On the validity of the van der Waals theory in Ising systems with long range interactions

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Abstract. We consider an Ising system in $d \geq 2$ dimensions with ferromagnetic spin-spin interactions $-J_\gamma(x,y)\sigma(x)\sigma(y)$, x, $y \in \mathbb{Z}^d$, where $J_\gamma(x,y)$ scales like a Kac potential. We prove that when the temperature is below the mean field critical value, for any γ small enough (i.e. when the range of the interaction is long but finite), there are only two pure homogeneous phases, as stated by the van der Waals theory. The proof follows that in [12] on the translationally invariant states at low temperatures for nearest neighbor interactions, supplemented by a "relative uniqueness criterion for Gibbs fields" which yields uniqueness in a restricted ensemble of measures, in a context where there is a phase transition. This criterion is derived by introducing special couplings as in [2] which reduce the proof of relative uniqueness to the absence of percolation of "bad events".

Keywords: *Kac potentials, Translationally invariant Gibbs states, Couplings.*

1. Introduction.

To prove the validity of the van der Waals theory in an equilibrium Statistical Mechanics setting, Kac, Uhlenbeck and Hemmer, [14], have introduced interactions that depend on a scaling parameter $\gamma > 0$ studying the limit $\gamma \to 0^+$ where the range of the interaction becomes infinite. This program has been carried out by Kac, Uhlenbeck and Hemmer in some particular models and then by Lebowitz and Penrose in a more general class of systems, [16]. These results prove that the phase diagram (of the free energy versus the magnetization at fixed temperature) converges for any temperature to the phase diagram of the van der Waals theory. Notice however that this does not imply that a phase transition in the limit theory occurs also before the limit, when the interaction range, maybe very long, is still finite. Strangely enough such a basic question (whose answer is obviously in the negative for $d = 1$ dimensions) has not been investigated in $d \geq 2$ till recently when in [7]

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it has been proved that for ferromagnetic Ising systems with "pure" Kac potentials (those defined in the next section) and in $d \geq 2$ dimensions, the true inverse critical temperature $\beta_c(\gamma)$ converges, as $\gamma \to 0^+$, to its van der Waals value (that in our setup is equal to 1). This result has then been extended in [4] to non trivial "reference interactions" while in [5] an upper bound on $\beta_c(\gamma)$ is derived for pure Kac potentials while a lower bound in $d = 2$ is proved in [6]. We complete here the proof of the validity of the van der Waals theory for pure Kac potentials by showing that not only the critical temperature but also the whole phase diagram is qualitatively the same at $\gamma > 0$ (and small) as in the van der Waals theory. Namely for any $\beta > 1$ (i.e. below the van der Waals critical temperature) there is $\gamma_\beta > 0$ so that for any $\gamma \leq \gamma_\beta$ there are only two pure homogeneous phases as in the van der Waals theory. The precise statement is that any translationally invariant Gibbs measure is a convex combination of the two Gibbs measures obtained with plus and minus boundary conditions and that their magnetizations converge to the magnetizations of the two pure phases in the van der Waals theory. In the course of the proof we introduce two ensembles of Gibbs measures obtained by taking thermodynamic limits with "mostly" $+$, respectively $-$, spins in the boundary conditions. We then prove uniqueness of Gibbs states in each one of these restricted ensembles by extending the Dobrushin condition for the absence of phase transitions, [9], following a coupling scheme introduced by van den Berg and Maes, [2], see also [1], [3] and [13] for other versions of the coupling and applications.

In the next section we give the main definitions and results, the proofs are reported in Sections 3 and 4.

2. Main definitions and results.

Ising systems with Kac potentials.

We consider an Ising system in $\mathbb{Z}^d$, $d \geq 2$. $\mathcal{X} = \{-1, 1\}^{\mathbb{Z}^d}$ is the space of all the spin configurations σ and for any subset Λ of $\mathbb{Z}^d$, $\sigma_\Lambda \in \mathcal{X}_\Lambda = \{-1, 1\}^\Lambda$ is the restriction of σ to Λ. We denote by

$$\nu_{\gamma,\Lambda}(\sigma_\Lambda|\tau) = Z_{\gamma,\Lambda}(\tau)^{-1} \exp\big[-\beta H_\gamma(\sigma_\Lambda|\tau)\big] \tag{2.1}$$

the Gibbs distribution at the inverse temperature β, in a finite region Λ with boundary conditions $\tau \in [-1, 1]^{\Lambda^c}$, Λ^c the complement of Λ in $\mathbb{Z}^d$, (notice that the outside spins are allowed to take values in the whole interval $[-1, 1]$). $Z_{\gamma,\Lambda}(\tau)$ is the partition function (i.e. the normalization constant which makes $\nu_{\gamma,\Lambda}(\cdot|\tau)$ a probability on $\mathcal{X}_\Lambda$). The hamiltonian is

$$H_\gamma(\sigma_\Lambda|\tau) = -\frac{1}{2} \sum_{\substack{x,y\in\Lambda \\ x\neq y}} J_\gamma(x, y)\sigma(x)\sigma(y) - \sum_{\substack{x\in\Lambda \\ y\in\Lambda^c}} J_\gamma(x, y)\sigma(x)\tau(y) \tag{2.2}$$

with γ the Kac scaling parameter (for simplicity we take $\gamma = 2^{-n}$, $n \in \mathbb{N}$) and

$$J_\gamma(x, y) \doteq \gamma^d J(\gamma|x - y|) \tag{2.3}$$

the Kac potential. We suppose $J(s)$ a non-negative, smooth function supported by $s \in [0,1]$ and normalized so that

$$\int_{\mathbb{R}^d} dr\, J(|r|) = 1 \tag{2.4}$$

The Lebowitz-Penrose theorem.

Let $F_\gamma(\beta, m)$ be the free energy density (in the thermodynamic limit) when γ is positive, with β the inverse temperature and m the magnetization density. In [16] it is proved that

$$\lim_{\gamma \to 0^+} F_\gamma(\beta, m) = \mathrm{CE}\Big\{ -\frac{m^2}{2} - \beta^{-1} i(m) \Big\} \doteq F_0(\beta, m) \tag{2.5}$$

where $\mathrm{CE}\{f\}$ is the convex envelope of f and $i(m)$ is the Bernoulli entropy:

$$i(m) = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2} \tag{2.6}$$

$F_0(\beta, m)$ is a strictly convex function of $m \in [-1,1]$ for $\beta \le 1$, while for $\beta > 1$ it is flat in the interval $[-m_\beta, +m_\beta]$, where $m_\beta > 0$ solves

$$m_\beta = \tanh\{\beta m_\beta\} \tag{2.7}$$

The mean field inverse critical temperature is thus $\beta_c^{\mathrm{mf}} = 1$ and for $\beta > 1$ there are two and only two pure (homogeneous) phases with magnetizations $\pm m_\beta$.

Let $\beta_{\mathrm{c}}(\gamma)$ be the inverse critical temperature (in the sense of the unicity of Gibbs states) for the system at $\gamma > 0$. In [7] and in [5] it is proved that $\lim_{\gamma \to 0^+} \beta_{\mathrm{c}}(\gamma) = \beta_c^{\mathrm{mf}}$. Let $\nu_\gamma^\pm$ be the translationally invariant Gibbs states obtained in the thermodynamic limit from finite volume Gibbs states with boundary condition τ identically equal to 1, respectively -1 (their existence follows from ferromagnetic inequalities). Our main result is:

2.1 Theorem.

For any $\beta > 1$ there is $\gamma_\beta > 0$ so that for any $\gamma \le \gamma_\beta$ any translationally invariant Gibbs measure is a convex combination of ν_γ^+ and ν_γ^-.

Since the translationally invariant Gibbs states are identified to the (homogeneous) phases, [18], Theorem 2.1 proves that like in the van der Waals theory there are only two pure phases when the temperature is below the mean field critical value and γ is small enough. It follows from our estimates, but also from the analysis in Lebowitz and Penrose, [16], that the magnetization of the two pure phases converges to $\pm m_\beta$ as $\gamma \to 0^+$, thus completing the proof of the validity of the van der Waals theory, as discussed in the Introduction.

Contours and the Peierls estimates.

As in the original work of Lebowitz and Penrose our proofs are based on a coarse graining procedure that, following [7], uses the notion of block spins. These are defined via empirical spin averages with the help of two spatial scales.

The partitions $\mathcal{D}^{(p)}$. We denote by $C^{(p)}$, $p \in \mathbb{Z}$, the cube in $\mathbb{Z}^d$ of side $2^p\gamma^{-1}$ and center the origin and then call $\mathcal{D}^{(p)} = \{C_j^{(p)};\, j \in \mathbb{Z}^d\}$ the partition of $\mathbb{Z}^d$ into the translates of the original cube $C^{(p)}$. Namely $C_j^{(p)}$ is the cube with center $(2^p\gamma^{-1}+1)j$ and side $2^p\gamma^{-1}$.

Block spins. Let $k \in \mathbb{N}$ and given $x \in \mathbb{Z}^d$ let $C_j^{(-k)}$ be the cube of the partition $\mathcal{D}^{(-k)}$ that contains x. We then set

$$\sigma^{(k)}(x) \doteq \frac{1}{|C_j^{(-k)}|} \sum_{y \in C_j^{(-k)}} \sigma(y) \tag{2.8}$$

Given $\zeta > 0$ and $\gamma > 0$, we define for any $h \in \mathbb{N}$ the block spin variables $\eta_i^{(h)}$, $i \in \mathbb{Z}^d$, as

$$\eta_i^{(h)} = \begin{cases} \pm 1 & \text{if } \left|\sigma^{(k)}(x) \mp m_\beta\right| < \zeta \text{ for all } x \in C_i^{(h)} \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

By letting ζ small and k large, a spin configuration having $\eta_i^{(h)} = 1$ becomes close on $C_i^{(h)}$ to the function constantly equal to $+m_\beta$. This is the closest possible analogue of a sup norm, given that the spins have only values ± 1.

After the introduction of the block spins, we can follow Pirogov and Sinai (see [17], [5], [19]) and define the notion of (generalized Ising) contours.

Correct blocks. A block $C_i^{(h)}$ is positive correct if $\eta_i^{(h)} = 1$ and $\eta_j^{(h)} = 1$ for all $C_j^{(h)}$ that are $\star$-connected to $C_i^{(h)}$. Similarly a block $C_i^{(h)}$ is negative correct if $\eta_i^{(h)} = \eta_j^{(h)} = -1$. $C_i^{(h)}$ is correct if it is either positive or negative correct; it is incorrect if it is not correct.

Support of a contour and contours. The support of a contour is a maximal, $\star$-connected component of the incorrect blocks. A contour is defined by its support and the specification of the values of the block spins on the support. By an abuse of notation we sometimes call contour the support of the contour.

$\pm$ block spin boundary conditions. A $\mathcal{D}^{(h)}$-measurable region Λ has $+$ $(-)$ block spin boundary conditions if τ is such that $|\tau^{(k)}(x) - m_\beta| < \zeta$ (respectively $|\tau^{(k)}(x) + m_\beta| < \zeta$) for all $x \in \Lambda^c$ such that $\text{dist}(x, \Lambda) \le \gamma^{-1}$. In such a case we write $\nu_{\gamma,\Lambda}(\cdot|\tau) \in G^{\pm}(\Lambda)$.

Contours in finite regions. Let Λ be a bounded, $\mathcal{D}^{(h)}$-measurable region and τ a $+$ $(-)$ block spin boundary condition. We then use the convention that all the blocks in Λ^c are considered positive (negative) correct and defining those inside Λ according to the previous rules. Contours are defined consequently and their supports are contained in Λ. Limited to Section 4, where $\nu_{\gamma,\Lambda}(\cdot|\tau)$ is not necessarily in $G^{\pm}(\Lambda)$, we use on the contrary the convention that all the blocks in Λ^c are incorrect, those inside are defined normally.

The typical configurations of a state in $G^+(\Lambda)$ for small γ is made of a sea of $+$ block spins with rare islands of $-$ block spins, very much alike to the Ising nearest neighbor system at low temperatures: this is a consequence of the following Peierls estimates, proved in [7].

2.2 Theorem. ([7])

There are functions $\zeta^\star(\beta) > 0$, $k^\star(\beta,\zeta) \in \mathbb{N}$, $h^\star(\beta,\zeta,k) \in \mathbb{N}$, $c'(\beta,\zeta,k,h) > 0$ and $c(\beta,\zeta,k,h) > 0$ so that the following holds. Let $\gamma > 0$, $\beta > 1$, $\zeta \le \zeta^\star(\beta)$, $k \ge k^\star(\beta,\zeta)$, $h \ge h^\star(\beta,\zeta,k)$, Λ a bounded, $\mathcal{D}^{(h)}$-measurable region, $\nu_{\gamma,\Lambda}(\cdot|\tau) \in G^{\pm}(\Lambda)$, $n \ge 1$, $C_{i_j}^{(h)}$, $j = 1,...,n$, in Λ, a n-tuple of distinct, $\mathcal{D}^{(h)}$-measurable cubes in Λ. Then, calling $c' = c'(\beta,\zeta,k,h)$, $c = c(\beta,\zeta,k,h)$, $\eta_i = \eta_i^{(h)}$,

$$\nu_{\gamma,\Lambda}\Big(\{C_{i_j} \text{ is incorrect for all } j = 1,..,n\}\Big|\tau\Big) \le \Big[c' e^{-c\gamma^{-d}}\Big]^n \tag{2.10}$$

and, for any block C_i in Λ,

$$\nu_{\gamma,\Lambda}\Big(\{\eta_i = \mp 1\}\Big|\tau\Big) \le c' e^{-c\gamma^{-d}} \tag{2.11}$$

In the sequel we will use the following corollary of Theorem 2.2 whose simple proof is omitted.

2.3 Corollary.

The bounds (2.10) and (2.11) hold with β, ζ and k as in Theorem 2.2, $h = h_\gamma$, where $2^{h_\gamma} = \gamma^{-1}$ (recall that γ is in $\{2^{-n}; n \in \mathbb{N}\}$) and with c and c' positive parameters that depend only on β, ζ and k.

Notation. Unless otherwise stated we will hereafter take $h = h_\gamma$ and write $\mathcal{D}$ for $\mathcal{D}^{(h_\gamma)}$, C_i for ${C_i}^{(h_\gamma)}$ and η_i for ${\eta_i}^{(h_\gamma)}$ referring to the cubes C_i as "the blocks". Sometimes we simply write C for a block in $\mathcal{D}$.

By Theorem 2.2 and Corollary 2.3 two Gibbs measures in $G^+(\Lambda)$ are very similar to each other, they have both a lot of $+$ block spins with very few $-$'s, but they are much more alike than that, as proved in the next theorem which is the main ingredient in the proof of Theorem 2.1.

2.4 Theorem.

There are $c > 0$ and $\gamma' > 0$ so that for any $\gamma \le \gamma'$ the following holds. Let K be a finite set in $\mathbb{Z}^d$, Λ a bounded, $\mathcal{D}$-measurable region that contains K and ρ the distance of K from the boundary of Λ. Then, for any σ_K and any $\nu_{\gamma,\Lambda}(\cdot|\tau) \in G^{\pm}(\Lambda)$,

$$\left|\nu_{\gamma,\Lambda}(\sigma_K|\tau) - \nu_\gamma^{\pm}(\sigma_K)\right| \le |K| e^{-c\gamma^2\rho} \tag{2.12}$$

Outline of the proof of Theorem 2.1.

Using Theorem 2.4 we will adapt to the present case the proof of Gallavotti and Miracle-Solé, [12], on the characterization of the translationally invariant Gibbs states at low temperatures. Their starting point (in the context of ferromagnetic, nearest neighbor Ising systems) was a representation of the spin configurations as islands at whose boundaries the spins are either all $+1$ or all -1. At low temperatures these islands are typically very large, in the sense that, fixed a bounded set K in $\mathbb{Z}^d$, the fraction of its translates that are "well inside an island" goes to 1 in the thermodynamic limit. When considering a translationally invariant Gibbs state we can translate an observable without changing its expectation, so that by exploiting the largeness of the islands and the convergence (in the thermodynamic limit) of the Gibbs states with all $+$ or all $-$ boundary conditions, it is then possible to conclude that the probability of a configuration σ_K converges to the probability of σ_K in the $+$ or $-$ Gibbs states, weighted by the fraction of $+$ or $-$ islands that contain the translates of K. The proof that the islands are as large as required by the above argument uses a Peierls estimate based on the assumption that the temperature is low. The statement about the convergence of the conditional Gibbs measures with $+$ or $-$ boundary conditions follows from ferromagnetic inequalities.

In our case the Peierls estimates are proved in Corollary 2.3 and they yield a good lower bound on the width of the islands. Here γ small plays the role of the low temperature condition in [12]. By Theorem 2.2 we can then replace (in the thermodynamic limit) the expectation "inside the island" by the expectation in the $+$ or $-$ states and conclude the proof of Theorem 2.1 (actually our proof will be a little more devious). While the first part of the argument is mainly a combination of the techniques developed in [7] and in [12], the proof of Theorem 2.4 is more intriguing and maybe original. For this reason we have anticipated it to the next section where we prove the weaker version stated in Theorem 3.1. In Section 4 we characterize the typical spin configurations in terms of $\pm$ islands and using Theorem 3.1 we will prove first Theorem 2.1 and then Theorem 2.4, as a consequence of Theorem 3.1 and Theorem 2.1.

3. Thermodynamic limit with + or − block spin conditions.

In this section we prove a weaker version of Theorem 2.4. Let Λ_n be an increasing sequence of $\mathcal{D}$-measurable cubes and $\nu_{\gamma,\Lambda_n}(\cdot|\tau^\pm)$ the Gibbs measures in Λ_n with boundary conditions $\tau^\pm \equiv \pm m_\beta$. By compactness there are $\mu_\gamma^\pm$ and a sequence Λ_{n_k} so that $\nu_{\gamma,\Lambda_{n_k}}(\cdot|\tau^\pm)$ converges weakly to $\mu_\gamma^\pm$. $\mu_\gamma^\pm$ are evidently Gibbs states on $\mathcal{X}$ and $\mu_\gamma^+ \geq \mu_\gamma^-$, stochastically.

3.1 Theorem.

In the same context as in Theorem 2.4,

$$\left|\nu_{\gamma,\Lambda}(\sigma_K|\tau) - \mu_\gamma^\pm(\sigma_K)\right| \leq |K| e^{-c\gamma^2\rho} \tag{3.1}$$

Remarks. The factor $\gamma^2\rho$ will arise as (proportional to) the minimal number of blocks C_i in a $\star$-connected path from K to Λ^c. We however expect in (3.1) a dependence on $\gamma\rho$ rather than $\gamma^2\rho$, i.e. lengths measured in units of the interaction range. Recall that with γ fixed, $\rho \to +\infty$ as $\Lambda \nearrow \mathbb{Z}^d$ so that the right hand side of (3.1) vanishes and $G^\pm(\Lambda)$ shrinks, in the weak topology, to $\mu_\gamma^\pm$. To have lighter notation we will hereafter restrict to the + case, the − case following by the spin flip symmetry of the model.

The remaining of this section is devoted to the proof of Theorem 3.1. Its ultimate origin lies on the stability of the two phases $\pm m_\beta$ in the limit theory. Let

$$f(m,\beta,h) \doteq -\frac{m^2}{2} - \beta^{-1} i(m) - hm \tag{3.2}$$

be the mean field free energy density with external magnetic field h (that for $h = 0$ becomes equal to the expression in the argument of the convex envelope in (2.5)). For h small enough (and $\beta > 1$) f has two local minima, $m^\pm(\beta,h)$ which are stable in the sense that

$$\frac{\partial m^\pm(\beta,h)}{\partial h} < 1 \tag{3.3}$$

Thus, if for instance we want to increase the magnetization $m_\beta = m^+(\beta,0)$ by $\delta > 0$ we must apply a field h which is strictly larger than δ. Thus small magnetization fluctuations in the + state at $h = 0$ are not likely to self-sustain, an indication in favour of the uniqueness of the + state. The difficulty for the argument to apply to the proof of Theorem 3.1 is the presence of large fluctuations that do necessarily occur somewhere in the infinite space at $\gamma > 0$ and for which the above stability analysis does not apply. A state in $G^+(\Lambda)$ (for Λ very large) looks indeed like the + state ν_γ^+ with a "large sea" of + block spins by Theorem 2.2 and Corollary 2.4. In this sea, like for ν_γ^+, there are rare islands of − block spins, produced by the large fluctuations mentioned above. The problem is then to exclude that their statistics (in ν_γ^+ and $\nu_{\gamma,\Lambda}$) remains different also when $\Lambda \nearrow \mathbb{Z}^d$: locally the states are very similar but on a larger scale they may in principle have different behaviors. We will examine separately these two scales, starting from the former chosen as the scale of the blocks, i.e. proportional to γ^{-2}. It is so small that the probability of $\eta_i < 1$ is negligible, so that the above mean field stability analysis applies, but at the same time it is much larger than the range γ^{-1} of the interaction to have a weak dependence on of the boundaries when we look well inside the block.

The Dobrushin coupling.

Let $x \in \mathbb{Z}^d$, τ a configuration on $\mathbb{Z}^d \setminus \{x\}$, $\nu_{\gamma,x}(\cdot|\tau)$ the Gibbs conditional distribution given τ; τ_+ the configuration on $\mathbb{Z}^d$ identical to τ on $\mathbb{Z}^d \setminus \{x\}$ completed by putting $\tau_+(x) = 1$. We denote by A_x the set of configurations τ such that $|\tau^{(k)}(y) - m_\beta| < \zeta$ for all y such that $\mathrm{dist}(x,y) \le \gamma^{-1}$, see (2.8) for notation.

3.2 Lemma.

For any $\zeta > 0$ small enough and k large enough there are $\gamma' > 0$, $a < 1$ and $r_y \ge 0$, $y \neq 0$, with

$$\sum_{y \neq 0} r_y \le a < 1 \tag{3.4}$$

such that for any $x \in \mathbb{Z}^d$, τ and τ' in A_x and $\gamma \le \gamma'$

$$\Big| E^{\nu_{\gamma,x}(\cdot|\tau)}\big[\sigma(x)\big] - E^{\nu_{\gamma,x}(\cdot|\tau')}\big[\sigma(x)\big] \Big| \le \sum_{y \neq x} r_{y-x} |\tau(y) - \tau'(y)| \tag{3.5}$$

Proof.

By an explicit computation

$$E^{\nu_{\gamma,x}(\cdot|\tau)}\big[\sigma(x)\big] = \tanh\left\{ \beta \sum_{y \neq x} J_\gamma(x,y)\tau(y) \right\} \tag{3.6}$$

The l.h.s. of (3.5) is then bounded by

$$\frac{\beta}{\cosh^2\big(\beta m(\tau,\tau')\big)} \Big| \sum_y J_\gamma(x,y)\big(\tau(y) - \tau'(y)\big) \Big| \tag{3.7}$$

where $m(\tau,\tau')$ is a number in the interval with endpoints

$$\sum_{y \neq x} J_\gamma(x,y)\tau(y) \quad \text{and} \quad \sum_{y \neq x} J_\gamma(x,y)\tau'(y)$$

Recalling that τ and τ' are in A_x the first factor converges to

$$\frac{\beta}{\cosh^2(\beta m_\beta)} = \frac{\partial m^+(\beta,h)}{\partial h}\Big|_{h=0} < 1$$

when $\zeta \to 0$, $k \to \infty$ and $\gamma \to 0^+$. Since

$$\sum_{y\neq x} J_\gamma(x,y) \to 1$$

as $\gamma \to 0^+$, we obtain (3.5) from (3.7) by suitably choosing ζ, k and γ. Lemma 3.2 is proved. $\square$

The inequality (3.5) recalls conditions that are well known in the literature: were it holding for all τ and τ', it would be the Liggett contraction condition for spin flip processes, as well as the Dobrushin uniqueness condition for Gibbs fields. This indicates two possible strategies, one dynamical, the other one, which is followed here, in the context of equilibrium statistical mechanics. The former, which looks indeed harder, has, on the other hand, interest in its own right and is related to a line of research proposed by Durrett and Neuhauser, [11], who investigated properties of the Glauber+Kawasaki processes (which have the extra difficulty that their invariant measures are not Gibbsian) in terms of the limit behavior of their mean field limit, described by a reaction diffusion equation. Like in our case there are two locally stable phases, but the proof in [11] requires one of them to be metastable. This is used to prove that when the two phases are simultaneously present in different regions of space, then the stable one invades the other at finite speed and, as a consequence, there is a unique invariant measure for the spin process. Such a condition fails in our case where the two phases are completely symmetric, due to the spin flip symmetry of the model, and from this it follows that the interface dynamics is ruled by a motion by curvature (as proved in the limit $\gamma \to 0^+$ on a suitable space-time scaling, [8], [15] and references therein). This should be sufficient to prove uniqueness in an ensemble of measures characterized by the presence of a majority phase, as this will propagate at the expenses of the minority one (according to the motion by curvature) but the competition is much more balanced and the process very slow (diffusive): an argument like the one in [11] does not seem to apply, even in the simpler context of Gibbsian states.

The Dobrushin approach is based on the construction of good couplings between Gibbs measures with different boundary conditions.

Some basic notation. The space where the coupling is defined is $\Omega = \mathcal{X} \times \mathcal{X}$. We call $\omega = (\sigma, \sigma')$ its elements and η_i and η'_i the corresponding block spins. We write Ω_Λ and ω_Λ when we restrict to a region $\Lambda \subset \mathbb{Z}^d$. σ and σ' are respectively the first and the second marginal of ω and given two probabilities ν and ν' on $\mathcal{X}$ we say that a probability Q on Ω is a coupling, or a joint representation, of ν and ν' if its marginal laws are ν and ν'. Couplings on Ω_Λ are defined analogously.

Couplings of single spin distributions. Let $\Lambda = \{x\}$, $\nu = \nu_{\gamma,x}(\cdot|\tau)$, $\nu' = \nu_{\gamma,x}(\cdot|\tau')$. Since $\sigma(x)$ takes only the values ± 1, $\nu_{\gamma,x}(\cdot|\tau)$ and $\nu_{\gamma,x}(\cdot|\tau')$ are stochastically ordered and therefore there is a joint representation $q_{\gamma,x}(\cdot|\tau,\tau')$ on $\Omega_{\{x\}}$ which is supported by $\{\sigma(x) \ge \sigma'(x)\}$ if $\nu_{\gamma,x}(\cdot|\tau) \ge \nu_{\gamma,x}(\cdot|\tau')$ and by $\{\sigma(x) \le \sigma'(x)\}$ otherwise. Thus

$$E^{q_{\gamma,x}(\cdot|\tau,\tau')}\big[|\sigma(x)-\sigma'(x)|\big] = \Big|E^{\nu_{\gamma,x}(\cdot|\tau)}\big[\sigma(x)\big] - E^{\nu_{\gamma,x}(\cdot|\tau')}\big[\sigma(x)\big]\Big| \tag{3.8}$$

By (3.5), if both τ and τ' are in A_x, we have

$$E^{q_{\gamma,x}(\cdot|\tau,\tau')}\big[|\sigma(x)-\sigma'(x)|\big] \le \sum_{y\neq x} r_{y-x}|\tau(y)-\tau'(y)| \tag{3.9}$$

Extension to Gibbs states in finite regions. Following Dobrushin, [9], we next want to construct couplings on Ω_Λ (with Λ a bounded, $\mathcal{D}$-measurable region) where the first moments satisfy again an inequality like in (3.9). Once done it, by iteration we will get a good bound on the moments $|\sigma(x)-\sigma'(x)|$, if x is far from the boundaries of Λ. The approach requires uniformity on τ and τ' in (3.5) with the coefficients r_y that satisfy (3.4). This fails in our case if we allow either τ or τ' to be not in A_x, but we can use the Peierls estimates (2.11) to prove that this occurs with small probability. Let Δ be either a block or its union with some of its $\star$-neighbors. We set

$$e_\gamma \doteq 2\max_{\{\Delta\}} \sup_{\nu_{\gamma,\Delta}(\cdot|\tau)\in G^+(\Delta)} \nu_{\gamma,\Delta}\Big(\{\text{there is } C_i\subset\Delta \text{ such that } \eta_i<1\}\ \Big|\ \tau\Big) \le c'' e^{-c\gamma^{-d}} \tag{3.10}$$

as it follows from (2.11) with c'' equal to c' times the maximal number of blocks in Δ.

3.3 Lemma.

In the same context of Lemma 3.2 there exists a joint representation $Q_{\gamma,\Delta}(\cdot|\tau,\tau')$ of any two measures $\nu_{\gamma,\Delta}(\cdot|\tau)$ and $\nu_{\gamma,\Delta}(\cdot|\tau')$ both in $G^+(\Delta)$, so that, for any $x\in\Delta$,

$$E^{Q_{\gamma,\Delta}(\cdot|\tau,\tau')}\big[|\sigma(x)-\sigma'(x)|\big] \le \sum_{y\neq x} r_{y-x}\, E^{Q_{\gamma,\Delta}(\cdot|\tau,\tau')}\big[|\sigma(y)-\sigma'(y)|\big] + e_\gamma \tag{3.11}$$

where $\sigma(y)=\tau(y)$ and $\sigma'(y)=\tau'(y)$ for $y\notin\Delta$.

Proof.

Setting $\sigma=\tau$, $\sigma'=\tau'$ on Δ^c and given $x\in\Delta$ we denote by $\omega_{x^c}=(\sigma_{x^c},\sigma'_{x^c})$ a pair of configurations on $\mathbb{Z}^d\setminus\{x\}$ and calling $\omega_1(x)=(\sigma_1(x),\sigma_1'(x))$ and $\omega_2(x)=(\sigma_2(x),\sigma_2'(x))$ we suppose that $\omega_1(x)\neq\omega_2(x)$. We then define a jump process on Ω_Δ where the only transitions are of the type

$$\Big(\omega_1(x),\omega_{x^c}\Big) \longrightarrow \Big(\omega_2(x),\omega_{x^c}\Big)$$

and have intensity $q_{\gamma,x}(\omega_2(x)|\omega_{x^c})$, see (3.8). We denote by L the generator of this process.

We start the process from the product of $\nu_{\gamma,\Delta}(\cdot|\tau)$ and $\nu_{\gamma,\Delta}(\cdot|\tau')$. Then at any time $t \geq 0$ the law of the process is a joint representation of these two measures. By compactness and continuity, the Cesaro time averaged measure converges as $t \to +\infty$ to a stationary measure $Q_{\gamma,\Delta}(\cdot|\tau,\tau')$, which is also a joint representation. Then for any $x \in \Delta$

$$E^{Q_{\gamma,\Delta}(\cdot|\tau,\tau')}\big[L|\sigma(x)-\sigma'(x)|\big] = 0$$

After expressing the action of the generator we use (3.8) and Lemma 3.2 in A_x, while the probability of the complement of A_x is bounded in terms of the marginals of $Q_{\gamma,\Delta}$ which are the Gibbs measures $\nu_{\gamma,\Delta}(\cdot|\tau)$ and $\nu_{\gamma,\Delta}(\cdot|\tau')$, thus obtaining (3.11). Lemma 3.3 is proved. □

The Dobrushin coupling. We will call Dobrushin coupling the probability $Q_{\gamma,\Delta}(\cdot|\tau,\tau')$ of Lemma 3.3 when both $\nu_{\gamma,\Delta}(\cdot|\tau)$ and $\nu_{\gamma,\Delta}(\cdot|\tau')$ are in $G^+(\Delta)$, otherwise we define $Q_{\gamma,\Delta}(\cdot|\tau,\tau')$ as their product.

Remarks. There obviously exist several couplings of $\nu_{\gamma,\Delta}(\cdot|\tau)$ and $\nu_{\gamma,\Delta}(\cdot|\tau')$, it is less evident, but nonetheless true, that when $\tau \geq \tau'$ there are at least two couplings supported by the set $\{\sigma_\Delta \geq \sigma'_\Delta\}$. One is the Dobrushin coupling derived as above by starting from a configuration in $\{\sigma_\Delta \geq \sigma'_\Delta\}$. It can be seen that this coupling gives positive probability to any ω_Δ in such a set. A different coupling will be considered later for larger regions, its version in the present context would be supported by configurations ω_Δ such that if they have "a circuit of agreement" then they agree also inside it. This coupling could also be defined to preserve the order and from the above support property it would be different from the Dobrushin coupling.

Uniqueness is not evident even in the subclass of couplings defined as the invariant measures of the Markov process of Lemma 3.3, (observe in fact that its rates are not all strictly positive and the chain might not be ergodic). Unfortunately the process is not reversible which means that the corresponding DLR equations (i.e. find a measure which has $q_{\gamma,x}(\cdot|\tau,\tau')$ as its conditional probabilities) do not have solutions (such a property would make the proof of Theorem 3.1 much simpler). Due to the lack of reversibility even a simple change of the rates by factors that only depends on x would then change the invariant measure, for instance a different coupling is (in general) derived starting from the Markov chain at discrete times with updates $q_{\gamma,x}(\cdot|\tau,\tau')$, with the sites x ordered in some arbitrary way. This was the coupling originally proposed by Dobrushin in [9].

We conclude the subsection with a corollary of Lemma 3.3, Theorem 3.4 below, whose simple proof is omitted.

The set $R_\gamma(\Delta,\omega_B)$. Let Δ be as above and B^c a bounded set containing $\mathcal{D}$. $R_\gamma(\Delta,\omega_B)$ is then the set of all $\omega_\Delta = (\sigma_\Delta,\sigma'_\Delta)$ such that whenever there is $x \in \Delta$ with $\sigma_B(y) = \sigma'_B(y)$ for all y such that $\mathrm{dist}(x,y) \leq \gamma^{-1}|\log\gamma|^2$ then $\sigma_\Delta(x) = \sigma'_\Delta(x)$.

3.4 Theorem.

There is $c > 0$ and $\gamma' > 0$ so that for any $\gamma \le \gamma'$ if both $\nu_{\gamma,\Delta}(\cdot|\tau)$ and $\nu_{\gamma,\Delta}(\cdot|\tau')$ are in $G^+(\Delta)$ then

$$Q_{\gamma,\Delta}\Big(R_\gamma(\Delta,(\tau,\tau'))\Big|\tau,\tau'\Big) \ge 1 - e^{-c|\log\gamma|^2} \tag{3.12}$$

Possible strategies for an extension to larger regions.

To extend our analysis to arbitrarily large regions we will use a coupling which is a variant of that introduced by van den Berg and Maes, [2], to prove uniqueness of Gibbs states in models where the Dobrushin uniqueness condition is not necessarily satisfied, see also [1], [3] and [13]. The method is quite flexible and it can be adapted to cases where uniqueness fails and the interaction has long range, as here. But let us proceed by a step at a time discussing first the possible options open after the analysis of the previous subsection, as we will see the BM coupling comes out as the most natural, almost mandatory, choice.

For notational simplicity let us consider the whole space and two Gibbs measures ν and ν', which are limit points of $\{G^+(\Lambda)\}$ as $\Lambda \nearrow \mathbb{Z}^d$, the aim is to prove that $\nu = \nu'$. Calling $\bar C_0 = C_0 \cup \partial_0$, where ∂_0 is the corridor outside C_0 of width γ^{-1}, we define the partition $\mathcal{T} = \{\bar C_j;\ j \in \mathbb{Z}^d\}$, obtained by translates of $\bar C_0$, writing $\bar C_j = C_j \cup \partial_j$ with obvious meaning of the symbols. To extend the Dobrushin coupling to a coupling between ν and ν' we can for instance distribute the spins in the corridors ∂_j by using the product measure. Once the spins are specified on the corridors, those in the blocks C_j become conditionally independent and we can thus apply simultaneously the Dobrushin coupling. As a result we have with large probability agreement inside most of the cubes C_j. However no matter how many and how large they are (which is indeed the case if γ is small) the region of agreement is only made of disconnected islands, since the sets C_j are separated by the corridors ∂_j. To build bridges between the islands we apply repeatedly the same coupling procedure, but using partitions $\mathcal{T}^{(i)}$ obtained by translations the original one. At each step i we apply the coupling procedure relative to the partition $\mathcal{T}^{(i)}$ starting from the coupling realized at the step $i-1$. After a suitable number of iterations (2^d if the translations are well chosen) we get to a final coupling $Q^\star$ such that:

A property of $Q^\star$. With probability 1 there exist infinitely many, $\star$-connected, $\mathcal{D}$-measurable circuits where the spins σ and σ' agree and such that, given any bounded set K, there is a circuit of agreement that strictly contains K in its interior.

The proof of this statement is omitted because we have not been able to proceed further proving that $\nu = \nu'$, even though it seems we have got not too far from it. For instance an apparently "slightly" stronger property would do the job:

An unproved stronger property of $Q^\star$. With probability 1, the circuits of agreement can be chosen in a family $\{\mathcal{C}_i\}$ of increasing circuits.

Then if ν and ν' have trivial σ-algebra at infinity, by the martingale convergence theorem there is a.s. weak convergence of $\nu_{\gamma,\mathcal{C}_i^0}(\cdot|\sigma_{\mathcal{C}_i})$ to ν and of $\nu_{\gamma,\mathcal{C}_i^0}(\cdot|\sigma'_{\mathcal{C}_i})$ to ν', where $\mathcal{C}_i \in \{\mathcal{C}_i\}$ (the increasing family of circuits) and $\mathcal{C}_i^0$ is the interior of $\mathcal{C}_i$. By the validity of "the stronger property" there is a.s. a subsequence $\mathcal{C}_{i_j}$ in $\{\mathcal{C}_i\}$ where $\sigma_{\mathcal{C}_{i_j}} = \sigma'_{\mathcal{C}_{i_j}}$, hence $\nu = \nu'$. Unfortunately our analysis only yields the "property" and not the "stronger property", as it is based on the non existence of a percolating path of disagreement. Presumably $Q^\star$ does not satisfy the "stronger property" which is indeed truly stronger than the property we have proved.

Strong Markov property. An alternative idea, which is the one we actually pursue, relies on the strong Markov property of Gibbs measures. If we look at ν and ν' proceeding from the outside and we find a (thick) circuit $\mathcal{C}$ where the configurations agree, then by the strong Markov property the configurations will agree in the whole $\mathcal{C}^0$. In the previous setup we could choose a circuit of agreement $\mathcal{C}$ looking from the outside, but (in a sense) the coupling $Q^\star$ is constructed looking also at the inside and the strong Markov property does not hold. Another way to say it is that $Q^\star(\omega_{\mathcal{C}^0}|\omega_{\mathcal{C}})$, $\omega_{\mathcal{C}} = (\sigma_{\mathcal{C}}, \sigma'_{\mathcal{C}})$, is not in general a joint representation of the the conditional Gibbs measures given $\sigma_{\mathcal{C}}$ and $\sigma'_{\mathcal{C}}$. If it were, we would then be in business, because $\sigma_{\mathcal{C}} = \sigma'_{\mathcal{C}}$ as $\mathcal{C}$ is a circuit of agreement. The marginals of $Q^\star(\omega_{\mathcal{C}^0}|\omega_{\mathcal{C}})$ are not the conditional Gibbs measures because of the same pathology (lack of reversibility) observed in the Remarks after Lemma 3.3. We can however take advantage of this same circumstance in the following way. Given a bounded region Λ and calling $\omega_{\Lambda^c} = (\sigma_{\Lambda^c}, \sigma'_{\Lambda^c})$ we could from one side define a Dobrushin coupling $Q_{\gamma,\Lambda}(\omega_\Lambda|\omega_{\Lambda^c})$, but we may also proceed differently. Let $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, $Q_{\gamma,\Lambda}(\omega_{\Lambda_1}|\omega_{\Lambda^c})$ the marginal of $Q_{\gamma,\Lambda}(\omega_\Lambda|\omega_{\Lambda^c})$ on Ω_{Λ_1} and $Q_{\gamma,\Lambda_2}(\omega_{\Lambda_2}|\omega_{\Lambda_1}, \omega_{\Lambda^c})$ the Dobrushin coupling on Ω_{Λ_2}, given ω_{Λ_1} and ω_{Λ^c}. Then

$$Q_{\gamma,\Lambda}(\omega_{\Lambda_1}|\omega_{\Lambda^c})Q_{\gamma,\Lambda_2}(\omega_{\Lambda_2}|\omega_{\Lambda_1}, \omega_{\Lambda^c})$$

is not (in general) equal to the original coupling $Q_{\gamma,\Lambda}(\omega_{\Lambda_1}, \omega_{\Lambda_2}|\omega_{\Lambda^c})$ (it would be if the latter had the Markov property), yet it is still a coupling which may even be better than the original one! We can in fact choose Λ_1 and Λ_2 as functions of ω_{Λ^c} (this is all right because it only requires to look at the outside) in such a way that Λ_1 screens Λ_2 from the "bad part" of Λ^c, where there is disagreement. Then if there is agreement in that part of Λ_1 that faces Λ_2, Λ_2 only sees agreement at its boundaries and we can then take the coupling on Ω_{Λ_2} which is supported by the diagonal. If instead the coupling in Λ_1 is not succesful we can try again, splitting Λ_2 in two regions, and so on. This is the strategy proposed by van den Berg and Maes that we will follow here using block spins to reduce to nearest neighbor interactions as in [2] and distinguishing between bad and good conditioning (the dangerous – block spins) to obtain a uniqueness result in the restricted ensemble of states obtained as limit points of $\{G^+(\Lambda)\}$.

The BM coupling.

Let Λ be a bounded, $\mathcal{D}$-measurable set. We then take $\nu_{\gamma,\Lambda}(\cdot|\tau) \in G^+(\Lambda)$, $\nu_{\gamma,\Lambda_{n_k}}(\cdot|\tau^+)$, $\Lambda_{n_k} \supset \Lambda$, is an element of the sequence that defines μ_γ^+. Let τ' be a configuration outside Λ, chosen with distribution $\nu_{\gamma,\Lambda_{n_k}}(\cdot|\tau^+)$, $\tau'(x) = \tau^+(x) = m_\beta$ for $x \in \Lambda_{n_k}$. Given τ and τ', we want to construct a coupling on Ω_Λ between the two Gibbs measures $\nu_1(\sigma_\Lambda) = \nu_{\gamma,\Lambda}(\sigma_\Lambda|\tau)$ and $\nu_2(\sigma_\Lambda) = \nu_{\gamma,\Lambda}(\sigma_\Lambda|\tau')$. The coupling will be the limit law (attained after a finite number of steps) of a Markov chain.

The state space. The state space of the Markov chain is $\{\xi : \xi = (B, \omega_B)\}$ where B is a $\mathcal{D}$-measurable set that contains Λ^c, $\omega_B = (\sigma_B, \sigma'_B) \in \Omega_B$ and $\sigma_B(x) = \tau(x)$, $\sigma'_B(x) = \tau'(x)$ for $x \in \Lambda^c$.

Notation. We fix (arbitrarily) a one to one map $n(C)$, from $\{C \in \mathcal{D} : C \subset \Lambda\}$ onto the first N positive integers, where N is the cardinality of $\{C \in \mathcal{D} : C \subset \Lambda\}$. Let $\xi = (B, \omega_B)$ and $C \subset \Lambda$. We say that C is of type a) if there is a block $C_i \subset B$, $\star$-connected to C and such that $\eta_i + \eta'_i < 2$; C is of type b) if it is not of type a) and if there is $x \in B$ such that $\mathrm{dist}(x, C) \le \gamma^{-1}$ and $\sigma(x) \neq \sigma'(x)$. We denote by $F(\xi)$ a set which is the cube C of type a) with minimal $n(C)$ if this class is not empty, otherwise it is the cube of type b) with the minimal $n(C)$ unless also this set is empty, in which case $F = B^c$.

The transition probability of the Markov chain, denoted by $P(\xi, \xi^\star)$, $\xi = (B, \omega_B)$, $\xi^\star = (B^\star, \omega^\star_{B^\star})$ is defined as follows. $P(\xi, \xi) = 1$ if $B = \mathbb{Z}^d$, if $B \neq \mathbb{Z}^d$ $P(\xi, \xi^\star)$ is supported by $\xi^\star = (B^\star, \omega^\star_{B^\star})$ with $B^\star = B \cup F(\xi)$ and $\omega^\star_{B^\star}(x) = \omega_B(x)$ for all $x \in B$. We thus only need to specify the conditional distribution of $\omega^\star_{F(\xi)}$ that will be denoted by $\pi_\xi(\omega^\star_F)$, $F = F(\xi)$. $\pi_\xi(\omega^\star_F)$ is a coupling of the marginals on $\mathcal{X}_F$ of $\nu_{\gamma,B^c}(\cdot|\sigma_B)$ and $\nu_{\gamma,B^c}(\cdot|\sigma'_B)$. If F is of type a), calling $\omega^\star_F = (\sigma^\star_F, (\sigma^\star_F)')$,

$$\pi_\xi(\omega^\star_F) = \nu_{\gamma,B^c}(\sigma^\star_F|\sigma_B)\nu_{\gamma,B^c}((\sigma^\star_F)'|\sigma'_B) \tag{3.13}$$

If F is neither of type a) nor of type b) then $F = B^c$ and

$$\pi_\xi(\omega^\star_F) = \nu_{\gamma,B^c}(\sigma^\star_F|\sigma_B)\mathbf{1}_{\sigma^\star_F = (\sigma^\star_F)'} \tag{3.14}$$

Let finally F be of type b). Denote by Δ the union of F and of the cubes C in B^c $\star$-connected to F. $\partial\Delta$ is the union of all the cubes C not in Δ but $\star$-connected to Δ, $\partial^0\Delta = \partial\Delta \cap B^c$. Let $p_\xi(\cdot)$ be the probability on $\Omega_{\partial^0\Delta}$ defined as

$$p_\xi(\tilde\omega_{\partial^0\Delta}) = \nu_{\gamma,B^c}(\tilde\sigma_{\partial^0\Delta}|\sigma_B)\nu_{\gamma,B^c}(\tilde\sigma'_{\partial^0\Delta}|\sigma'_B) \tag{3.15}$$

that is the marginal on $\Omega_{\partial^0\Delta}$ of the product of the conditional Gibbs measures on $\mathcal{X}_{B^c}$. We then set

$$\pi_\xi(\omega^\star_F) \doteq E^{p_\xi}\left[Q_{\gamma,\Delta}(\omega^\star_F|\tilde\omega_{\partial^0\Delta},\omega_B)\right] \tag{3.16}$$

where $Q_{\gamma,\Delta}(\cdot|\tilde\omega_{\partial^0\Delta},\omega_B)$ is the Dobrushin coupling on Ω_Δ of the Gibbs measures with boundary conditions $(\tilde\sigma_{\partial^0\Delta},\sigma_B)$ and, respectively, $(\tilde\sigma'_{\partial^0\Delta},\sigma'_B)$. In (3.16) we then consider the marginal on Ω_F of this coupling (still dependent on $\tilde\omega_{\partial^0\Delta}$) which is then integrated by the measure $p_\xi(\cdot)$ defined in (3.15).

The van den Berg and Maes (BM) coupling. Denote by $\mathbb{P}$ the law and by $\mathbb{E}$ the expectation of the above Markov chain $\{\xi_n\}$ when it starts from $\xi_0=(\Lambda^c,\omega_{\Lambda^c})$, $\omega_{\Lambda^c}=(\tau,\tau')$. Let $n\ge N$, N the number of sets C in Λ, then

$$\mathbb{P}\left(\{\xi_n=\xi_N;\ B_N=\mathbb{Z}^d\}\right)=1 \tag{3.17}$$

The BM coupling Q is then defined as

$$Q(\omega)=\mathbb{P}\big(\xi_N=(\mathbb{Z}^d,\omega)\big) \tag{3.18}$$

and since $\omega(x)=\big(\tau(x),\tau'(x)\big)$ for all $x\in\Lambda^c$, $Q(\omega)$ may be identified to a probability on Ω_Λ which in the next lemma is proved to be a coupling of ν_1 and ν_2.

3.5 Lemma.

Q is a coupling of ν_1 and ν_2.

Proof.

Let $g(\omega)$, $\omega\in\Omega$, only depend on σ_Λ. We need to show that

$$E^Q[g]=E^{\nu_{\gamma,\Lambda}(\cdot|\tau)}[g] \tag{3.19}$$

the analysis of the other marginal being completely analogous.

Let $\xi=(B,\omega_B)$, $\omega_B=(\sigma_B,\sigma'_B)$ and

$$G(\xi)\doteq E^{\nu_{\gamma,B^c}(\cdot|\sigma_B)}[g] \tag{3.20}$$

We will prove that for all $n\ge 0$

$$E^Q[g]=\mathbb{E}[g(\xi_N)]=\mathbb{E}[G(\xi_n)] \tag{3.21}$$

which proves (3.19): in fact since the Markov chain starts from $\xi_0=\big(\Lambda^c,(\tau,\tau')\big)$, $\mathbb{E}[G(\xi_0)]=G(\xi_0)$ and by (3.20),

$$G(\xi_0)=E^{\nu_{\gamma,\Lambda}(\cdot|\tau)}[g]$$

which together with (3.21) with $n = 0$ proves (3.19).

We thus only need to prove (3.21). (3.21) is an identity for $n = N$ because $\xi_N = (\mathbb{Z}^d, \omega)$ and, by (3.20), $G(\xi_N) = g(\xi_N)$. It thus remains to prove that

$$\mathbb{E}[G(\xi_n)] = \mathbb{E}[G(\xi_{n-1})]$$

We have

$$\mathbb{E}[G(\xi_n)] = \mathbb{E}\left[\sum_{\xi^\star} P(\xi_{n-1}, \xi^\star) G(\xi^\star)\right] \tag{3.22}$$

Let $\xi_{n-1} = (B, \omega_B)$, then $P(\xi_{n-1}, \xi^\star) \neq 0$ only for $\xi^\star = (B^\star, \omega^\star_{B^\star})$, $B^\star = B \cup F$, $F = F(\xi_{n-1})$, and $\omega^\star_{B^\star}(x) = \omega_B(x)$ for all $x \in B$. By (3.20), for such $\xi^\star$,

$$G(\xi^\star) = E^{\nu_{\gamma,(B^\star)^c}(\cdot|\sigma^\star_{B^\star})}[g] \doteq h(\sigma^\star_F; \sigma_B) \tag{3.23}$$

Thus the sum in (3.22) involves only the dependence of h on its first variable $\sigma^\star_F$. We have

$$\sum_{\xi^\star} P(\xi_{n-1}, \xi^\star) G(\xi^\star) = E^{\nu_{\gamma,B^c}(\cdot|\sigma_B)}[h(\sigma^\star_F; \sigma_B)] \tag{3.24}$$

because the transition probability $P(\xi_{n-1}, \cdot)$ has been defined as a joint representation on Ω_F of the marginals on $\mathcal{X}_F$ of the Gibbs conditional probabilities on $\mathcal{X}_{B^c}$ given σ_B and σ'_B and, since σ_B is fixed, h is a function on $\mathcal{X}_F$. By the Gibbs property, the r.h.s. of (3.24) becomes

$$E^{\nu_{\gamma,B^c}(\cdot|\sigma_B)}\left[E^{\nu_{\gamma,B^c\setminus F}(\cdot|\sigma^\star_F,\sigma_B)}[g]\right] = E^{\nu_{\gamma,B^c}(\cdot|\sigma_B)}[g] = G(\xi_{n-1})$$

hence (3.21). Lemma 3.5 is proved. □

A percolation problem.

As in the van der Berg and Maes paper, [2], we will bound the difference between $\nu_{\gamma,\Lambda}(\cdot|\tau)$ and $\nu_{\gamma,\Lambda_{n_k}}(\cdot|\tau^+)$ in terms of a percolation probability.

Notation. We write $Q(\cdot) = Q(\cdot|\tau, \tau')$ to make explicit the dependence on the boundary conditions. Recall that τ is fixed while τ' is the marginal distribution of $\nu_{\gamma,\Lambda_{n_k}}(\cdot|\tau^+)$ on $\mathcal{X}_{\Lambda^c}$. τ' is then with probability 1 identical to τ^+ on $\Lambda^c_{n_k}$. It is convenient to consider $Q(\cdot|\tau, \tau')$ also as a probability on Ω supported by configurations $\omega = (\sigma, \sigma')$ such that $(\sigma, \sigma') = (\tau, \tau')$ on Ω_{Λ^c}.

For any $K \subset \Lambda$ we have

$$\left|\nu_{\gamma,\Lambda}(\sigma_K|\tau) - \nu_{\gamma,\Lambda_{n_k}}(\sigma_K|\tau^+)\right| \leq E^{\nu_{\gamma,\Lambda_{n_k}}(\cdot|\tau^+)}\left[Q(\sigma_K \neq \sigma'_K|\tau, \tau')\right] \doteq Q^\star\left[\sigma_K \neq \sigma'_K\right] \tag{3.25}$$

where $Q^\star$ is the probability on Ω defined by

$$E^{Q^\star}[f] = E^{\nu_{\gamma,\Lambda_{n_k}}(\cdot|\tau^+)}\left[E^{Q(\cdot|\tau,\tau')}[f]\right] \tag{3.26}$$

Further properties of the Markov chain. With $\mathbb{P}$ probability 1 there is only one trajectory $(\xi_0, \xi_1, ..., \xi_N)$ which ends at ξ_N. In fact the chain is just a sequence of specifications added to ξ_0 (which is not random) and that sequentially specify ξ_1, ξ_2...till ξ_N. The collection of all such specifications are encoded in ξ_N which thus allows to reconstruct the whole path of the chain.

Since the BM coupling has the same distribution as ξ_N under $\mathbb{P}$, by identifying $\xi_N = (\mathbb{Z}^d, \omega)$ with ω we can define on Ω the variables B_n as those corresponding to $\xi_n = (B_n, \omega_{B_n})$ and $F_n = B_n \setminus B_{n-1}$, the notation being consistent because ω_{B_n} in $\xi_n = (B_n, \omega_{B_n})$ is also the restriction of ω to B_n.

Unsuccessful cubes. $I_0(\omega)$ is the set of cubes $C \in \mathcal{D}$ (called unsuccessful relative to ω) such that the following holds: there is $n \le N$ so that $F_n = C$ is of type b) relative to $\xi_{n-1} = (B, \omega_B)$ and $\omega_C \notin R_\gamma(C, \omega_B)$, which is defined just before Theorem 3.4.

$\mathcal{D}$ contours. A block C is $\mathcal{D}_1$ ($\mathcal{D}_2$) positive, negative, correct, incorrect if it is so relative to the first (second) marginal of ω. $I_1(\omega)$ ($I_2(\omega)$) denotes the set of the $\mathcal{D}_1$ ($\mathcal{D}_2$) incorrect blocks. A block C is called $\mathcal{D}$ incorrect if it is either $\mathcal{D}_1$ or $\mathcal{D}_2$ incorrect. $\mathcal{D}$ contours are the maximal $\star$-connected components of the set of the $\mathcal{D}$ incorrect blocks.

The interior of a set, the internal boundary. The complement of $\star$-connected bounded set A has an unbounded, maximal, $\star$-connected component. Its complement is called the interior of A and it is denoted by $\mathrm{int}(A)$. The strict interior of A is the part of $\mathrm{int}(A)$ which is not in A. The internal boundary $\delta\Lambda$ of Λ is the union of all the cubes $C \subset \Lambda$ that are $\star$-connected to Λ^c.

3.6 Proposition.

If $\{\sigma_K \neq \sigma'_K\}$ then with Q probability 1 there is a $\star$-connected set which is union of $\mathcal{D}$ contours and unsuccessful cubes and whose interior has non void intersection with both K and $\delta\Lambda$.

Proof.

Without loss of generality we suppose K union of two or more cubes C. Postponing the proof of

Statement. *If $\sigma_K \neq \sigma'_K$ then, with Q probability 1, there are $(\mathcal{L}_1, ..., \mathcal{L}_n)$, $(C_1, ..., C_m)$, $m \le n$, and $(\mathcal{M}_1, ..., \mathcal{M}_\ell)$ where:*

i) $\mathcal{L}_k$ *is a* $\star$*-connected set of cubes* C_i *such that* $\eta_i + \eta_i' < 2$.

ii) $\mathcal{M}_k$ *is a* $\star$*-connected set of unsuccessful cubes.*

iii) The union of all the sets $\mathcal{L}_k$, $\mathcal{M}_k$ *and* C_k *is a* $\star$*-connected set with non empty intersection with* K *and* $\delta\Lambda$.

we observe that for each $\mathcal{L}_k$ there is a minimal contour Γ_k which contains $\mathcal{L}_k$ in its interior. Moreover the sets C_k are contained in the union of the sets Γ_k. Thus the set in the statement of the Proposition is the union of the contours Γ_k and $\mathcal{M}_k$. We are thus left with:

Proof of the Statement. A bounded $\mathcal{D}$-measurable, connected set $\mathcal{C}$ is called a path if $\mathcal{C}^c$ has only two connected components. A path is a path of agreement for ω if $\sigma(x) = \sigma'(x)$ for all $x \in \mathcal{C}$ which have distance $\leq \gamma^{-1}$ from the strict interior of $\mathcal{C}$. With Q probability 1 if ω has a path of agreement $\mathcal{C}$ then there is agreement also in the whole $\mathrm{int}(\mathcal{C})$.

A cube C is a successful cube for ω if, for some n, $C = F(\xi_n)$, C is of type b) and it is not unsuccessful. If $\mathcal{C}$ is a path of successful cubes then it is a path of agreement.

Given ω let n be such that $B_{n+1} = \mathbb{Z}^d$ and F_n not a single cube C, so that F_n is a region of agreement. Then if $\sigma_K \neq \sigma'_K$ there is no path made of successful cubes and cubes in F_n whose interior contains K. Therefore there is a $\mathcal{D}$-measurable, $\star$-connected set which has non empty intersection with both K and $\delta\Lambda$ made of cubes C which are either unsuccessful or $\star$-connected to a non positive cube C.

The Statement follows from the above observations. Proposition 3.6 is proved. $\square$

Notation. Let $\mathcal{A}_K$ be the family of all the collections A of $\star$-connected cubes in Λ_{n_k} such that

$$\mathrm{int}\,(\cup_{C\in A} C) \cap K \neq \emptyset$$

The set in Proposition 3.6 is the union of the cubes in some $A \in \mathcal{A}_K$ with the further property that

$$\mathrm{int}\,(\cup_{C\in A} C) \cap \delta\Lambda \neq \emptyset$$

For this reason the number n of cubes in A must be larger than L_γ with $L_\gamma = c\gamma^2 \mathrm{dist}(K, \Lambda^c)$, $c > 0$ a constant.

Going back to (3.25) and using Proposition 3.6 we then have

$$\begin{aligned} &\left|\nu_{\gamma,\Lambda}(\sigma_K|\tau) - \nu_{\gamma,\Lambda_{n_k}}(\sigma_K|\tau^+)\right| \\ &\qquad \leq \sum_{n\geq L_\gamma} \sum_{A\in\mathcal{A}_K:|A|=n} \sum_{\{A_0,A_1,A_2\}} Q^\star\left(\{\omega : A_i \subset I_i(\omega),\, i=0,1,2\}\right) \end{aligned} \tag{3.27}$$

where the last sum is over all the triples (A_0, A_1, A_2) of disjoint sets whose union is A and such that A_0 and A_1 are in Λ, and A_2 is in Λ_{n_k}.

By Cauchy-Schwartz

$$Q^\star\left(\{\omega : A_i \subset I_i(\omega),\, i=0,1,2\}\right)^4 \le \prod_{i=0}^{2} Q^\star\left(\{\omega : A_i \subset I_i(\omega)\}\right) \tag{3.28}$$

When $i=1$ ($i=2$) the event on the right hand side is measurable w.r.t. σ (σ') and since $Q^\star$ is a coupling the probability becomes the Gibbs probability on Λ conditioned on τ and respectively the Gibbs probability on Λ_{n_k} conditioned on τ^+. By Corollary 2.3 we then have

$$\prod_{i=1}^{2} Q^\star\left(\{\omega : A_i \subset I_i(\omega)\}\right) \le \left[c' e^{-c\gamma^{-d}}\right]^{|A_1|+|A_2|} \tag{3.29}$$

We are then left with the estimate of $Q^\star\left(\{\omega : A_i \subset I_0(\omega)\}\right)$. We need a preliminary lemma:

3.7 Lemma.

There are c and c' positive so that, for any $\xi = (B, \omega_B)$ with $F \equiv F(\xi)$ a cube of type b),

$$\sum_{\xi^\star} P(\xi, \xi^\star) \mathbf{1}_{\omega_F^\star \notin R_\gamma(F,\omega_B)} \le c' e^{-c|\log\gamma|^2} \tag{3.30}$$

Proof.

By definition

$$\sum_{\xi^\star} P(\xi, \xi^\star) \mathbf{1}_{\omega_F^\star \notin R_\gamma(F,\omega_B)} = \sum_{\omega_F^\star} \pi_\xi(\omega_F^\star) \mathbf{1}_{\omega_F^\star \notin R_\gamma(F,\omega_B)} \tag{3.31}$$

Calling Δ and $\partial^0\Delta$ the sets that enter in the definition of $P(\xi,\xi^\star)$, we denote by $\mathcal{G}$ the set of all $\tilde\omega_{\partial^0\Delta}$ such that there is a block C_i in $\partial^0\Delta$ with $\tilde\eta_i + \tilde\eta_i' < 2$. Then, recalling (3.16), the right hand side of (3.31) is bounded by

$$p_\xi\left(\tilde\omega_{\partial^0\Delta} \in \mathcal{G}\right) + \sup_{\tilde\omega_{\partial^0\Delta} \notin \mathcal{G}} Q_{\gamma,\Delta}\left(\{\omega_F^\star \notin R_\gamma(F,\omega_B)\} \big| \tilde\omega_{\partial^0\Delta}, \omega_B\right) \tag{3.32}$$

$\mathcal{G} \subset \mathcal{G}_1 \cup \mathcal{G}_2$, where $\mathcal{G}_1$ is the set with some $\tilde\eta_i < 1$ and $\mathcal{G}_2$ the set with some $\tilde\eta_i' < 1$. Then, recalling (3.15) and the Peierls estimate in Corollary 2.3, the first term in (3.32) is bounded by $c' e^{-c\gamma^{-d}}$ for some positive c' and c. The second term in (3.32) is bounded using Theorem 3.4: in fact, by the definition of b) cubes, the block spins on the boundary of B are all positive, hence the assumptions of Theorem 3.4 are satisfied. Lemma 3.7 is proved. $\square$

3.8 Proposition.

Let c and c' be as in Lemma 3.7, then

$$Q^\star\Big(A_0 \subset I_0(\omega)\Big) \le \Big[c' e^{-c|\log\gamma|^2}\Big]^{|A_0|} \tag{3.33}$$

Proof.

We fix τ' in Λ^c (the estimates will be independent of τ') and consider the Markov chain $\{\xi_n\}$. Let $\xi_n = (B_n, \omega_{B_n})$ and $I_0(\xi_n)$ the set of unsuccessful cubes C that are in B_n. We have

$$Q\Big(A_0 \subset I_0(\omega)\big|\tau,\tau'\Big) = \mathbb{P}\Big(A_0 \subset I_0(\xi_N)\Big)$$

We shorthand $\epsilon = c' e^{-c|\log\gamma|^2}$ and, writing $\xi = (B, \omega_B)$, we define

$$g(\xi) = \epsilon^{|A_0| - |A_0 \cap B|} \mathbf{1}_{A_0 \cap B \subset I_0(\xi)}$$

Since $\xi_N = (\mathbb{Z}^d, \omega)$, we have

$$\mathbb{P}\Big(A_0 \subset I_0(\xi_N)\Big) = \mathbb{E}[g(\xi_N)]$$

We will next prove that for all n

$$\mathbb{E}[g(\xi_n)] \le \mathbb{E}[g(\xi_{n-1})] \tag{3.34}$$

In fact

$$\mathbb{E}[g(\xi_n)] = \mathbb{E}\left[\sum_{\xi^\star} P(\xi_{n-1}, \xi^\star) g(\xi^\star)\right]$$

If $F(\xi_{n-1}) \cap A_0 \ne \emptyset$ then $g(\xi_n) = 0$ unless $F(\xi_{n-1})$ is a b) cube C and $\omega^\star_C \notin R_\gamma(C, \omega_{B_{n-1}})$. Then using Lemma 3.7 the sum over $\xi^\star$ is bounded by $g(\xi_{n-1})$. If on the other hand $F(\xi_{n-1}) \cap A_0 = \emptyset$, then $g(\xi^\star) = g(\xi_{n-1})$ hence (3.34). Proposition 3.8 is proved. □

By (3.28), (3.29) and (3.33) we get, for some constant $\bar c > 0$,

$$Q^\star\left(\{\omega : A_i \subset I_i(\omega),\ i = 0,1,2\}\right) \le e^{-\bar c \sum_{i=0}^2 |A_i| |\log\gamma|^2} \tag{3.35}$$

Since the number of partitions of a set $A \in \mathcal{A}_K$ into (A_1, A_2, A_3) is bounded by $3^{|A|}$, from (3.27) and (3.35) we get, for some $c > 0$,

$$\Big|\nu_{\gamma,\Lambda}(\sigma_K|\tau) - \nu_{\gamma,\Lambda_{n_k}}(\sigma_K|\tau^+)\Big| \le \sum_{n \ge L_\gamma} \left(3 e^{-c|\log\gamma|^2}\right)^n N_K(n) \tag{3.36}$$

where $N_K(n)$ is the cardinality of $\{A \in \mathcal{A}_K : |A| = n\}$. To estimate $N_K(n)$ we proceed as follows. If G is any $\star$-connected, $\mathcal{D}$-measurable region, the number of its translates whose interior has non void intersection with K is bounded by $|K|n(G)^d$, $n(G)^d$ the number of cubes in G. By applying this argument to the sets in $\mathcal{A}_K$, we get $N_K(n) \le n^d \tilde{N}_K(n)$ where $\tilde{N}_K(n)$ is the number of $\star$-connected, $\mathcal{D}$-measurable regions G with $n(G) = n$ and which contain a fixed cube. By using a counting argument in [10] we get $\tilde{N}_K(n) \le a^n$ for some $a > 0$. Collecting all the previous estimates we then get Theorem 3.1.

4. Proof of Theorem 2.1.

In this section we prove Theorem 2.1 following the proof of Gallavotti and Miracle-Solé for nearest neighbor interactions, [12].

We will first prove that Theorem 2.1 follows from Proposition 4.1 below which, together with Theorem 3.1, implies Theorem 2.4. We will denote in the sequel by $\mu(g)$ the integral of g with respect to μ while f will always denote a cylinder function measurable on a bounded set.

4.1 Proposition.

There is $\gamma' > 0$ so that for any $\gamma \le \gamma'$ and any translationally invariant Gibbs measure ν there is $\alpha \in [0,1]$ such that

$$\nu = \alpha \mu_\gamma^+ + (1-\alpha)\mu_\gamma^- \tag{4.1}$$

Proof of Theorems 2.1 and Corollary 2.4.

We just need to prove that $\nu_\gamma^\pm = \mu_\gamma^\pm$:

(1) $\mu_\gamma^+ \ge \mu_\gamma^-$, stochastically, by the definition of $\mu_\gamma^\pm$.

(2) $\nu_\gamma^+ \ge \mu_\gamma^+$ since $\nu_\gamma^+ \ge \nu$ for any Gibbs measure ν; moreover $\nu_\gamma^+ \le \mu_\gamma^+$ by (4.1) taking $\nu = \nu_\gamma^+$ and using (1). Thus $\nu_\gamma^+ = \mu_\gamma^+$. An analogous argument proves that $\nu_\gamma^- = \mu_\gamma^-$. □

In the sequel Λ will always denote a $\mathcal{D}$-measurable cube with center the origin, τ a spin configuration in Λ^c, ν a translationally invariant Gibbs measure, $\nu_{\Lambda^c}(d\tau)$ the corresponding marginal.

We will first rephrase Proposition 4.1 in a more convenient way, Proposition 4.2 below, that will then be proved in the rest of the section. By the DLR equations

$$\nu(f) = \int \nu_{\Lambda^c}(d\tau)\, \nu_{\gamma,\Lambda}(f|\tau) \tag{4.2}$$

Since ν is translationally invariant, for any $a \in \mathbb{Z}^d$

$$\nu(f) = \nu(f_a) \tag{4.3}$$

where f_a is f shifted by a.

Let $\mathcal{A}_\Lambda$ be the set of a for which f_a is σ_Λ-measurable and define

$$\bar{f}_\Lambda \doteq \frac{1}{|\mathcal{A}_\Lambda|} \sum_{a \in \mathcal{A}_\Lambda} f_a \tag{4.4}$$

Then, since $\nu(f) = \nu(\bar{f}_\Lambda)$,

$$\nu(f) = \int \nu_{\Lambda^c}(d\tau)\, \nu_{\gamma,\Lambda}(\bar{f}_\Lambda|\tau) \tag{4.5}$$

4.2 Proposition.

For any $\gamma > 0$ small enough there are $\epsilon_{\gamma,\Lambda,f}$ and $\alpha_{\gamma,\Lambda,\tau}$ such that $\epsilon_{\gamma,\Lambda,f} \to 0$ as $\Lambda \nearrow \mathbb{Z}^d$ for any γ and f fixed; moreover

$$\left|\nu_{\gamma,\Lambda}(\bar{f}_\Lambda|\tau) - \alpha_{\gamma,\Lambda,\tau}\mu_\gamma^+(f) - (1-\alpha_{\gamma,\Lambda,\tau})\mu_\gamma^-(f)\right| \le \epsilon_{\gamma,\Lambda,f} \tag{4.6}$$

Proof that Proposition 4.2 implies Proposition 4.1.

By (4.6), after integration over $\nu_{\Lambda^c}(d\tau)$, (ν a translationally invariant Gibbs measure) we have

$$\left|\nu(f) - \mu_\gamma^-(f) - [\mu_\gamma^+(f) - \mu_\gamma^-(f)] \int \nu_{\Lambda^c}(d\tau)\alpha_{\gamma,\Lambda,\tau}\right| \le \epsilon_{\gamma,\Lambda,f} \tag{4.7}$$

By letting $\Lambda \nearrow \mathbb{Z}^d$ the right hand side vanishes and we conclude that if $\mu_\gamma^+ = \mu_\gamma^-$ then $\nu = \mu_\gamma^-$. But we know from [7] that $\nu_\gamma^\pm$ are two distinct, translationally invariant Gibbs measures, hence $\mu_\gamma^+ \neq \mu_\gamma^-$. Thus there is f so that $\mu_\gamma^+(f) \neq \mu_\gamma^-(f)$ and we then conclude that

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \int \nu_{\Lambda^c}(d\tau)\alpha_{\gamma,\Lambda,\tau}$$

exists. Calling it α we then obtain (4.1) as the limit of (4.7) and conclude the proof of Proposition 4.1 which is therefore reduced to that of Proposition 4.2. □

The proof of Proposition 4.2 exploits that most of the points $x \in \Lambda$ are surrounded by a far away connected circuit of blocks C_i where the block spins η_i have a definite sign (either all $+1$ or all -1). By using Theorem 3.1 we then obtain (4.6) with $\alpha_{\gamma,\Lambda,\tau}$ the fraction of points in a plus circuit. The argument that shows that most of the points are surrounded by the above circuits is taken from Gallavotti and Miracle-Solé after generalizing the notion of contour and using a Peierls estimate proved in [7]. Instead of starting from x and looking for circuits that surround it, we start from the outside and characterize the region where these circuits do not exist. Such a region will be an "external contour", here we rely on the notion of contours introduced by Pirogov and Sinai that we adapt to the present context in the following way:

We use the notion of correct blocks with the convention that those outside Λ are incorrect, see the subsection "Contours and Peierls estimates" in Section 2.

Each spin configuration σ_Λ defines a contour configuration $\underline{\Gamma} = \{\Gamma_1, ..., \Gamma_n\}$. By definition the set

$$\partial\Lambda_{\text{int}} \doteq \{C_i \subseteq \Lambda \,:\, \text{dist}(C_i, \Lambda^c) = 0\} \tag{4.8}$$

is incorrect, thus there is an element Γ^{e} in $\underline{\Gamma}$ that contains $\partial\Lambda_{\text{int}}$. Let $K^{(1)}$ be the union of all the blocks C_i in $\Lambda \setminus \Gamma^{\text{e}}$ that are $\star$-connected to Γ^{e} and $K^{(2)}$ the union of all the blocks C_i in $\Lambda \setminus [\Gamma^{\text{e}} \cup K^{(1)}]$ that are $\star$-connected to $\Gamma^{\text{e}} \cup K^{(1)}$. Let $K_j^{(1)}$, $j \in J^{(1)}$ and $K_j^{(2)}$, $j \in J^{(2)}$, be the $\star$-connected components respectively of $K^{(1)}$ and $K^{(2)}$. We need two circuits $K^{(1)}$ and $K^{(2)}$ for measurability reasons, see Lemma 4.3 below whose proof is omitted being a straightforward consequence of the above definitions.

4.3 Lemma.

Each one of the sets $K_j^{(2)}$, $j \in L^{(2)}$, is a closed connected circuit made of blocks C_i where the block spin η_i has a constant, non zero value. Moreover, calling $\bar{\Gamma} \doteq \Gamma^{\text{e}} \cup K^{(1)} \cup K^{(2)}$, $K^{(2)}$ is $\sigma_{\bar{\Gamma}}$-measurable.

We call M_j, $j \in J^{(2)}$, the "islands" (strictly) inside the circuits $K_j^{(2)}$. We also denote by $|\Gamma^{\text{e}}|$ the number of cubes C_i that are in Γ^{e} and by L the side of Λ. Following Gallavotti and Miracle-Solé we distinguish the cases when $|\Gamma^{\text{e}}| < L^{d-2/3}$ and $|\Gamma^{\text{e}}| \geq L^{d-2/3}$. Using the DLR equations and the measurability stated in Lemma 4.3 we have

$$\begin{aligned} &\left| \nu_{\gamma,\Lambda}(\bar{f}_\Lambda | \tau) - E^{\nu_{\gamma,\Lambda}(\cdot|\tau)} \left[\mathbf{1}_{|\Gamma^{\text{e}}| < L^{d-2/3}} \prod_{j \in J^{(2)}} \nu_{\gamma, M_j}\left(\bar{f}_\Lambda | \sigma_{K_j}\right) \right] \right| \\ &\qquad\qquad \leq \|f\|_\infty \nu_{\gamma,\Lambda}\left(\left\{ |\Gamma^{\text{e}}| \geq L^{d-2/3} \right\} \Big| \tau \right) \end{aligned} \tag{4.9}$$

We denote by M_j^0 the sites of $\mathbb{Z}^d$ that are "well inside" the island M_j, namely those which have distance larger than $L^{1/3}$ from the boundary K_j of M_j. Calling $\mathcal{A}_j$ the set of $a \in \mathbb{Z}^d$ such that f_a is $\sigma_{M_j^0}$-measurable and $\bar{f}_{M_j^0}$ the corresponding average, we have in $|\Gamma^{\text{e}}| < L^{d-2/3}$

$$\left| \bar{f}_\Lambda - \frac{1}{|J^{(2)}|} \sum_{j \in J^{(2)}} \bar{f}_{M_j^0} \right| \leq \frac{1}{|\mathcal{A}_\Lambda|} \|f\|_\infty \left\{ L^{d-2/3} (\gamma^{-2})^d 2 L^{1/3} + \left(|\mathcal{A}_\Lambda| - \sum_{j \in J^{(2)}} |\mathcal{A}_j| \right) \right\} \tag{4.10}$$

which is obtained by separating from the sum in the definition of $\bar{f}_\Lambda$ the contribution due to a in $\mathcal{A}_j$. The number of the remaining a's is bounded for L large enough by the number of sites in Γ^{e} times $2L^{1/3}$ (the factor 2 to compensate for the addition of the two strips $K(i)$, $i = 1, 2$, hence the request that L is large enough). This gives the first factor on the right hand side of (4.10), the second one comes when reconstructing the right normalization in the averages $\bar f_{M_j^0}$. Thus there is $c_{\gamma,f}>0$ so that in $|\Gamma^{\rm e}|<L^{d-2/3}$

$$\Big|\bar f_\Lambda - \frac{1}{|J^{(2)}|}\sum_{j\in J^{(2)}} \bar f_{M_j^0}\Big| \le c_{\gamma,f}\gamma^{-d}L^{-1/3} \tag{4.11}$$

The right hand side vanishes as $L\to\infty$ with $\gamma>0$ and f fixed.

Going back to (4.9) and using (4.11) we get

$$\begin{aligned}\Big|\nu_{\gamma,\Lambda}(\bar f_\Lambda|\tau) - E^{\nu_{\gamma,\Lambda}(\cdot|\tau)}\Big[\mathbf{1}_{|\Gamma^{\rm e}|<L^{d-2/3}}\frac{1}{|J^{(2)}|}\sum_{j\in J^{(2)}}\nu_{\gamma,M_j}\big(\bar f_{M_j^0}|\sigma_{K_j}\big)\Big]\Big| \\ \le c_{\gamma,f}\gamma^{-d}L^{-1/3} + \|f\|_\infty \nu_{\gamma,\Lambda}\Big(\Big\{|\Gamma^{\rm e}|\ge L^{d-2/3}\Big\}\Big|\tau\Big)\end{aligned} \tag{4.12}$$

Calling $|J^{(2)}_\pm|$ the number of $j\in J^{(2)}$ for which $K_j^{(2)}$ is a $+$, respectively $-$, circuit we conclude using Theorem 3.1 that there is $\delta_{\gamma,f,L}$ that vanishes as $L\to+\infty$ and such that

$$\begin{aligned}\Big|\nu_{\gamma,\Lambda}(\bar f_\Lambda|\tau) - E^{\nu_{\gamma,\Lambda}(\cdot|\tau)}\Big[\frac{|J^{(2)}_-|}{|J^{(2)}|}\mu_\gamma^-(f) + \frac{|J^{(2)}_+|}{|J^{(2)}|}\mu_\gamma^+(f)\Big]\Big| \\ \le \delta_{\gamma,f,L} + c_{\gamma,f}\gamma^{-d}L^{-1/3} + 2\|f\|_\infty \nu_{\gamma,\Lambda}\Big(\Big\{|\Gamma^{\rm e}|\ge L^{d-2/3}\Big\}\Big|\tau\Big)\end{aligned} \tag{4.13}$$

To prove Proposition 4.2 we thus need to show that the probability of $\big\{|\Gamma^{\rm e}|\ge L^{d-2/3}\big\}$ vanishes as $L\to+\infty$, which will follow using the Peierls estimates proved in [7] and a counting argument from [10]. We first observe that there is $c>0$ such that for any boundary condition τ, any spin configuration σ_Λ and any $\gamma>0$

$$\big|H_\gamma(\sigma_\Lambda|\tau) - H_\gamma(\sigma_\Lambda|m_\beta)\big| \le cL^{d-1}\gamma^{-d} \tag{4.14}$$

Then, recalling the notation $\tau^+\equiv m_\beta$, we have

$$\nu_{\gamma,\Lambda}\Big(\Big\{|\Gamma^{\rm e}|\ge L^{d-2/3}\Big\}\Big|\tau\Big) \le e^{2cL^{d-1}\gamma^{-d}}\nu_{\gamma,\Lambda}\Big(\Big\{|\Gamma^{\rm e}|\ge L^{d-2/3}\Big\}\Big|\tau^+\Big) \tag{4.15}$$

The number of external contours of length k is bounded by 3^{3kd}, as it follows from a counting argument in [10]. In [7] it is proved that there is $c_0>0$ so that

$$\nu_{\gamma,\Lambda}\big(\Gamma^{\rm e}\big|\tau^+\big) \le e^{-c_0\gamma^{-d}|\Gamma^{\rm e}|} \tag{4.16}$$

then from (4.15)

$$\nu_{\gamma,\Lambda}\Big(\Big\{|\Gamma^{\rm e}|\ge L^{d-2/3}\Big\}\Big|\tau\Big) \le e^{2cL^{d-1}\gamma^{-d}}\sum_{k=L^{d-2/3}}^{\infty}3^{3kd}e^{-c_0\gamma^{-d}k}$$

which for all γ small enough vanishes as $L\to+\infty$. We have therefore proved Proposition 4.2. $\square$

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