NONINVERTIBILITY AND "SEMI-" ANALOGS OF (SUPER) MANIFOLDS, FIBER BUNDLES AND HOMOTOPIES

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Abstract

Supersymmetry contains initially noninvertible objects, but it is common to deal with the invertible ones only, factorizing former in some extent. We propose to reconsider this ansatz and try to redefine such fundamental notions as supermanifolds, fiber bundles and homotopies using some weakening invertibility conditions. The prefix semi-reflects the fact that the underlying morphisms form corresponding semigroups consisting of a known group part and a new ideal non-invertible part. We found that the absence of invertibility gives us the generalization of the cocycle conditions for transition functions of supermanifolds and fiber bundles in a natural way, which can lead to construction of noninvertible analogs of Čech cocycles. We define semi-homotopies, which can be noninvertible and describe mappings into the semi-supermanifolds introduced.

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1 Introduction

The noninvertible extension of the notion of a supermanifold seems intuitively natural in connection with the well known theory of Hopf spaces which in general have no conditions on existence of inverses (in homotopy sense) [61]. Several guesses were made in the past concerning inner noninvertibility inherent in the supermanifold theory, e.g. "...a general SRS needs not have a body" [12], "...there may be no inverse projection (body map [57]) at all" [46], or "...a body may not even exist in the most extreme examples" [8]. In particular, while investigating noninvertibility properties of superconformal symmetry [19, 17] it was assumed [15, 18] the possible existence of supersymmetric objects analogous to super Riemann surfaces, but without body, and shown preliminary how to construct them [16]. The noninvertibility in supermanifold theory actually arises from odd nilpotent elements and zero divisors of underlying Grassmann-Banach algebras (see [28, 49, 60] for nontrivial examples). In the infinite dimensional case there exist (topologically) quasinilpotent odd elements which are not really nilpotent [50], and, moreover, in some superalgebras one can construct pure soul elements which are not nilpotent even topologically [48]. We should also mention the possibility of definition of a supermanifold without the notion of topological space [42]. Other supermanifold problems with odd directions (and therefore connected with noninvertibility in either event) are described in [1, 9, 10, 27, 34].

It is well known that the standard patching definition of a supermanifold [54, 57, 66] does not essentially differ from one of an ordinary manifold [29, 32]. Mostly the word "super-" makes them different (as however many other definitions too [68]). Nothing in principle was changed from the time of Gauss who used the concept of manifold in carthography of earth's surface. And now while constructing super generalizations of manifolds one thinks about this picture only and imagine intuitively the earth only, in spite of the fact that in supercase there are much more abstract possibilities to construct such and similar objects. The interpretation of the nature of anticommuting variables can be also dramatically changed in future (see e.g. [69]). Therefore, it is logical from the very beginning to weaken (but as fine as possible) the initial invertibility restrictions. After consistent building of an object which has a more complicated and nontrivial structure in this general case, the requirement of invertibility can lead to several nonequivalent projection including new ones invisible previously.

2 Standard patch definition of supermanifold

Let us consider the standard patch definition [54, 53, 66] of a supermanifold M. We cover it by a collection of superdomains U_α such that $M = \sum U_\alpha$. Then we take in every domain some functions (*coordinate maps*) $\varphi_\alpha : U_\alpha \to D^{n|m} \subset \mathbb{R}^{n|m}$, where $\mathbb{R}^{n|m}$ is a superspace in which our super "ball" lives and $D^{n|m}$ is an open domain in $\mathbb{R}^{n|m}$. Next we call the pair $\{U_\alpha, \varphi_\alpha\}$ a *local chart* and claim that the union of charts $\cup \{U_\alpha, \varphi_\alpha\}$ is an atlas of a *supermanifold*.

Next we introduce gluing *transition functions* as follows. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ and

$$\begin{array}{c} \varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{n|m}, \\ \varphi_\beta : U_\beta \to V_\beta \subset \mathbb{R}^{n|m}. \end{array} \tag{1}$$

Then the above morphisms are restricted to $\varphi_\alpha : U_{\alpha\beta} \to V_{\alpha\beta} = V_\alpha \cap \varphi_\alpha (U_{\alpha\beta})$ and $\varphi_\beta : U_{\alpha\beta} \to V_{\beta\alpha} = V_\beta \cap \varphi_\beta (U_{\alpha\beta})$. The maps $\Phi_{\alpha\beta} : V_{\beta\alpha} \to V_{\alpha\beta}$ which are called to make the following diagram

$$\begin{tikzcd} U_{\alpha\beta} \arrow[r, "\varphi_\beta"] \arrow[dr, "\varphi_\alpha"'] & V_{\beta\alpha} \arrow[d, "\Phi_{\alpha\beta}"] \\ & V_{\alpha\beta} \end{tikzcd} \tag{2}$$

to be commuted are transition functions of a manifold in a given atlas. Here we stress, first, that $U_{\alpha\beta} \subset M$ and $V_{\alpha\beta}, V_{\beta\alpha} \subset \mathbb{R}^{n|m}$. Second, from (2) one usually concludes that

$$\Phi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} \tag{3}$$

The transition (super) functions $\Phi_{\alpha\beta}$ give us possibility to restore the whole (super) manifold from individual charts and coordinate maps. Indeed they contain all information about the (super) manifold. They may belong to different functional classes, which gives possibility to specify more narrow classes of manifolds and supermanifolds, for instance (super) smooth, analytic, Lipschitz and others [29, 45]. Mostly the prefix "super-" only distinguishes the patch definitions of a manifold and supermanifold (which gives us possibility to write it in brackets) and the properties of $\Phi_{\alpha\beta}$ [9, 57, 68]. Here we do not discuss them in detail and try to put minimum restrictions on $\Phi_{\alpha\beta}$, concentrating our attention on their abstract properties and generalizations following from them.

Additionally, from (3) it follows that transition functions satisfy the cocycle conditions

$$\Phi_{\alpha\beta}^{-1} = \Phi_{\beta\alpha} \tag{4}$$

on $U_\alpha \cap U_\beta$ and

$$\Phi_{\alpha\beta}\Phi_{\beta\gamma}\Phi_{\gamma\alpha} = 1_{\alpha\alpha} \tag{5}$$

on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$, where $1_{\alpha\alpha} \stackrel{def}{=} \mathrm{id}\ (U_\alpha)$.

Usually it is implied that all the maps φ_α are homeomorphisms, and they can be described by one-to-one invertible continuous (super) smooth functions (i.e. one wants "to jump" in both directions between any two intersecting domains $U_\alpha \cap U_\beta \neq \emptyset$). First, it is reasonable (from 19th century carthography viewpoint) not to distinguish between U_α and $D^{n|m}$, i.e. locally supermanifolds are as the whole superspace $\mathbb{R}^{n|m}$. For earth it is right, but for superspace – questionable. The matter is not only in more rich fiber bundle [6, 7, 21] and sheaf [30, 33, 35] structures due to consideration of all constructions over underlying Grassmann (or more general [48, 60, 66, 25]) algebra. The problem lies in another abstract level of the constructions, if the invertibility conditions are weakened in some extent.

3 Noninvertible extension of supermanifold

Earlier there was the following common prescription: one had ready objects (e.g. real manifolds which can be investigated almost visually), and then using various methods and guesses one found restrictions on transition functions. Notwithstanding, noninvertible functions were simply excluded (saying magic words "factorizing by nilpotents we again derive the well-known result") from consideration, because of desire to be in the nearest analogy with intuitively clear and understandable nonsupersymmetric case.

Here we go in opposite direction: we know that in supermathematics noninvertible variables and functions *do exist*. Which objects could be constructed by means of them? What gives "factorizing by non-nilpotents", i.e. consideration of non-group features of theory? How changes the general abstract meaning of the most important notions, e.g. manifolds and fiber bundles? We now try to leave aside inner structure of noninvertible objects analogous to supermanifolds and concentrate our attention on there general abstract definitions. Further we think to work them out in more detail.

3.1 Problem of division in superanalysis

We begin with old division problem of superanalysis [4, 55]. Can we solve the simplest equation $ax = b$ with even noninvertible a , i.e. when $a^n = 0$? Here $a, b, x \in \Lambda_0$, where Λ is a commutative Banach $\mathbb{Z}_2$-graded superalgebra over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$, or $\mathbb{C}$) with a decomposition into the direct sum: $\Lambda = \Lambda_0 \oplus \Lambda_1$ (the elements a from Λ_0 and Λ_1 are homogeneous and have the fixed even and odd parity defined as $|a| \stackrel{def}{=} \{i \in \{0,1\} = \mathbb{Z}_2 | \, a \in \Lambda_i\}$). The answer is "yes": we should expand both sides in series of generators of Λ and equal the coefficients before the same power of generators. The simplest example is $\alpha x = 2\alpha\beta\gamma$ which has the solution $x = 2\beta\gamma + \operatorname{Ann}\alpha$. This means that the extended division over noninvertible variable *can be defined*, i.e. the solution of $ax = b$ when $a^n = 0$ *exists*, but it can be found by indirect way (without notorious factorization by nilpotents). But how to solve $\alpha x = 2\beta$ or $\alpha\beta x = 2\gamma\rho$? For instance, one can use the technique of the nilpotent semigroup theory [22, 31, 62]. Another possibility to solve such equations is to exploit the abstract semigroup methods [36] and define solutions as equivalent classes of variables or functions. We only mark now that the solution of $ax = b$ exists when b is also nilpotent $b^k = 0$. The same conclusion can be applied to functional equations containing superfunctions: they *can* be solved, and so they *have to be considered on a par* with invertible ones. We stress that among ordinary functions there exist noninvertible ones as well [38, 39], but the kind of noninvertibility considered here is very special: it appears only due to the existence of nilpotents in underlying superalgebra [28, 50, 60]. Here we do not consider concrete equations and methods of their solving, we only use the fact of their existence to reformulate some definitions and extend well-known notions.

3.2 Definition of a semi-supermanifold

Now we formulate a patch definition of an object analogous to supermanifold, i.e. try to weaken demand of invertibility of coordinate maps (1). Let us consider a (super) space M covered by open sets U_α as $M = \sum U_\alpha$. Let the maps $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{n|m}$ are not all homeomorphisms, i.e. among them there are noninvertible maps.

Definition 1 *A chart is a pair* $(U_\alpha,\varphi_\alpha)$ *,where* φ_α *is invertible. A semi-chart is a pair* $\left(\tilde{U}_\alpha,\tilde{\varphi}_\alpha\right)$ *, where* $\tilde{\varphi}_\alpha$ *is noninvertible.*

Definition 2 *A semi-atlas is a union of charts and semi-charts.*

Definition 3 *A semi-supermanifold is a superspace* M *represented as a semi-atlas.*

How to define an analog of transition functions? We should consider the same diagram (2), but we cannot use (3) through noninvertibility of some of φ_α's.

Definition 4 *Gluing semi-transition functions of a semi-supermanifold are defined by the equations*

$$\Phi_{\alpha\beta} \circ \varphi_\beta = \varphi_\alpha \tag{6}$$

$$\Phi_{\beta\alpha} \circ \varphi_\alpha = \varphi_\beta \tag{7}$$

We stress that to determine $\Phi_{\alpha\beta}$ the equation (6) cannot be solved by (3). Instead we should find artificial methods of its solving, e.g. as in previous subsection, expanding in superalgebra generator series, or using abstract semigroup methods [36] which consider the solutions of noninvertible equations as equivalence classes.

The functions $\Phi_{\beta\alpha}$ are now determined not from (4) in which the left hand side is not well defined, but from the commutative diagram

$$\begin{tikzcd} U_{\alpha\beta} \arrow[r, "\varphi_\beta"] \arrow[dr, "\varphi_\alpha"'] & V_{\beta\alpha} \\ & V_{\alpha\beta} \arrow[u, "\Phi_{\beta\alpha}"'] \end{tikzcd} \tag{8}$$

and the equation (7) following from it. They are also can be noninvertible and therefore the cocycle conditions should be modified not to use only invertible functions.

Remark. Even in the standard case the cocycle conditions (5) for supermanifolds are not automatically satisfied when (3) holds, and therefore they should be imposed by hand [44].

So instead of (4) and (5) we have

Conjecture 5 *The semi-transition functions of a semi-supermanifold satisfy the following relations*

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta} \tag{9}$$

on $U_\alpha \cap U_\beta$ *overlaps and*

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta}, \tag{10}$$

$$\Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} = \Phi_{\beta\gamma}, \tag{11}$$

$$\Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} = \Phi_{\gamma\alpha} \tag{12}$$

on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ *and*

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\rho} \circ \Phi_{\rho\alpha} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta}, \tag{13}$$

$$\Phi_{\beta\gamma} \circ \Phi_{\gamma\rho} \circ \Phi_{\rho\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} = \Phi_{\beta\gamma}, \tag{14}$$

$$\Phi_{\gamma\rho} \circ \Phi_{\rho\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\rho} = \Phi_{\gamma\rho}, \tag{15}$$

$$\Phi_{\rho\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\rho} \circ \Phi_{\rho\alpha} = \Phi_{\rho\alpha} \tag{16}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\rho$ *.*

Here the first relation (9) is called to generalize the first cocycle condition (4), while other relations correspond (5). We call (9)–(16) *tower relations*.

Definition 6 *A semi-supermanifold is called* reflexive *if, in addition to (9)–(16), the semi-transition functions satisfy to the reflexivity conditions*

$$\Phi_{\beta\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = \Phi_{\beta\alpha} \tag{17}$$

on $U_\alpha \cap U_\beta$ *overlaps and*

$$\Phi_{\alpha\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} \circ \Phi_{\alpha\gamma} = \Phi_{\alpha\gamma}, \tag{18}$$

$$\Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} \circ \Phi_{\alpha\gamma} \circ \Phi_{\gamma\beta} = \Phi_{\gamma\beta}, \tag{19}$$

$$\Phi_{\beta\alpha} \circ \Phi_{\alpha\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} = \Phi_{\beta\alpha} \tag{20}$$

on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ *and*

$$\Phi_{\alpha\rho} \circ \Phi_{\rho\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} \circ \Phi_{\alpha\rho} = \Phi_{\alpha\rho}, \tag{21}$$

$$\Phi_{\rho\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} \circ \Phi_{\alpha\rho} \circ \Phi_{\rho\gamma} = \Phi_{\rho\gamma}, \tag{22}$$

$$\Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} \circ \Phi_{\alpha\rho} \circ \Phi_{\rho\gamma} \circ \Phi_{\gamma\beta} = \Phi_{\gamma\beta}, \tag{23}$$

$$\Phi_{\beta\alpha} \circ \Phi_{\alpha\rho} \circ \Phi_{\rho\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} = \Phi_{\beta\alpha} \tag{24}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\rho$ *.*

Remark. One could think that the reflexivity conditions differ from (9)–(16) by index permutations only. This is true. But the functions $\Phi_{\alpha\beta}$ entering in (9)–(16) and in (17)–(24) *are the same*, therefore the latter are independent equations imposed on $\Phi_{\alpha\beta}$.

Assertion 7 *The relations analogous to (9)–(24), but having two or more multipliers in the right hand side are consequences of them.*

Proof. For instance, consider

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma}. \tag{25}$$

Multiplying from the right on $\Phi_{\alpha\beta}$ we derive

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\alpha\beta}. \tag{26}$$

Then using (9) we obtain

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta} \tag{27}$$

which coincides with (10). □

Remark. In any actions with noninvertible functions $\Phi_{\alpha\beta}$ we are not allowed to cancellate, because the semigroup of $\Phi_{\alpha\beta}$'s is a semigroup without cancellation, and we are forced to exploit the corresponding methods [13, 23, 40, 65].

Corollary 8 *The relations (9)–(24) satisfy identically in the standard invertible case, i.e. when the conditions (3),(4) and (5) hold valid.*

So that, $\Phi_{\alpha\beta}$ satisfying the relations (9)–(24) can be viewed as some noninvertible generalization of the transition functions as cocycles in the Čech cohomology of coverings [37, 63].

3.3 Orientation of semi-manifolds

It is well known that orientation of ordinary manifolds is determined by the Jacobian sign of transition functions $\Phi_{\alpha\beta}$ written in terms of local coordinates on $U_\alpha \cap U_\beta$ overlaps [29, 58]. Since this sign belong to $\mathbb{Z}_2$, there exist two orientations on U_α. Two overlapping charts are *consistently oriented* (or *orientation preserving*) if $\Phi_{\alpha\beta}$ has positive Jacobian, and a manifold is *orientable* if it can be covered by such charts, thus there are two kinds of manifolds: orientable and nonorientable. In supersymmetric case the role of Jacobian plays Berezinian [4] which has a "sign" belonging to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ [67, 64], and so there are four orientations on U_α and five corresponding kinds of supermanifold orientability [59].

Definition 9 *In case a nonvanishing Berezinian of $\Phi_{\alpha\beta}$ is nilpotent (and so has no definite sign in the previous sense) there exists additional nilpotent orientation on U_α of a semi-supermanifold.*

A degree of nilpotency of Berezinian allows us to classify semi-supermanifolds having nilpotent orientability.

3.4 Obstructedness and semi-manifolds

The semi-supermanifolds defined above belong to a class of so called obstructed semi-manifolds. Let us rewrite (3), (4) and (5) as the following (infinite) series

$$n = 1: \quad \Phi_{\alpha\alpha} = 1_{\alpha\alpha}, \tag{28}$$

$$n = 2: \quad \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = 1_{\alpha\alpha}, \tag{29}$$

$$n = 3: \quad \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} = 1_{\alpha\alpha}, \tag{30}$$

$$n = 4: \quad \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\delta} \circ \Phi_{\delta\alpha} = 1_{\alpha\alpha} \tag{31}$$

$$\cdots \quad \cdots$$

Definition 10 *A semi-manifold is called obstructed if some of the cocycle conditions (28)–(31) are broken.*

Remark. The introduced notion of obstructed manifold should not be mixed with the notion of obstruction for ordinary manifolds [2] and supermanifolds [4] or obstruction to extensions [37] and in the theory of characteristic classes [41, 26].

It can happen that starting from some $n = n_m$ all higher cocycle conditions hold valid.

Definition 11 *Obstructedness degree of a semi-manifold is a maximal n_m for which the cocycle conditions (28)–(31) are broken. If all of them hold valid, then $n_m \stackrel{def}{=} 0$.*

Corollary 12 *Ordinary manifolds (with invertible transition functions) have vanishing obstructedness, and the obstructedness degree for them is equal to zero, i.e. $n_m = 0$.*

Conjecture 13 *The obstructed semi-manifolds may have non-vanishing ordinary obstruction which can be calculated extending the standard methods [4] to the non-invertible case.*

Therefore, using the obstructedness degree n_m, we have possibility to classify semi-manifolds properly.

3.5 Tower identity semigroup

Let us consider a series of the selfmaps $\mathrm{e}_{\alpha\alpha}^{(n)} : U_\alpha \to U_\alpha$ of a semi-manifold defined as

$$\mathrm{e}_{\alpha\alpha}^{(1)} = \Phi_{\alpha\alpha}, \tag{32}$$

$$\mathrm{e}_{\alpha\alpha}^{(2)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha}, \tag{33}$$

$$\mathrm{e}_{\alpha\alpha}^{(3)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha}, \tag{34}$$

$$\mathrm{e}_{\alpha\alpha}^{(4)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\delta} \circ \Phi_{\delta\alpha} \tag{35}$$

$$\ldots \quad \ldots$$

We will call $\mathrm{e}_{\alpha\alpha}^{(n)}$'s *tower identities*. From (28)–(31) it follows

Assertion 14 *For an ordinary supermanifolds all tower identities coincide with the usual identity map*

$$\mathrm{e}_{\alpha\alpha}^{(n)} = 1_{\alpha\alpha}. \tag{36}$$

The obstructedness degree can be treated as a maximal $n = n_m$ for which tower identities differ from the identity, i.e. (36) is broken. So the tower identities give the measure of distinction of a semi-supermanifold from an ordinary supermanifold. Being an important inner characteristic the tower identities play a deep fundamental role in description of semi-supermanifolds. Therefore, we will study some of their properties in detail.

Proposition 15 *The tower identities are units for the semi-transition functions*

$$\mathrm{e}_{\alpha\alpha}^{(n)} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta}, \tag{37}$$

$$\Phi_{\alpha\beta} \circ \mathrm{e}_{\beta\beta}^{(n)} = \Phi_{\alpha\beta}. \tag{38}$$

Proof. It follows directly from the tower relations (9)–(16) and the definition (9)–(16). □

Proposition 16 *The tower identities are idempotents*

$$\mathrm{e}_{\alpha\alpha}^{(n)} \circ \mathrm{e}_{\alpha\alpha}^{(n)} = \mathrm{e}_{\alpha\alpha}^{(n)}. \tag{39}$$

Proof. We prove the statement for $n = 2$ and for other n it can be proved by induction. We write (39) as

$$\mathrm{e}_{\alpha\alpha}^{(2)} \circ \mathrm{e}_{\alpha\alpha}^{(2)} = \mathrm{e}_{\alpha\alpha}^{(2)} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = \left(\mathrm{e}_{\alpha\alpha}^{(2)} \circ \Phi_{\alpha\beta}\right) \circ \Phi_{\beta\alpha}.$$

Then using (37) we obtain

$$\left(\mathrm{e}_{\alpha\alpha}^{(2)} \circ \Phi_{\alpha\beta}\right) \circ \Phi_{\beta\alpha} = \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = \mathrm{e}_{\alpha\alpha}^{(2)}.$$

□

The functional nonsupersymmetric equations of the above kind were studied in [3].

Definition 17 *Conjugate tower identities correspond to the same partition of the semi-supermanifold and consists of the semi-transition functions taken in opposite order*

$$\tilde{\mathrm{e}}_{\alpha\alpha}^{(1)} = \mathrm{e}_{\alpha\alpha}^{(1)}, \tag{40}$$

$$\tilde{\mathrm{e}}_{\alpha\alpha}^{(2)} = \mathrm{e}_{\alpha\alpha}^{(2)}, \tag{41}$$

$$\tilde{\mathrm{e}}_{\alpha\alpha}^{(3)} = \Phi_{\alpha\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha}, \tag{42}$$

$$\tilde{\mathrm{e}}_{\alpha\alpha}^{(4)} = \Phi_{\alpha\delta} \circ \Phi_{\delta\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} \tag{43}$$

$$\cdots \quad \cdots$$

The conjugate tower identities play the role of tower identities, but for reflexivity conditions (17)–(24). By analogy with (37)–(38) we have

Proposition 18 *The conjugate tower identities are reflexive units for the semi-transition functions*

$$\tilde{\mathrm{e}}^{(n)}_{\beta\beta} \circ \Phi_{\beta\alpha} = \Phi_{\beta\alpha}, \tag{44}$$

$$\Phi_{\beta\alpha} \circ \tilde{\mathrm{e}}^{(n)}_{\alpha\alpha} = \Phi_{\beta\alpha}. \tag{45}$$

Assertion 19 *For the same partition the conjugate tower identities annihilate the tower identities in the following sense*

$$\mathrm{e}^{(n)}_{\alpha\alpha} \circ \tilde{\mathrm{e}}^{(n)}_{\alpha\alpha} = \mathrm{e}^{(2)}_{\alpha\alpha}. \tag{46}$$

Proof. Let us consider the case $n = 3$. Using the definitions we derive

$$\mathrm{e}^{(3)}_{\alpha\alpha} \circ \tilde{\mathrm{e}}^{(3)}_{\alpha\alpha} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha}$$

$$= \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ (\Phi_{\gamma\alpha} \circ \Phi_{\alpha\gamma}) \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \mathrm{e}^{(2)}_{\gamma\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha}$$

$$= \Phi_{\alpha\beta} \circ (\Phi_{\beta\gamma} \circ \Phi_{\gamma\beta}) \circ \Phi_{\beta\alpha} = \Phi_{\alpha\beta} \circ \mathrm{e}^{(2)}_{\beta\beta} \circ \Phi_{\beta\alpha} = \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = \mathrm{e}^{(2)}_{\alpha\alpha}.$$

For other n the statement can be proved by induction. □

Definition 20 *A semi-supermanifold is nice if the tower identities do not depend on a given partition.*

The multiplication of the tower identities of a nice semi-supermanifold can be defined as follows

$$\mathrm{e}^{(n)}_{\alpha\alpha} \circ \mathrm{e}^{(m)}_{\alpha\alpha} = \mathrm{e}^{(n+m)}_{\alpha\alpha}. \tag{47}$$

Assertion 21 *The multiplication (47) is associative.*

Therefore, we are able to give

Definition 22 *The tower identities of a nice semi-supermanifold form a tower semigroup under the multiplication (47).*

So we obtained the quantitative description of inner noninvertibility properties of semi-supermanifolds. The introduced tower semigroup plays the same role for semi-supermanifolds as the fundamental group for ordinary manifolds [20, 37, 63].

3.6 Semicommutative diagrams and n-regularity

The above constructions have the general importance for any set of noninvertible mappings.

The extension of $n = 2$ cocycle given by (9) can be viewed as some analogy with regular elements in semigroups [11, 51] or generalized inverses in matrix theory [47, 56], category theory [14] and theory of generalized inverses of morphisms [43].

The relations (10)–(16) and with other n can be considered as noninvertible analogue of regularity for higher cocycles. Therefore, by analogy with (9)–(16) it is natural to formulate the general

Definition 23 *An noninvertible mapping $\Phi_{\alpha\beta}$ is n-regular, if it satisfies the following conditions*

$$\overbrace{\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \ldots \circ \Phi_{\rho\alpha} \circ \Phi_{\alpha\beta}}^{n} = \Phi_{\alpha\beta} \tag{48}$$

$$+ perm$$

on overlaps $\overbrace{U_\alpha \cap U_\beta \cap \ldots \cap U_\rho}^{n}$.

The formula (9) describes 3-regular mappings, the relations (10)–(12) correspond to 4-regular ones, and (13)–(16) give 5-regular mappings.

Remark. The 3-regularity coincides with the ordinary regularity.

Another definition of n-regularity can be given by the formulas (37)–(38).

The higher regularity conditions change dramatically the general diagram technique of morphisms, when we turn to noninvertible ones. Indeed, the commutativity of invertible morphism diagrams is based on the relations (28)–(31), i.e. on the fact that the tower identities are ordinary identities (36). When morphisms are noninvertible (a semi-supermanifold has a nonvanishing obstructedness), we cannot "return to the same point", because $\mathrm{e}_{\alpha\alpha}^{(n)} \neq 1_{\alpha\alpha}$, and we have to consider "nonclosed" diagrams due to the fact that the relation $\mathrm{e}_{\alpha\alpha}^{(n)} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta}$ is noncancellative now.

Summarizing the above statements we propose the following intuitively consistent changing of the standard diagram technique as applied to noninvertible morphisms. In every case we get a new arrow which corresponds to the additional multiplier in (37). Thus, for $n = 2$ we obtain

	Invertible		Noninvertible

$$n{=}2 \qquad \begin{tikzcd} {} \arrow[r, shift left, "\Phi_{\alpha\beta}"] & {} \arrow[l, shift left, "\Phi_{\beta\alpha}"] \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} {} \arrow[r, shift left=2, "\Phi_{\alpha\beta}"] \arrow[r] & {} \arrow[l, shift left=2, "\Phi_{\beta\alpha}"] \end{tikzcd}$$

which describes the transition from (29) to (9) and presents the ordinary regularity condition for morphisms [14, 43]. The most intriguing semicommutative diagram is the triangle one

$$n{=}3 \qquad \begin{tikzcd} {} \arrow[r, "\Phi_{\alpha\beta}"] & {} \arrow[d, "\Phi_{\beta\gamma}"] \\ & {} \arrow[ul, "\Phi_{\gamma\alpha}"] \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} {} \arrow[r, shift left, "\Phi_{\alpha\beta}"] \arrow[r, shift right] & {} \arrow[d, "\Phi_{\beta\gamma}"] \\ & {} \arrow[ul, "\Phi_{\gamma\alpha}"] \end{tikzcd} \quad + \text{ perm}$$

which generalizes the cocycle condition (5). By analogy one can write higher order diagrams.

4 Noninvertibility and semi-bundles

A similar approach can be applied to the noninvertible extension of fiber bundles, while defining them globally in terms of open coverings and transition functions.

Following the standard definitions [41, 45, 58] and weakening invertibility we now construct new objects analogous to fiber bundles.

4.1 Definition of semi-bundles

Let E and M be a *total* (*bundle*) superspace and *base* semi-supermanifold respectively, and $\pi : E \to M$ be a *semi-projection map* which is not necessarily invertible (but can be smooth). Denote by F_b the set of points of E that are mapped to $b \in M$ (a *pre-image* of b), i.e. the *semi-fiber over* b is $F_b \stackrel{def}{=} \{x \in E \,|\, \pi(x) = b\}$. Then, $F = \cup F_b$ is a *semi-fiber.*

Definition 24 *A semi-bundle is* $\mathcal{L} \stackrel{def}{=} (E, M, F, \pi)$.

A *section* $s : M \to F$ of the fiber bundle (E, M, F, π) is usually defined by $\pi(s(b)) = b$ which in the form $\pi \circ s = 1_M$ is very similar to (4) and (29) and holds valid for invertible maps π and s only. Therefore, a very few ordinary nontrivial fiber bundles admit corresponding sections [41].

Thus, using analogy with (9), we come to the following

Definition 25 *A semi-section of the semi-bundle $\mathcal{L} = (E, M, F, \pi)$ is defined by*

$$\pi \circ s \circ \pi = \pi. \tag{49}$$

A reflexive semi-section satisfies to the additional condition

$$s \circ \pi \circ s = s. \tag{50}$$

Let $\tilde{\pi} : M \times F \to M$ is the canonical semi-projection on the first factor $\tilde{\pi}(b, f) = b, \;\; f \in F$, then $\tilde{\pi}$ gives rise to a *product fiber bundle*. If $\lambda : E \to M \times F$ is a morphism (called a *trivialization*), then $\tilde{\pi} \circ \lambda = \pi$, and the semi-bundle $\mathcal{L} = (E, M, F, \pi)$ is *trivial*. If there exists a continuous map $\eta : M \to F$, then the semi-bundle $(M \times F, M, F, \tilde{\pi})$ admits the section $s : M \to M \times F$ given by $s(b) = (b, \eta(b))$.

Let $E_\alpha \stackrel{def}{=} \{x \in E \,|\, \pi_\alpha(x) = b, \; b \in U_\alpha \subset M\}$ (here we do not use the standard notion $\pi^{-1}(U_\alpha)$ for E_α intentionally, because now π_α is allowed to be noninvertible), where $\pi_\alpha : E_\alpha \to U_\alpha$ is a restriction, i.e. $\pi_\alpha \stackrel{def}{=} \pi \mid_{U_\alpha}$. Then the semi-bundle $\mathcal{L} = (E, M, F, \pi)$, is *locally trivial*, if $\forall b \in M \, \exists U_\alpha \ni b$ such that there exists the trivializing morphisms $\lambda_\alpha : E_\alpha \to U_\alpha \times F$ satisfying $\tilde{\pi}_\alpha \circ \lambda_\alpha = \pi_\alpha$. That is, the diagram

$$\begin{tikzcd} E_\alpha \arrow[r, "\lambda_\alpha"] \arrow[dr, "\pi_\alpha"'] & U_\alpha \times F \arrow[d, "\tilde{\pi}_\alpha"] \\ & U_\alpha \end{tikzcd} \tag{51}$$

commutes.

Definition 26 *A semi-section of a locally trivial semi-bundle $\mathcal{L}$ is given by the maps $s_\alpha : U_\alpha \to E$ which satisfy the compatibility conditions*

$$\lambda_\alpha \circ s_\alpha \mid_b = \lambda_\beta \circ s_\beta \mid_b, \;\; b \in U_\alpha \cap U_\beta. \tag{52}$$

Now let $\{U_\alpha, \lambda_\alpha\}$ be a trivializing covering of π such that $\cup U_\alpha = M$ and $U_\alpha \cap U_\beta \neq \emptyset \Rightarrow E_\alpha \cap E_\beta \neq \emptyset$. Then we demand the trivializing morphisms λ_α to be agree, which means that the diagrams

$$\begin{tikzcd} E_\alpha \cap E_\beta \arrow[r, "\lambda_\beta"] \arrow[dr, "\lambda_\alpha"'] & U_\alpha \cap U_\beta \times F \arrow[d, "\Lambda_{\alpha\beta}"] \\ & U_\alpha \cap U_\beta \times F \end{tikzcd} \tag{53}$$

and

$$\begin{tikzcd} E_\alpha \cap E_\beta \arrow[r, "\lambda_\beta"] \arrow[dr, "\lambda_\alpha"'] & U_\alpha \cap U_\beta \times F \\ & U_\alpha \cap U_\beta \times F \arrow[u, "\Lambda_{\beta\alpha}"'] \end{tikzcd} \tag{54}$$

should commute, where $\Lambda_{\alpha\beta}$ and $\Lambda_{\beta\alpha}$ are maps acting along a semi-fiber F.

Definition 27 *Gluing semi-transition functions of a locally trivial semi-bundle $\mathcal{L} = (E, M, F, \pi)$ are defined by the equations*

$$\Lambda_{\alpha\beta} \circ \lambda_\beta = \lambda_\alpha, \tag{55}$$

$$\Lambda_{\beta\alpha} \circ \lambda_\alpha = \lambda_\beta. \tag{56}$$

Conjecture 28 *The semi-transition functions of a semi-bundle $\mathcal{L}$ satisfy the following relations*

$$\Lambda_{\alpha\beta} \circ \Lambda_{\beta\alpha} \circ \Lambda_{\alpha\beta} = \Lambda_{\alpha\beta} \tag{57}$$

on $U_\alpha \cap U_\beta$ overlaps and

$$\Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\alpha} \circ \Lambda_{\alpha\beta} = \Lambda_{\alpha\beta}, \tag{58}$$

$$\Lambda_{\beta\gamma} \circ \Lambda_{\gamma\alpha} \circ \Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} = \Lambda_{\beta\gamma}, \tag{59}$$

$$\Lambda_{\gamma\alpha} \circ \Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\alpha} = \Lambda_{\gamma\alpha} \tag{60}$$

on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ and

$$\Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \Lambda_{\rho\alpha} \circ \Lambda_{\alpha\beta} = \Lambda_{\alpha\beta}, \tag{61}$$

$$\Lambda_{\beta\gamma}\circ\Lambda_{\gamma\rho}\circ\Lambda_{\rho\alpha}\circ\Lambda_{\alpha\beta}\circ\Lambda_{\beta\gamma}=\Lambda_{\beta\gamma}, \tag{62}$$

$$\Lambda_{\gamma\rho}\circ\Lambda_{\rho\alpha}\circ\Lambda_{\alpha\beta}\circ\Lambda_{\beta\gamma}\circ\Lambda_{\gamma\rho}=\Lambda_{\gamma\rho}, \tag{63}$$

$$\Lambda_{\rho\alpha}\circ\Lambda_{\alpha\beta}\circ\Lambda_{\beta\gamma}\circ\Lambda_{\gamma\rho}\circ\Lambda_{\rho\alpha}=\Lambda_{\rho\alpha} \tag{64}$$

on $U_\alpha\cap U_\beta\cap U_\gamma\cap U_\rho$.

Definition 29 *A semi-bundle* $\mathcal{L}$ *is called reflexive if, in addition to (57)-(64), the semi-transition functions satisfy to the reflexivity conditions*

$$\Lambda_{\beta\alpha}\circ\Lambda_{\alpha\beta}\circ\Lambda_{\beta\alpha}=\Lambda_{\beta\alpha} \tag{65}$$

on $U_\alpha\cap U_\beta$ *overlaps and*

$$\Lambda_{\alpha\gamma}\circ\Lambda_{\gamma\beta}\circ\Lambda_{\beta\alpha}\circ\Lambda_{\alpha\gamma}=\Lambda_{\alpha\gamma}, \tag{66}$$

$$\Lambda_{\gamma\beta}\circ\Lambda_{\beta\alpha}\circ\Lambda_{\alpha\gamma}\circ\Lambda_{\gamma\beta}=\Lambda_{\gamma\beta}, \tag{67}$$

$$\Lambda_{\beta\alpha}\circ\Lambda_{\alpha\gamma}\circ\Lambda_{\gamma\beta}\circ\Lambda_{\beta\alpha}=\Lambda_{\beta\alpha} \tag{68}$$

on triple overlaps $U_\alpha\cap U_\beta\cap U_\gamma$ *and*

$$\Lambda_{\alpha\rho}\circ\Lambda_{\rho\gamma}\circ\Lambda_{\gamma\beta}\circ\Lambda_{\beta\alpha}\circ\Lambda_{\alpha\rho}=\Lambda_{\alpha\rho}, \tag{69}$$

$$\Lambda_{\rho\gamma}\circ\Lambda_{\gamma\beta}\circ\Lambda_{\beta\alpha}\circ\Lambda_{\alpha\rho}\circ\Lambda_{\rho\gamma}=\Lambda_{\rho\gamma}, \tag{70}$$

$$\Lambda_{\gamma\beta}\circ\Lambda_{\beta\alpha}\circ\Lambda_{\alpha\rho}\circ\Lambda_{\rho\gamma}\circ\Lambda_{\gamma\beta}=\Lambda_{\gamma\beta}, \tag{71}$$

$$\Lambda_{\beta\alpha}\circ\Lambda_{\alpha\rho}\circ\Lambda_{\rho\gamma}\circ\Lambda_{\gamma\beta}\circ\Lambda_{\beta\alpha}=\Lambda_{\beta\alpha} \tag{72}$$

on $U_\alpha\cap U_\beta\cap U_\gamma\cap U_\rho$.

For a fixed $b\in U_\alpha\cap U_\beta$ gluing transition function $\Lambda_{\alpha\beta}$ describe morphisms of a semi-fiber F to itself by the condition

$$\Lambda_{\alpha\beta}:(b,f)\to(b,L_{\alpha\beta}f)\,, \tag{73}$$

where $L_{\alpha\beta}:U_\alpha\cap U_\beta\to F$ and $f\in F$. The functions $L_{\alpha\beta}$ satisfy to the generalized cocycle conditions similar to (57)–(72).

Remark. The sections and transition functions of a (super) fiber bundle are noninvertible even in the standard definitions [24, 32, 64]. But this noninvertibility has different nature as compare with our assumptions. The transition functions are implied to be homeomorphisms and sections should be in 1-1 correspondence with maps from the base to the fiber [52, 55]. Our definitions (9-24) and (49-72) extend them, allowing to include in consideration properly noninvertible superfunctions as well.

4.2 Morphisms of semi-bundles

Let $\mathcal{L} = (E, M, F, \pi)$ and $\mathcal{L}' = (E', M', F', \pi')$ be two semi-bundles.

Definition 30 *A semi-bundle morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ consists of two morphisms $f = (f_E, f_M)$, where $f_E : E \to E'$ and $f_M : M \to M'$,satisfying $f_M \circ \pi = \pi' \circ f_E$ such that the diagram*

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f_M"] & M' \end{tikzcd} \tag{74}$$

is commutative.

Let $E_b = \{x \in E \,|\, \pi(x) = b,\ b \in U \subset M\}$, then $f_E(E_b) \subset E'_{f_M(b)}$ for each b, and so the semi-fiber over $b \in M$ is carried into the semi-fiber over $f(b) \in M'$ by f_E being a fiber morphism. If a semi-bundle has a section, f_E acts as follows $s(b) \to s'(f_M(b))$.

In most applications of fiber bundles the morphism f_M is identity, and $f_0 = (f_E, \mathrm{id})$ is called *B-morphism* [24]. Nevertheless, in case of semi-bundles an opposite extreme situation can take place, when f_M is a noninvertible morphism.

For each fixed $b \in M$ there exist trivializing maps $\lambda : E_b \to U \times F$ and $\lambda' : E_{f_M(b)} \to U' \times F'$, $f_M(U) \subset U'$ which lead to a map of semi-fibers h_b determined by the commutative diagram

$$\begin{tikzcd} E_b \arrow[r, "f_E(b)"] \arrow[d, "\lambda"'] & E'_{f_M(b)} \arrow[d, "\lambda'"] \\ U \times F \arrow[r, "h_b"] & U' \times F' \end{tikzcd} \tag{75}$$

To describe a semi-bundle morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ locally we choose open coverings $M = \cup U_\alpha$ and $M' = \cup U'_{\alpha'}$ together with trivializations λ_α and $\lambda'_{\alpha'}$ (see (51)). Then the connection between semi-transition functions $\Lambda_{\alpha\beta}$

and $\Lambda'_{\alpha'\beta'}$ (55)–(56) of two semi-bundles $\mathcal{L}$ and $\mathcal{L}'$ can be found from the commutative diagram

$$\begin{tikzcd}
U_{\alpha\beta} \times F \arrow[r, "\Lambda_{\alpha\beta}"] \arrow[d, "h_\alpha"'] & U_{\alpha\beta} \times F \arrow[d, "h_\beta"] \\
U'_{\alpha'\beta'} \times F' \arrow[r, "\Lambda'_{\alpha'\beta'}"] & U'_{\alpha'\beta'} \times F'
\end{tikzcd} \tag{76}$$

where morphisms h_α are defined by the diagram

$$\begin{tikzcd}
E \arrow[r, "f_E"] \arrow[d, "\lambda_\alpha"'] & E' \arrow[d, "\lambda'_{\alpha'}"] \\
U_\alpha \times F \arrow[r, "h_\alpha"] & U'_{\alpha'} \times F'
\end{tikzcd} \tag{77}$$

¿From (76) we have the relation between semi-transition functions

$$h_\alpha \circ \Lambda_{\alpha\beta} = \Lambda'_{\alpha'\beta'} \circ h_\beta \tag{78}$$

which holds valid for noninvertible h_α as well, while in the invertible case [24, 32] the equation (78) is solved with respect to $\Lambda'_{\alpha'\beta'}$, as follows $\Lambda'_{\alpha'\beta'} = h_\alpha \circ \Lambda_{\alpha\beta} \circ h_\beta^{-1}$ (it can be considered as an equivalence of cocycles). However, in general (78) is a system of superequations which should be solved by the standard [4] or extended [5] methods of superanalysis.

Let M admits two trivializing coverings $\{U_\alpha, \lambda_\alpha\}$ and $\{U'_{\alpha'}, \lambda'_{\alpha'}\}$. In general they are not connected and semi-transition functions $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha'\beta'}$ are independent. However, if M is the base superspace for two semi-bundles $\mathcal{L}$ and $\mathcal{L}'$ which are connected by a B-morphism $\mathcal{L} \overset{f_\circ}{\to} \mathcal{L}'$, then $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha'\beta'}$ should agree properly.

Proposition 31 *The semi-transition functions $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha'\beta'}$ agree if there exist additional maps $\tilde{\Lambda}_{\alpha'\beta} : U'_{\alpha'} \cap U_\beta$ and $\tilde{\Lambda}_{\alpha\beta'} : U_\alpha \cap U'_{\beta'}$ connected between themselves by the relations*

$$\tilde{\Lambda}_{\alpha'\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} = \tilde{\Lambda}_{\alpha'\beta} \tag{79}$$

on $U'_{\alpha'} \cap U_\beta$ *and*

$$\tilde{\Lambda}_{\alpha\beta'} \circ \tilde{\Lambda}_{\beta'\alpha} \circ \tilde{\Lambda}_{\alpha\beta'} = \tilde{\Lambda}_{\alpha\beta'} \tag{80}$$

on $U_\alpha \cap U'_{\beta'}$ *overlaps.*

Then the agreement conditions for $\Lambda_{\alpha\beta}$ *and* $\Lambda'_{\alpha'\beta'}$ *are*

$$\tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} = \tilde{\Lambda}_{\alpha'\beta}, \tag{81}$$

$$\Lambda_{\beta\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} = \Lambda_{\beta\gamma}, \tag{82}$$

$$\tilde{\Lambda}_{\gamma\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} = \tilde{\Lambda}_{\gamma\alpha'} \tag{83}$$

on triple overlaps $U'_{\alpha'} \cap U_\beta \cap U_\gamma$ *and*

$$\Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} \circ \Lambda'_{\alpha'\beta'} = \Lambda'_{\alpha'\beta'}, \tag{84}$$

$$\tilde{\Lambda}_{\beta'\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} = \tilde{\Lambda}_{\beta'\gamma}, \tag{85}$$

$$\tilde{\Lambda}_{\gamma\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} = \tilde{\Lambda}_{\gamma\alpha'} \tag{86}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U_\gamma$ *overlaps. Then*

$$\tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} = \tilde{\Lambda}_{\alpha'\beta}, \tag{87}$$

$$\Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} = \Lambda_{\beta\gamma}, \tag{88}$$

$$\Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} = \Lambda_{\gamma\rho}, \tag{89}$$

$$\tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} = \tilde{\Lambda}_{\rho\alpha'} \tag{90}$$

on $U'_{\alpha'} \cap U_\beta \cap U_\gamma \cap U_\rho$ *and*

$$\Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha\beta'} = \Lambda'_{\alpha'\beta'}, \tag{91}$$

$$\tilde{\Lambda}_{\beta'\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} = \tilde{\Lambda}_{\beta'\gamma}, \tag{92}$$

$$\Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \Lambda_{\gamma\rho} = \Lambda_{\gamma\rho}, \tag{93}$$

$$\tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} = \tilde{\Lambda}_{\rho\alpha'} \tag{94}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U_\gamma \cap U_\rho$ *and*

$$\Lambda'_{\alpha'\beta'} \circ \Lambda'_{\beta'\gamma'} \circ \tilde{\Lambda}_{\gamma'\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} = \Lambda'_{\alpha'\beta'}, \tag{95}$$

$$\Lambda'_{\beta'\gamma'} \circ \tilde{\Lambda}_{\gamma'\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} = \tilde{\Lambda}_{\beta'\gamma}, \tag{96}$$

$$\tilde{\Lambda}_{\gamma'\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \Lambda'_{\beta'\gamma'} \circ \tilde{\Lambda}_{\gamma'\rho} = \tilde{\Lambda}_{\gamma'\rho}, \tag{97}$$

$$\tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \Lambda'_{\beta'\gamma'} \circ \tilde{\Lambda}_{\gamma'\rho} \circ \tilde{\Lambda}_{\rho\alpha'} = \tilde{\Lambda}_{\rho\alpha'} \tag{98}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U'_{\gamma'} \cap U_\rho$.

Proof. Construct a sum of trivializing coverings $\{U_\alpha, \lambda_\alpha\}$ and $\{U'_{\alpha'}, \lambda'_{\alpha'}\}$ and then use (57)–(64). $\square$

Proposition 32 *The semi-transition functions $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha'\beta'}$ reflexively agree if there exist additional reflexive maps $\tilde{\Lambda}_{\alpha'\beta} : U'_{\alpha'} \cap U_\beta$ and $\tilde{\Lambda}_{\alpha\beta'} : U_\alpha \cap U'_{\beta'}$ connected between themselves (in addition to (79)–(80)) by the reflexive relations*

$$\tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \tilde{\Lambda}_{\beta\alpha'} = \tilde{\Lambda}_{\beta\alpha'} \tag{99}$$

on $U'_{\alpha'} \cap U_\beta$ and

$$\tilde{\Lambda}_{\beta'\alpha} \circ \tilde{\Lambda}_{\alpha\beta'} \circ \tilde{\Lambda}_{\beta'\alpha} = \tilde{\Lambda}_{\beta'\alpha} \tag{100}$$

on $U_\alpha \cap U'_{\beta'}$ overlaps. The reflexive semi-transition functions $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha'\beta'}$ should satisfy (in addition to (81)–(98)) the following reflexivity agreement relations

$$\tilde{\Lambda}_{\alpha'\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} = \tilde{\Lambda}_{\alpha'\gamma}, \tag{101}$$

$$\Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} \circ \Lambda_{\gamma\beta} = \Lambda_{\gamma\beta}, \tag{102}$$

$$\tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} = \tilde{\Lambda}_{\beta\alpha'} \tag{103}$$

on $U'_{\alpha'} \cap U_\beta \cap U_\gamma$ and

$$\tilde{\Lambda}_{\alpha'\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} = \tilde{\Lambda}_{\alpha'\gamma}, \tag{104}$$

$$\tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} = \tilde{\Lambda}_{\gamma\beta'}, \tag{105}$$

$$\Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} \circ \Lambda'_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} = \Lambda'_{\beta'\alpha'} \tag{106}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U_\gamma$ overlaps. Then

$$\tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} = \tilde{\Lambda}_{\alpha'\rho}, \tag{107}$$

$$\Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} = \Lambda_{\rho\gamma}, \tag{108}$$

$$\Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} = \Lambda_{\gamma\beta}, \tag{109}$$

$$\tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} = \tilde{\Lambda}_{\beta\alpha'} \tag{110}$$

on $U'_{\alpha'} \cap U_\beta \cap U_\gamma \cap U_\rho$ and

$$\tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} = \tilde{\Lambda}_{\alpha'\rho}, \tag{111}$$

$$\Lambda_{\rho\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} = \Lambda_{\rho\gamma}, \tag{112}$$

$$\tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} = \tilde{\Lambda}_{\gamma\beta'}, \tag{113}$$

$$\Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} = \Lambda'_{\beta'\alpha'} \tag{114}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U_{\gamma} \cap U_{\rho}$ *and*

$$\tilde{\Lambda}_{\alpha'\rho} \circ \tilde{\Lambda}_{\rho\gamma'} \circ \Lambda'_{\gamma'\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} = \tilde{\Lambda}_{\alpha'\rho}, \tag{115}$$

$$\tilde{\Lambda}_{\rho\gamma'} \circ \Lambda'_{\gamma'\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \tilde{\Lambda}_{\rho\gamma'} = \tilde{\Lambda}_{\rho\gamma'}, \tag{116}$$

$$\Lambda'_{\gamma'\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \tilde{\Lambda}_{\rho\gamma'} \circ \Lambda'_{\gamma'\beta'} = \Lambda'_{\gamma'\beta'}, \tag{117}$$

$$\Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \tilde{\Lambda}_{\rho\gamma'} \circ \Lambda'_{\gamma'\beta'} \circ \Lambda'_{\beta'\alpha'} = \Lambda'_{\beta'\alpha'} \tag{118}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U'_{\gamma'} \cap U_{\rho}$.

Analogously we can define and study a principal and associated semi-bundles with a structure semigroup, but this is a subject of a separate paper which will appear elsewhere.

5 Noninvertibility and semi-homotopies

Here we briefly dwell on some possibilities to extend directly the notion of homotopy to noninvertible continuous mappings.

A *homotopy* [20, 37] is a continuous mapping between two maps $f : X \to Y$ and $g : X \to Y$ in the space $C(X, Y)$ of maps $X \to Y$ such that $\gamma_{t=0}(x) = f(x)$, $\gamma_{t=1}(x) = g(x)$. Such maps are called *homotopic*. In other words [63] a homotopy from X to Y is a continuous function $\Gamma : X \times I \to Y$ where $I = [0, 1]$ is a unit interval. For a fixed $t \in I$ one has *stages* $\gamma_t : X \to Y$ defined by $\gamma_t(x) = \Gamma(x, t)$. The relation of homotopy divides $C(X, Y)$ into a set of equivalent classes $\pi(X, Y)$ called *homotopy classes* which are a set of connected components of $C(X, Y)$. Therefore, $\pi(*, Y)$ ($*$ is a point) the homotopy classes correspond to connected components of Y. If $C(X, Y)$ is connected, then the homotopy between $f(x)$ and $g(x)$ can be chosen as their average, i.e.

$$\gamma_t(x) = tf(x) + (1 - t) g(x). \tag{119}$$

Two maps f and g are *homotopically equivalent* if $f \circ g$ and $g \circ f$ are homotopic to the identity.

Now let X and Y are supermanifolds in some of the definitions [9, 12, 57] or semi-supermanifold in our above definition. Then there exist a possibility to extend the notion of homotopy. The idea is in extending the definition of the parameter t. In the standard case the unit interval $I = [0, 1]$ was taken for simplicity, because any two intervals on a real axis are homeomorphic, and so they are topologically equal.

In supercase the situation is totally different. Let X and Y are defined over Λ , a commutative $\mathbb{Z}_2$-graded superalgebra admitting a decomposition into direct sum $\Lambda = \Lambda_0 \oplus \Lambda_1$ of the even Λ_0 and odd Λ_1 parts and into the direct sum $\Lambda = \mathbb{B} \oplus S$ of the body $\mathbb{B}$ and soul $\mathbb{S}$ (see [54, 57] for details). The body map $\varepsilon : \Lambda \to \mathbb{B}$ can be viewed as discarding all nilpotent superalgebra generators, which gives a number part. So we have three topologically disjoint cases:

1. The parameter $t \in \Lambda_0$ is even and has a body, i.e. $\varepsilon(t) \neq 0$.
2. The parameter $t \in \Lambda_0$ is even and has no body, i.e. $\varepsilon(t) = 0$.
3. The parameter $\tau \in \Lambda_1$ is odd (any odd element has no body).

The first choice can be reduced to the standard case by means of a corresponding homeomorphism, and such t can always be considered in the unit interval $I = [0, 1]$. However, the following two possibilities are topologically disjoint from the first one and between themselves.

Definition 33 *An even semi-homotopy between two supermaps $f : X \to Y$ and $g : X \to Y$ is a noninvertible (in general) mapping $X \to Y$ depending on a nilpotent bodyless even parameter $t \in \Lambda_0$ and two bodyless even constants $a, b \in \Lambda_0$ such that*

$$\begin{aligned} \Delta I^{ab} \gamma_{t=a}^{even} &= \Delta I^{ab} f(x), \\ \Delta I^{ab} \gamma_{t=b}^{even} &= \Delta I^{ab} g(x), \end{aligned} \tag{120}$$

where

$$\begin{gathered} \gamma_t^{even}(x) = \Gamma^{even}(x, t), \ \Gamma^{even} : X \times I^{ab} \to Y, \\ I^{ab} = [a, b], \ \Delta I^{ab} = b - a. \end{gathered} \tag{121}$$

Definition 34 *An odd semi-homotopy between two supermaps $f: X \to Y$ and $g: X \to Y$ is a noninvertible (in general) mapping $X \to Y$ depending on a nilpotent odd parameter $\tau \in \Lambda_1$ and two odd constants $\mu, \nu \in \Lambda_1$ such that*

$$\begin{array}{c} \Delta\mathcal{I}^{\alpha\beta}\gamma^{odd}_{\tau=\alpha} = \Delta\mathcal{I}^{\alpha\beta} f(x), \\ \Delta\mathcal{I}^{\alpha\beta}\gamma^{odd}_{\tau=\beta} = \Delta\mathcal{I}^{\alpha\beta} g(x). \end{array} \tag{122}$$

$$\begin{array}{c} \gamma^{odd}_{\tau}(x) = \Gamma^{odd}(x,\tau), \Gamma^{odd} : X \times \mathcal{I}^{\alpha\beta} \to Y, \\ \mathcal{I}^{\alpha\beta} = [\alpha, \beta], \ \Delta\mathcal{I}^{\alpha\beta} = \beta - \alpha. \end{array} \tag{123}$$

Remark. In (121) and (123) I^{ab} and $\mathcal{I}^{\alpha\beta}$ are not intervals in any sense, because among bodyless variables there is no possibility to establish an order relation [9, 10, 55], and so ΔI^{ab} and $\Delta\mathcal{I}^{\alpha\beta}$ are only notations.

Nevertheless, we can give an example of an analog of the average (119) for an odd semi-homotopy

$$(\beta - \alpha)\gamma^{odd}_{\tau}(x) = (\beta - \tau) f(x) + (\tau - \alpha) g(x) \tag{124}$$

which can satisfy some supersmooth conditions.

Remark. In (120) and (122) it is not possible to cancel the left and right hand parts by I^{ab} and $\mathcal{I}^{\alpha\beta}$ correspondingly, because the solutions for semi-homotopies γ^{even}_t and γ^{odd}_{τ} are viewed as equivalence relations. This is clearly seen from (124) where the division by $(\beta - \alpha)$ is impossible, nevertheless a solution for $\gamma^{odd}_{\tau}(x)$ exists.

The most important property of semi-homotopies is their possible noninvertibility which follows from the nilpotency of t and τ and the definitions (120) and (122). Therefore, Y cannot be a supermanifold, it can be a semi-supermanifold only.

It can be assumed that semi-homotopies play the same or similar role in the study of continuous properties and classification of semi-supermanifolds, as ordinary homotopies play for ordinary manifolds. So that it is worthwhile to study their properties thoroughly and in more detail, which will be done elsewhere.

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References

[1] C. Bartocci, U. Bruzzo, and D. Hernandez Ruiperez, *The Geometry of Supermanifolds*, Kluwer, Dordrecht, 1991.

[2] H. J. Baues, *Obstruction Theory*, Springer-Verlag, Berlin, 1977.

[3] A. Bednarek and A. Wallace, *The functional equation* $(xy)(yz) = xz$, Rev. Remaine Math. Pures Appl. **16** (1971), 3–6.

[4] F. A. Berezin, *Introduction to Superanalysis*, Reidel, Dordrecht, 1987.

[5] M. J. Bergvelt and J. M. Rabin, *Super curves, their Jacobians, and super KP equations*, San Diego preprint, alg-geom/9601012, 1996.

[6] U. Bruzzo and R. Cianci, *Mathematical theory of super fibre bundles*, Class. Q. Grav. **1** (1984), 213–226.

[7] U. Bruzzo and J. A. Dominguez Pérez, *Line bundles over families of (super) Riemann surfaces. II: The graded case*, J. Geom. Phys. **10** (1993), 269–286.

[8] P. Bryant, *Supermanifolds, supersymmetry and Berezin integration*, in Complex Differential Geometry and Supermanifolds in Strings and Fields, (P. J. M. Bongaarts and R. Martini, eds.), Springer-Verlag, Berlin, 1988, pp. 150–167.

[9] ———, *Global properties of supermanifolds and their bodies*, Math. Proc. Cambridge Phil. Soc. **107** (1990), 501–523.

[10] R. Catenacci, C. Reina, and P. Teofilatto, *On the body of supermanifolds*, J. Math. Phys. **26** (1985), 671–674.

[11] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. 1, Amer. Math. Soc., Providence, 1961.

[12] L. Crane and J. M. Rabin, *Super Riemann surfaces: uniformization and Teichmüller theory*, Comm. Math. Phys. **113** (1988), 601–623.

[13] A. Császár and E. Thümmel, *Multiplicative semigroups of continuous mappings*, Acta Math. Hung. **56** (1990), 189–204.

[14] D. L. Davis and D. W. Robinson, *Generalized inverses of morphisms*, Linear Algebra Appl. **5** (1972), 329–338.

[15] S. Duplij, *On semigroup nature of superconformal symmetry*, J. Math. Phys. **32** (1991), 2959–2965.

[16] ———, *Second N=1 superanalog of complex structure*, Univ. Kaiserslautern, preprint KL-TH-95/23, alg-geom-9510013, 1995.

[17] ———, *Some abstract properties of semigroups appearing in superconformal theories*, University of Kaiserslautern, preprint KL-TH-95/11, hep-th-9505179 (to appear in *Semigroup Forum*), 1995.

[18] ———, *Ideal structure of superconformal semigroups*, Theor. Math. Phys. **106** (1996), 355–374.

[19] ———, *On an alternative supermatrix reduction*, Lett. Math. Phys. **37** (1996), 385–396.

[20] A. T. Fomenko, D. B. Fuchs, and V. L. Gutenmacher, *Homotopic Topology*, Akadémiai Kiadó, Budapest, 1986.

[21] M. Grasso and P. Teofilatto, *Gauge theories, flat superforms and reduction of super fiber bundles*, Rep. Math. Phys. **25** (1987), 53–71.

[22] P. A. Grillet, *A construction of finite commutative nilsemigroups*, Commun. Algebra **19** (1991), 3145–3172.

[23] R. D. Hofer, *Restrictive semigroups of continuous selfmaps on connected spaces*, Proc. London Math. Soc. **25** (1972), 358–384.

[24] D. Husemoller, *Fibre Bundles*, Springer-Verlag, Berlin, 1966.

[25] A. Jadczyk and K. Pilch, *Superspaces and supersymmetries*, Comm. Math. Phys. **78** (1981), 373–390.

[26] F. W. Kamber and P. Tondeur, *Foliated Bunles and Characteristic Classes*, Springer-Verlag, Berlin, 1975.

[27] A. Yu. Khrennikov, *Functional superanalysis*, Russian Math. Surv. **43** (1988), 87–114.

[28] ———, *Pseudotopological commutative superalgebras with nilpotent ghosts*, Math. Notes **48** (1990), 114–122.

[29] A. A. Kosinski, *Differential Manifolds*, Academic Press, San Diego, 1993.

[30] B. Kostant, *Graded manifolds, graded Lie theory and prequantization*, Lett. Math. Phys. **570** (1977), 177–300.

[31] G. Lallement, *On nilpotency in semigroups*, Pacific J. Math. **42** (1972), 693–700.

[32] S. Lang, *Differential and Riemannian Manifolds*, Springer-Verlag, Berlin, 1995.

[33] D. Leites, *Supermanifold Theory*, Math. Methods Sci. Invest., Petrozavodsk, 1983.

[34] ———, *Selected problems of supermanifold theory*, Duke Math. J. **54** (1987), 649–656.

[35] D. A. Leites, *Introduction to the theory of supermanifolds*, Russian Math. Surv. **35** (1980), 1–64.

[36] E. S. Ljapin, *Semigroups*, Amer. Math. Soc., Providence, 1968.

[37] S. MacLane, *Homology*, Springer-Verlag, Berlin, 1967.

[38] K. D. Magill, *On restrictive semigroups of continuous functions*, Fund. Math. **71** (1971), 131–137.

[39] ———, *A survey of semigroups of continuous selfmaps*, Semigroup Forum **11** (1975), 1–189.

[40] ———, *Homomorphisms of semigroups of continuous selfmaps*, Bull. Alld. Math. Soc. **2** (1987), 1–36.

[41] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, 1974.

[42] V. V. Minachin, *Berezinians in substitution structures*, Func. Anal. Appl. **22** (1988), 90–91.

[43] *Generalized Inverses and Applications*, (M. Z. Nashed, ed.), Academic Press, New York, 1976.

[44] P. Nelson, *Lectures on supermanifolds and strings*, in Particles, Strings and Supernovae, (A. Jevicki and C.-I. Tan, eds.), World Sci., Teaneck, 1989, pp. 997–1073.

[45] T. Okubo, *Differential Geometry*, Dekker, New York, 1987.

[46] I. B. Penkov, *$\mathcal{D}$–modules on supermanifolds*, Inv. Math. **71** (1981), 501–512.

[47] R. Penrose, *A generalized inverse for matrices*, Math. Proc. Cambridge Phil. Soc. **51** (1955), 406–413.

[48] V. Pestov, *Ground algebras for superanalysis*, Rep. Math. Phys. **29** (1991), 275–287.

[49] ———, *Nonstandard hulls of normed Grassmannian algebras and their application in superanalysis*, Soviet Math. Dokl. **317** (1991), 565–569.

[50] ———, *A contribution to nonstandard superanalysis*, J. Math. Phys. **33** (1992), 3263–3273.

[51] M. Petrich, *Inverse Semigroups*, Wiley, New York, 1984.

[52] R. D. Porter, *Introduction to Fibre Bundles*, Dekker, New York, 1977.

[53] J. M. Rabin, *Supermanifolds and super Riemann surfaces*, in Super Field Theories, (H. C. Lee, V. Elias, G. Kunstatter, et al., eds.), Plenum Press, New York, 1987, pp. 557–569.

[54] J. M. Rabin and L. Crane, *Global properties of supermanifolds*, Comm. Math. Phys. **100** (1985), 141–160.

[55] ———, *How different are the supermanifolds of Rogers and DeWitt?*, Comm. Math. Phys. **102** (1985), 123–137.

[56] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and its Application*, Wiley, New York, 1971.

[57] A. Rogers, *A global theory of supermanifolds*, J. Math. Phys. **21** (1980), 1352–1365.

[58] A. S. Schwarz, *Topology for Physicists*, Springer-Verlag, Berlin, 1994.

[59] V. N. Shander, *Orientation of supermanifolds*, Func. Anal. Appl. **22** (1988), 91–92.

[60] I. P Shestakov, *Superalgebras as a building material for constructing counterexamples*, in Hadronic Mechanics and Nonpotential Interaction, Nova Sci. Publ., Commack, NY, 1992, pp. 53–64.

[61] J. Stasheff, *H-Spaces from a Homotopy Point of View*, Springer-Verlag, Berlin, 1970.

[62] R. P. Sullivan, *Semigroups generated by nilpotent transformations*, J. Algebra **110** (1987), 324–343.

[63] R. M. Switzer, *Algebraic Topology—Homotopy and Homology*, Springer-Verlag, Berlin, 1975.

[64] G. M. Tuynman, *An introduction to supermanifolds*, Univ. de Lille preprint, 1995.

[65] E. M. Vechtomov, *On semigroups of continuous partial functions in topological spaces*, Russian Math. Surv. **45** (1990), 143–144.

[66] V. S. Vladimirov and I. V. Volovich, *Superanalysis. 1. Differential calculus*, Theor. Math. Phys. **59** (1984), 3–27.

[67] T. Voronov, *Geometric Integration on Supermanifolds*, Gordon and Breach, New York, 1991.

[68] B. S. De Witt, *Supermanifolds*, Cambridge Univ. Press, Cambridge, 1984.

[69] Yu. A. Yappa, *On the interpretation of anticommuting variables in the theory of superspace*, Lett. Math. Phys. **14** (1987), 157–159.