Deformation Quantization and Nambu Mechanics

Giuseppe Dito[1]*, Moshé Flato[1,2], Daniel Sternheimer[1,2], Leon Takhtajan[3]†

[1] Research Institute for Mathematical Sciences
Kyoto University
Kitashirakawa, Oiwake-cho
Sakyo-ku, Kyoto 606-01 Japan

[2] Département de Mathématiques
Université de Bourgogne
BP 138, F-21004 Dijon Cedex France

[3] Department of Mathematics
State University of New York at Stony Brook
Stony Brook, NY 11794–3651 USA

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Abstract: Starting from deformation quantization (star-products), the quantization problem of Nambu Mechanics is investigated. After considering some impossibilities and pushing some analogies with field quantization, a solution to the quantization problem is presented in what we call the Zariski quantization of fields (observables, functions, in this case polynomials). This quantization is based on the factorization over $\mathbb{R}$ of polynomials in several real variables. We quantize the algebra of fields generated by the polynomials by defining a deformation of this algebra which is Abelian, associative and distributive. This procedure is then adapted to derivatives (needed for the Nambu brackets), which ensures the validity of the Fundamental Identity of Nambu Mechanics also at the quantum level. Our construction is in fact more general than the particular case considered here: it can be utilized for quite general defining identities and for much more general star-products.

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1 Introduction

1.1 Nambu Mechanics

Nambu proposed his generalization of Hamiltonian Mechanics [16] by having in mind a generalization of the Hamilton equations of motion which allows the formulation of a statistical mechanics on $\mathbb{R}^3$. He stressed that the only feature of Hamiltonian Mechanics that one needs to retain for that purpose, is the validity of Liouville theorem. In that spirit, he considered the following equation of motion:

$$\frac{d\vec{r}}{dt} = \nabla g(\vec{r}) \wedge \nabla h(\vec{r}) \ , \quad \vec{r} = (x, y, z) \in \mathbb{R}^3 \ , \tag{1}$$

where x, y, z are the dynamical variables and g, h are two Hamiltonian functions of $\vec{r}$. Then Liouville theorem follows directly from the identity:

$$\nabla \cdot (\nabla g(\vec{r}) \wedge \nabla h(\vec{r})) = 0 \ ,$$

which tells us that the velocity field in Eq. (1) is divergenceless.

As a physical motivation for Eq. (1), Nambu has shown that the Euler equations for the angular momentum of a rigid body can be put into that form if the dynamical variables are taken to be the components of the angular momentum vector $\vec{L} = (L_x, L_y, L_z)$, and g and h are taken to be, respectively, the total kinetic energy and the square of the angular momentum.

Moreover, he noticed that the evolution equation for a function f on $\mathbb{R}^3$ induced by the equation of motion (1) can be cast into the form:

$$\frac{df}{dt} = \frac{\partial(f, g, h)}{\partial(x, y, z)} \ , \tag{2}$$

where the right-hand side is the Jacobian of (f, g, h) with respect to (x, y, z). This expression was easily generalized to $n-1$ Hamiltonian functions on $\mathbb{R}^n$. The Jacobian can be interpreted as a kind of generalized Poisson bracket: it is skew-symmetric with respect to f, g and h; it is a derivation of the algebra of smooth functions on $\mathbb{R}^3$, i.e., the Leibniz rule is verified in each argument. Hence there is a complete analogy with the Poisson bracket formulation of Hamilton equations except, at first sight, for the equivalent of Jacobi identity which seems to be lacking. In fact, in the usual Poisson formulation, the Jacobi identity is the infinitesimal form of Poisson theorem which states that the bracket of two integrals of motion is also an integral of motion. If we want a similar theorem for Nambu Mechanics there must be an infinitesimal form of it which will provide a generalization of Jacobi identity. Denote by $\{f, g, h\}$ the Jacobian appearing in Eq. (2). Let $\phi_t: \vec{r} \mapsto \phi_t(\vec{r})$ be the flow for Eq. (1). Then a generalization of Poisson theorem would imply that ϕ_t is a "canonical transformation" for the generalized bracket:

$$\{f_1 \circ \phi_t, f_2 \circ \phi_t, f_3 \circ \phi_t\} = \{f_1, f_2, f_3\} \circ \phi_t \ .$$

Differentiation of this equality with respect to t yields the desired generalization of Jacobi identity:

$$\{\{g,h,f_1\},f_2,f_3\}+\{f_1,\{g,h,f_2\},f_3\}+\{f_1,f_2,\{g,h,f_3\}\} \\ =\{g,h,\{f_1,f_2,f_3\}\}\ ,\quad \forall g,h,f_1,f_2,f_3\in C^\infty(\mathbb{R}^3)\ .$$

This identity and its generalization to $\mathbb{R}^n$, called Fundamental Identity (FI), was introduced by Flato, Frønsdal [9] and Takhtajan [20] as a consistency condition for Nambu Mechanics (this consistency condition was also formulated in [18]) and allows a generalized Poisson theorem: the generalized bracket of n integrals of motion is an integral of motion. It turns out that the Jacobian on $\mathbb{R}^n$ satisfies the FI.

Since the publication of Nambu's paper in 1973, different aspects of this new geometrical structure have been studied by several authors. In [1], it is shown that Nambu Mechanics on $\mathbb{R}^n$ can be viewed, through Dirac's constraints theory, as an embedding into a singular Hamiltonian system on $\mathbb{R}^{2n}$. An invariant geometrical formulation of Nambu Mechanics has recently been given in [20] leading to the notion of Nambu-Poisson manifolds. Several physical systems have been formulated within the Nambu framework: in [5], it is shown, among others, that the $SU(n)$-isotropic harmonic oscillator and the $SO(4)$-Kepler systems admit a Nambu structure. Other examples are discussed in [20].

1.2 An Overlook of Zariski Quantization

Nambu also discussed the quantization of this new structure. This turns out to be a non-straightforward task [1], [20] and the usual approaches to quantization failed to give an appropriate solution. See Sect. 2.1 for further details.

The aim of this paper is to present a solution for the quantization of Nambu structures. This solution is based on deformation quantization and involves arithmetic aspects in its construction related to factorization of polynomials in several real variables. For that reason, the quantization scheme we shall present here is called Zariski Quantization. We shall give here a brief overlook of this solution.

Consider the Nambu bracket on $\mathbb{R}^3$ given by the Jacobian:

$$\{f_1,f_2,f_3\}=\sum_{\sigma\in S_3}\epsilon(\sigma)\frac{\partial f_1}{\partial x_{\sigma_1}}\frac{\partial f_2}{\partial x_{\sigma_2}}\frac{\partial f_3}{\partial x_{\sigma_3}}\ , \tag{3}$$

where S_3 is the permutation group of $\{1,2,3\}$ and $\epsilon(\sigma)$ is the sign of the permutation σ. When one verifies that the Jacobian is actually a Nambu bracket, all one needs are some specific properties of the pointwise product of functions appearing in the right-hand side of Eq. (3). Namely, it is Abelian, associative, distributive (with respect to addition) and satisfies the Leibniz rule. The idea here is to look for a deformation of the usual product which enjoys the previously stated properties and to define a deformed Nambu bracket by replacing the usual product by the deformed product. Denote by $\times$ such a deformed product. Then the deformed bracket:

$$[f_1, f_2, f_3] \equiv \sum_{\sigma \in S_3} \epsilon(\sigma) \frac{\partial f_1}{\partial x_{\sigma_1}} \times \frac{\partial f_2}{\partial x_{\sigma_2}} \times \frac{\partial f_3}{\partial x_{\sigma_3}} , \tag{4}$$

will define a deformation of the classical Nambu structure expressed by (3).

In this desired context, the whole problem of quantizing Nambu structures reduces to the construction of the deformed product $\times$. Some trivial deformations of the usual product provide such deformed products, but of course these are not interesting. Also one has to bear in mind a theorem by Gelfand which states that an Abelian involutive Banach algebra $\mathcal{B}$ is isomorphic to an algebra of continuous functions on the spectrum (maximal ideals) of $\mathcal{B}$, endowed with the pointwise product. Hence we cannot expect to find a non-trivial deformation of the usual product on a dense subspace of $C^0(\mathbb{R}^n)$ with all the desired properties. At best we would deform the spectrum. Moreover, Abelian algebra deformations of Abelian algebras are classified by the Harrison cohomology and it turns out that the second Harrison cohomology space is trivial for an algebra of polynomials [12]. Hence it is not possible to find a non-trivial Abelian algebra deformation (in the sense of Gerstenhaber [12]) of the algebra of polynomials on $\mathbb{R}^n$.

We shall see in Sect. 3.1 what are the difficulties met when one tries to construct a deformed Abelian associative algebra consisting of functions on $\mathbb{R}^3$. It is possible to construct an Abelian associative deformation of the usual pointwise product on the space of real polynomials on $\mathbb{R}^3$ of the following form:

$$f \times_\beta g = T(\beta(f) \otimes \beta(g)) , \tag{5}$$

where β maps a real polynomial on $\mathbb{R}^3$ to the symmetric algebra constructed over the polynomials on $\mathbb{R}^3$. T is an "evaluation map" which allows to go back to (deformed) polynomials. It replaces the (symmetric) tensor product $\otimes$ by a symmetrized form of a given "partial" Moyal product on $\mathbb{R}^3$ (Moyal product on a hyperplane in $\mathbb{R}^3$ with deformation parameter $\hbar$). The extension of the map β to $\hbar$-deformed polynomials by requiring that it annihilates (non-zero) powers of $\hbar$, will give rise to an Abelian deformation of the usual product (T restores a $\hbar$-dependence). In general (5) does not define an associative product and we look for a β which makes the product $\times_\beta$ associative. Consider a real (normalized) polynomial P on $\mathbb{R}^3$: it can be uniquely factored into irreducible factors $P = P_1 \cdots P_n$. Define α on the space of real (normalized) polynomials by: $\alpha(P) = P_1 \otimes \cdots \otimes P_n$. With the choice $\beta = \alpha$ in (5), it can be easily shown that the product $\times_\alpha$ is associative. But the map α is not a linear map, hence the product $\times_\alpha$ is not distributive and Leibniz rule is not verified.

These difficulties are related to the fact that, from the physical point of view, the dynamical variables with respect to which the Nambu bracket is expressed do not necessarily represent point-particles (see the example for Euler equations mentioned in Sect. 1.1). As a matter of fact, the point-particle interpretation in Hamiltonian Mechanics is based on the following feature: one can construct dynamical systems with phase-space of arbitrarily (even) dimension by composing systems with phase-spaces of smaller dimensions. Remember that $\mathbb{R}^{2n}$ endowed with its canonical Poisson bracket is nothing but the direct sum of 2-dimensional spaces ($\mathbb{R}^2$) endowed with their canonical Poisson brackets. In this situation it is possible to interpret a system of n free particles as n systems of one free particle. Such a situation no longer prevails in Nambu Mechanics. The FI imposes strong constraints on Nambu structures and the linear superposition of two Nambu structures does not define in general a Nambu structure (see [20]). In that sense, it seems hopeless to have some notion of point-particles in Nambu Mechanics and this fact suggests that quantization here will have more to do with a field-like approach than with a quantum-mechanical one, and we shall have to quantize the observables (functions) rather than the dynamical variables themselves.

In relation with what has been said above about field-like quantization for Nambu Mechanics and in order to go around Gelfand theorem and cohomological difficulties, we are led to consider an algebra $\mathcal{A}_0$ (a kind of Bosonic Fock space) on which is defined the classical Nambu structure: quantization is interpreted as a (generalized) deformation $\mathcal{A}_\hbar$ of the algebra $\mathcal{A}_0$. More precisely, let $\mathcal{N}$ be an Abelian associative algebra with product $(f,g) \mapsto f \cdot g$; the algebraic structure of Nambu Mechanics is given by a trilinear map on $\mathcal{N}$ taking values in $\mathcal{N}$, $[\cdot,\cdot,\cdot]\colon (f,g,h) \mapsto [f,g,h] \in \mathcal{N}$ such that $\forall f_0, f_1, f_2, f_3, f_4, f_5 \in \mathcal{N}$:

a) $[f_1, f_2, f_3] = \epsilon(\sigma)[f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}], \quad \sigma \in S_3;$

b) $[f_0 \cdot f_1, f_2, f_3] = f_0 \cdot [f_1, f_2, f_3] + [f_0, f_2, f_3] \cdot f_1;$

c) $[f_1, f_2, [f_3, f_4, f_5]] = [[f_1, f_2, f_3], f_4, f_5] + [f_3, [f_1, f_2, f_4], f_5] + [f_3, f_4, [f_1, f_2, f_5]].$

This is the setting for classical Nambu Mechanics when the algebra $\mathcal{N}$ is the algebra of smooth functions on $\mathbb{R}^3$ with the pointwise product, and the bracket is the Jacobian. Since we are looking for a field-like quantization, the classical Nambu Mechanics (and hence the Nambu bracket (3)) will be defined on a kind of Fock space algebra $\mathcal{A}_0$ incorporating derivatives with product $\bullet$, described in Sect. 3.3. The map α is extended to $\mathcal{A}_0$ by linearity (with respect to the addition in $\mathcal{A}_0$) and the classical evaluation map here will take values in $\mathcal{A}_0$ and will simply replace the symmetric tensor product by the usual product and the tensor sum by the addition in $\mathcal{A}_0$.

Then quantization will consist in "deforming" the algebra $(\mathcal{A}_0, \bullet)$ to an Abelian associative algebra $(\mathcal{A}_\hbar, \bullet_\hbar)$, by requiring that α annihilates $\hbar$ and by using the evaluation map which replaces the symmetric tensor product by a symmetrized product given by a star-product. The quantum Nambu bracket $[\cdot,\cdot,\cdot]_{\bullet_\hbar}$ will be given by expression (4) where the $\times$-product is replaced by the $\bullet_\hbar$-product and where the derivatives are defined on $\mathcal{A}_\hbar$. This extension will permit the FI and the Leibniz rule (with respect to the bracket) to be satisfied. Hence this deformed bracket on the algebra $\mathcal{A}_\hbar$ will define a quantization of the classical Nambu structure on $\mathcal{A}_0$. By the same procedure, one gets immediately generalizations to $\mathbb{R}^n$, $n \geq 2$.

The paper is organized as follows. Here below we review briefly Nambu-Poisson manifolds. In Sect. 2 we discuss the problems encountered in quantization of Nambu Mechanics and recall the deformation quantization approach. Section 3 is devoted to the construction of a solution for the quantization of Nambu-Poisson structures on $\mathbb{R}^n$, $n \geq 2$, by introducing the Zariski quantization scheme. The paper is concluded by several remarks about possible extensions of this work and related mathematical problems.

1.3 Nambu-Poisson Manifolds

Let us first review some basic notions on Nambu-Poisson manifolds (the reader is referred to [20] for further details). Let M be a m-dimensional C^∞-manifold. Denote by A the algebra of smooth real-valued functions on M. S_n stands for the group of permutations of the set $\{1, \ldots, n\}$. We shall denote by $\epsilon(\sigma)$ the sign of the permutation $\sigma \in S_n$.

Definition 1 *A Nambu bracket of order n ($2 \leq n \leq m$) on M is defined by a n-linear map on A taking values in A:*

$$\{\cdot, \ldots, \cdot\} \colon A^n \to A \ ,$$

such that the following statements are satisfied $\forall f_0, \ldots, f_{2n-1} \in A$:

a) Skew-symmetry

$$\{f_1, \ldots, f_n\} = \epsilon(\sigma)\{f_{\sigma_1}, \ldots, f_{\sigma_n}\} \ , \quad \forall \sigma \in S_n;$$

b) Leibniz rule

$$\{f_0 f_1, f_2, \ldots, f_n\} = f_0\{f_1, f_2, \ldots, f_n\} + \{f_0, f_2, \ldots, f_n\} f_1 \ ; \tag{6}$$

c) Fundamental Identity

$$\begin{aligned} &\{f_1, \ldots, f_{n-1}, \{f_n, \ldots, f_{2n-1},\}\} \\ &\quad = \{\{f_1, \ldots, f_{n-1}, f_n\}, f_{n+1}, \ldots, f_{2n-1}\} \\ &\qquad + \{f_n, \{f_1, \ldots, f_{n-1}, f_{n+1}\}, f_{n+2}, \ldots, f_{2n-1}\} \\ &\qquad + \cdots + \{f_n, f_{n+1}, \ldots, f_{2n-2}, \{f_1, \ldots, f_{n-1}, f_{2n-1}\}\} \ . \end{aligned} \tag{7}$$

Properties a) and b) imply that there exists a n-vector field η on M such that:

$$\{f_1, \ldots, f_n\} = \eta(df_1, \ldots, df_n) \ , \quad \forall f_1, \ldots, f_n \in A. \tag{8}$$

Of course the FI imposes constraints on η, analyzed in [20]. A n-vector field on M is called a Nambu tensor, if its associated Nambu bracket defined by Eq. (8) satisfies the FI.

Definition 2 *A Nambu-Poisson manifold (M, η) is a manifold M on which is defined a Nambu tensor η. Then M is said to be endowed with a Nambu-Poisson structure.*

The dynamics associated with a Nambu bracket on M is specified by $n-1$ Hamiltonians $H_1, \ldots, H_{n-1} \in A$ and the time evolution of $f \in A$ is given by:

$$\frac{df}{dt} = \{H_1, \ldots, H_{n-1}, f\} \,. \tag{9}$$

Suppose that the flow ϕ_t associated with Eq. (9) exists and let U_t be the one-parameter group acting on A by $f \mapsto U_t(f) = f \circ \phi_t$. It follows from the FI that:

Theorem 1 *The one-parameter group U_t is an automorphism of the algebra A for the Nambu bracket.*

Definition 3 *$f \in A$ is called an integral of motion for the system defined by Eq. (9) if it satisfies $\{H_1, \ldots, H_{n-1}, f\} = 0$.*

It follows from the FI that a Poisson-like theorem exists for Nambu-Poisson manifolds:

Theorem 2 *The Nambu bracket of n integrals of motion is also an integral of motion.*

For the case $n = 2$, the FI is Jacobi identity and one recovers the usual definition of Poisson manifold. On $\mathbb{R}^2$, the canonical Poisson bracket of two functions $\mathcal{P}(f, g)$ is simply their Jacobian and Nambu defined his bracket on $\mathbb{R}^n$ as a Jacobian of n functions $f_1, \ldots, f_n \in C^\infty(\mathbb{R}^n)$ of n variables $x_1, \ldots, x_n$:

$$\{f_1, \ldots, f_n\} = \sum_{\sigma \in S_n} \epsilon(\sigma) \frac{\partial f_1}{\partial x_{\sigma_1}} \cdots \frac{\partial f_n}{\partial x_{\sigma_n}} \,,$$

which gives the canonical Nambu bracket of order n on $\mathbb{R}^n$. Other examples of Nambu-Poisson structures have been found [6]. One of them is a generalization of linear Poisson structures and is given by the following Nambu bracket of order n on $\mathbb{R}^{n+1}$:

$$\{f_1, \ldots, f_n\} = \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \frac{\partial f_1}{\partial x_{\sigma_1}} \cdots \frac{\partial f_n}{\partial x_{\sigma_n}} x_{\sigma_{n+1}} \,.$$

In general any manifold endowed with a Nambu structure of order n is locally foliated by Nambu-Poisson manifolds of dimension n endowed with the canonical Nambu structure [11]. In particular, it is shown in [11] that any Nambu tensor is decomposable (this fact, conjectured in [20], was eventually discovered to be a consequence of an old result [21] reproduced in a textbook by Schouten [19] Chap. II Sects. 4 and 6, formula (6.7)).

2 The Quantization Problem

2.1 Difficulties with Usual Quantizations

In his 1973 paper Nambu has also studied the quantization of his generalized mechanics. He was looking for an operator representation of a trilinear bracket which is skew-symmetric and satisfies Leibniz rule (several combinations of conditions weaker than the preceding were discussed as well). The main difficulty encountered was to conciliate skew-symmetry and Leibniz rule at the same time. It is interesting to note that Nambu suggested the use of non-associative algebras in order to overcome the problems appearing with operatorial techniques.

Other aspects of operatorial quantization of Nambu Mechanics were discussed in [1], [6], [20]. In [20], a representation of the ($n = 3$ case) Nambu-Heisenberg commutation relations:

$$[A_1, A_2, A_3] \equiv \sum_{\sigma \in S_3} \epsilon(\sigma) A_{\sigma_1} A_{\sigma_2} A_{\sigma_3} = cI ,$$

where c is a constant and I is the unit operator, was constructed. The operators A_1, A_2, A_3 act on a space of states parametrized by a ring of algebraic integers $\mathbb{Z}[\rho]$ in the quadratic number field $\mathbb{Q}[\rho]$ (where $1 + \rho + \rho^2 = 0$). The cases $n = 5$ and $n = 7$ are studied in [6].

A possible alternative to quantize Nambu bracket by deformation quantization [2], [3] was discussed in [20] (see Sect. 2.2 for a brief review on star-products). If one looks at the canonical Nambu bracket on $\mathbb{R}^3$ as a trilinear differential operator D on $A = C^\infty(\mathbb{R}^3)$, then one can define a $\hbar$-deformed trilinear product on A by:

$$(f_1, f_2, f_3)_\hbar = \exp(\hbar D)(f_1, f_2, f_3) , \quad f_1, f_2, f_3 \in A. \tag{10}$$

The "deformed bracket" associated with the product (10) would naturally be defined by:

$$[f_1, f_2, f_3]_\hbar = \frac{1}{3!} \sum_{\sigma \in S_3} \epsilon(\sigma) (f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3})_\hbar , \tag{11}$$

leading to a deformation of Nambu bracket. But (11) is not a deformation of a Nambu-Poisson structure: *it does not satisfy the FI.* Furthermore, it is not clear what kind of associativity conditions one should impose on a trilinear product for the Leibniz rule to be valid. Anyhow, if F is a nonlinear analytic function of one variable, we know [14] that there is no deformation of the Nambu bracket satisfying the FI of the form:

$$(f_1, f_2, f_3)_\hbar = F(\hbar D)(f_1, f_2, f_3) , \quad f_1, f_2, f_3 \in A.$$

Note that the previous negative result does not mean that there is no (differentiable) deformation of Nambu-Poisson structures since general deformations of the form:

$$[f_1, f_2, f_3]_\hbar = \{f_1, f_2, f_3\} + \sum_{r \geq 1} \hbar^r D_r(f_1, f_2, f_3) ,$$

where the D_r's are trilinear differential operators on A, have to be considered, but it shows that deformation quantization will not provide a straightforward solution to the quantization problem of Nambu-Poisson structures. Nevertheless we shall present a solution in Sect. 3.3 that relies heavily on deformation quantization.

Another possible avenue for the quantization problem is to apply Feynman Path Integral techniques. A canonical formalism and an action principle have been defined for Nambu Mechanics permitting the definition of an action functional [20]. Within this formalism, it would be possible to formally define the path integral for Nambu Mechanics, but this approach is essentially equivalent to usual deformation quantization since the Feynman Path Integral is given by the star-exponential (see the end of Sect. 2.2).

2.2 Deformation Quantization

For completeness we give here a brief review on deformation quantization and star-products; a full treatment can be found in [2], [3] and a recent review in [10]. Let M be a Poisson manifold. We denote by A the algebra of C^∞-functions on M and by $\mathcal{P}(f,g)$ the Poisson bracket of $f,g \in A$. Let $A[[\nu]]$ be the space of formal power series in the parameter ν with coefficients in A. A star-product $*_\nu$ on M is an associative (generally non-abelian) deformation of the usual product of the algebra A, and is defined as follows:

Definition 4 *A star-product on M is a bilinear map $(f,g) \mapsto f *_\nu g$ from $A \times A$ to $A[[\nu]]$, taking the form:*

$$f *_\nu g = \sum_{r \geq 0} \nu^r C_r(f,g) \ , \quad \forall f,g \in A,$$

where $C_0(f,g) = fg$, $f,g \in A$, and $C_r : A \times A \to A$ $(r \geq 1)$ are bidifferential operators (bipseudodifferential operators can sometimes be considered) on A satisfying:

a) $C_r(f,c) = C_r(c,f) = 0$, $r \geq 1$, $c \in \mathbb{R}$, $f \in A$;

b) $C_1(f,g) - C_1(g,f) = 2\mathcal{P}(f,g)$, $f,g \in A$;

c) $\sum_{\substack{r+s=t \\ r,s\geq 0}} C_r(C_s(f,g),h) = \sum_{\substack{r+s=t \\ r,s\geq 0}} C_r(f,C_s(g,h))$, $\forall t \geq 0$, $f,g,h \in A$.

By linearity, $*_\nu$ is extended to $A[[\nu]] \times A[[\nu]]$. Condition a) ensures that $c *_\nu f = f *_\nu c = cf$, $c \in \mathbb{R}$ (and may be omitted, in which case an equivalent star-product will verify it). Condition c) is equivalent to the associativity equation $(f *_\nu g) *_\nu h = f *_\nu (g *_\nu h)$. Condition b) implies that the star-bracket

$$[f,g]_{*_\nu} \equiv (f *_\nu g - g *_\nu f)/2\nu \ ,$$

is a deformation of the Lie-Poisson algebra on M. Hence a star-product on M deforms at once the two classical structures on A, i.e. the Abelian associative algebra for the pointwise product of functions and the Lie algebra structure given by the Poisson bracket. This leads to:

Definition 5 *A deformation quantization of the Poisson manifold* $(M, \mathcal{P})$ *is a star-product on* M*.*

Definition 6 *Two star-products* $*$ *and* $*'$ *are said to be equivalent if there exists a map* $T\colon A[[\nu]] \to A[[\nu]]$ *having the form:*

$$T = \sum_{r \geq 0} \nu^r T_r \ ,$$

where the T_r*'s* $(r \geq 1)$ *are differential operators vanishing on constants and* $T_0 = Id$*, such that*

$$Tf * Tg = T(f *' g) \ , \quad f, g \in A[[\nu]].$$

A star-product which is equivalent to the pointwise product of functions is said to be trivial.

For physical applications, the deformation parameter ν is taken to be $i\hbar/2$. On $\mathbb{R}^{2n}$ the basic example of star-product is the Moyal product defined by:

$$f *_M g = \exp\left(\frac{i\hbar}{2}\mathcal{P}\right)(f, g) \ . \tag{12}$$

It corresponds to the Weyl (totally symmetric) ordering of operators in Quantum Mechanics. On $\mathbb{R}^{2n}$ endowed with its canonical Poisson bracket, other orderings can be considered as well and they correspond to star-products equivalent to the Moyal product. For example, the normal star-product (which is the exponential of "half of the Poisson bracket" in the variables $p \pm iq$) is equivalent to Moyal product. From now on, we implicitly set $\nu = i\hbar/2$.

A given Hamiltonian $H \in A$ determines the time evolution of an observable $f \in A$ by the Heisenberg equation:

$$\frac{df_t}{dt} = [H, f_t]_{*_\nu} \ . \tag{13}$$

The one-parameter group of time evolution associated with Eq. (13) is given by the star-exponential defined by:

$$\exp_*\left(\frac{tH}{i\hbar}\right) \equiv \sum_{r \geq 0} \frac{1}{r!}\left(\frac{t}{i\hbar}\right)^r (*H)^r \ , \tag{14}$$

where $(*H)^r = H * \cdots * H$ (r factors). Then the solution to Eq. (13) can be expressed as:

$$f_t = \exp_*\left(\frac{tH}{i\hbar}\right) * f * \exp_*\left(\frac{-tH}{i\hbar}\right) \ .$$

In many examples, the star-exponential is convergent as a series in the variable t in some interval ($|t| < \pi$ for the harmonic oscillator in the Moyal case) and converges as a distribution on M for fixed t. Then it makes sense to consider a Fourier-Dirichlet expansion of the star-exponential:

$$\exp_*\left(\frac{tH}{i\hbar}\right)(x) = \int \exp(\lambda t/i\hbar)d\mu(x;\lambda)\ , \quad x \in M, \tag{15}$$

the "measure" μ being interpreted as the Fourier transform (in the distribution sense) of the star-exponential in the variable t. Equation (15) permits to define [3] the spectrum of the Hamiltonian H as the support Λ of the measure μ. In the discrete case where

$$\exp_*\left(\frac{tH}{i\hbar}\right)(x) = \sum_{\lambda\in\Lambda} \exp(\lambda t/i\hbar)\pi_\lambda(x)\ , \quad x \in M,$$

the functions π_λ on M are interpreted as eigenstates of H associated with the eigenvalues λ, and satisfy

$$H * \pi_\lambda = \pi_\lambda * H = \lambda\pi_\lambda\ , \quad \pi_\lambda * \pi_{\lambda'} = \delta_{\lambda\lambda'}\ , \quad \sum_{\lambda\in\Lambda} \pi_\lambda = 1\ .$$

In the Moyal case, the Feynman Path Integral can be expressed [15] as the Fourier transform over momentum space of the star-exponential. In field theory, where the normal star-product is relevant, the Feynman Path Integral is given (up to a multiplicative factor) [8] by the star-exponential.

From the preceding, it should be clear that deformation quantization provides a completely autonomous quantization scheme of a classical Hamiltonian system and we shall use it for the quantization of Nambu-Poisson structures.

3 A New Quantization Scheme: Zariski Quantization

We saw in Sect. 2.1 that a direct application of deformation quantization to Nambu-Poisson structures is not possible. Instead of looking at the deformed Nambu bracket as some skew-symmetrized form of a n-linear product, we deform directly the Nambu bracket. Then it turns out that a solution to the quantization problem can be constructed in this way. The construction that will be presented here is based on the following simple remark: the Jacobian of n functions on $\mathbb{R}^n$ is a Nambu bracket because the usual product of functions is Abelian, associative, distributive and respects the Leibniz rule. If we replace the usual product in the Jacobian by any product having the preceding properties, we get a "modified Jacobian" which is still a Nambu bracket. That is to say, the "modified Jacobian" is skew-symmetric, it satisfies the Leibniz rule with respect to the new product and the FI is verified. Now if we suppose that the new product is a deformation of the usual product, then the "modified Jacobian" will be a deformation of the Nambu bracket providing a deformation quantization of the Nambu-Poisson structure.

3.1 Quantization of Nambu-Poisson Structure of Order 3: The Setting

This section is devoted to preliminaries needed for the construction of an Abelian associative deformed product on $\mathbb{R}^3$. The generalization to $\mathbb{R}^n$ will be discussed later.

First we shall make some general comments on possible candidates that one can consider for an Abelian deformed product. Even though $\mathbb{R}^3$ is not a symplectic manifold, we can define a "partial" Moyal product between functions in $A = C^\infty(\mathbb{R}^3)$. Denote by (x_1, x_2, x_3) the coordinates in $\mathbb{R}^3$. Let $\mathcal{P}_{12}$ be the Poisson bracket with respect to the variables (x_1, x_2), i.e. for $f, g \in A$, it is defined by $\mathcal{P}_{12}(f,g) = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}$. Then denote by $*_{12}$ the Moyal product constructed with $\mathcal{P}_{12}$ and with deformation parameter $\hbar$, that is:

$$f *_{12} g = \sum_{r \geq 0} \frac{\hbar^r}{r!} \mathcal{P}_{12}^r(f,g) \ , \quad f, g \in A.$$

Then $A[[\hbar]]$ endowed with the product $*_{12}$ is a non-abelian associative deformation of A endowed with the usual product. If, in order to get an Abelian algebra, one simply applies the "Jordan trick" to the non-abelian algebra $(A[[\hbar]], *_{12})$ by defining a product by $f \times g = \frac{1}{2}(f *_{12} g + g *_{12} f)$, one will get a non-associative algebra. Here associativity is lacking, because the product $\times$ does not make a complete symmetrization with respect to (f_1, f_2, f_3) in the expression $(f_1 \times f_2) \times f_3$.

Somehow a kind of symmetrization, not necessarily with respect to the factors appearing in the product, is needed for associativity and the product we are looking for should share some features of the tensor product of particle-states in the Bosonic Fock space as is done in second quantization. It suggests to look at a map sending $f \in A$ to the symmetric tensor algebra $\mathrm{Symm}(A)$ of A and then go back to $A[[\hbar]]$ by an "evaluation map" which replaces the symmetric tensor product in $\mathrm{Symm}(A)$ by a completely symmetrized form of the Moyal product $*_{12}$.

Let us make precise the previous remark. Start with any map:

$$\beta \colon A \to \mathrm{Symm}(A) \ ,$$

such that $\beta(1) = I$ and extend it to the map from $A[[\hbar]]$ into $\mathrm{Symm}(A)$ (denoted by the same symbol β) by requiring that it vanishes on the non-zero powers of $\hbar$. Define the evaluation map $T \colon \mathrm{Symm}(A) \to A[[\hbar]]$ as a canonical linear map whose restriction on $A^{\overset{n}{\otimes}}$ is given by:

$$f_1 \otimes \cdots \otimes f_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma_1} *_{12} \cdots *_{12} f_{\sigma_n} \ , \tag{16}$$

where $\otimes$ stands for the *symmetric* tensor product. Then we define a map $\times_\beta \colon A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]]$ — the β-product — by the following formula:

$$f \times_\beta g = T(\beta(f) \otimes \beta(g)) \ , \quad f, g \in A[[\hbar]]. \tag{17}$$

It is clear that the β-product is always Abelian. However, for general β, it is neither associative, nor distributive, nor a deformation of the usual pointwise product on A, or has 1 as unit element. Thus associativity $(f \times_\beta g) \times_\beta h = f \times_\beta (g \times_\beta h)$ of the β-product reads

$$T(\beta(T(\beta(f) \otimes \beta(g))) \otimes \beta(h)) = T(\beta(f) \otimes \beta(T(\beta(g) \otimes \beta(h)))) \ , \quad \forall f, g, h \in A[[h]],$$

and it is an equation for the map β. Before giving a non-trivial example of the associative β-product which is a deformation of the usual product, we summarize simple basic facts regarding this construction in the following theorem.

Theorem 3 *i) The standard unit* 1 *is the unit element of the* β*-product:* $f \times_\beta 1 = f$, $\forall f \in A$, *if and only if* $T \circ \beta = id_A$.

ii) If, in addition to i), $\beta \colon A \to \mathrm{Symm}(A)$ *is an algebra homomorphism, then the* β*-product on* A *coincides with the usual pointwise product.*

iii) If the β*-product is a deformation of the usual product, then the associativity condition reduces to*

$$T(\beta(fg) \otimes \beta(h)) = T(\beta(f) \otimes \beta(gh)) \ , \quad \forall f, g, h \in A.$$

iv) If, in addition to i), the β*-product is an associative deformation of the usual product, then it coincides with the usual product.*

v) If β *is an algebra homomorphism and the* β*-product is a deformation of the usual product, then the* β*-product is associative.*

Proof. Part i) is obvious, since it is equivalent to

$$f \times_\beta 1 = T(\beta(f)) = f \ , \quad \forall f \in A.$$

For the part ii), we have

$$f \times_\beta g = T(\beta(f) \otimes \beta(g)) = T(\beta(fg)) = fg \ , \quad \forall f, g \in A.$$

To prove iii), simply note that if β-product is a deformation of the usual product, then $\beta(f \times_\beta g) = \beta(fg)$ (fg stands for the usual product), and the equation follows.

Part iv) follows from part iii) by setting $h = 1$ in the associativity condition, then we get:

$$f \times_\beta g = T(\beta(fg)) = fg \ , \quad \forall f, g \in A.$$

Finally, part v) follows from part iii) and associativity of the $\otimes$-product in $\mathrm{Symm}(A)$, since

$$T(\beta(fg) \otimes \beta(h)) = T((\beta(f) \otimes \beta(g)) \otimes \beta(h)) = T(\beta(f) \otimes (\beta(g) \otimes \beta(h))) = T(\beta(f) \otimes \beta(gh)).$$

We shall give an example of such map β for which condition v) of the Theorem 3 is satisfied, so that it gives an Abelian associative deformation of the usual product. For that purpose, we need to restrict A to the algebra N of polynomials on $\mathbb{R}^3$, which will allow a more refined decomposition in the symmetric algebra, thus avoiding the triviality of the product. In fact, we shall factorize polynomials on $\mathbb{R}^3$ into irreducible factors $P = P_1 \cdots P_n$ and send them to elements of the form $P_1 \otimes \cdots \otimes P_n$ in the symmetric algebra. This will give the desired Abelian associative deformation of the usual product.

Remark 1 *The standard embedding of the polynomial algebra into its symmetric algebra by elements of degree 1 (i.e. without any decomposition at all) gives rise to a non-associative product because of the incompatibility between the usual product and the Moyal product: associativity would require that $(PQ) * R + R * (PQ) = P * (QR) + (QR) * P$, which fails in general.*

Remark 2 *Another extreme case is when every polynomial is embedded into the symmetric algebra via complete symmetrization (i.e. by replacing every monomial by the corresponding $\otimes$-monomials in the symmetric algebra). In this example β-product again gives the usual product. Indeed the corresponding map β is obviously a homomorphism and, according to Theorem 3, part ii), one needs to verify that $T(\beta(P)) = P$ for all polynomials P, that is to say, $T(Q_1 \otimes \cdots \otimes Q_n) = Q_1 \cdots Q_n$, where Q_i stands for x_1, x_2 or x_3. This fact represents a well-known property of the Moyal quantization, and its proof is left to the reader.*

The choice of β we shall present makes a non-trivial compromise between commutativity and associativity though at this stage it lacks the property of being distributive. This β-product is constructed as follows. Let $N = \mathbb{R}[x_1, x_2, x_3]$ be the algebra of polynomials in the variables x_1, x_2, x_3 with real coefficients and let

$$\mathcal{S}(N) = \bigoplus_{n=1}^{\infty} N^{\overset{n}{\otimes}} ,$$

be its symmetric tensor algebra without scalars. Next, for any $P \in N$ define its *maximal monomial* to be a monomial of the highest total degree in P, maximal with respect to the lexicographical ordering induced by (x_1, x_2, x_3). We call $P \in N$ a *normalized* polynomial, if its maximal monomial has coefficient 1. Since the product of normalized polynomials is again normalized, normalized polynomials form a semi-group that we shall denote by N_1. We should also include 0 as a normalized polynomial, so that $0 \in N_1$.

Also consider the algebra $N[\hbar]$ (polynomials in $\hbar$ with coefficients in N) and call $P \in N[\hbar]$ a normalized polynomial if the coefficient of its lowest degree term in $\hbar$ is normalized in N. All normalized polynomials in $N[\hbar]$ form a semi-group $N_1^\hbar$ (under the usual product).

Every polynomial in N_1 can be uniquely factored into a product of irreducible normalized polynomials:

$$P = P_1 \cdots P_n .$$

Note that this factorization, as well as the set of all irreducible polynomials, depend on the choice of the ground field (in our case $\mathbb{R}$, the field of real numbers). Since we are dealing with polynomials in several variables, even over the field of complex numbers irreducible polynomials need not to be linear. In fact, the set of all irreducible polynomials in n variables over a field k plays a fundamental role in algebraic geometry over k: it defines the so-called Zariski topology in the space k^n (and in the corresponding projective space as well). This is why we call the concrete realization of the β-product, based on the factorization of polynomials, Zariski quantization.

We define a map $\tilde{\alpha}: N_1 \to \mathcal{S}(N)$ by:

$$\tilde{\alpha}(P) = P_1 \otimes \cdots \otimes P_n \ , \quad P \in N_1.$$

Denote by $\pi: N_1^\hbar \to N_1$ the homomorphism which attaches to a polynomial in $N_1^\hbar$ its coefficient of degree 0 in $\hbar$; it is always an element in N_1 (and may be zero as well). This "projection onto the classical part" allows to extend $\tilde{\alpha}$ to the homomorphism

$$\alpha = \tilde{\alpha} \circ \pi: N_1^\hbar \to \mathcal{S}(N) \ ,$$

which takes into account only the classical part of the polynomial in $N_1^\hbar$. Finally, denote the restriction of the evaluation map T from $\mathrm{Symm}(A)$ to $\mathcal{S}(N)$ by the same symbol $T: \mathcal{S}(N) \to N[\hbar]$. Then specializing our general construction of β-products to the case $\beta = \alpha$ we get the map $\times_\alpha: N_1^\hbar \times N_1^\hbar \to N[\hbar]$, given by the following formula:

$$P \times_\alpha Q = T(\alpha(P) \otimes \alpha(Q)) \ , \quad \forall P, Q \in N_1^\hbar.$$

Theorem 4 *The map $\times_\alpha$ defines an Abelian associative product on $N_1^\hbar$ which is a deformation of the usual product on N_1.*

Proof. First, the classical part of $P \times_\alpha Q$ is equal to $\pi(P)\pi(Q) \in N_1$ (it may be zero as well), since the classical part of the Moyal product is the usual product and π is a homomorphism. This shows that indeed the map $\times_\alpha$ maps $N_1^\hbar \times N_1^\hbar$ into $N_1^\hbar$. In particular, if $P, Q \in N_1$, then

$$P \times_\alpha Q|_{\hbar=0} = PQ \ ,$$

so that $\times_\alpha$ is some deformation of the usual product. By this we mean nothing more than the above formula; due to the projection onto the classical part and the decomposition into irreducible factors, what we get is more general than a deformation in the sense of Gerstenhaber; in particular "Gerstenhaber" deformations are defined on the base field $\mathbb{R}[[\hbar]]$ while here (at least in the present construction) we do not have $\hbar$-linearity. Second, α is a homomorphism, so that associativity follows from Theorem 3, part v).

Remark 3 *Note that in the definition of the evaluation map T the Moyal product $*_{12}$ can be replaced by any star-product on $\mathbb{R}^3$ without affecting the associativity and deformation* *properties of the product $\times_\alpha$. In particular, one also has products $\times_\alpha^{(ij)}$ constructed from partial Moyal products on (ij)-planes in $\mathbb{R}^3$. It is easy to show that the totally symmetrized product: $(f,g)\mapsto \frac{1}{3}(f\times_\alpha^{(12)} g + f\times_\alpha^{(23)} g + f\times_\alpha^{(31)} g)$, is an Abelian, associative deformation of the usual product.*

Remark 4 *Note that 1 is not a unit element for the product $\times_\alpha$. Indeed, in general it is not true that $P\times_\alpha 1 = P$, $\forall P\in N_1^\hbar$. However, it is true when P is either an irreducible polynomial, or reduces completely into a product of linear factors.*

The space $N_1^\hbar$ endowed with the product $\times_\alpha$ is then an Abelian semi-group. The following example shows that $(N_1^\hbar, \times_\alpha)$ cannot be extended to an algebra in $N[\hbar]$. Consider the polynomials $P = x_1^2+\epsilon^2 x_2^2$, $\epsilon\in\mathbb{R}$, and $Q = x_2^2$. P is irreducible, then $\alpha(P) = x_1^2+\epsilon^2x_2^2$ (considered as an element of $N^{\overset{1}{\otimes}}$), while $\alpha(Q) = x_2\otimes x_2 \in N^{\overset{2}{\otimes}}$. One has (for notation simplicity, we write here $*$ instead of $*_{12}$)

$$\begin{aligned}
&P\times_\alpha Q\\
&\quad= T((x_1^2+\epsilon^2x_2^2)\otimes x_2\otimes x_2)\\
&\quad= \frac{1}{3}[(x_1^2+\epsilon^2x_2^2)*x_2*x_2 + x_2*(x_1^2+\epsilon^2x_2^2)*x_2 + x_2*x_2*(x_1^2+\epsilon^2x_2^2)]\\
&\quad= (x_1^2+\epsilon^2x_2^2)x_2^2 + \frac{2}{3}\hbar^2\ .
\end{aligned}$$

It is easy to verify that $x_1^2\times_\alpha x_2^2 = x_1^2x_2^2$ and $x_2^2\times_\alpha x_2^2 = x_2^4$, so that $(x_1^2+\epsilon^2x_2^2)\times_\alpha x_2^2 \neq x_1^2\times_\alpha x_2^2 + \epsilon^2(x_2^2\times_\alpha x_2^2)$. Hence $\times_\alpha$ is *not a distributive* product with respect to the addition in $N[\hbar]$. Moreover the preceding example shows that: $\lim_{\epsilon\to 0}((x_1^2+\epsilon^2x_2^2)\times_\alpha x_2^2) \neq x_1^2\times_\alpha x_2^2$, i.e. $\times_\alpha$ is *not a continuous* product.

These special aspects of $\times_\alpha$ imply the following: if we replace the usual product in the canonical Nambu bracket of order 3 by the product $\times_\alpha$ in order to get a deformed Nambu bracket:

$$[f,g,h]_\hbar \equiv \sum_{\sigma\in S_3}\epsilon(\sigma)\frac{\partial f}{\partial x_{\sigma_1}}\times_\alpha \frac{\partial g}{\partial x_{\sigma_2}}\times_\alpha \frac{\partial g}{\partial x_{\sigma_3}}\ ,$$

we will not get a deformation of the Nambu-Poisson structure. It can be easily verified that the Leibniz rule (with respect to $\times_\alpha$) and the FI are not satisfied. At this point, these facts should not be too surprising: as mentioned in Sect. 1.2, we know that we cannot expect to find a non-trivial deformation of the usual product on N with all the nice properties.

To summarize, we have some space N_1 with the usual product, and a deformed product on $N_1^\hbar$. Along the lines of what is done for topological quantum groups [4] and in second quantization, let us look at "functions" on N_1 (e.g. formal series). Intuitively we get a deformed coproduct and the dual of this space of "functions" (polynomials on polynomials) will then have a product and a deformed product, both of which will be distributive with respect to the vector space addition. Now the product of polynomials is again a polynomial. So in fact we are getting some deformed product on an algebra generated by the polynomials. We shall make this heuristic view precise in the next section.

3.2 Zariski Product

The product $\times_\alpha$ on $N_1^\hbar$ defined in Sect. 3.1 is Abelian and associative, but is not distributive with respect to the addition in $N[\hbar]$. Hence $(N_1^\hbar, \times_\alpha)$ is only a semi-group. We shall extend the product $\times_\alpha$ to an algebra $\mathcal{Z}_\hbar$ and get an Abelian algebra deformation of an Abelian algebra $\mathcal{Z}_0$ generated by the irreducible polynomials in N_1. The algebra $\mathcal{Z}_0$ is actually a kind of Fock space constructed from the irreducible polynomials considered as building blocks.

Let $N_1^{irr} \subset N_1$ be the set of real irreducible normalized polynomials. Let $\mathcal{Z}_0$ be a real vector space having a basis indexed by products of elements of N_1^{irr}, we denote the basis by $\{Z_{u_1\cdots u_m}\}$, where $u_1, \ldots, u_m \in N_1^{irr}$, and $m \geq 1$. The vector space $\mathcal{Z}_0$ is made into an algebra by defining a product $\bullet^z\colon \mathcal{Z}_0 \times \mathcal{Z}_0 \to \mathcal{Z}_0$ by:

$$Z_{u_1\cdots u_m} \bullet^z Z_{v_1\cdots v_n} = Z_{u_1\cdots u_m v_1\cdots v_n} \ , \quad \forall u_1, \ldots, u_m, v_1, \ldots v_n \in N_1, \forall m, n \geq 1.$$

$\mathcal{Z}_0$ endowed with the product $\bullet^z$ is the free Abelian algebra generated by the set of irreducible polynomials or equivalently the algebra of the semi-group N_1. Note that the addition in $\mathcal{Z}_0$ is *not* related to the addition in N, i.e. $Z_{u+v} \neq Z_u + Z_v$.

Every $u \in N$ can be uniquely factored as follows: $u = cu_1 \cdots u_m$, where $c \in \mathbb{R}$ and $u_1, \ldots, u_m \in N_1$, and we shall sometimes write Z_u for $cZ_{u_1\cdots u_m}$. This provides a multiplicative (but non additive) injection of N into the algebra $\mathcal{Z}_0$.

Let $\mathcal{Z}_\hbar = \mathcal{Z}_0[\hbar]$ be the vector space of polynomials in $\hbar$ with coefficients in $\mathcal{Z}_0$. Let the map $\zeta\colon N_1^\hbar \to \mathcal{Z}_\hbar$ be the injection of $N_1^\hbar$ into $\mathcal{Z}_\hbar$ defined by:

$$\zeta(\sum_{r\geq 0} \hbar^r u_r) = \sum_{r\geq 0} \hbar^r Z_{u_r} \ , \quad \forall u_0 \in N_1, u_i \in N, i \geq 1. \tag{18}$$

Using the injection ζ we can extend the product $\times_\alpha$ on $N_1^\hbar$ to $\mathcal{Z}_\hbar$ by first defining the product on the basis elements:

$$Z_{u_1\cdots u_m} \bullet^z_\hbar Z_{v_1\cdots v_n} = \zeta((u_1 \cdots u_m) \times_\alpha (v_1 \cdots v_n)) \ , \quad \forall u_1, \ldots, u_m, v_1, \ldots v_n \in N_1, \forall m, n \geq 1, \tag{19}$$

and then extend it to all of $\mathcal{Z}_\hbar$ by requiring that the product $\bullet^z_\hbar$ annihilates the non-zero powers of $\hbar$:

$$(\sum_{r\geq 0} \hbar^r A_r) \bullet^z_\hbar (\sum_{s\geq 0} \hbar^s B_s) = A_0 \bullet^z_\hbar B_0 \ , \quad \forall A_r, B_s \in \mathcal{Z}_0, r, s \geq 0.$$

Theorem 5 *The vector space $\mathcal{Z}_\hbar$ endowed with the product $\bullet_\hbar^z$ is an Abelian algebra which is some deformation of the Abelian algebra $(\mathcal{Z}_0, \bullet^z)$.*

Proof. By definition the product $\bullet_\hbar^z$ is distributive and Abelian. The associativity of $\bullet_\hbar^z$ follows directly from the associativity for the product $\times_\alpha$. For $\hbar = 0$, the product $\times_\alpha$ is the usual product, and Eq. (19) becomes:

$$Z_{u_1 \cdots u_m} \bullet_\hbar^z Z_{v_1 \cdots v_n}|_{\hbar=0} = \zeta(u_1 \cdots u_m v_1 \cdots v_n) = Z_{u_1 \cdots u_m v_1 \cdots v_n} = Z_{u_1 \cdots u_m} \bullet^z Z_{v_1 \cdots v_n} \ ,$$

showing that the product $\bullet_\hbar^z$ is some deformation of the product $\bullet^z$.

The next step would be to define derivatives δ_i, $1 \le i \le 3$, on $\mathcal{Z}_0$ and then extend them to $\mathcal{Z}_\hbar$. This would allow to define first the classical Nambu bracket on $\mathcal{Z}_0$, and the quantum one on $\mathcal{Z}_\hbar$. The trivial definition $\delta_i Z_u = Z_{\partial_i u}$, $\forall u \in N$, where ∂_i is the usual derivative with respect to x^i, does not satisfy the Leibniz rule (except on the diagonal) because of the different nature of the addition in N and in $\mathcal{Z}_0$. Unfortunately, what seems to be another very natural definition of derivative on $\mathcal{Z}_0$ does not satisfy the Frobenius property (commutativity of the derivatives in several variables). These derivatives would be linear maps $\delta_i \colon \mathcal{Z}_0 \to \mathcal{Z}_0$, $1 \le i \le 3$, defined as follows. For $u \in N_1^{irr}$, we let $\delta_i Z_u = Z_{\partial_i u}$, where ∂_i denotes the usual partial derivative of u with respect to x^i. The action of δ_i on a general basis element Z_v, $v \in N_1$ is given by postulating Leibniz rule on the product of irreducible polynomials $v = v_1 v_2 \cdots v_m$:

$$\delta_i \ Z_{v_1 v_2 \cdots v_m} = Z_{(\partial_i v_1) v_2 \cdots v_m} + Z_{v_1 (\partial_i v_2) \cdots v_m} + \cdots + Z_{v_1 v_2 \cdots (\partial_i v_m)} \ .$$

Obviously, the maps δ_i are derivations on the algebra $\mathcal{Z}_0$, but one can easily show that they are not commuting maps, i.e. $\delta_i \delta_j \ne \delta_j \delta_i$, $i \ne j$. This comes from the fact that when one takes the derivatives of an irreducible polynomial u, the polynomials $\partial_i u$, $1 \le i \le 3$, do not necessarily factorize out into the same number of factors. An example is given in $\mathbb{R}^2$ by $u = (x^3 + x^2 y + 4xy^2 + 5y^3 + 5xy + \frac{17}{2} y^2 + 4y) \in N_1^{irr}$. A consequence of this fact is the following: If one defines the classical Nambu bracket on $\mathcal{Z}_0$ by replacing, in the Jacobian, the usual product by $\bullet^z$ and the usual partial derivatives by the maps δ_i, this new bracket will not satisfy the FI. There will be anomalies in the FI (even at this classical, or "prequantized" level) due to terms which can not cancel out each other because the Frobenius property is not satisfied on $\mathcal{Z}_0$. In order to have a family of commuting derivations which can naturally be related to the usual derivatives of a polynomial, we need to extend the algebra on which will be defined the classical Nambu bracket. This algebra will consist of Taylor series in the variables (y^1, y^2, y^3) of the translated polynomials $u(x + y)$. One can look at this algebra as a jet space over the polynomials and it will be constructed in the next section.

Nonetheless the algebra $\mathcal{Z}_\hbar$ with product $\bullet_\hbar^z$ provides an Abelian deformation of the algebra $\mathcal{Z}_0$ and this is interesting *per se* because it gives an example of a non-trivial Abelian deformation, however generalized and therefore not necessarily classified by the Harrison cohomology (defined on the sub-complex of the Hochschild complex consisting of symmetric cochains [12], [13]).

3.3 Quantization of Nambu-Poisson Structure of order 3: A Solution

Let us construct the space $\mathcal{A}_0$ on which will be defined the classical Nambu structure. On this space we will have an injection of the semi-group N_1 (normalized polynomials in the variables (x^1, x^2, x^3)) which will allow a natural definition of the derivative of an element of $\mathcal{A}_0$. We shall consider a space of "Taylor series" in the variables (y^1, y^2, y^3) of translated polynomials $x \mapsto u(x+y)$ with coefficients in the algebra $\mathcal{Z}_0$ introduced in Sect. 3.2.

Let $\mathcal{E} = \mathcal{Z}_0[y^1, y^2, y^3]$ be the algebra of polynomials in the variables (y^1, y^2, y^3) with coefficients in $\mathcal{Z}_0$. Instead of the usual Taylor series

$$u(x+y) = u(x) + \sum_i y^i \partial_i u(x) + \frac{1}{2} \sum_{i,j} y^i y^j \partial_{ij} u(x) + \cdots ,$$

which we multiply by $(uv)(x+y) = u(x+y)v(x+y)$ we look at "Taylor series" in $\mathcal{E}$, for $u \in N_1$:

$$J(Z_u) = Z_u + \sum_i y^i Z_{\partial_i u} + \frac{1}{2} \sum_{i,j} y^i y^j Z_{\partial_{ij} u} + \cdots = \sum_n \frac{1}{n!} (\sum_i y^i \partial_i)^n (Z_u), \qquad (20)$$

where $\partial_i u$, $\partial_{ij} u$, etc. are the usual derivatives of $u \in N_1 \subset N$ with respect to the variables x^i, x^i and x^j, etc., $\partial_i Z_u \equiv Z_{\partial_i u}$ and, since in general the derivatives of $u \in N_1$ are in N, one has to factor out the appropriate constants in $Z_{\partial_i u}$, $Z_{\partial_{ij} u}$, etc. (i.e. $Z_{\lambda u} \equiv \lambda Z_u$, $u \in N_1$, $\lambda \in \mathbb{R}$). J defines an additive map from $\mathcal{Z}_0$ to $\mathcal{E}$ (to say that J is multiplicative is tantamount to the Leibniz property).

Let $\mathcal{A}_0$ be the sub-algebra of $\mathcal{E}$ generated by elements of the form (20). We shall denote by $\bullet$ the product in $\mathcal{A}_0$ which is naturally induced by the product in $\mathcal{E}$. In order to define the (classical) Nambu structure on $\mathcal{A}_0$, we need to make precise what is meant by the derivative of an element of $\mathcal{A}_0$. Remember that the derivative $\partial_i u(x+y)$ is again a Taylor series of the form $\partial_i u(x) + \sum_j y^j \partial_{ij} u(x) + \cdots$. We shall define thus the derivative Δ_a, $1 \le a \le 3$, of an element of the form (20) by:

$$\Delta_a(J(Z_u)) = J(Z_{\partial_a u}) = Z_{\partial_a u} + \sum_i y^i Z_{\partial_{ai} u} + \frac{1}{2} \sum_{i,j} y^i y^j Z_{\partial_{aij} u} + \cdots , \quad u \in N_1, 1 \le a \le 3. \qquad (21)$$

One can look at definition (21) of Δ_a as the restriction, to the subset of elements of the form $J(Z_u)$, of the formal derivative with respect to y^a in the ring $\mathcal{E} = \mathcal{Z}_0[y^1, y^2, y^3]$. Since $\Delta_a(J(Z_u)) = J(Z_{\partial_a u})$, we have $\Delta_a(\mathcal{A}_0) = \mathcal{A}_0$ and we get a family of maps $\Delta_a : \mathcal{A}_0 \to \mathcal{A}_0$, $1 \le a \le 3$, restriction to $\mathcal{A}_0$ of the derivations with respect to y^a, $1 \le a \le 3$, in $\mathcal{E}$. We can summarize the properties of Δ_a in the:

Lemma 1 *The maps $\Delta_a : \mathcal{A}_0 \to \mathcal{A}_0$, $1 \le a \le 3$, defined by Eq. (21) constitute a family of commuting derivations of the algebra $\mathcal{A}_0$.*

Proof. Follows directly from the fact that Δ_a, $1 \leq a \leq 3$, are the restriction to the subalgebra $\mathcal{A}_0$ of the formal derivatives on the ring $\mathcal{E} = \mathcal{Z}_0[y^1, y^2, y^3]$.

The definition of derivatives on $\mathcal{A}_0$ leads to the following natural definition of the classical Nambu bracket on the Abelian algebra $\mathcal{A}_0$:

Definition 7 *The classical Nambu bracket on $\mathcal{A}_0$ is the trilinear map taking values in $\mathcal{A}_0$ given by:*

$$(A, B, C) \mapsto [A, B, C]_\bullet \equiv \sum_{\sigma \in S_3} \epsilon(\sigma) \Delta_{\sigma_1} A \bullet \Delta_{\sigma_2} B \bullet \Delta_{\sigma_3} C \ , \quad \forall A, B, C \in \mathcal{A}_0. \tag{22}$$

Theorem 6 *The classical Nambu bracket given in Def. 7 defines a Nambu structure on $\mathcal{A}_0$.*

Proof. It follows trivially from the fact that $(\mathcal{A}_0, \bullet)$ is an Abelian algebra and from Lemma 1.

Now that we have a classical Nambu structure on $\mathcal{A}_0$, we shall construct a quantum Nambu structure by defining some Abelian deformation $(\mathcal{A}_\hbar, \bullet_\hbar)$ of $(\mathcal{A}_0, \bullet)$. The construction is based on the map α introduced in Sect. 3.1 and we shall extend the definition of the product $\bullet^z_\hbar$ defined in Sect. 3.2 to the present setting for the Nambu structure on $\mathcal{A}_0$.

Let $\mathcal{E}[\hbar]$ be the algebra of polynomials in $\hbar$ with coefficients in $\mathcal{E}$. We consider the subspace $\mathcal{A}_\hbar$ of $\mathcal{E}[\hbar]$ consisting of series $\sum_{r \geq 0} \hbar^r A_r$ for which the coefficient A_0 is in $\mathcal{A}_0$. Then we define a map $\bullet_\hbar$: $\mathcal{A}_\hbar \times \mathcal{A}_\hbar \to \mathcal{E}[\hbar]$ by extending the product $\bullet^z_\hbar$ defined by (19) (it is sufficient to define it on $\mathcal{A}_0$ since $\bullet^z_\hbar$ annihilates the non-zero powers of $\hbar$):

$$J(Z_u) \bullet_\hbar J(Z_v) = Z_u \bullet^z_\hbar Z_v + \sum_i y^i (Z_{\partial_i u} \bullet^z_\hbar Z_v + Z_u \bullet^z_\hbar Z_{\partial_i v}) + \cdots \ , \quad \forall u, v \in N_1. \tag{23}$$

Actually $\bullet_\hbar$ defines a product on $\mathcal{A}_\hbar$ and we have:

Theorem 7 *The vector space $\mathcal{A}_\hbar$ endowed with the product $\bullet_\hbar$ is an Abelian algebra which is some Abelian deformation of the Abelian algebra $(\mathcal{A}_0, \bullet)$.*

Proof. For $A = \sum_{r \geq 0} \hbar^r A_r$ and $B = \sum_{s \geq 0} \hbar^s B_s$ in $\mathcal{A}_\hbar$, we have $A \bullet_\hbar B = A_0 \bullet_\hbar B_0$ and the coefficient of $\hbar^0$ of the latter is $A_0 \bullet B_0$ which is in $\mathcal{A}_0$ since $A_0, B_0 \in \mathcal{A}_0$. This shows that $\bullet_\hbar$ is actually a product on $\mathcal{A}_\hbar$. By definition this product is Abelian. Hence $(\mathcal{A}_\hbar, \bullet_\hbar)$ is an Abelian algebra.

It is clear from the preceding that for $\hbar = 0$, we have $A \bullet_\hbar B|_{\hbar=0} = A_0 \bullet B_0$, which shows that the product $\bullet_\hbar$ is some deformation of the product $\bullet$.

The derivatives Δ_a, $0 \leq a \leq 3$, are naturally extended to $\mathcal{A}_\hbar$. Every element $A \in \mathcal{A}_\hbar$ can be written as $A = \sum_I y^I A_I$, where $I = (i_1, \ldots, i_n)$ are multi-indexes and $A_I \in \mathcal{Z}_\hbar$. Then the product $A \bullet_\hbar B$, $A, B \in \mathcal{A}_\hbar$, reads:

$$A \bullet_\hbar B = \sum_{I,J} y^I y^J A_I \bullet^z_\hbar B_I \ .$$

Since $(\mathcal{Z}_\hbar, \bullet_\hbar^z)$ is an Abelian algebra and the derivative Δ_a acts as a formal derivative with respect to y^a on the product $A \bullet_\hbar B$, the usual properties (linearity, Leibniz, Frobenius) of a derivative are still satisfied on $\mathcal{A}_\hbar$. So we can now define the quantum Nambu bracket on $\mathcal{A}_\hbar$.

Definition 8 *The quantum Nambu bracket on $\mathcal{A}_\hbar$ is the trilinear map taking values in $\mathcal{A}_\hbar$ defined by:*

$$(A, B, C) \mapsto [A, B, C]_{\bullet_\hbar} \equiv \sum_{\sigma \in S_3} \epsilon(\sigma) \Delta_{\sigma_1} A \bullet_\hbar \Delta_{\sigma_2} B \bullet_\hbar \Delta_{\sigma_3} C \ , \quad \forall A, B, C \in \mathcal{A}_\hbar. \tag{24}$$

Theorem 8 *The quantum Nambu bracket endows $\mathcal{A}_\hbar$ with a Nambu structure which is some deformation of the classical Nambu structure on $\mathcal{A}_0$*

Proof. The proof that the quantum Nambu bracket endows $\mathcal{A}_\hbar$ with a Nambu structure is similar to the one of Theorem 6. That the quantum Nambu bracket is some deformation of the classical Nambu bracket follows from Theorem 7.

3.4 Generalizations

What has been done in the previous two sections can be easily generalized to $\mathbb{R}^n$, $n \geq 2$. The only non-straightforward modification to be done appears in the evaluation map (16). One has to distinguish two cases: when n is even and when n is odd. If $n = 2p$, $p \geq 1$, then one replaces the partial Moyal product in (16) by the usual Moyal product on $\mathbb{R}^{2p}$. If $n = 2p+1$, $p \geq 1$, one uses the partial Moyal product $*_{1 \dots 2p}$ on the hyperplane defined by $x_{2p+1} = 0$ (as for the case $n = 3$, other possibilities can be considered). The other definitions and properties are directly generalized to $\mathbb{R}^n$. Note that the canonical Nambu-Poisson structure of order 2 on $\mathbb{R}^2$ is the usual Poisson structure; there our procedure gives a quantization of the Poisson bracket $\mathcal{P}$ different from Moyal, however not on $N[\hbar]$ but on $\mathcal{A}_\hbar$; this quantization will in a sense be somewhat like in field theory. The same applies to $\mathbb{R}^{2n}$ by starting with a sum of Poisson brackets on the various $\mathbb{R}^2$.

Our construction can be generalized to any orbit of the coadjoint action of a Lie algebra on its dual (the case of $\mathbb{R}^3$ corresponds to $\mathfrak{su}(2)^\star$). In that case, instead of the Moyal product appearing in the evaluation map, one can use a covariant star-product on the orbit [10].

4 Concluding Remarks

We have found a quantized version of Nambu Mechanics and we shall end this article with a few remarks concerning some related physical and mathematical points. We would like to stress that many features of the solution proposed can be of direct relevance for other quantization problems.

4.1 Sesqui-quantization

One should notice that here we quantize a linear span of polynomials which are in a way our "fields". In this scheme the irreducible polynomials play a very special rôle: they generate all the polynomials and are kind of building blocks in the quantum case. For example on $\mathbb{R}^2$ the harmonic oscillator Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ cannot be considered as the sum of the two observables p^2 and q^2; it has to be considered as an irreducible element of the algebra. The same thing is true for the anharmonic oscillator with Hamiltonian $\frac{1}{2}(p^2 + q^2 + \lambda q^4)$, $\lambda > 0$, which is not considered here as the sum of a free Hamiltonian with an interaction term. In usual Quantum Mechanics the Hermitian operator $H = P^2 + Q^2$ is the sum of two operators, but the physically measurable quantities (spectrality) related to these operators seem to ignore that the Hamiltonian is the sum of two observables. To make it precise, the spectrum of the harmonic oscillator Hamiltonian is discrete, while P^2 and Q^2 have both continuous positive spectra; hence a priori there is no way to relate these spectra. Before going further, let us mention that, as in the usual deformation quantization case, we have a natural definition of the spectrum of an observable in the Zariski Quantization. Consider the polynomial $H \in N_1$ in the variables p and q, and map it to its Taylor series $J(Z_H) \in \mathcal{A}_0$ given by (20), and build the deformed exponential function:

$$\exp_{\bullet_\hbar}\left(\frac{tH}{i\hbar}\right) = \sum_{n\geq 0} \frac{1}{n!}\left(\frac{t}{i\hbar}\right)^n J(Z_H) \bullet_\hbar \cdots \bullet_\hbar J(Z_H)\ . \tag{25}$$

In (25) let $y = 0$, then:

$$J(Z_H) \bullet_\hbar \cdots \bullet_\hbar J(Z_H)|_{y=0} = Z_H \bullet_\hbar^z \cdots \bullet_\hbar^z Z_H = \zeta(H \times_\alpha \cdots \times_\alpha H) \in \mathcal{Z}_\hbar,$$

where ζ is defined in (18). As for the star-exponential (14) we define the spectrum of H to be the support of the measure appearing in the Fourier-Dirichlet expansion of (25) with $y = 0$. For H irreducible, it is easy to see that $J(Z_H) \bullet_\hbar \cdots \bullet_\hbar J(Z_H)|_{y=0} = \zeta(H * \cdots * H)$, where $*$ is the Moyal product on $\mathbb{R}^2$ with deformation parameter $\frac{1}{2}i\hbar$. In that case we get the same spectrum as for the Moyal case. For completely reducible elements like p^2 and q^2, the exponential (25) reduces to the usual exponential, in which case the spectrum is continuous. So these three observables have the same spectra as in the usual case, and the Zariski Quantization scheme makes a distinction among them from the very beginning. Somehow this new scheme is halfway between first and second quantizations (hence the name "sesqui-quantization"): it is not quite a field theory (though a field-like formulation is possible) but shares many features with it (Fock space, irreducible polynomials seen as "1-particle" states, etc.).

4.2 Zariski Star-Products

For the Poisson case, Zariski Quantization gives a quantization which differs from the usual one in many respects. The most important one is that the quantum Poisson bracket is *not* the skew-symmetrized form of an associative product. But this quantized bracket can be seen as the "classical" part of another quantum bracket coming from an associative algebra. For $\mathbb{R}^{2n}$, consider the Zariski-Poisson bracket $\mathcal{P}_{\bullet_\hbar}$ built as indicated in Sect. 3.4; in definition (12) of the Moyal product replace $\mathcal{P}$ by $\mathcal{P}_{\bullet_\hbar}$; we get the Zariski-Moyal product $f \bullet_M g = \exp(\nu \mathcal{P}_{\bullet_\hbar})$, where ν is at first seen as a different parameter and later identified with $\frac{1}{2}i\hbar$. Due to the properties of $\bullet_\hbar$, one gets another associative deformation of the usual product. The corresponding deformed bracket will then start with the (Zariski) quantum Poisson bracket and provides a Lie algebra deformation. Then a theory of "star-products" constructed with the product $\bullet_\hbar$ can be developed in a straightforward way.

4.3 General Poisson Manifolds: An Overlook

The quantization presented here was done in an algebraic setting, the product $\bullet_\hbar$ being defined on the algebra $\mathcal{A}_\hbar$ constructed from polynomials on $\mathbb{R}^n$. One can consider extensions to an algebraic variety S. It should be possible to define a similar Abelian deformed product between polynomials on S using an embedding of S into $\mathbb{R}^n$ by polynomial Dirac constraints [7] that will induce on S a Poisson structure. Furthermore we know from Nash [17] that compact real analytic Riemannian manifolds can be analytically and isometrically embedded into some $\mathbb{R}^n$; the proof follows from his previous result on differentiable embeddings by showing that there are "arbitrarily close" analytic and differentiable manifolds. In this context, it is thus reasonable to expect that the procedure developed here can be extended (at least in the compact case) to arbitrary differentiable manifolds. Eventually, as for Nambu Mechanics, similar techniques may be applied to the quantization of not necessarily regular Poisson structures on algebraic varieties, real analytic manifolds and differential manifolds.

4.4 Cohomology

From a mathematical point of view, it would be interesting to study general Abelian deformations of $\mathcal{Z}_0$ and $\mathcal{A}_0$ and look for associated cohomology complexes. A more detailed study of the kind of "deformation" obtained here for these algebras, both as associative algebras and as Nambu bracket algebras, is certainly worthwhile. In view of Sect. 4.2, "quantum" cohomology versions of the relevant cohomologies should also be of interest.

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References

[1] Bayen, F., Flato, M.: Remarks concerning Nambu's generalized mechanics. Phys. Rev. D **11**, 3049–3053 (1975)

[2] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization: I. Deformations of Symplectic Structures. Ann. Phys. **111**, 61–110 (1978)

[3] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization: II. Physical Applications. Ann. Phys. **111**, 111–151 (1978)

[4] Bonneau, Ph., Flato, M., Gerstenhaber, M., Pinczon, G.: The Hidden Group Structure of Quantum Groups. Commun. Math. Phys. **161**, 125–156 (1994)

[5] Chatterjee, R.: Dynamical Symmetries and Nambu Mechanics. Lett. Math. Phys. **36** 117–126 (1996)

[6] Chatterjee, R., Takhtajan, L.: Aspects of Classical and Quantum Nambu Mechanics. Preprint hep-th: 9507125 (1995)

[7] Dirac, P. A. M.: Lectures on Quantum Mechanics, Belfer Graduate School of Sciences Monograph Series No. 2, Yeshiva University, New-York, 1964

[8] Dito, J.: Star-Product Approach to Quantum Field Theory: The Free Scalar Field. Lett. Math. Phys. **20**, 125–134 (1990)

[9] Flato, M., Frønsdal, C.: Unpublished (1992)

[10] Flato, M., Sternheimer, D.: Closedness of Star-Products and Cohomologies. In: Brylinsky, J. L. et al. (eds.) Lie Theory and Geometry: In Honor of Bertram Kostant. Progress in Mathematics, pp. 241–259. Boston: Birkhäuser 1994

[11] Gautheron, Ph.: Some Remarks Concerning Nambu Mechanics. To appear in Lett. Math. Phys. (1996)

[12] Gerstenhaber, M., Schack, S.: Algebraic Cohomology and Deformation Theory. In: Hazewinkel, M. and Gerstenhaber, M. (eds.) Deformation Theory of Algebras and Structures and Applications. pp. 11–264. Dordrecht: Kluwer 1988. See also: Barr, M.: Harrison Homology, Hochschild Homology, and Triples. J. Alg., **8** 314–323 (1968)

[13] Gerstenhaber, M., Schack, S.: Algebras, Bialgebras, Quantum Groups, and Algebraic Deformations. In: Gerstenhaber, M. and Stasheff, J. (eds.) Deformation Theory and Quantum Groups with Applications to Mathematical Physics. Contemporary Mathematics **134**, pp.51–92. Providence (RI): American Mathematical Society 1992.

[14] Gutt, S.: Unpublished (1992)

[15] Pankaj Sharan.: $*$-Product Representation of Path Integrals. Phys. Rev. D **20**, 414–426 (1979)

[16] Nambu, Y.: Generalized Hamiltonian Dynamics. Phys. Rev. D **7**, 2405–2412 (1973)

[17] Nash, J. F.: Analyticity of the solutions of implicit function problems with analytic data. Ann. of Math. **84**, 345–355 (1966)

[18] Sahoo, D., Valsakumar, M. C.: Nambu mechanics and its quantization. Phys. Rev. A **46**, 4410–4412 (1992)

[19] Schouten, J. A.: Tensor Analysis for Physicists. Second edition. New-York: Dover 1989.

[20] Takhtajan, L.: On Foundation of the Generalized Nambu Mechanics. Commun. Math. Phys. **160**, 295–315 (1994)

[21] Weitzenböck, R.: Invariantentheorie. Groningen: P. Noordhoff 1923.