Geometric Approach to Inverse Scattering for the Schrödinger Equation with Magnetic and Electric Potentials

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Abstract

We consider the Hamiltonian $H = (p - A(x))^2/(2m) + V(x)$ of a quantum particle in a magnetic field $B = rotA$ and a potential V in space dimensions $\nu \geq 2$. If V is of short range then the high velocity limit of the scattering operator uniquely determines the magnetic field B and the potential V. If, in addition, long–range potentials V^l are present, some knowledge of (the far out tail of) V^l is needed to define a modified Dollard wave operator and a scattering operator S^D. Again its high velocity limit uniquely determines B and $V = V^s + V^l$. Moreover, we give explicit error bounds which are inverse proportional to the velocity.

1 Notations and Main Results

We study a 2–body quantum mechanical system in $\nu \geq 2$ dimensions. Here the interactions are given by a vector potential $\mathbf{A}$ and a scalar potential $V = V^s + V^l$. V^s is assumed to be short–range, i.e. $V^s \in \mathcal{V}_{SR}$ with

$$\begin{aligned} \mathcal{V}_{SR} \ :=\ & \{V^s \,|\, V^s \text{ is Kato–small with relative bound } < 1, \\ & \int_0^\infty dR \, \|V^s(\mathbf{y}) \, F(|\mathbf{y}| \geq R) \, (-\Delta + 1\!\!1)^{-1}\| < \infty \Big\}. \end{aligned}$$

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V^l is assumed to be long–range (i.e., $V^l \in C^1(\mathbb{R}^\nu)$, V^l tends to zero asymptotically and $|\nabla V^l(\mathbf{y})| \leq \text{const}\,(1+|\mathbf{y}|)^{-\gamma}$ with $\gamma > 3/2$). Each component of the field strength tensor $F = d\mathbf{A}$ is Kato–small with relative bound < 1, and the decay for each component is assumed as

$$\|F_{ij}(\mathbf{y})\, F(|\mathbf{y}| \geq R)\, (-\Delta + \mathbb{1})^{-1}\| \leq \text{const}\,(1+R)^{-(2+\delta)},\ \delta > 0. \tag{1.1}$$

In addition we assume

$$\|\partial_i F_{ij}(\mathbf{y})\, F(|\mathbf{y}| \geq R)\, (-\Delta + \mathbb{1})^{-1}\| \leq \text{const}\,(1+R)^{-(2+\tilde{\delta})},\ \tilde{\delta} > 0. \tag{1.2}$$

In $\nu = 3$ dimensions the vector $(\sum_{i=1}^3 \partial_i F_{ij})_{j=1,2,3} = -\,\mathbf{rot}\,\mathbf{B}$ and up to a constant equal to the current density. Each component of the vector potential is supposed to be a continuous, short–range L^2–function with short–range $\operatorname{div}\mathbf{A}$. For $\mathbf{A}_i$ and F_{ij} a regularization with $(-\Delta + \mathbb{1})^{-1/2}$ is sufficient. We remark that for the proof of the reconstruction formulas the decay assumptions on the vector potential can be weakened (see Generalizations). By $H_0 = \frac{1}{2m}\mathbf{p}^2$ we denote the free Hamiltonian and by

$$\begin{aligned} H &= \frac{1}{2m}(\mathbf{p} - \mathbf{A}(\mathbf{x}))^2 + V(\mathbf{x}) \\ &:= H_0 - \frac{1}{m}\mathbf{A}(\mathbf{x})\mathbf{p} + \frac{i}{2m}\operatorname{div}\mathbf{A} + \frac{1}{2m}|\mathbf{A}|^2(\mathbf{x}) + V(\mathbf{x}) \end{aligned}$$

the interacting Hamiltonian with reduced mass m and momentum $\mathbf{p} = -i\nabla_{\mathbf{x}}$. Then the interacting Hamiltonian is self–adjoint on the domain $\mathcal{D}(H) = \mathcal{D}(H_0) = W^{2,2}(\mathbb{R}^\nu)$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^\nu)$.
In the special case, when V is of short range the wave operators

$$\Omega_\pm := \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt}e^{-iH_0t} \tag{1.3}$$

exist and are complete (see, e.g., [E 92, Theorem 4.1] and [A 94, Theorem 3.7]). The scattering operator S is

$$S := (\Omega_+)^*\Omega_-. \tag{1.4}$$

Let $\Phi_0 \in \mathcal{H}$ be a normalized asymptotic configuration with compact momentum support in the ball of radius $m\eta$, $\eta > 0$, and momentum space wave function $\hat{\phi}_0 \in C_0^\infty(B_{m\eta}(0))$. We can find a function $f \in C_0^\infty(\mathbb{R}^\nu)$ such that $\Phi_0 = f(\mathbf{p})\Phi_0$. By $\Phi_{\mathbf{v}}$ we denote the boosted configuration translated by $m\mathbf{v}$ in momentum space, where $\mathbf{v}$ is the velocity:

$$\Phi_{\mathbf{v}} = e^{im\mathbf{v}\mathbf{x}}\Phi_0 \;\Leftrightarrow\; \hat{\phi}_{\mathbf{v}}(\mathbf{p}) = \hat{\phi}_0(\mathbf{p} - m\mathbf{v}), \tag{1.5}$$

which has compact velocity support in $B_\eta(\mathbf{v})$. Let $v := |\mathbf{v}|$ and without loss of generality let $v \geq 16\eta$. We will obtain asymptotics of S for the high velocity limit in an arbitrary direction $\hat{\mathbf{v}} := \mathbf{v}/v$, $v \to \infty$.

Theorem 1 *Suppose that $V = V^s$. Then the following formula for the asymptotics of S holds for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as in Eq. (1.5):*

$$\lim_{v\to\infty} (S\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) = \left(e^{i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi} \Phi_0, \Psi_0 \right). \tag{1.6}$$

In particular, the scattering operator S uniquely determines the magnetic field $\mathbf{B}$.

Regarding S as a mapping from the set of short–range potentials $\mathcal{V}_{SR}$ into the set of bounded operators $L(\mathcal{H})$ we have the following

Theorem 2 *Suppose that $V = V^s$. Then the following reconstruction formula holds for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as in Eq. (1.5):*

$$\begin{aligned}
&\lim_{v\to\infty} iv \left(\left(e^{-i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi} S - \mathbb{1} \right) \Phi_{\mathbf{v}}, \Psi_{\mathbf{v}} \right) - (F(\mathbf{B})\Phi_0, \Psi_0) \\
= \;& \int_{-\infty}^{\infty} (V^s(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\, d\tau,
\end{aligned} \tag{1.7}$$

where $F(\mathbf{B})$ is an explicit operator which is independent of V^s, see (2.24) and (2.27). In particular, for a given magnetic field $\mathbf{B}$ the scattering map

$$S(\mathbf{B}, \cdot) : \mathcal{V}_{SR} \to L(\mathcal{H})$$

is injective.

For $V = V^s + V^l$, the modified Dollard wave operators

$$\Omega_\pm^D := \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt} U^D(t,0) \;\text{ with }\; U^D(t,0) = e^{-iH_0 t} e^{-i\int_0^t V^l(s\mathbf{p}/m)\,ds} \tag{1.8}$$

exist and are complete (see, e.g., [E 92, Theorem 4.1] and [A 94, Theorem 3.14]).

$$S^D := (\Omega_+^D)^* \Omega_-^D \tag{1.9}$$

is the scattering operator S^D. Since the splitting of V into a short– and a long–range part is not unique the Dollard wave operators are not uniquely defined. Nevertheless, the magnetic field and the scalar potential are uniquely obtained from *any* of the scattering operators S^D.

Theorem 3 *The same formula for the asymptotics of S^D as in the short–range case holds for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as in Eq. (1.5):*

$$\lim_{v\to\infty} (S^D\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) = \left(e^{i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi} \Phi_0, \Psi_0 \right). \tag{1.10}$$

In particular, the scattering operator S^D uniquely determines the magnetic field.

Theorem 4 *The following reconstruction formula holds for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as in Eq. (1.5):*

$$\begin{aligned} &\lim_{v\to\infty} iv\left(\left(e^{-i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}S^D - \mathbb{1}\right)\Phi_{\mathbf{v}},\Psi_{\mathbf{v}}\right) - (F(\mathbf{B},V^l)\Phi_0,\Psi_0) \\ = &\int_{-\infty}^{\infty}(V^s(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0,\Psi_0)\,d\tau, \end{aligned} \tag{1.11}$$

where $F(\mathbf{B},V^l)$ is an explicit operator which is independent of V^s, see (2.24) and (3.11). In particular, for a given magnetic field $\mathbf{B}$ and a long–range potential V^l the scattering map

$$S^D(\mathbf{B},V^l,\cdot):\mathcal{V}_{SR}\to L(\mathcal{H})$$

is injective.
Moreover, the following formula holds for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as in Eq. (1.5):

$$\begin{aligned} &\lim_{v\to\infty} iv\left(\left[e^{-i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}S^D,\mathbf{p}_1\right]\Phi_{\mathbf{v}},\Psi_{\mathbf{v}}\right) - (F(\mathbf{B})\Phi_0,\Psi_0) \\ = &\int_{-\infty}^{\infty}\{(V^s(\mathbf{x}+\tau\hat{\mathbf{v}})\mathbf{p}_1\Phi_0,\Psi_0) - (V^s(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0,\mathbf{p}_1\Psi_0) \\ &+ i((\partial_1 V^l)(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0,\Psi_0)\}\,d\tau, \end{aligned} \tag{1.12}$$

where $F(\mathbf{B})$ is an explicit operator which is independent of V. In particular, if $\mathbf{B}$ is given any of the scattering operators S^D uniquely determines the total potential V.

All formulas hold in the high velocity limit. For large, but fixed, v we obtain an error term of order $\mathcal{O}(v^{-1})$, see (2.29) and (3.12) for details.

Remark 5 *As each component of $\mathbf{A}$ is of short range, we can find a function $\lambda\in C^2(\mathbb{R}^\nu)$ tending to zero asymptotically with $\mathbf{B}=\mathbf{rot}\,\mathbf{A}=\mathbf{rot}(\mathbf{A}+\nabla\lambda)$. Then $S^{(D)}$ and $\exp(\pm\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi)$ are gauge invariant under these gauge transformations and all formulas are independent of the special choice of the gauge.*

Our proofs use a geometric, time–dependent method developed by Enss and Weder. This method has been used in [EW 95] to prove Theorems 2 and 4 in the case without magnetic field.
To our knowledge the case with magnetic fields was previously treated by Novikov and Khenkin in [NK 87]. Eskin and Ralston proved in [ER 95] that the scattering operator at fixed energy uniquely determines the magnetic field and the scalar potential by stationary phase methods for exponentially decaying $\mathbf{A}$ and V^s. The results of Nicoleau in [N 96a] and [N 96b] are similar to our Theorems

1–4: By stationary methods he proves that the high energy limit uniquely determines the magnetic field and the scalar potential, where the potentials have to be C^∞–functions with stronger decay assumptions on higher derivatives, but not necessarily short–range vector potential. In these papers singularities of the short–range part are not included.

2 Proof in the short–range case $V = V^s$

We first compute $\Omega_-\Phi_{\mathbf{v}}$ using the abbreviation

$$\begin{aligned} I(\mathbf{x},\mathbf{p}) &:= H - H_0 \\ &= V^s(\mathbf{x}) + \frac{i}{2m}\operatorname{div}\mathbf{A} + \frac{1}{2m}|\mathbf{A}|^2(\mathbf{x}) - \frac{1}{m}\mathbf{A}(\mathbf{x})\mathbf{p} \end{aligned} \tag{2.1}$$

for the short–range interaction by the scalar and the vector potentials.

$$\begin{aligned} &\Omega_-\Phi_{\mathbf{v}} \\ &= \Phi_{\mathbf{v}} - \int_{-\infty}^0 dt\,\frac{d}{dt}\left(e^{iHt}e^{-iH_0t}\right)\Phi_{\mathbf{v}} \\ &= \Phi_{\mathbf{v}} - \int_{-\infty}^0 dt\,e^{iHt} i\,I(\mathbf{x},\mathbf{p})\,e^{-iH_0t}\Phi_{\mathbf{v}} \\ &= e^{im\mathbf{v}\mathbf{x}}\Phi_0 - \int_{-\infty}^0 dt\,e^{im\mathbf{v}\mathbf{x}}e^{i(H+(\mathbf{p}-\mathbf{A}(\mathbf{x}))\mathbf{v})t} i\,I(\mathbf{x},\mathbf{p}+m\mathbf{v})\,e^{-i(H_0+\mathbf{p}\mathbf{v})t}\Phi_0, \end{aligned} \tag{2.2}$$

where the relation $e^{-im\mathbf{v}\mathbf{x}}f(\mathbf{p})\,e^{im\mathbf{v}\mathbf{x}} = f(\mathbf{p}+m\mathbf{v})$ is used. With the abbreviations

$$H_2 := H/v + (\mathbf{p} - \mathbf{A}(\mathbf{x}))\hat{\mathbf{v}} \quad \text{and} \quad H_1 := H_0/v + \mathbf{p}\hat{\mathbf{v}} \tag{2.3}$$

and the substitution $\tau = vt$ we obtain:

$$\Omega_-\Phi_{\mathbf{v}} = e^{im\mathbf{v}\mathbf{x}}\Omega_-(H_2, H_1)\Phi_0.$$

If there is an integrable bound of the integrand in (2.2) uniformly in $\mathbf{v}$ we can interchange the high velocity limit and the integral by the dominated convergence theorem. Since $\mathbf{A}$, $\operatorname{div}\mathbf{A}$, $|\mathbf{A}|^2$ and V^s are of short range the existence of such an integrable bound can be shown as in the case without magnetic field (see [EW 95, Lemma 2.2]). We conclude

$$\begin{aligned} \lim_{v\to\infty} e^{-im\mathbf{v}\mathbf{x}}\Omega_-\Phi_{\mathbf{v}} &= \Phi_0 - \int_{-\infty}^0 d\tau\,e^{i(\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}\tau}(-i\mathbf{A}(\mathbf{x})\hat{\mathbf{v}})e^{-i\mathbf{p}\hat{\mathbf{v}}\tau}\Phi_0 \\ &= \lim_{\tau\to-\infty} e^{i(\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}\tau}e^{-i\mathbf{p}\hat{\mathbf{v}}\tau}\Phi_0 \\ &=: \Omega_-((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}})\Phi_0, \end{aligned} \tag{2.4}$$

where $\text{s-lim}_{v\to\infty} \exp(iH_2\tau) = \exp(i(\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}\tau)$ and $\text{s-lim}_{v\to\infty}\exp(iH_1\tau) = \exp(i\mathbf{p}\hat{\mathbf{v}}\tau)$ for fixed τ is used. It is obvious that the same computation can be done for Ω_+ (by replacing $-\infty$ with $+\infty$). Analogously we obtain

$$\Omega_+((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}}) := \underset{\tau\to+\infty}{\text{s-lim}}\, e^{i(\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}\tau}e^{-i\mathbf{p}\hat{\mathbf{v}}\tau}. \tag{2.5}$$

We can prove that

$$e^{i(\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}\tau}e^{-i\mathbf{p}\hat{\mathbf{v}}\tau}e^{i\int_0^\tau \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi} = \mathbb{1}$$

holds for all $\tau\in\mathbb{R}$ by showing that the derivatives $d/d\tau$ of both sides are equal. Consequently we get the following explicit expressions for (2.4) and (2.5):

$$\Omega_-((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})\Phi_0 = e^{+i\int_{-\infty}^0 \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\Phi_0 \tag{2.6}$$

$$\Omega_+((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})\Phi_0 = e^{-i\int_0^\infty \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\Phi_0. \tag{2.7}$$

Therefore the scattering operator S is equal to $\exp(i\int_{-\infty}^\infty \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi)$ in the high velocity limit $v\to\infty$ and we obtain formula (1.6):

$$\lim_{v\to\infty}(S\Phi_{\mathbf{v}},\Psi_{\mathbf{v}}) = \left(e^{i\int_{-\infty}^\infty \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}\Phi_0,\Psi_0\right).$$

So far we have only used that $\operatorname{div}\mathbf{A}$ and the components of $\mathbf{A}$ are in $\mathcal{V}_{SR}$.
As $\mathbf{A}$ is continuous we infer $\int_{-\infty}^\infty \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi$ from above. This integral transform uniquely determines the magnetic field $\mathbf{rot}\,\mathbf{A}=\mathbf{B}$ (see [J 96, Lemma 3.4] for $\mathbf{A}\in(L^2(\mathbb{R}^\nu))^\nu\cap(C^0(\mathbb{R}^\nu))^\nu$ and [I 95, Lemma 3.4] for $\mathbf{A}\in(C^2(\mathbb{R}^\nu))^\nu$ and more decay assumptions on $\mathbf{A}$). Hence Theorem 1 is proved. ■

Now $\mathbf{B}$ is given and after choosing a short–range $\mathbf{A}$ with $\mathbf{rot}\,\mathbf{A}=\mathbf{B}$ especially $\Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})$ is known. Because of (2.4) we are interested in the difference

$$\Omega_\pm(H_2,H_1)-\Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}}). \tag{2.8}$$

We need the following intertwining properties of $\Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})$ in order to show that the norm of this difference is of order $\mathcal{O}(v^{-1})$:

$$\begin{aligned}
&(\mathbf{p}-\mathbf{A}(\mathbf{x}))\,\Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})\\
= {}& \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})\left(\mathbf{p}-\mathbf{A}(\mathbf{x})-\nabla\int_0^{\pm\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi\right) && (2.9)\\
= {}& \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})\left(\mathbf{p}-\int_0^{\pm\infty}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi\right), && (2.10)
\end{aligned}$$

and for finite $t\in\mathbb{R}$:

$$\begin{aligned}
&(\mathbf{p}-\mathbf{A}(\mathbf{x}))\,e^{-i\int_0^t \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\\
= {}& e^{-i\int_0^t \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\left(\mathbf{p}-\mathbf{A}(\mathbf{x})-\nabla\int_0^t\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi\right) && (2.11)\\
= {}& e^{-i\int_0^t \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\left(\mathbf{p}-\mathbf{A}(\mathbf{x}+t\hat{\mathbf{v}})-\int_0^t(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi\right). && (2.12)
\end{aligned}$$

Equalities (2.9) and (2.11) can be proven by showing

$$\begin{aligned}
& \frac{d}{ds}\left(e^{i\int_0^t \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi\, s}(\mathbf{p}-\mathbf{A}(\mathbf{x}))e^{-i\int_0^t \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi\, s}\right) \\
= \ & \frac{d}{ds}\left(\mathbf{p}-\mathbf{A}(\mathbf{x})-\nabla\int_0^t \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi\, s\right).
\end{aligned}$$

An easy calculus gives $\nabla\int_0^{\pm\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi = \int_0^{\pm\infty}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi - \mathbf{A}(\mathbf{x})$, respectively $\nabla\int_0^t\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi = \int_0^t(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi+\mathbf{A}(\mathbf{x}+t\hat{\mathbf{v}})-\mathbf{A}(\mathbf{x})$ and thus equalities (2.10) and (2.12), where we use the 3–dimensional notation $(\hat{\mathbf{v}}\times\mathbf{B})$ for all dimensions with $(\hat{\mathbf{v}}\times\mathbf{B})$ as a notation for

$$(\hat{\mathbf{v}}\times\mathbf{B})_i = \sum_{j=1}^{\nu} F_{ij}\hat{\mathbf{v}}_j. \tag{2.13}$$

In the following we use the abbreviation

$$\hat{\mathbf{F}}_{\pm}(\mathbf{x}) := \int_0^{\pm\infty}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi. \tag{2.14}$$

Lemma 6 *For H_2 and H_1 as defined in (2.3)*

$$\|\{\Omega_{\pm}(H_2,H_1)-\Omega_{\pm}((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})\}\,\Phi_0\| = \mathcal{O}(v^{-1}). \tag{2.15}$$

Proof: We formally compute the difference

$$\begin{aligned}
& \Omega_{\pm}(H_2,H_1)-\hat{\Omega}_{\pm} \\
= \ & \lim_{t\to\pm\infty} e^{iH_2t}e^{-iH_1t} - e^{-i\int_0^t\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi} \\
= \ & \lim_{t\to\pm\infty} e^{iH_2t}\left\{\mathbb{1}-e^{-iH_2t}e^{-i\int_0^t\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}e^{iH_1t}\right\}e^{-iH_1t} \\
= \ & -\lim_{t\to\pm\infty} e^{iH_2t}\int_0^t d\tau\,\frac{d}{d\tau}\left\{e^{-iH_2\tau}e^{-i\int_0^\tau\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}e^{iH_1\tau}\right\}e^{-iH_1t} \\
= \ & \lim_{t\to\pm\infty} e^{iH_2t}\int_0^t d\tau\, e^{-iH_2\tau}i\left\{H_2e^{-i\int_0^\tau\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\right. \\
& \left. -e^{-i\int_0^\tau\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}[H_1-\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\tau\hat{\mathbf{v}})]\right\}e^{iH_1\tau}e^{-iH_1t} \\
= \ & \lim_{t\to\pm\infty}\int_0^t d\tau\, e^{iH_2\tau}i\left\{H_2e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\right. \\
& \left. -e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}[H_1-\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+(t-\tau)\hat{\mathbf{v}})]\right\}e^{-iH_1\tau}
\end{aligned} \tag{2.16}$$

Now we calculate the following difference with the help of (2.12):

$$
\begin{aligned}
&\left\{H_2 e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi} - e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}[H_1 - \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+(t-\tau)\hat{\mathbf{v}})]\right\} \\
= \;& v^{-1}\, e^{-i\int_0^{t-\tau}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\left\{-\frac{1}{2m}\mathbf{p}\left(\int_0^{t-\tau}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+(t-\tau)\hat{\mathbf{v}})\right)\right. \\
&-\frac{1}{2m}\left(\int_0^{t-\tau}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+(t-\tau)\hat{\mathbf{v}})\right)\mathbf{p} \\
&\left.+\frac{1}{2m}\left(\int_0^{t-\tau}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+(t-\tau)\hat{\mathbf{v}})\right)^2 + V^s(\mathbf{x})\right\} \\
\overset{t\to\pm\infty}{\approx}\;& v^{-1}\, e^{-i\int_0^{\pm\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\left(-\frac{1}{2m}\mathbf{p}\hat{\mathbf{F}}_\pm(\mathbf{x}) - \frac{1}{2m}\hat{\mathbf{F}}_\pm(\mathbf{x})\mathbf{p} + \frac{1}{2m}\hat{\mathbf{F}}_\pm^2(\mathbf{x}) + V^s(\mathbf{x})\right),
\end{aligned}
$$

where we used $(\hat{\mathbf{v}}\times\mathbf{B})\hat{\mathbf{v}} = 0$ which comes from the anti–symmetry of the field strength tensor F. Since $\|(1+\mathbf{x}^2)^{3/2}\Phi_0\| \le \text{const} < \infty$ the norm of (2.16) is bounded by

$$
\begin{aligned}
& v^{-1}\int_0^t d\tau\, \left\|\left\{-\frac{1}{2m}(\mathbf{p}-m\mathbf{v})\left(\int_0^{t-\tau}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+t\hat{\mathbf{v}})\right)\right.\right. \\
&-\frac{1}{2m}\left(\int_0^{t-\tau}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+t\hat{\mathbf{v}})\right)(\mathbf{p}-m\mathbf{v}) \\
&\left.+\frac{1}{2m}\left(\int_0^{t-\tau}(\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+t\hat{\mathbf{v}})\right)^2 + V^s(\mathbf{x})\right\} \\
&\left. e^{-iH_0\tau/v} f(\mathbf{p}-m\mathbf{v})(1+\mathbf{x}^2)^{-3/2}\right\| \qquad (2.17) \\
\overset{!}{\le}\;& v^{-1}\int_0^t d\tau\, h(|\tau|).
\end{aligned}
$$

If there is such an integrable majorant $h \in L^1([0,\pm\infty), d\tau)$ for the norm (2.17) uniformly in $\mathbf{v}$ we finished the proof.
The integrability of the summand with V^s is proven in [EW 95, Lemma 2.2]. We know that for a function $\bar{f} \in C_0^\infty(\mathbb{R}^\nu)$ with $\bar{f} \equiv 1$ on the support of f $\|\int_0^{t-\tau} d\xi\, (\hat{\mathbf{v}}\times\mathbf{B})_i(\mathbf{x}+\xi\hat{\mathbf{v}})\, F(|\mathbf{x}-\tau\hat{\mathbf{v}}| \le |\tau|/4)\bar{f}(\mathbf{p})\| \le \text{const}\,(1+|\tau|)^{-(1+\delta)}$, $\delta > 0$, because of

$$
\begin{aligned}
& \int_0^{t-\tau} d\xi\, \left\|(\hat{\mathbf{v}}\times\mathbf{B})_i(\mathbf{x}+\xi\hat{\mathbf{v}})\, F(|\mathbf{x}-\tau\hat{\mathbf{v}}| \le |\tau|/4)\bar{f}(\mathbf{p})\right\| \\
\le\;& \int_\tau^t d\xi\, \left\|(\hat{\mathbf{v}}\times\mathbf{B})_i(\mathbf{x}+\xi\hat{\mathbf{v}})\, F(|\mathbf{x}| \le |\tau|/4)\bar{f}(\mathbf{p})\right\| \\
\le\;& \int_\tau^t d\xi\, \left\|(\hat{\mathbf{v}}\times\mathbf{B})_i(\mathbf{x})\, F(|\mathbf{x}-\xi\hat{\mathbf{v}}| \le |\xi|/4)\bar{f}(\mathbf{p})\right\| \\
\le\;& \sum_{j=1}^{\nu}\int_\tau^t d\xi\, \left\|\hat{\mathbf{v}}_j F_{ij}(\mathbf{x})\, F(|\mathbf{x}| \ge 3|\xi|/4)\bar{f}(\mathbf{p})\right\| \\
\le\;& \text{const}\sum_{j=1}^{\nu}\int_\tau^t d\xi\,(1+|\xi|/2)^{-(2+\delta)} \le \text{const}\,(1+|\tau|)^{-(1+\delta)}, \qquad (2.18)
\end{aligned}
$$

where we used the decay assumption on each component F_{ij}. Besides

$$\begin{aligned}
& \left\| \mathbf{A}_i(\mathbf{x}+(t-\tau)\hat{\mathbf{v}})\, F(|\mathbf{x}-\tau\hat{\mathbf{v}}| \leq |\tau|/4)\bar{f}(\mathbf{p}) \right\| \\
= \; & \left\| \mathbf{A}_i(\mathbf{x})\, F(|\mathbf{x}-t\hat{\mathbf{v}}| \leq |\tau|/4)\bar{f}(\mathbf{p}) \right\| \\
\leq \; & \left\| \mathbf{A}_i(\mathbf{x})\, F(|\mathbf{x}| \geq 3|\tau|/4)\bar{f}(\mathbf{p}) \right\| \leq h_1(|\tau|),
\end{aligned} \tag{2.19}$$

where we used that each component of $\mathbf{A}$ is of short range. For the terms in (2.17) with $\mathbf{p}-m\mathbf{v}$ on the left side we have to calculate the commutator with $\mathbf{p}$ and infer $\sum_{i,j=1}^{\nu} \partial_i F_{ij}\hat{\mathbf{v}}_j$ and $\operatorname{div}\mathbf{A}$. In the same way as above we can show that

$$\left\| \int_0^{t-\tau} d\xi \sum_{i,j=1}^{\nu} \partial_i F_{ij}\hat{\mathbf{v}}_j(\mathbf{x}+\xi\hat{\mathbf{v}})\, F(|\mathbf{x}-\tau\hat{\mathbf{v}}| \leq |\tau|/4)\bar{f}(\mathbf{p}) \right\| \leq h_2(|\tau|) \text{ and} \tag{2.20}$$

$$\left\| \operatorname{div}\mathbf{A}(\mathbf{x}+(t-\tau)\hat{\mathbf{v}})\, F(|\mathbf{x}-\tau\hat{\mathbf{v}}| \leq |\tau|/4)\bar{f}(\mathbf{p}) \right\| \leq h_3(|\tau|) \tag{2.21}$$

by using the decay assumptions on $\partial_i F_{ij}$ and $\operatorname{div}\mathbf{A}$. We remark that in $\nu=3$ dimensions $\sum_{i,j=1}^{3} \partial_i F_{ij}\hat{\mathbf{v}}_j = -\,\mathbf{rot}\,\mathbf{rot}\,\mathbf{A}\hat{\mathbf{v}} = -\,\mathbf{rot}\,\mathbf{B}\hat{\mathbf{v}} = -\mu_0\,\mathbf{j}\hat{\mathbf{v}}$ with permeability constant μ_0 and current density $\mathbf{j}$. Then the integrable bound can be shown as in [EW 95, Proof of Lemma 2.2] by using

$$\left\| F(|\mathbf{x}| \geq |\tau|/8)(1+\mathbf{x}^2)^{-3/2} \right\| \leq \text{const}\,(1+|\tau|)^{-2} \text{ and} \tag{2.22}$$

$$\left\| F(|\mathbf{x}-\tau\hat{\mathbf{v}}| \geq |\tau|/4) e^{-iH_0\tau/v} f(\mathbf{p}-m\mathbf{v}) F(|\mathbf{x}| \leq |\tau|/8) \right\| \leq C_l\,(1+|\tau|)^{-l} \tag{2.23}$$

uniformly in $\mathbf{v}$ for $l=1,2,\ldots$, which is a propagation property expressing the fact that the solutions of the free Schrödinger equation have rapid decay away from the classically forbidden region (see [E 83, Proposition 2.10] or [EW 95, Proposition 2.1]). ■

To get shorter formulas we define

$$K_\pm(\mathbf{p},\mathbf{x}) := -\frac{1}{2m}\mathbf{p}\hat{\mathbf{F}}_\pm(\mathbf{x}) - \frac{1}{2m}\hat{\mathbf{F}}_\pm(\mathbf{x})\mathbf{p} + \frac{1}{2m}\hat{\mathbf{F}}_\pm^2(\mathbf{x}) \tag{2.24}$$

with $\hat{\mathbf{F}}_\pm$ as defined in (2.14). In particular the proof of Lemma 6 shows

$$\begin{aligned}
& v\left\{\Omega_\pm(H_2,H_1) - \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}})\right\}\Phi_0 \\
\overset{v\to\infty}{\approx} \; & i\int_0^{\pm\infty} d\tau\, e^{-i\int_0^{\pm\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\left(V^s(\mathbf{x}+\tau\hat{\mathbf{v}}) + K_\pm(\mathbf{p},\mathbf{x}+\tau\hat{\mathbf{v}})\right)\Phi_0,
\end{aligned} \tag{2.25}$$

where $\text{s-lim}_{v\to\infty}\exp(iH_2\tau) = \exp(i(\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}\tau)$ and $\text{s-lim}_{v\to\infty}\exp(iH_1\tau) = \exp(i\mathbf{p}\hat{\mathbf{v}}\tau)$ for fixed τ is used. Hence we already know that

$$\Omega_\pm(H_2,H_1) - \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}},\mathbf{p}\hat{\mathbf{v}}) =: R_\pm = \mathcal{O}(v^{-1})$$

and $\text{s-lim}_{v\to\infty} vR_\pm$ is explicitly known. Now we complete the proof of the reconstruction formula for V^s and calculate the following limit:

$$
\begin{aligned}
& \lim_{v\to\infty} iv\left(\left(S - e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\right)\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right) \\
= \; & \lim_{v\to\infty}\left\{iv\left(S\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right) - iv\left(e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right)\right\} \\
= \; & \lim_{v\to\infty}\{iv(R_-\Phi_0, \Omega_+((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}})\Psi_0) \\
& + iv(\Omega_-((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}})\Phi_0, R_+\Psi_0) \\
& + iv(R_-\Phi_0, R_+\Psi_0) \qquad (2.26)\\
& + iv\,\underbrace{(\Omega_-((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}})\Phi_0, \Omega_+((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}})\Psi_0)}_{=\left(e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\Phi_0, \Psi_0\right)} \\
& - iv\left(e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\Phi_0, \Psi_0\right)\Big\} \\
= \; & \lim_{v\to\infty}\left\{iv\left(R_-\Phi_0, e^{-i\int_0^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\Psi_0\right) + iv\left(e^{i\int_{-\infty}^{0}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\Phi_0, R_+\Psi_0\right)\right\}.
\end{aligned}
$$

We subtract the terms of $R_\pm$ which only depend on $\mathbf{B}$, multiply in the scalar product with $\exp(-i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi)$ and obtain

$$
\begin{aligned}
& \lim_{v\to\infty} iv\left(e^{-i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}\left(S - e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}\right)\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right) \\
& - \int_0^{\infty}\left(e^{-i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}K_+(\mathbf{p}, \mathbf{x}+\tau\hat{\mathbf{v}})e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}\Phi_0, \Psi_0\right)\,d\tau \\
& - \int_{-\infty}^{0}(K_-(\mathbf{p}, \mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\,d\tau \qquad (2.27)\\
= & \int_{-\infty}^{\infty}(V^s(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\,d\tau, \qquad (2.28)
\end{aligned}
$$

which is the desired reconstruction formula in Theorem 2. The error term can be calculated from (2.26) and is bounded by

$$
v^{-1}\|vR_-\Phi_0\|\,\|vR_+\Psi_0\|. \qquad (2.29)
$$

Therefore the error bounds are of order $\mathcal{O}(v^{-1})$.
S is known and $\mathbf{B}$ is known. Then the terms only depending on $\mathbf{B}$ can be computed and (2.28) is known for all Φ_0, Ψ_0 in a dense set. The injectivity results from the fact that the reconstruction of the potential V^s can be reduced to the inversion of the Radon transform of a bounded, continuous, square integrable function on $\mathbb{R}^2$ (see [EW 95, Proof of Theorem 1.1]) which uniquely determines the function (see [H 84, Chapter I, Theorem 2.17]).
Hence Theorem 2 is proved. ■

3 Proof in the long–range case $V = V^s + V^l$

Now $H = (\mathbf{p} - \mathbf{A}(\mathbf{x}))/(2m) + V(\mathbf{x})$ with $V = V^s + V^l$. We use the abbreviation

$$I^l(\mathbf{x},\mathbf{p},t) := V^l(\mathbf{x}) - V^l(t\mathbf{p}/m) \tag{3.1}$$

for the long–range interaction and compute $\Omega_-^D\Phi_{\mathbf{v}}$:

$$\begin{aligned}
&\Omega_-^D\Phi_{\mathbf{v}} \\
= \;& \Phi_{\mathbf{v}} - \int_{-\infty}^0 dt\,\frac{d}{dt}\left(e^{iHt}U^D(t)\right)\Phi_{\mathbf{v}} \\
= \;& e^{im\mathbf{v}\mathbf{x}}\Phi_0 - \int_{-\infty}^0 dt\, e^{im\mathbf{v}\mathbf{x}} e^{i(H+(\mathbf{p}-\mathbf{A}(\mathbf{x}))\mathbf{v})t} i\left\{I(\mathbf{x},\mathbf{p}+m\mathbf{v}) + I^l(\mathbf{x},\mathbf{p}+m\mathbf{v},t)\right\} \\
& e^{-i\left(H_0t+\mathbf{p}\mathbf{v}t+\int_0^t V^l(s(\mathbf{p}+m\mathbf{v})/m)ds\right)}\Phi_0.
\end{aligned} \tag{3.2}$$

With the abbreviations

$$H_2 := H/v + (\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}} \quad\text{and}\quad H_1^D(\tau) := H_0/v + \mathbf{p}\hat{\mathbf{v}} + v^{-1}V^l\left(\frac{p+m\mathbf{v}}{m}\frac{\tau}{v}\right), \tag{3.3}$$

where $H_1^D(\tau)$ generates

$$\bar{U}^D(\tau/v,0) := \exp\left\{-i\left(H_0\tau/v + \mathbf{p}\hat{\mathbf{v}}\tau + v^{-1}\int_0^\tau V^l\left(\frac{\mathbf{p}+m\mathbf{v}}{m}\frac{s}{v}\right)ds\right)\right\}, \tag{3.4}$$

we obtain the following relation:

$$\Omega_-^D\Phi_{\mathbf{v}} = e^{im\mathbf{v}\mathbf{x}}\Omega_-(H_2, H_1^D)\Phi_0.$$

If there is an integrable bound of the integrand in (3.2) uniformly in $\mathbf{v}$ we can interchange the high velocity limit and the integral. Since $\mathbf{A}$, div $\mathbf{A}$, $|\mathbf{A}|^2$ and V^s are of short range the existence of such an integrable bound for I can be shown as in the case without magnetic field (see [EW 95, Lemma 3.2]). The integrable bound for I^l is given in [EW 95, Lemma 3.3]. Similar to the short–range case we conclude

$$\lim_{v\to\infty} e^{-im\mathbf{v}\mathbf{x}}\Omega_-^D\Phi_{\mathbf{v}} = \Omega_-((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}})\Phi_0, \tag{3.5}$$

where s-lim$_{v\to\infty}\exp(iH_2\tau) = \exp(i(\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}\tau)$ and s-lim$_{v\to\infty}\exp(iH_1^D(\tau)) = \exp(i\mathbf{p}\hat{\mathbf{v}}\tau)$ for fixed τ is used. It is obvious that the same computation can be done for Ω_+^D (by replacing $-\infty$ with $+\infty$).
Then the scattering operator S^D yields the same high velocity limit as in the short–range case, that is $S^D \overset{v\to\infty}{\approx} \exp(i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi)$, and we obtain formula (1.10):

$$\lim_{v\to\infty}(S^D\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) = \left(e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}\Phi_0, \Psi_0\right).$$

So far we have only used that div $\mathbf{A}$ and the components of $\mathbf{A}$ are in $\mathcal{V}_{SR}$.
For $\mathbf{A} \in (L^2(\mathbb{R}^\nu))^\nu \cap (C^0(\mathbb{R}^\nu))^\nu$ formula (1.10) uniquely determines the magnetic field $\mathbf{B}$, which has been noted above. Hence Theorem 3 is proved. ∎

Now $\mathbf{B}$ is given and after choosing a short–range $\mathbf{A}$ with $\mathbf{rot}\,\mathbf{A} = \mathbf{B}$ especially $\Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}})$ is known. We are again interested in the difference

$$\Omega_\pm(H_2, H_1^D) - \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}}) \tag{3.6}$$

and show

Lemma 7 *For H_2 and H_1^D as defined in (3.3)*

$$\left\| \left\{ \Omega_\pm(H_2, H_1^D) - \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}}) \right\} \Phi_0 \right\| = \mathcal{O}(v^{-1}). \tag{3.7}$$

Proof: We compute the difference and obtain similar to (2.16)

$$\begin{aligned}
&\Omega_\pm(H_2, H_1^D) - \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}}) \\
= & - \lim_{t\to\pm\infty} e^{iH_2 t} \int_0^t d\tau \, \frac{d}{d\tau} \left\{ e^{-iH_2\tau} e^{-i\int_0^\tau \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi} \bar{U}^D((t-\tau)/v, t/v) \right\} \bar{U}^D(t/v, 0) \\
= & \lim_{t\to\pm\infty} \int_0^t d\tau \, e^{iH_2\tau} i \left\{ H_2 e^{-i\int_0^{t-\tau} \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi} - e^{-i\int_0^{t-\tau} \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi} \right. \\
& \left. [H_1^D(t-\tau) - \hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+(t-\tau)\hat{\mathbf{v}})] \right\} \bar{U}^D(\tau/v, 0)
\end{aligned} \tag{3.8}$$

with $\bar{U}^D(\tau/v, 0)$ as defined in (3.4). Then (3.8) is bounded by

$$\begin{aligned}
& v^{-1} \lim_{t\to\pm\infty} \int_0^t d\tau \left\| \left\{ -\frac{1}{2m}(\mathbf{p}-m\mathbf{v}) \left(\int_0^{t-\tau} (\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+t\hat{\mathbf{v}}) \right) \right.\right. \\
& - \frac{1}{2m} \left(\int_0^{t-\tau} (\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+t\hat{\mathbf{v}}) \right) (\mathbf{p}-m\mathbf{v}) \\
& + \frac{1}{2m} \left(\int_0^{t-\tau} (\hat{\mathbf{v}}\times\mathbf{B})(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi + \mathbf{A}(\mathbf{x}+t\hat{\mathbf{v}}) \right)^2 + V^s(\mathbf{x}) \\
& \left.\left. + V^l(\mathbf{x}) - V^l\left(\frac{\mathbf{p}}{m}\frac{\tau}{v}\right) \right\} U^D(\tau/v, 0) f(\mathbf{p}-m\mathbf{v})(1+\mathbf{x}^2)^{-2} \right\| \\
\overset{!}{\leq} \; & v^{-1} \int_0^{\pm\infty} d\tau \, h(|\tau|).
\end{aligned} \tag{3.9}$$

If we find an integrable majorant $h \in L^1([0, \pm\infty), d\tau)$ for the norm (3.9) uniformly in $\mathbf{v}$ we finished the proof.
The integrability of the summand with V^s is proven in [EW 95, Lemma 3.2] and of the summand with $V^l(\mathbf{x}) - V^l(\frac{\mathbf{p}}{m}\frac{\tau}{v})$ in [EW 95, Lemma 3.3]. With (2.18)–(2.21) the integrability of the remaining terms can be shown as in [EW 95, Proof of Lemma 3.2] with the help of (2.23) and

$$\left\| F(|\mathbf{x}| \geq |\tau|/8) e^{-i\int_0^{\tau/v} V^l(s\mathbf{p}/m)ds} f(\mathbf{p}-m\mathbf{v})(1+\mathbf{x}^2)^{-2} \right\| \leq C\,(1+|\tau|)^{-(2+\varepsilon)} \tag{3.10}$$

with $\varepsilon > 0$. For the propagation property of the Dollard correction (3.10) see [EW 95, Proposition 3.1]. ∎

Similar to (2.25) the proof of Lemma 7 shows with $K_\pm$ as defined in (2.24)

$$\begin{aligned}
& v\left\{\Omega_\pm(H_2, H_1^D) - \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}})\right\}\Phi_0 \\
\overset{v\to\infty}{\approx}\ & i\int_0^{\pm\infty} d\tau\, e^{-i\int_0^{\pm\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})d\xi}\left\{V^s(\mathbf{x}+\tau\hat{\mathbf{v}}) + V^l(\mathbf{x}+\tau\hat{\mathbf{v}}) - V^l(\tau\hat{\mathbf{v}})\right\}\Phi_0 \\
& + i\int_0^{\pm\infty} d\tau\, e^{-i\int_0^{\pm\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi} K_\pm(\mathbf{p}, \mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0,
\end{aligned}$$

where $\text{s-lim}_{v\to\infty}\exp(iH_2\tau) = \exp(i(\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}\tau)$ and $\text{s-lim}_{v\to\infty}\exp(iH_1^D(\tau)) = \exp(i\mathbf{p}\hat{\mathbf{v}}\tau)$ for fixed τ is used. Hence we already know that

$$\Omega_\pm(H_2, H_1^D) - \Omega_\pm((\mathbf{p}-\mathbf{A}(\mathbf{x}))\hat{\mathbf{v}}, \mathbf{p}\hat{\mathbf{v}}) =: R_\pm^D = \mathcal{O}(v^{-1})$$

and $\text{s-lim}_{v\to\infty} vR_\pm^D$ is explicitly known. As in the short–range case we complete the proof of the reconstruction formula for V^s and obtain

$$\begin{aligned}
& \lim_{v\to\infty} iv\left(e^{-i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}\left(S^D - e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}\right)\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right) \\
& - \int_0^\infty\left(e^{-i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi} K_+(\mathbf{p}, \mathbf{x}+\tau\hat{\mathbf{v}}) e^{i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi}\Phi_0, \Psi_0\right)d\tau \\
& - \int_{-\infty}^0 (K_-(\mathbf{p}, \mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\,d\tau \qquad (3.11)\\
& - \int_{-\infty}^\infty\left(\left\{V^l(\mathbf{x}+\tau\hat{\mathbf{v}}) - V^l(\tau\hat{\mathbf{v}})\right\}\Phi_0, \Psi_0\right)d\tau \\
= & \int_{-\infty}^\infty (V^s(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\,d\tau,
\end{aligned}$$

which is the reconstruction formula (1.11) in Theorem 4. The injectivity already has been shown in the short–range case above. Again the error term is bounded by

$$v^{-1}\|vR_-^D\Phi_0\|\,\|vR_+^D\Psi_0\| \qquad (3.12)$$

and therefore of order $\mathcal{O}(v^{-1})$.

A short computation for the commutator of $\exp(-i\int_{-\infty}^{\infty}\hat{\mathbf{v}}\mathbf{A}(\mathbf{x}+\xi\hat{\mathbf{v}})\,d\xi)S^D$ with the first component of the momentum $\mathbf{p}_1$ gives (1.12). Then the total potential V can be computed from (1.12) (see [EW 95, Proof of Theorem 1.2]). Hence Theorem 4 is proved. ∎

4 Generalizations

For magnetic fields with compact support we can construct suitable gauges for the vector potential. In this case we do not need decay assumptions of the vector potential for proving the reconstruction formulas. In the general case the behaviour of the magnetic field inside an arbitrary, but fixed ball has no influence for the formulas. It suffices that the regularized magnetic field and the regularized current density outside the ball decays as $|\mathbf{y}|^{-(2+\delta)}$, $\delta > 0$. See [A 96] for details.

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