MEL'NIKOV 'S VECTOR AND SINGULARITY ANALYSIS OF N D.O.F HAMILTONIAN SYSTEMS

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Abstract

We investigate periodically perturbed, integrable 2 degree of freedom (d.o.f.) Hamiltonian systems as a paradigm of the dynamics of 3 d.o.f. Hamiltonian systems. We derive the Mel'nikov vector, determining transversal intersections of the invariant manifolds of a saddle fixed point of the associated (time-periodic) map and investigate its connection with the breakdown of the Painlevé property of the unperturbed solutions in complex time.

1 Introduction

More than a century ago H. Poincaré and P. Painlevé introduced new concepts in order to study nonlinear systems of differential equations. Consider a diffeomorphism in $\mathbb{R}^2$ with a hyperbolic fixed point p. A point q is called a *homoclinic point* of p if q is in the intersection of the stable and unstable manifold of p. The point q is called a transversal homoclinic point of p if the intersection of the stable and unstable manifolds is transversal, i.e. the tangent spaces at q to the stable and unstable manifolds span the whole space.

Poincaré [11] observed that the existence of homoclinic points implies complexity of the dynamics of the diffeomorphism. Later G.D.Birkhoff [9] proved that every transversal homoclinic point of a two–dimensional diffeomorphism is accumulated by priodic orbits. The results by Smale [10], now called the Smale–Birkhoff theorem, extend the Birkhoff 's results both in two dimensional and to higher dimensional case s and assert that the existence of a transversal homoclinic point implies the existence of a transversal homoclinic point implies the existence of an invariant Cantor set in which the periodic orbits are dense. A criterion for the existence of an homoclinic intersection in the Poincaré map defined by the suspended flow of the perturbed system needs to be defined. Melnikov [4] gave a computable criterion for the existence of transverse homoclinic intersection in perturbed integrable planar systems by defining a function

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which gives an approximation of the splitting distance between the stable and the unstable manifolds of the perturbed system.

Painlevé and co–workers [8] found that nonlinear differential equations can be classified according to the types of singularities exhibited by the solutions in the complex plane. Painlevé was specifically interested in the simplest case where the solution has no movable critical points, the so–called *Painlevé property.*

Recently, Chow and Yamashita [1] extended the theories for the two–dimensional case to higher dimensional cases, derived the Mel'nikov vector and examined conditions for a transversal homoclinic point. Goriely [2] showed, for integrable systems under perturbations, that under suitable assumptions on the existence of homoclinic solutions, the leading order behavior of the Ψ–series (Ψ–series generalizes the Laurent series and introduce expansions in powers of $log(t-t_*)$, where t_* is a movable singularity) is directly connected to the Mel'nikov vector. A new way to compute the Mel'nikov vector in higher dimensions is based on consideration of the compatibility conditions for the formal existence of the Laurent expansion for nonintegrable systems.

The plan of the paper is as follows. In Section 2, we give the expression of the Mel'nikov vector of a n-d.o.f Hamiltonian system due to Chow and Yamashita. In Section 3, we describe the new way to compute the Melnikov vector which base on the singularity analysis. The method of residues is applied to estimate the Mel'nikov vector. In Section 4, we derive the Mel'nikov vector of a 3 d.o.f Hamiltonian system, determining transversal intersections of the invariant manifolds of a saddle fixed point of the associated time–periodic map and investigate its connection with the breakdown of the Painlevé property of the unperturbed solution in the complex time.

2 Mel'nikov 's Vector in Hamiltonian Systems

We consider the following completely integrable Hamiltonian system

$$\dot{x} = X_H(x), \ x \in \mathbb{R}^{2n} \tag{2.1}$$

where $X_H(x) = J\nabla H(X), \ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and its time periodic non–Hamiltonian and Hamiltonian perturbations respectively:

$$\dot{x} = X_H(x) + \epsilon g(t,x) \tag{2.2}$$

$$\dot{x} = X_H(x) + \epsilon X_G(t,x) \tag{2.3}$$

In this paper, we apply the Mel'nikov vector for systems (2.2) and (2.3) (see [1]) to a specific example of a periodically driven 2 d.o.f Hamiltonian of the Hénon–Heiles type. We also show explicitly how this vector is related to the singularity analysis of the solutions of (2.2), (2.3), in complex time (see [2]).

We suppose that system (2.1) has two hyperbolic critical points x_- and x_+ (not necessarily distinct) joined by an orbit $\gamma(t)$ of system (2.1): $\lim_{t\to\pm\infty}\gamma(t) = x_\pm$ Thus the linearized system of (2.1) along the orbit γ is given by

$$\dot{z} = A(t)z \tag{2.4}$$

Since $A(t) = DX_H(\gamma(t)) = JD^2H(\gamma(t))$, $A(t)$ is infinitesimally symplectic for each $t \in \mathbb{R}$, i.e. $A^*(t)J + JA(t) \equiv 0$ where $A^*(t)$ is the transpose of $A(t)$. Now lets us define the adjoint system of (2.4) by

$$\dot{\phi} = -A^*(t)\phi \tag{2.5}$$

We note that $z(t)$ is a solution of (2.4) if and only if $\phi(t) = J^{-1}z(t)$ is a solution of (2.5). Then the following is known:

Proposition 2.1 *Let $F \in C^\infty(\mathbb{R}^{2n})$. If $\{F, H\} = 0$, then $X_F(\gamma(t))$ is a solution of (2.4) and hence $\nabla F(\gamma(t))$ is a solution of (2.5).*

Now we suppose that there exist n C^∞–functions $F_1 = H$, $F_2, \ldots, F_n$ on $\mathbb{R}^{2n}$ which are in involution, and ∇F_i $i = 1, 2, \ldots, n$ are linearly independent everywhere in $\mathbb{R}^{2n} - \{x_\pm\}$. Let $F_i(\gamma(t)) = f_i \in \mathbb{R}$ $i = 1, 2, \ldots, n$, and define the set $M_f = \{x \in \mathbb{R}^{2n} - \{x_\pm\} : F_i(x) = f_i, i = 1, 2, \ldots, n\}$, an n-dimensional smooth manifold in $\mathbb{R}^{2n}$which is invariant under the flow of each Hamiltonian vector field X_{F_i}. It is easy to prove:

Proposition 2.2 *Suppose that Hamiltonian system (2.1) is completely integrable. Then $\nabla F_i(\gamma(t))$ forms a complete set of bounded solutions of the adjoint system (2.5).*

Suppose that $\{\gamma(t, \nu) : t \in \mathbb{R}, \nu \in S \subset \mathbb{R}^{k-1}\} \subset W^s(\bar{x}_+) \cap W^u(\bar{x}_-)$ where $k = dim(kerL)$ and L is defined by: $(Lz)(t) := \dot{z} - A(t)z$, S is an open subset of $\mathbb{R}^{k-1}$ and $\gamma(t, \nu)$ is a homoclinic or heteroclinic orbit connecting x_- and x_+ for each $\nu \in S$.

Now we give the special forms for the Mel'nikov vector of systems (2.2) and (2.3) according to the theory developed in [1]:

Theorem 2.1 [1] *For system (2.2), the Mel'nikov vector $M(t_0, \nu)$ is given by*

$$M_i(t_0, \nu) = \int_{-\infty}^{\infty} \nabla F_i(\gamma(t, \nu))g(t - t_0, \gamma(t, \nu))\, dt \tag{2.6}$$

and for (2.3)

$$M_i(t_0, \nu) = \int_{-\infty}^{\infty} \{F_i, G(t - t_0, \cdot)\}(\gamma(t, \nu))\, dt \tag{2.7}$$

where $\{\,\cdot, \cdot\,\}$ denotes the usual Poisson bracket, $i = 1, 2, \ldots, n$.

Remark 2.1 *The dimension of the Mel'nikov vector for a Hamiltonian system is the same as the number of degrees of freedom of that system, since it equals the number of bounded solutions of the variational problems (2.4) and (2.5).*

Proposition 2.3 [1] *Suppose that there exist t_0 and ν_0 such that*

$$\mathbf{M}(t_0, \nu_0) = 0 \tag{2.8}$$

$$rank(\frac{\partial}{\partial t_0}\mathbf{M}(t_0, \nu_0)\frac{\partial}{\partial \nu}\mathbf{M}(t_0, \nu_0)) = m \tag{2.9}$$

Then the stable and unstable manifolds of system (2.2) or (2.3) have a point of transversal intersection.

3 Logarithmic Branch Points and the Mel'nikov Theory

We consider systems of n first order ODEs in $\mathbb{R}^n$:

$$\dot{x} = f(x) + \epsilon g(x, t) \tag{3.1}$$

where $x \in \mathbb{R}^n$, $f \in C^2$, $g \in C \times C$. The system obtained by setting $\epsilon = 0$ is referred to as the unperturbed system.

A new way to compute the Mel'nikov vector is based on the consideration of the compatibility conditions for the formal existence of the Laurent expansion for nonintegrable systems [6]. We need some assumptions about the unperturbed system:

- **A1:** *Painlevé integrability.* The unperturbed system is completely integrable and has the Painlevé property. The Painlevé property is defined as follows:the general solution has no movable critical points in the complex plane of the independent variable t. We assume that around a singularity $t = t_*$, there exists an expansion of the solution in Laurent series:

$$x = a(t - t_*)^p \sum_{j=0}^{\infty} \alpha_j (t - t_*)^j \tag{3.2}$$

with a, $\alpha_j \in \mathbb{C}^n$ $\forall j$; $\alpha_0 = 1$; $p \in Z^n$. The Laurent series includes $(n-1)$ arbitrary coefficients and at least one of the components of the vector p is negative. The first step of the Painlevé test consists in finding all the truncations of $\hat{f}$ of the vector field

$$\dot{x} = f(x) = \hat{f}(x) + \bar{f}(x) \tag{3.3}$$

such that the leading behavior $x = a(t - t_*)^p$, $a \in \mathbb{C}_0^n$ is an exact solution of the truncated system

$$\dot{x} = \hat{f}(x) \tag{3.4}$$

where $p \in Z^n$ with at least one negative component. It is also required thet $\hat{f}(x)$ is not dominant, that is, at the singularity:

$$\bar{f}(a(t - t_*)^p) \sim \bar{a}(t - t_*)^{p + \bar{p} - 1} \tag{3.5}$$

as $t \longrightarrow t_*$ with $p \in \mathbb{N}_0^n$. Each balance (a, p) defined a different expansion.

The second step is the computation of the *resonances*. Each balance defines a new set of resonances. These resonances are the indices j of the coefficients α_j in the Laurent series (3.2) at which arbitrary constants first appear. These resonances are given by the eigenvalues of the matrix R:

$$R = D\hat{f}(a) - diag(p) \tag{3.6}$$

where $D\hat{f}(a)$ is the Jacobian matrix evaluated in a. The resonances are labeled r_i, $i = 1, \cdots, n$ with $r_1 = -1$. The necessary condition for the existence of the Laurent series is that all resonances are integer ($r_i \in Z$).

The third and the last step of the Painlevé test consists in checking that the arbitrariness of the coefficient α_r for the full system (3.3) does not introduce incompatible constraints on the coefficients α_j ($j \leq r$). The coefficients are computed by inserting the full Laurent series (3.2) in the original system (3.1) and by determining explicitly the recursion relation for the coefficients α_j:

$$R\alpha_j = j\alpha_j - P_j(\alpha_1, \cdots, \alpha_{n-1}) \tag{3.7}$$

where P_j is polynomial in its variables. If $j = r$ is an eigenvalue of R, then in order to solve the linear system for α_r, we obtain by the Fredholm alternative the *compatibility conditions*:

$$C_r = \bar{\beta}^{(r)} P_r \quad \forall r \geq 0$$

where $\bar{\beta}^{(r)}$ is the eigenvector of $R^\top$ of eigenvalue r.

- **A2:** *Hyperbolicity of the fixed point.* The unperturbed system has a hyperbolic fixed point at $x_0 = 0$ and the Jacobian matrix of the unperturbed vector field can be diagonalized.
- **A3:** *Periodicity of the homoclinic solution in Imt (Imaginary time axis).* This property will eventually lead to the estimation of the Mel'nikov integral through knowledge of the local expansion in the complex t plane.
- **A4:** *Analytic continuation of the solution of the unpeturbed system* in the strip S:

$$S = \{t \in \mathbb{C}; 0 \leq Im(t) \leq w\}$$

  Moreover, the homoclinic solution has a finite number N of poles in the strip S. This assumption is necessary for the estimation of the Mel'nikov integral by the method of residues. The number N of poles in the strip S should be finite in order to have finite sums of residues for the computation of the Mel'nikov vector.
- **A5:** *Periodicity of the perturbation in* $t \in \mathbb{R}$.
- **A6:** *Non dominant perturbations* The assumption about the behavior of the perturbation at the singularity is also central since it allows one to consider the perturbation $g(x,t)$ as a perturbation centering at higher powers of $(t - t_*)$ in the Laurent expansion.

$$g(x(t), t) \sim \delta\tau^{p+p_g-1}, \ as\ \tau = t - t_* \to 0$$

  where $p_g \in \mathbf{N}_0^n$, $\delta \in \mathbb{C}_0^n$ and t_* is the singularity of the homoclinic solution.

The Mel'nikov vector can be computed according to the following algorithmic procedure [2]:

- **Step 1:** Show the existence of a homoclinic loop which satisfies **A1**. Find w, the period in the imaginary direction and the position of the singularities t_* for the homoclinic solution.

- **Step 2:** Obtain the formal Laurent expansion (about t_*) with a logarithmic correction for the perturbed system up to the first positive resonance r:

$$x = \sum_{i=0}^{r} \alpha_i \tau^{p+i} + b_0 \tau^{p+r} log\tau + \mathcal{O}(\tau^{p+r+1}, log\tau^2) \tag{3.8}$$

  and define

$$c_{r_i} = \frac{\partial C_{r_i}}{\partial \epsilon}|_{\epsilon=0} \tag{3.9}$$

  where C_r is the inner product of the vector b_0 with the eigenvector $\bar{\beta}_r$ of $R^{\perp}$ of eigenvalue r, (the bounded solutions of the adjoint system) and R is the resonant matrix.

- **Step 3:** The Mel'nikov vector is then given by:

$$M_i(t_0) = M_i^{(0)} + 2\pi i \sum_{k=-\infty, k\neq 0}^{+\infty} \frac{e^{ikt_0}}{1 - e^{-kw}} \sum_{t_* \in S} c_{r_i}^{(k)}(t_*) e^{ikt_*} \tag{3.10}$$

  where $c_{r_i} = \sum_{k=-\infty}^{\infty} c_{r_i}^{(k)}(t) e^{ikt_0}$, t_0 the parameters of the Mel'nikov vector and

$$\mathbf{M}^{(0)} = \frac{-2\pi}{w} \sum_{t_* \in S} Res[(\bar{Q} g^{(0)}(x_{\pm}(t)))(t - t_*)] \tag{3.11}$$

  $\bar{Q}$ is the bounded solutions of the adjoint system and $x_{\pm}(t)$ is the homoclinic solution of the unperturbed system and $g^{(0)}$ the leading order term of g near $t = t_*$.

4 Example: A Perturbed Hénon–Heiles Hamiltonian

Consider the integrable Hénon–Heiles Hamiltonian :

$$H(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{v^2}{2}(x^2 + y^2) - yx^2 - \frac{y^3}{3} \tag{4.1}$$

where v is a parameter. Employing the symplectic transformation [7]

$$x = \frac{1}{\sqrt{2}}(q_1 + q_2), \quad y = \frac{1}{\sqrt{2}}(q_1 - q_2), \quad \dot{x} = \frac{1}{\sqrt{2}}(p_1 + p_2), \quad \dot{y} = \frac{1}{\sqrt{2}}(p_1 - p_2)$$

(4.1) becomes a separable Hamiltonian in q_1, q_2. Translating now the saddle points at the origin and adding coupling, dissipation and periodic driving terms we have the following perturbed system of differential equations

$$\ddot{q}_1 = v^2 q_1 + \sqrt{2}{q_1}^2 + \epsilon[-\mu_1(q_1 - q_2) + \mu_2 cos\omega t - \mu_3 p_1] \tag{4.2}$$

$$\ddot{q}_2 = v^2 q_2 - \sqrt{2}{q_2}^2 + \epsilon[\mu_1(q_1 - q_2) + \mu_2 cos\omega t - \mu_3 p_2] \tag{4.3}$$

For $\epsilon = 0$ system (4.2), (4.3) has two integrals:

$$F_1(q_1, p_1) = \frac{1}{2}p_1^2 - \frac{v^2}{2}q_1^2 - \frac{\sqrt{2}}{3}q_1^3, \quad F_2(q_2, p_2) = \frac{1}{2}p_2^2 - \frac{v^2}{2}q_2^2 + \frac{\sqrt{2}}{3}q_2^3$$

The unperturbed system of (4.2), (4.3) has an homoclinic orbit $\gamma(t, 0)$ to the origin

$$\gamma(t, 0) = (q(t), -q(t), \dot{q}(t), -\dot{q}(t)) \quad q(t) = \frac{-3v^2}{2\sqrt{2}} sech^2(\frac{vt}{2}) \tag{4.4}$$

The solutions on the homoclinic manifold can be expressed as:

$$\gamma(t, \theta) = (q(t - \theta_1), -q(t - \theta_2), \dot{q}(t - \theta_1), -\dot{q}(t - \theta_2)) \tag{4.5}$$

where $\theta = (\theta_1, \theta_2)$ and $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ and $q_i = q(t - \theta_i)$, $i = 1, 2$. Bounded solutions of the adjoint system along $\gamma(t, \theta)$ are given by $\nabla F_i(\gamma(t, \theta))$, (F_1, F_2) and are linearly independent.

The components of the Mel'nikov vector are

$$M_i(t_0, \theta, \mu) = \int_{-\infty}^{\infty} \nabla F_i(\gamma(t, \theta)) g(t - t_0, \gamma(t, \theta))\, dt \quad i = 1, 2 \tag{4.6}$$

while the linearized system (4.2), (4.3) and its adjoint have exponential dichotomy on $(-\infty, t_0]$ and $[t_0, \infty)$. To compute the integrals (4.6) we will make use of the method of residues, following the path $\gamma = \bigcup_{i=1}^{4} \gamma_i$ in the complex plane, given by

$$\gamma_1 = [-R, R] \ \gamma_2 = [R, R + \pi i] \ \gamma_3 = [R + \pi i, -R + \pi i] \ \gamma_4 = [-R + \pi i, -R]$$

in the limit $R \longrightarrow \infty$.

Finally (see Appendix) we obtain

$$M_k(t_0, \theta, \mu) = (-1)^{k+1} \mu_1 J + (-1)^k C \mu_2 s_k - \frac{3}{5} \mu_3 v^5, \qquad k = 1, 2 \tag{4.7}$$

where $s_k = sin\omega(t_0 - \theta_k)$, $C := 3\sqrt{2}\pi\omega^2 csch(\frac{\pi\omega}{v})$

$$J = \frac{9v^4}{4(sinh\frac{(\theta_1 - \theta_2)v}{2})^4} (3sinhv(\theta_1 - \theta_2) - v(\theta_1 - \theta_2)(coshv(\theta_1 - \theta_2) + 2)) \tag{4.8}$$

Since, in this example, the stable and unstable manifolds of the unperturbed system coincide and constitute a two dimensional manifold in $\mathbb{R}^4$,

$$rank(\frac{\partial}{\partial t_0} \mathbf{M}(t_0, \theta_0, \mu_0) \frac{\partial}{\partial \theta} \mathbf{M}(t_0, \theta_0, \mu_0)) = 2$$

implies a transversal intersection (see Proposition 1.3) at $t_0, \mu_0, \theta_0 = (\theta_1^0, \theta_2^0)$.

Finally, turning to the singularity analysis of the above system, we note that (4.2), (4.3) for $\epsilon = 0$ is completely integrable and has the Painlevé property possessing Laurent series with dominant behavior at the singularities $p = (-2, -3)$ and resonances $r = (-1, 6)$. The periodicity of the homoclinic solution in the imaginary direction is given by $w = \frac{2\pi}{v}$. The only singularity of the homoclinic solution in its fundamental domain is located on the imaginary axis: $t_* = \frac{i\pi}{v}$.

The formal Laurent expansion with a logarithmic correction for the perturbed system (4.2), (4.3) up to the first positive resonance $r = 6$, yields:

$$q_1(\tau) = \frac{3\sqrt{2}}{\tau^2} - \frac{v^2}{2\sqrt{2}} + \frac{v^4\tau^2}{40\sqrt{2}} + \sqrt{2}\alpha\tau^4 + A\tau^4 log(\tau) \tag{4.9}$$

$$q_2(\tau) = \frac{-3\sqrt{2}}{\tau^2} + \frac{v^2}{2\sqrt{2}} - \frac{v^4\tau^2}{40\sqrt{2}} + \sqrt{2}\beta\tau^4 + B\tau^4 log(\tau) \tag{4.10}$$

where in the first 3 terms on the rhs we have neglected $\mathcal{O}(\epsilon)$ terms, α, β are free constants, while the coefficients A, B can be computed by inserting (4.9), (4.10) in (4.2), (4.3). We compute the adjoint eigenvector $\bar{\beta}_r$ and the quantities c_{r_i} from the compatibility conditions. The residues c_{r_1}, c_{r_2} of the Mel'nikov vector around the singularity $t_* = \frac{i\pi}{v}$ are given by:

$$c_{r_k}(t_0, \theta, \mu) = (-1)^{k+1} 3\sqrt{2}\mu_2\omega^2 cosi\omega(\frac{\pi}{\nu} - \theta_k) \qquad k = 1, 2 \tag{4.11}$$

According to Eq.(3.11) the computation of $\mathbf{M}^{(0)}$ reduces to the residues computations of $(\bar{Q}g^{(0)}(x_\pm(t)))(t - t_*)$. We have

$$M_1^{(0)} = -\frac{3}{5}\mu_3 v^5 + \mu_1 J, \qquad M_2^{(0)} = -\frac{3}{5}\mu_3 v^5 - \mu_1 J \tag{4.12}$$

and (see Appendix)

$$J = -\frac{2\pi}{w}\sum Res[-\mu_1 q_1(t-\theta_2)\dot{q}_1(t-\theta_1)(t-t_*)] = -\frac{9v^4}{8}\frac{16e^{v(\theta_1-\theta_2)}}{(e^{v(\theta_1-\theta_2)}-1)^4}$$

$$\times(3-3e^{2v(\theta_1-\theta_2)}+v(\theta_1-\theta_2)+4v(\theta_1-\theta_2)e^{v(\theta_1-\theta_2)}+v(\theta_1-\theta_2)e^{2v(\theta_1-\theta_2)})$$

$$= \frac{9v^4}{4(sinh\frac{(\theta_1-\theta_2)v}{2})^4}(3sinhv(\theta_1-\theta_2)-v(\theta_1-\theta_2)(coshv(\theta_1-\theta_2)+2))$$

cf. (4.8). According to Eq.(3.10)–(3.11) we have for M_1

$$\begin{aligned} M_1(t_0,\theta,\mu) &= 2\pi i 3\sqrt{2}\mu_2\omega^2(\frac{e^{i\omega t_0}e^{i(i\omega(\frac{\pi}{\nu}-\theta_1))}}{1-e^{-\frac{2\pi\omega}{\nu}}}+\frac{e^{-i\omega t_0}e^{-i(i\omega(\frac{\pi}{\nu}-\theta_1))}}{1-e^{\frac{2\pi\omega}{\nu}}})+M_1^{(0)} \\ &= -C\mu_2 s_1 + M_1^{(0)} \end{aligned}$$

and

$$\begin{aligned} M_2(t_0,\theta,\mu) &= -2\pi i 3\sqrt{2}\mu_2\omega^2(\frac{e^{i\omega t_0}e^{i(i\omega(\frac{\pi}{\nu}-\theta_2))}}{1-e^{-\frac{2\pi\omega}{\nu}}}+\frac{e^{-i\omega t_0}e^{-i(i\omega(\frac{\pi}{\nu}-\theta_2))}}{1-e^{\frac{2\pi\omega}{\nu}}})+M_2^{(0)} \\ &= C\mu_2 s_2 + M_2^{(0)} \end{aligned}$$

which are identical with (4.7) cf. (4.12). Thus, a direct comparison between the results of singularity analysis and that of Chow and Yamashita can be made, concerning the form of the Mel'nikov vector and its dependence on the various parameters of the system.

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Appendix

The goal of this Appendix is to outline the proof of equation (4.7). First let us note that

$$\nabla F_i(\gamma(t,\theta)) = \left(\frac{\partial F_i}{\partial q_1}, \frac{\partial F_i}{\partial q_2}, \frac{\partial F_i}{\partial p_1}, \frac{\partial F_i}{\partial p_2}\right)(\gamma(t,\theta)) \quad i = 1,2 \tag{A.1}$$

and

$$g(t-t_0, \gamma(t,\theta)) = \left(0, 0, -\mu_1(q_1 - q_2) + \mu_2 cos\omega t - \mu_3 p_1, \mu_1(q_1 - q_2) + \mu_2 cos\omega t - \mu_3 p_2\right)^{\perp}$$

Equation (4.6) after some algebra becomes:

$$M_k(t_0, \theta, \mu) = (-1)^{k+1}\mu_1 J + (-1)^k C \mu_2 s_k - \frac{3}{5}\mu_3 v^5, \qquad k = 1,2 \tag{A.2}$$

where $s_k = sin\omega(t_0 - \theta_k)$, $C := 3\sqrt{2}\pi\omega^2 csch(\frac{\pi\omega}{v})$ and J is given by the following integral:

$$J = \frac{9v^4}{4}\int_{-\infty}^{\infty} sech^2\frac{v}{2}(t-\theta_1) tanh\frac{v}{2}(t-\theta_1) sech^2\frac{v}{2}(t-\theta_2)\, d(\frac{tv}{2})$$

$$= \frac{9v^4}{4}\int_{-\infty}^{\infty} sech^2(x-A) tanh(x-A) sech^2(x-B)\, dx = \frac{9v^4}{4} I(A,B) \tag{A.3}$$

with $A = \frac{v\theta_1}{2}$, $B = \frac{v\theta_2}{2}$

$$I(A,B) = \int_{-\infty}^{\infty} sech^2(x) tanh(x) sech^2(x+C)\, dx$$

$C \equiv A - B$. The hyperbolic functions satisfy the following expressions:

$$sech^2(x+C) + tanh^2(x+C) = 1$$

$$tanh^2(x+C) = \frac{tanhx + tanhC}{1 + tanhC tanhx}$$

Finally we obtain

$$J = \frac{9v^4}{4(sinh\frac{(\theta_1-\theta_2)v}{2})^4}\left(3sinhv(\theta_1-\theta_2) - v(\theta_1-\theta_2)(coshv(\theta_1-\theta_2)+2)\right) \tag{A.4}$$

To compute

$$\sum Res[-\mu_1 q_1(t-\theta_2)\dot{q}_1(t-\theta_1)(t-t_*)]$$

we consider the following analytic function f:

$$f(z) = -\frac{9v^5}{8}\frac{sinh\frac{v}{2}(z-\theta_1)}{cosh^3\frac{v}{2}(z-\theta_1)cosh^2\frac{v}{2}(z-\theta_2)}(z-t_*)$$

which has $z_1 = \frac{i\pi}{v} + \theta_1$, $z_2 = \frac{i\pi}{v} + \theta_2$ poles third and second order, respectively.