temperature correlation function in the small mass limit of the Bose particle. In the small mass limit, the Fredholm determinant

$\det\left(1+\hat{V}_{0,T}\right)$ is a τ-function of the Maxwell-Bloch equation, taking values in a commutative subalgebra of the quantum operator algebra. It is easily seen that after introducing new variables, the auxiliary field $\hat{y}$ and the scaled chemical potential β,

$$\hat{y} = y + \hat{y}_p + \hat{y}_q,\ y = \sqrt{\frac{mT}{2}}x,\ \hat{y}_p = \sqrt{\frac{mT}{2}}\hat{x}_p,\ \hat{y}_q = \sqrt{\frac{mT}{2}}\hat{x}_q, \quad \beta = \frac{h}{T}, \tag{4.1}$$

the Fredholm determinant $\det\left(1+\hat{V}_{0,T}\right)$ can be rewritten, after the corresponding change $\lambda \to \frac{\lambda}{\sqrt{2mT}}$ of the spectral parameter, as

$$\det\left(1+\hat{V}_{0,T}\right) = \det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}. \tag{4.2}$$

We want to emphasize that $\hat{y}$ is an operator in the auxiliary space. The integral operator $\hat{W}$ is defined by

$$\left(\hat{W}f\right)(\lambda) = \int_{-\infty}^{\infty} W(\lambda,\mu)f(\mu)d\mu, \tag{4.3}$$

where the integral kernel $W(\lambda,\mu)$ is given by

$$W(\lambda,\mu) = \frac{\sin\hat{y}(\lambda-\mu)}{\lambda-\mu}\vartheta_0(\mu,\beta). \tag{4.4}$$

The algebraic structure of the Fredholm determinant $\det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}$ has been investigated in the context of correlation functions for the impenetrable Bose gas [1]. It is convenient to introduce the function σ defined by

$$\sigma\left(\hat{y},\beta,\hat{\alpha}\right) = \ln\det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}. \tag{4.5}$$

The operator σ satisfies the Maxwell-Bloch equation in the case that $\hat{y}$ and $\hat{\alpha}$ are real numbers [1]. In our case, $\hat{y}$ and $\hat{\alpha}$ are quantum operator, but due to the fact that they commute with each other, we can follow the derivation in [1]. Therefore we arrive at the following results. In the sequel we use the following operator-derivation notation :

$$\frac{\partial}{\partial\hat{y}}F(\hat{y}) := \frac{\partial}{\partial z}F(z)\Big|_{z=\hat{y}}, \tag{4.6}$$

where $F = F(z)$ is a function of z.

Corollary 4.1 *The operator* $\sigma(\hat{y},\beta,\hat{\alpha}) = \ln\det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}$ *obeys the following nonlinear partial differential equation :*

$$\left(\frac{\partial}{\partial\beta}\frac{\partial^2}{\partial\hat{y}^2}\sigma\right)^2 = -4\left(\frac{\partial^2}{\partial\hat{y}^2}\sigma\right)\left(2\hat{y}\frac{\partial}{\partial\beta}\frac{\partial}{\partial\hat{y}}\sigma + \left(\frac{\partial}{\partial\beta}\frac{\partial}{\partial\hat{y}}\sigma\right)^2 - 2\frac{\partial}{\partial\beta}\sigma\right), \tag{4.7}$$

with the initial conditions :

$$\sigma = -\left(\frac{1-e^{\hat{\alpha}}}{\pi}d(\beta)\right)\ \hat{y} - \left(\frac{1-e^{\hat{\alpha}}}{\pi}d(\beta)\right)^2 \frac{\hat{y}^2}{2} + O(\hat{y}^3), \tag{4.8}$$

$$\lim_{\beta\to-\infty}\sigma(\hat{y},\beta,\hat{\alpha}) = 0, \tag{4.9}$$

where the scalar function $d(\beta)$ is defined in (3.26).

This initial data fixes the solution uniquely. The nonlinear differential equation (4.7) is called the Maxwell-Bloch equation [5]. Algebraically, it is known that at $T = 0$ the operator σ depends only on product of variables $\hat{y}\sqrt{\beta}$ [6]. We set $\tau = \hat{y}\sqrt{\beta} = \sqrt{\frac{mh}{2}}\hat{x}$. Equation (4.7) is rewritten at $T = 0$ for the operator

$$\sigma_0(\tau) = \tau\frac{d}{d\tau}\ln\det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}, \tag{4.10}$$

as

$$\left(\tau\frac{d^2\sigma_0}{d\tau^2}\right)^2 = -4\left(\tau\frac{d\sigma_0}{d\tau}-\sigma_0\right)\left(4\tau\frac{d\sigma_0}{d\tau}+\left(\frac{d\sigma_0}{d\tau}\right)^2-4\sigma_0\right). \tag{4.11}$$

This ordinary differential equation is the fifth Painlevé equation in [7]. Actually, rewriting (4.11) in terms of the function $y_0(\tau)$ defined by

$$\sigma_0(\tau) = -4i\tau u_0(\tau) + \frac{u_0(\tau)^2}{y_0(\tau)}(y_0(\tau)-1)^2, \quad u_0(\tau) = \frac{4i\tau y_0(\tau) - \tau\frac{dy_0(\tau)}{d\tau}}{2(y_0(\tau)-1)^2}, \tag{4.12}$$

we can get the familiar formula of the fifth Painlevé differential equation for the function $w(\tau) = y_0\left(\frac{\tau}{2}\right)$:

$$\frac{d^2w}{d\tau^2} = \left(\frac{dw}{d\tau}\right)^2\frac{3w-1}{2w(w-1)} + \frac{2w(w+1)}{w-1} + \frac{2iw}{\tau} - \frac{1}{\tau}\frac{dw}{d\tau}. \tag{4.13}$$

Next we derive the asymptotics of $\sigma(\hat{y},\beta,\hat{\alpha}) = \ln\det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}$. By means of the Riemann-Hilbert method, asymptotics of σ are derived for the case when $\hat{y}$ and $\hat{\alpha}$ are real number [1]. The idea of the Riemann-Hilbert method is due to Professor A.R. Its. In our case, $\hat{y}$ and $\hat{\alpha}$ are quantum operator, but due to the fact that they commute, we can follow the derivation in [1]. We arrive at the following asymptotics.

Corollary 4.2 *The asymptotics of the operator $\sigma(\hat{y},\beta,\hat{\alpha})$ for large $\hat{y}$ become as follows :*

$$\sigma(\hat{y},\beta,\hat{\alpha}) \quad = \quad -\hat{y}\ C(\beta,\hat{\alpha}) + \frac{1}{2}\int_{-\infty}^{\beta}\left(\frac{\partial C(b,\hat{\alpha})}{\partial b}\right)^2 db + \tag{4.14}$$

$$
\begin{aligned}
- & \frac{1}{8}\frac{(e^{-\hat{\alpha}}-1)^2}{r_1(\hat{\alpha})^4\ |a(\lambda_1(\hat{\alpha}),\hat{\alpha})|^4} e^{-4r_1(\hat{\alpha})\sin\varphi_1(\hat{\alpha})\ \hat{y}} \\
\times & \left(\frac{1}{\sin^2\varphi_1(\hat{\alpha})} + \cos\left\{4\hat{y}r_1(\hat{\alpha})\cos\varphi_1(\hat{\alpha}) - 4\arg a(\lambda_1(\hat{\alpha}),\hat{\alpha}) - 4\varphi_1(\hat{\alpha})\right\}\right) \\
+ & o\left(e^{-4r_1(\hat{\alpha})\sin\varphi_1(\hat{\alpha})\ \hat{y}}\right).
\end{aligned}
$$

Here we set

$$C(\beta,\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} \ln\left(\frac{1+e^{\mu^2-\beta}}{e^{\alpha}+e^{\mu^2-\beta}}\right) d\mu, \tag{4.15}$$

$$\lambda_1(\alpha) = \sqrt{\alpha+\beta+\pi i},\ \ r_1(\alpha) = |\lambda_1(\alpha)|,\ \ \varphi_1(\alpha) = \arg\lambda_1(\alpha), \tag{4.16}$$

$$a(\lambda,\alpha) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\ln\left(\frac{1+e^{\mu^2-\beta}}{e^{\alpha}+e^{\mu^2-\beta}}\right)\right\}. \tag{4.17}$$

5 Evaluation of the mean value

In this section we evaluate the vacuum expectation value of the operator $\det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}$ for $y=\sqrt{\frac{mT}{2}}x\to+\infty$. ¿From Corollary 4.2, we deduce

$$
\begin{aligned}
& \langle 0|\ \det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}|0\rangle \\
= & \ \langle 0|A\left(\beta,\hat{\alpha}\right)e^{-C(\beta,\hat{\alpha})\hat{y}} + B\left(\beta,\hat{\alpha}\right)e^{-\{C(\beta,\hat{\alpha})+4r_1(\hat{\alpha})\sin\varphi_1(\hat{\alpha})\}\hat{y}} \\
& + G\left(\beta,\hat{\alpha}\right)e^{\{-C(\beta,\hat{\alpha})+4i\lambda_1(\hat{\alpha})\}\hat{y}} + H\left(\beta,\hat{\alpha}\right)e^{\{-C(\beta,\hat{\alpha})-4i\lambda_1^*(\hat{\alpha})\}\hat{y}}|0\rangle + \cdots.
\end{aligned} \tag{5.1}
$$

Here we set

$$A(\beta,\alpha) = \exp\left\{\frac{1}{2}\int_{-\infty}^{\beta}\left(\frac{\partial C(b,\alpha)}{\partial b}\right)^2 db\right\}, \tag{5.2}$$

$$B\left(\beta,\alpha\right) = -\frac{(e^{-\alpha}-1)^2}{8r_1(\alpha)^4\sin^2\varphi_1(\alpha)}\left(\frac{a(\lambda_1^*(\alpha),\alpha)}{a(\lambda_1(\alpha),\alpha)}\right)^2 A(\beta,\alpha), \tag{5.3}$$

$$G(\beta,\alpha) = -\frac{(e^{-\alpha}-1)^2}{16}\frac{1}{\lambda_1(\alpha)^4 a(\lambda_1(\alpha),\alpha)^4}A(\beta,\alpha), \tag{5.4}$$

$$H(\beta,\alpha) = -\frac{(e^{-\alpha}-1)^2}{16}\frac{a(\lambda_1^*(\alpha),\alpha)^4}{\lambda_1^*(\alpha)^4}A(\beta,\alpha), \tag{5.5}$$

where $C(\beta,\alpha), \lambda_1(\alpha), r_1(\alpha), \varphi_1(\alpha)$ and $a(\lambda,\alpha)$ are defined in (4.15), (4.16) and (4.17). $\lambda_1^*(\alpha)$ is the complex conjugation of $\lambda_1(\alpha)$, i.e.

$$\lambda_1^*(\alpha) = \sqrt{\alpha+\beta-\pi i}. \tag{5.6}$$

$H(\beta,\alpha)$ is the complex conjugation of $G(\beta,\alpha)$.

In this section we evaluate the right-hand of the above vacuum expectation value. For the reader's convenience we summarize the commutation relations of the quantum operators :

$$\hat{y} = y + \hat{y}_p + \hat{y}_q, \quad y = \sqrt{\frac{mT}{2}}x, \ \hat{y}_p = \sqrt{\frac{mT}{2}}\hat{x}_p, \ \hat{y}_q = \sqrt{\frac{mT}{2}}\hat{x}_q, \tag{5.7}$$

$$\hat{\alpha} = \alpha + \hat{\alpha}_p + \hat{\alpha}_q, \quad \beta = \frac{h}{T}, \tag{5.8}$$

$$[\hat{y}_p, \hat{\alpha}_q] = \frac{\sqrt{2mT}}{c} = [\hat{\alpha}_p, \hat{y}_q], \quad \hat{y}_p|0\rangle = 0 = \hat{\alpha}_p|0\rangle, \quad \langle 0|\hat{y}_q = 0 = \langle 0|\hat{\alpha}_q. \tag{5.9}$$

The following Proposition is the key to calculating the vacuum expectation value.

Proposition 5.1 *The following asymptotic formula holds at large $y \to +\infty$:*

$$\langle 0|e^{\hat{y}E(\hat{\alpha})}F(\hat{\alpha})|0\rangle = F\left(\alpha + \frac{\sqrt{2mT}}{c}E(\alpha)\right)e^{yE(\alpha)} + \cdots. \tag{5.10}$$

Here $E(\alpha)$ and $F(\alpha)$ are meromorphic functions of α.

Proof. In this proof we use the following abbreviations :

$$\delta = \frac{\sqrt{2mT}}{c}, \quad A_0 + A_1\hat{\alpha}_q + A_2\hat{\alpha_q}^2 + \cdots = E(\alpha + \hat{\alpha}_q). \tag{5.11}$$

First we expand the exponential function and use the relations $\langle 0|\hat{y}_q = 0, \ \hat{\alpha}_p|0\rangle = 0$ and $[\hat{\alpha}_p, \hat{\alpha}_q] = 0$. We obtain

$$(e.v.) := \langle 0|F(\hat{\alpha})\exp\{\hat{y}E(\hat{\alpha})\}|0\rangle = \langle 0|\sum_{n=0}^{\infty}\frac{1}{n!}(y+\hat{y}_p)^n(E(\alpha+\hat{\alpha}_q))^nF(\hat{\alpha})|0\rangle. \tag{5.12}$$

We expand

$$(E(\alpha+\hat{\alpha}_q))^n = (A_0 + A_1\hat{\alpha}_q + A_2\hat{\alpha_q}^2 + \cdots)^n, \tag{5.13}$$

and using the commutation relation :

$$\langle 0|[f(\hat{y}_p), \hat{\alpha}_q^k] = \delta^k\langle 0|f^{(k)}(\hat{y}_p), \tag{5.14}$$

we obtain

$$\begin{aligned}
(e.v.) &= \langle 0|\sum_{n=0}^{\infty}\frac{1}{n!}(y+\hat{y}_p)^n \sum_{\substack{m_0+m_1+m_2+\cdots=n \\ m_j\geq 0}} \frac{n!}{m_0!m_1!m_2!\cdots}A_0^{m_0}A_1^{m_1}A_2^{m_2}\cdots \\
&\times \hat{\alpha}_q^{m_1+2m_2+3m_3+\cdots}F(\hat{\alpha})|0\rangle \qquad (5.15) \\
&= \langle 0|\sum_{n=0}^{\infty}\sum_{\substack{m_0+m_1+m_2+\cdots=n \\ m_j\geq 0}} \frac{n!}{m_0!m_1!m_2!\cdots}\,\delta^{m_1+2m_2+3m_3+\cdots}(y+\hat{y}_p)^{n-(m_1+2m_2+3m_3+\cdots)} \\
&\times A_0^{m_0}A_1^{m_1}A_2^{m_2}\cdots\frac{n(n-1)\cdots(n+1-(m_1+2m_2+3m_3+\cdots))}{n!}F(\hat{\alpha})|0\rangle. \qquad (5.16)
\end{aligned}$$

Using the relation

$$\frac{1}{2\pi i}\oint \frac{e^t}{t^{n-k+1}}dt = \frac{n(n-1)\cdots(n-k+1)}{n!} = \frac{1}{(n-k)!}, \tag{5.17}$$

we can factor as follows :

$$\begin{aligned}(e.v.) &= \langle 0|\sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint\frac{e^t}{t}\left(\frac{A_0(y+\hat{y}_p)}{t}+A_1\delta+\frac{A_2\delta^2 t}{y+\hat{y}_p}+\frac{A_3\delta^3t^2}{(y+\hat{y}_p)^2}+\cdots\right)^n F(\hat{\alpha})|0\rangle \quad (5.18)\\ &= \langle 0|\sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint\frac{e^t}{t}\left(\frac{A_0(y+\hat{y}_p)}{t}\right)^n F(\hat{\alpha})|0\rangle+\cdots \quad \text{for} \quad y\to+\infty. \quad (5.19)\end{aligned}$$

Using the relations :

$$\frac{1}{1-z} = 1+z+z^2+z^3+\cdots, \quad f(z) = \frac{1}{2\pi i}\oint\frac{f(t)}{t-z}dt, \tag{5.20}$$

we get the following :

$$(e.v.) = \langle 0|\frac{1}{2\pi i}\oint\frac{e^t}{t-A_0(y+\hat{y}_p)}F(\hat{\alpha})|0\rangle+\cdots = \langle 0|e^{A_0(y+\hat{y}_p)}F(\hat{\alpha})|0\rangle+\cdots. \tag{5.21}$$

Using the relation $e^A B e^{-A} = e^{ad(A)}(B)$, we obtain

$$e^{E(\alpha)\hat{y}_p}F(\hat{\alpha})e^{-E(\alpha)\hat{y}_p} = e^{E(\alpha)ad(\hat{y}_p)}F(\hat{\alpha}) = e^{E(\alpha)\frac{\sqrt{2mT}}{c}\frac{\partial}{\partial a}}F(a)\Big|_{a=\hat{\alpha}} \tag{5.22}$$

Therefore we can drop the quantum operators in the expectation value.

$$(e.v.) = e^{yE(\alpha)}\langle 0|\ e^{E(\alpha)\frac{\sqrt{2mT}}{c}\frac{\partial}{\partial a}}F(a)\Big|_{a=\hat{\alpha}}\ e^{E(\alpha)\hat{y}_p}|0\rangle+\cdots = e^{E(\alpha)\frac{\sqrt{2mT}}{c}\frac{\partial}{\partial a}}F(a)\Big|_{a=\alpha}+\cdots. \tag{5.23}$$

Here we have used the relations $\langle 0|\hat{\alpha}_q = 0 = \hat{\alpha}_p|0\rangle$, $\hat{y}_p|0\rangle = 0$.

Because the exponential of derivation is a shift operator :

$$e^{w\frac{\partial}{\partial z}}f(z) = f(z+w), \tag{5.24}$$

we arrive at (5.10).

□

Now, we arrive at the following theorem.

Theorem 5.2 *The leading terms of the asymptotics of the expectation value behave exponentially as follows.*

$$\begin{aligned}&\langle 0|\ \det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=\frac{1-\exp(\hat{\alpha})}{\pi}}|0\rangle \quad (5.25)\\ =\ & A\left(\beta,\alpha-\frac{\sqrt{2mT}}{c}C(\beta,\alpha)\right)e^{-C(\beta,\alpha)y}\end{aligned}$$

$$
\begin{aligned}
&+ \quad B\left(\beta, \alpha - \frac{\sqrt{2mT}}{c}\left\{C(\beta,\alpha) + 4r_1(\alpha)\sin\varphi_1(\alpha)\right\}\right) e^{-\{C(\beta,\alpha)+4r_1(\alpha)\sin\varphi_1(\alpha)\}y} \\
&+ \quad G\left(\beta, \alpha + \frac{\sqrt{2mT}}{c}\left\{-C(\beta,\alpha) + 4i\lambda_1(\alpha)\right\}\right) e^{\{-C(\beta,\alpha)+4i\lambda_1(\alpha)\}y} \\
&+ \quad H\left(\beta, \alpha + \frac{\sqrt{2mT}}{c}\{-C(\beta,\alpha) - 4i\lambda_1^*(\alpha)\}\right) e^{\{-C(\beta,\alpha)-4i\lambda_1^*(\alpha)\}y} + \cdots .
\end{aligned}
$$

Here $A(\beta,\alpha), B(\beta,\alpha), G(\beta,\alpha)$ and $H(\beta,\alpha)$ are defined in (5.2),(5.3),(5.4) and (5.5).

Proof Applying Proposition 5.1 to (5.1), we arrive at the result. □

When we consider $c = \infty$, Theorem 5.2 coincides with the asymptotics results for the impenetrable Bose gas case [1].

Corollary 5.3 *At the limit $m \to 0$, $g \to \infty$, $x \to \infty$ such that $c = 2mg$ fixed and $\sqrt{m}x \to \infty$, the leading terms of asymptotics of the expectation value become :*

$$
\begin{aligned}
\langle j(x)j(0)\rangle_T \quad &\to \quad D^2 + \\
&+ \quad \frac{mT}{2}\left(B_0(\beta) + B_1(\beta)\sqrt{\frac{mT}{2}}x + B_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\exp\left\{-4r_1(0)\sin\varphi_1(0)\sqrt{\frac{mT}{2}}x\right\} \\
&+ \quad \frac{mT}{2}\left(G_0(\beta) + G_1(\beta)\sqrt{\frac{mT}{2}}x + G_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\exp\left\{4i\lambda_1(0)\sqrt{\frac{mT}{2}}x\right\} \\
&+ \quad \frac{mT}{2}\left(H_0(\beta) + H_1(\beta)\sqrt{\frac{mT}{2}}x + H_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\exp\left\{-4i\lambda_1^*(0)\sqrt{\frac{mT}{2}}x\right\} + \cdots .
\end{aligned}
\tag{5.26}
$$

Here $D = \frac{N}{L}$ is the density of the thermodynamics and $\beta = \frac{h}{T}$. Here $B_j(\beta)$, $G_j(\beta)$ and $H_j(\beta)$, $(j = 0,1,2)$ are functions of β. $H_j(\beta)$ $(j = 0,1,2)$ is the complex conjugation of $G_j(\beta)$ $(j = 0,1,2)$, i.e. $H_j(\beta) = G_j^*(\beta)$. Explicit formulas for $B_j(\beta)$, $G_j(\beta)$ and $H_j(\beta)$ $(j = 0,1,2)$ are summarized in the Appendix.

Proof. From Corollary 3.3, Corollary 5.3 and the relation

$$
\langle j(x)j(0)\rangle_T = \frac{1}{2}\frac{\partial^2}{\partial x^2}\langle Q(x)^2\rangle_T = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left.\frac{\partial^2}{\partial\alpha^2}\langle\exp\left(\alpha Q(x)\right)\rangle_T\right|_{\alpha=0}, \tag{5.27}
$$

we can derive the result. For example the constant D^2 is derived by :

$$
D^2 = \left(1 + \frac{2}{c}D\right)\frac{1}{2}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial\alpha^2}\left.A\left(\beta, \alpha - \frac{\sqrt{2mT}}{c}C(\beta,\alpha)\right)e^{-C(\beta,\alpha)y}\right|_{\alpha=0} + \cdots . \tag{5.28}
$$

□

V. Korepin [8] proposed a method of presenting correlation functions in the form of special series. This method is useful in the calculation of the long distance asymptotics. N. Bogoliubov and V. Korepin [9] considered the asymptotics of correlation functions for the penetrable Bose gas by the special series method. Corollary 5.3 coincides with the result of [9]. For the impenetrable Bose gas case ($c = \infty$), V. Korepin and N. Slavnov [10] calculated higher order corrections and derived pre-exponential polynomials by the special series method. In this paper we derived pre-exponential polynomials for penetrable Bose gas case ($0 < c < +\infty$) by using the determinant representation.

Appendix

In this Appendix we summarize the asymptotics of the density-density correlation function. We use the notation given in Corollary 5.3. In the sequel, we use the following abbreviations :

$$\lambda_1 = \lambda_1(0) = \sqrt{\beta + \pi i},\ r_1 = |\lambda_1| = |\lambda_1(0)|,\ \varphi_1 = \arg\lambda_1 = \arg\lambda_1(0). \tag{5.29}$$

First we summarized the coefficients of $\exp\left\{-4r_1\sin\varphi_1\sqrt{\frac{mT}{c}}x\right\}$:

$$\frac{mT}{2}\left(B_0(\beta) + B_1(\beta)\sqrt{\frac{mT}{2}}x + B_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\exp\left\{-4r_1\sin\varphi_1\sqrt{\frac{mT}{2}}x\right\}. \tag{5.30}$$

The functions $B_j(\beta)$ are given by

$$B_j(\beta) = \tilde{B}_j(\beta)\left(1 + \frac{2}{c}D\right), \quad (j = 0, 1, 2), \tag{5.31}$$

where $\tilde{B}_j(\beta)$ are given by

$$\tilde{B}_2(\beta) = 16r_1^2\sin^2\varphi_1\left(\frac{d(\beta)}{\pi} + \frac{2\sin\varphi_1}{r_1}\right)^2 \times B\left(\beta, -4\Delta r_1\sin\varphi_1\right), \tag{5.32}$$

$$\begin{aligned}\tilde{B}_1(\beta) &= 32r_1^2\sin^2\varphi_1\left(\frac{d(\beta)}{\pi} + \frac{2\sin\varphi_1}{r_1}\right)\left(1 + \Delta\left(\frac{d(\beta)}{\pi} + \frac{2\sin\varphi_1}{r_1}\right)\right) \times \left(\frac{\partial B}{\partial\alpha}\right)(\beta, -4\Delta r_1\sin\varphi_1) \\ &- 4r_1\sin\varphi_1\left\{\left(\frac{2d(\beta)}{\pi}\right)^2 + \frac{16\sin\varphi_1}{\pi r_1}d(\beta) - \frac{4r_1\sin\varphi_1}{\pi}\left(\frac{\partial d}{\partial\beta}\right)(\beta)\right. \\ &+ \left.\frac{4\sin^2\varphi_1}{r_1^2}(5 + \cos 2\varphi_1)\right\} \times B(\beta, -4\Delta r_1\sin\varphi_1)\end{aligned} \tag{5.33}$$

$$\tilde{B}_0(\beta) = 16r_1^2\sin^2\varphi_1\left\{1 + \Delta\left(\frac{d(\beta)}{\pi} + \frac{2\sin\varphi_1}{r_1}\right)\right\}^2 \times \left(\frac{\partial^2 B}{\partial\alpha^2}\right)(\beta, -4\Delta r_1\sin\varphi_1)$$

$$
\begin{aligned}
&- \quad 4r_1 \sin\varphi_1 \left[\frac{4d(\beta)}{\pi} + \frac{8\sin\varphi_1}{r_1} + \Delta \left\{ \left(\frac{2d(\beta)}{\pi} \right)^2 + \frac{16\sin\varphi_1}{\pi r_1} d(\beta) \right. \right. \qquad (5.34)\\
&- \quad \left. \left. \frac{4r_1\sin\varphi_1}{\pi} \left(\frac{\partial d}{\partial \beta} \right)(\beta) + \frac{4\sin^2\varphi_1}{r_1^2}(5+\cos 2\varphi_1) \right\} \right] \times \left(\frac{\partial B}{\partial \alpha} \right)(\beta, -4\Delta r_1 \sin\varphi_1)\\
&+ \quad 2\left\{ \left(\frac{d(\beta)}{\pi} \right)^2 + \frac{4\sin\varphi_1}{\pi r_1} d(\beta) - \frac{4r_1\sin\varphi_1}{\pi} \left(\frac{\partial d}{\partial \beta} \right)(\beta) + \frac{4\sin^2 2\varphi_1}{r_1^2} \right\} \times B\left(\beta, -4\Delta r_1 \sin\varphi_1\right).
\end{aligned}
$$

Here we used the abbreviations :

$$
\Delta = \frac{\sqrt{2mT}}{c},\ d(\beta) = \int_{-\infty}^{\infty} \frac{1}{1+e^{\mu^2-\beta}} d\mu,\ \beta = \frac{h}{T}, \qquad (5.35)
$$

and the function $B(\beta,\alpha)$ as defined in (5.3).

Next we summarize the coefficients of $\exp\left\{4i\lambda_1\sqrt{\frac{mT}{2}}x\right\}$:

$$
\frac{mT}{2}\left(G_0(\beta) + G_1(\beta)\sqrt{\frac{mT}{2}}x + G_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2 \right) \exp\left\{ 4i\lambda_1\sqrt{\frac{mT}{2}}x \right\} \qquad (5.36)
$$

The functions $G_j(\beta)$ are given by

$$
G_j(\beta) = \tilde{G}_j(\beta)\left(1+\frac{2}{c}D\right), \quad (j=0,1,2). \qquad (5.37)
$$

where $\tilde{G}_j(\beta)$ are given by

$$
\begin{aligned}
\tilde{G}_2(\beta) &= -16\lambda_1^2 \left(\frac{d(\beta)}{\pi} + \frac{2i}{\lambda_1} \right)^2 \times G\left(\beta, 4\Delta i\lambda_1\right), \qquad (5.38)\\
\tilde{G}_1(\beta) &= -32\lambda_1^2 \left(\frac{d(\beta)}{\pi} + \frac{2i}{\lambda_1} \right)\left(1 + \Delta\left(\frac{d(\beta)}{\pi} + \frac{2i}{\lambda_1} \right) \right) \times \left(\frac{\partial G}{\partial \alpha} \right)(\beta, 4\Delta i\lambda_1) \qquad (5.39)\\
&+ \quad 4i\lambda_1 \left\{ \left(\frac{2d(\beta)}{\pi} \right)^2 + \frac{16i}{\pi\lambda_1} d(\beta) + \frac{4i\lambda_1}{\pi}\left(\frac{\partial d}{\partial \beta} \right)(\beta) - \frac{12}{\lambda_1^2} \right\} \times G(\beta, 4\Delta i\lambda_1),\\
\tilde{G}_0(\beta) &= -16\lambda_1^2 \left\{ 1 + \Delta\left(\frac{d(\beta)}{\pi} + \frac{2i}{\lambda_1} \right) \right\}^2 \times \left(\frac{\partial^2 G}{\partial \alpha^2} \right)(\beta, 4\Delta i\lambda_1) \qquad (5.40)\\
&+ \quad 4i\lambda_1 \left[\frac{4d(\beta)}{\pi} + \frac{8i}{\lambda_1} + \Delta\left\{ \left(\frac{2d(\beta)}{\pi} \right)^2 + \frac{16i}{\pi\lambda_1} d(\beta) + \frac{4i\lambda_1}{\pi}\left(\frac{\partial d}{\partial \beta} \right)(\beta) - \frac{12}{\lambda_1^2} \right\} \right] \times \left(\frac{\partial G}{\partial \alpha} \right)(\beta, 4\Delta i\lambda_1)\\
&+ \quad 2\left\{ \left(\frac{d(\beta)}{\pi} \right)^2 + \frac{4i}{\pi\lambda_1} d(\beta) + \frac{4i\lambda_1}{\pi}\left(\frac{\partial d}{\partial \beta} \right)(\beta) \right\} \times G\left(\beta, 4\Delta i\lambda_1\right),
\end{aligned}
$$

Here the function $G(\beta,\alpha)$ is defined in (5.4). Next we summarize the coefficients of $\exp\left\{-4i\lambda_1^*\sqrt{\frac{mT}{2}}x\right\}$:

$$
\frac{mT}{2}\left(H_0(\beta) + H_1(\beta)\sqrt{\frac{mT}{2}}x + H_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2 \right) \exp\left\{ -4i\lambda_1^*\sqrt{\frac{mT}{2}}x \right\} \qquad (5.41)
$$

where $\lambda_1^* = \sqrt{\beta - \pi i}$. The functions $H_j(\beta)$ are given by the complex conjugation of $G_j(\beta)$.

$$H_j(\beta) = G_j^*(\beta), \quad (j = 0, 1, 2). \tag{5.42}$$

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