LIMIT AT RESONANCES OF LINEARIZATIONS=20 OF SOME COMPLEX ANALYTIC DYNAMICAL SYSTEMS

ALBERTO BERRETTI AND STEFANO MARMI

Abstract. We consider the behaviour near resonances of linearizations of=20 germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ and of=20 the semi–standard map.=20

We prove that there exists suitable scalings under which the=20 linearizations converge uniformly to some analytic function as the=20 multiplier, or rotation number, tends non–tangentially to a=20 resonance. This limit functions are computed analytically in the case=20 of germs and are related to the formal classifications of germs with a=20 parabolic fixed point. In the semi–standard map case we give a=20 heuristic argument to compute the limit.

1. Introduction

Problems in which series with small denominators appear are quite=20 frequent in the theory of dynamical systems: they typically appear=20 when one tries to conjugate the dynamics of the system to the one of=20 the corresponding linearized problem. As such, they appear in the=20 theory of Hamiltonian systems, in the theory of area–preserving twist=20 maps, in the theory of circle maps, in the costruction of normal forms=20 of differential equations in the neighborhood of a fixed point, in the=20 theory of iterated maps of $\mathbb{C}^n$, and in many other, often=20 unexpected, cases. We refer the reader to the relevant, huge=20 literature on the topic for comprehensive references and background.

Small denominators have the effect that the convergence of the series=20 which expresses the linearization is problematic, and the=20 linearization itself has a complicated analytic structure as a=20 function of the dynamical variables, of the perturbative parameter,=20 and of the characteristic frequencies of the motion we are trying to=20 conjugate to linear. In some specific cases, this analytic structure=20 has been the subject of numerical studies (*e. g.* [1] and=20 references quoted therein, [2], [3]).=20

In this article we prove some simple results for the linearization of=20 germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ (see *e. g.* =20 [4]) and for the so called "semi-standard map"=20 [5], [6].

In particular, for each of the systems we take into account, we prove=20 that the linearizations have well–defined limits as the multiplier,=20 or rotation number, tends to a rational value p/q, *provided=20 the dynamical variable is suitably rescaled*. In the case of germs,=20 we actually compute this limit: this generalizes a result contained in=20 [4], as this reference computes the limit only for the case=20 of quadratic polynomials. In the case of the semi-standard map, we=20 provide a heuristic argument for the limit function, obtaining a=20 result which fully explains the observed numerical evidence. We note=20 also that our results explain the numerical results of [3].

Of course, it would be quite interesting to generalize these results=20 to more interesting, and complicated, systems like the standard map:=20 [1] contains some partial results in this direction.=20

2. Germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$

Let G denote the group of germs of holomorphic diffeomorphisms=20 of $(\mathbb{C}, 0)$ with a fixed point at 0. An important problem=20 in complex dynamics is to describe the conjugacy classes of this group.=20 Since 0 is a fixed point, one has some natural conjugacy invariants like= =20 the multiplier $\lambda = 3Df'(0)$ of the germ f at 0 and=20 the holomorphic index:=20

$$i(f,0) = 3D\frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z - f(z)},$$

where we integrate on a small loop in the positive direction around=20 0. If $\lambda \neq 1$, 0 is a simple fixed point and one clearly has:

$$i(f,0) = 3D\frac{1}{1-\lambda}.$$

Let G_λ denote the set of $f \in G$ such that=20 $f'(0) = 3D\lambda$. f is *linearizable* if it belongs to the=20 same conjugacy class of the rotation $R_\lambda(z) = 3D\lambda z$.=20 If $|\lambda| \neq 1$, by the Poincaré–Kœnigs linearization theore= m=20 [7], [8], one knows that there exists a unique germ $h_{\lambda,f} \in G_1$ such th= at:

$$= 09 f \circ h_{\lambda,f} = 3Dh_{\lambda,f} \circ R_\lambda. = 09 \tag{2.1}$$

The function $h_{\lambda,f}$ is called the *linearization* of=20 f: attracting ($|\lambda| < 1$) and repelling ($|\lambda| > 1$)=20 fixed points of holomorphic diffeomorphisms are linearizable. If one=20 keeps $f - R_\lambda$ fixed as λ varies, the linearization=20 $h_{\lambda,f}$ depends holomorphically on the multiplier λ. =20 If f_λ is an entire function and $|\lambda| > 1$ then=20 $h_{\lambda,f}$ is an entire function.

When $|\lambda| = 3D1$ the fixed point at the origin is *indifferent*=20 and one must distinguish three different situations:

=09*parabolic* or *resonant case*:=20 =09=09=09$\lambda = 3D\Lambda = 3D\exp\left(2\pi i\frac{p}{q}\right)$, =09=09=09where $(p|q)$=3D1; =09 *Brjuno's case*: $\lambda = 3D\exp(2\pi i\omega)$ and ω=20 =09=09=09is a *Brjuno number* [9]), [10]: if=20 =09=09=09$\{p_k/q_k\}$ denotes the sequence of partial fractions of=20 =09=09=09the continued fraction expansion of ω, one requires the=20 =09=09=09convergence of the series $\sum_{k=3D0}^{\infty}\frac{\log=20=09=09=09 q_{k+1}}{q_k} <$ $+\infty$; =09 *Cremer's case*: $\lambda = 3D = 09\exp(2\pi i = 09\omega)$ =09=09=09and=09ω is *not*=09a Brjuno number. Actually, Cremer proved [11] that G_λ is not a=20 conjugacy class if $\sup_n \frac{\log q_{n+1}}{q_n} = 3D\infty$. =20 However we think that it is quite fair to give his name to the=20 complement of the Brjuno set.

In case (1) Ecalle [12] and Voronin=20 [13] have given a complete classification of the=20 conjugacy classes of G contained in G_Λ; R_Λ=20 belongs to the class of G_Λ with all the elements of order=20 q, but other classes are also possible.=20

In 1987 Yoccoz [4] proved that in case=20 (2) above G_λ is a conjugacy class of G,=20 whereas in case (3) it is not a conjugacy class and=20 there exists at least a non-linearizable germ $f \in G_\lambda$. A=20 remarkable example is given by the quadratic polynomial:=20

$$P_\lambda(z) = 3De^{2\pi i\omega}(z - z^2)$$

=20 which is linearizable if and only if ω is a Brjuno number.

An interesting property of the linearization $h_{\lambda,f}$ is that,=20 keeping $f - R_\lambda$ fixed as λ varies in the unit disk=20 $\mathbb{D} = 3D\{z \in \mathbb{C} | |z| < 1\}$, one can prove that it has=20 non-tangential limits at all Brjuno numbers. In what follows we shall=20 show the existence of suitable scalings under which $h_{\lambda,f}$=20 has non-tangential limits at resonances (*i. e.* rational values of ω=20 or values equal to roots of unity for λ). We will also compute=20 these limits and show how they are reletad to the *formal*=20 classification of conjugacy classes of G_Λ.=20

2.1. **The main theorem.** Let us thus consider the one-parameter family of germs $f_\lambda = 20 \in G_\lambda$:

$$f_\lambda(z) = 3D\lambda z + \sum_{n=3D2}^{\infty} f_n z^n,$$

where we keep the coefficients $\{f_n\}_{n=3D2}^{\infty}$, *i. e.* =20 $f_\lambda - R_\lambda$, fixed. We denote simply by h_λ=20 the corresponding linearization. We have the following theorem.

3. **Theorem 2.1.** *Let $V \subset \mathbb{D}$ be a non-tangential sector with vertex=20 $\Lambda = 3De^{2\pi ip/q}$; there exists a positive integer k and a=20 complex number $A \in \mathbb{C}^*$ such that, if we let:*

$$= 09\tilde{h}_\lambda(z) = 3D\frac{1}{(\lambda - \Lambda)^{1/kq}} = 09 = 09 = 09 = 09 = 09 = 09 = 09 h_{(\lambda-\Lambda)^{1/kq}\lambda}(z), = 09 \tag{2.2}$$

then, as $\lambda \to \Lambda$ in V, $\tilde{h}_\lambda$=20 tends uniformly on some small open disk around 0 *to the function:*

$$= 09\tilde{h}_\Lambda(z) = 3Dz\left(= 09 = 09 = 09 = 09 = 09 = 09 = 09 = 09 = 09 1 - \frac{A}{q\Lambda^{q-1}} z^{kq} = 09 = 09 = 09 = 09 = 09 = 09 = 09\right)^{-1/kq}. = 09 \tag{2.3}$$

This theorem generalizes to germs a result of Yoccoz [4] for the= =20 linearization of the quadratic polynomial, obtained by different=20 techniques.=20

Remark 2.1. Note that one may use any scaling $z \to= 20s((\lambda-\Lambda)^{\frac{1}{kq}})z$ instead of $z \to= 20(\lambda-\Lambda)^{\frac{1}{kq}}z$, provided s is analytic,=20 $s(0) = 3D0$ and $s'(0) \neq 0$. In this case one finds:

$$\tilde{h}_\Lambda = 3Dz\left(= 09 = 09 = 09 = 09 = 09 = 09 = 091 - \frac{As'(0)}{q\Lambda^{q-1}}z^{kq} = 09 = 09 = 09 = 09 = 09 = 09\right)^{-1/kq}.$$

Before proving this theorem we need to recall some properties of=20 parabolic fixed points.=20

2.2. **Parabolic Fixed Points.** It is wel known [14] that if f_Λ is a rational=20 function of degree $d \geq 2$ then 0 belongs to the Julia set of=20 f_Λ. In a neighborhood of 0 the dynamics is described in=20 terms of attracting and repelling petals.=20

We actually need a much weaker result, which holds for all germs=20 $f_\Lambda \in G_\Lambda$. In the following, we denote with=20 $f^{\circ q}$ the composition of the function f with itself q=20 times.

Lemma 2.1. *There exists a positive integer $k \geq 1$ and a nonzero complex=20 number $A \in \mathbb{C}^*$ such that:*

$$= 09 f_\Lambda^{\circ q}(z) - z + Az^{kq+1} + O(z^{kq+2}). = 09 \tag{2.4}$$

If $f_k = 3D0$, $k = 3D2, \ldots, n$, then $kq+1 \geq n$.=20

Proof. This is quite easily proved by comparing the series developments of=20 both sides in the equation $f_\Lambda \circ f_\Lambda^{\circ q} = 3D = 20 f_\Lambda^{\circ q} \circ f_\Lambda$.=20 ☐

Remark 2.2. If f_λ is a polynomial of degree d, then, by a result of=20 Douady and Hubbard (see=20 ([15]), sect. XX, prop. XX), $1 \leq k \leq d-1$.=20

Remark 2.3. There exists a unique $B \in \mathbb{C}^*$ such that $f_\Lambda^{\circ q}$=20 belongs to the same analytic conjugacy class of a germ g_B of the=20 form:

$$g_B(z) = 3Dz + z^{kq+1} + Bz^{2kq+1} + O(z^{2kq+2}).$$

Moreover $f_\Lambda^{\circ q}$ is *formally* and=20 *topologically* conjugated to the polynomial $z + z^{kq+1} + Bz^{2kq+1}$.= =20

Indeed, the existence of an analytic conjugation of $f_\Lambda^{\circ q}$= =20 to g_B is proved by induction over a finite number of steps, starting with f_λ and iterating conjugations by=20 polynomials $\phi_j(z) = 3Dz + \beta_j z^{kq+j+1}$, with=20 $j = 3D1, \ldots, kq-1$ and suitably chosen β_j. The formal=20 conjugation is proved by iterating conjugations by polynomials=20 $\phi_l(z) = 3Dz + \beta_l z^l$, with $l \geq 2k+2$ and suitably=20 chosen β_l.=20

Remark 2.4. Let:

$$f_\Lambda^{\circ q} = 3Dz + Az^{kq+1} + \sum_{j=3Dkq+2}^{\infty} A_j z^{=j}.$$

Then, since $i(f_\Lambda^{\circ q},0)=3Di(g_B,0)=3DB$, we have:

$$B=3D-\frac{1}{A}\sum_{j=3D1}^{kq}\frac{(-1)^j}{A^j}=09\sum_{n_1+\ldots+n_j=3Dkq}A_{n_1+kq+1}\cdots A_{n_j+kq+1}.$$

2.3. **Germs with Almost Resonant Linear Part.** We now consider the one-parameter family of germs $f_\lambda\in=20G_\lambda$, with λ close to Λ, and their iterates=20 $f_\lambda^{\circ q}$. Without any loss of generality, we can assume the= =20 following normalization condition: there exists $r>0$ and $\rho>0$=20 such that for all $\lambda\in\mathbb{C}$, $|\lambda-\Lambda|<\rho$:

=09f_λ, $f_\lambda^{\circ q}:\mathbb{D}_r\mapsto\mathbb{C}$=20 =09injectively, where $\mathbb{D}_r=3D\{z\in\mathbb{C}||z|<r\}$;
=09 $f_\lambda^{\circ q}=3D\lambda^q z+=20=09\sum_{j=3D2}^\infty f_j^{(q)}(\lambda)z^j$ and=20 =09$|f_j^{(q)}(\lambda)|\leq r^{1-j}$.

2. **Remark 2.5.** $f_j^{(q)}(\lambda)$ is a polynomial in λ, with a zero at=20 Λ if $2\leq j\leq kq$. Thus we can assume that there exists=20 $c_1>0$ such that for all $|\lambda-\Lambda|<\rho$:

$$|f_j^{(q)}(\lambda)|\leq c_1|\lambda-\Lambda|r^{1-j}$$

for all $j=3D2,\ldots,kq$.

Let $\alpha\in(0,\pi/2)$ and let $C(\Lambda,\alpha)$ denote the=20 complex cone with vertex at Λ defined by:

$$=09C(\Lambda,\alpha)=3D\{\lambda|\arg\Lambda+\pi-\alpha<=09=09=09=09=09\arg(\lambda-\Lambda)<=09=09=09=09=09\arg\lambda+\pi+\alpha.=09 \tag{2.5}$$

Let $C(\Lambda,\alpha,\rho)=3DC(\Lambda,\alpha)\cap=20\{|\lambda-\Lambda|<\rho\}$, where $\rho>0$. We then have the=20 following.

Lemma 2.2. *For all $\alpha\in(0,\pi/2)$ and for all sufficiently small $\rho>0$=20 there exists $c_2>c_1>0$ such that, if $\lambda\in=20C(\Lambda,\alpha,\rho)$, then:*

$$=09|\lambda^{jq}-\lambda^q|\geq c_2|\lambda-\Lambda|=09 \tag{2.6}$$

for all $j\geq 2$.

Proof. If $j=3D2$ one has:

$$\frac{|\lambda^{2q}-\lambda^q|}{|\lambda-\Lambda|}=3D|\lambda^q|\frac{|\lambda^q-1|}{|\lambda-\Lambda|}\geq c_3(1-\rho)^q|q\Lambda^{q-1}|\geq c_2;$$

if $j>2$ instead:

$$\frac{|\lambda^{jq}-\lambda^q|}{|\lambda-\Lambda|}=3D|\lambda^q|\frac{|\lambda^q-1|}{|\lambda-\Lambda|}|1+\lambda^q+\ldots+\lambda^{(j-2)q}|\geq c_4(1-\rho)^q|q\Lambda^{q-1}|(j-2)\geq c_2,$$

where c_3 and c_4 are some positive constants. □

Note that one needs the assumption that $\lambda\to\Lambda$=20 non-tangentially in order to control the sum=20 $|1+\lambda^q+\ldots+\lambda^{(j-2)q}|$ which is clearly not=20 bounded below if $|\lambda|=3D1$.=20

The power series expansion:

$$= 09h_\lambda(z) = 3Dz + \sum_{j=3D2}^{\infty} h_j(\lambda)z^j = 09 \tag{2.7}$$

for the linearization h_λ of f_λ can be=20 recursively determined by means of (2.1). However one=20 also has:

$$= 09f_\lambda^{\circ q} \circ h_\lambda = 3Dh_\lambda \circ f_\lambda^{\circ q=}, = 09 \tag{2.8}$$

thus one gets:

$$= 09h_j(\lambda) = 3D\frac{1}{\lambda^{jq} - \lambda^q} = 09 = 09 \sum_{i=3D2}^{j} f_j^{\circ q} = 20 = 09 = 09 \sum_{j_1+\ldots+j_i=3Dj} = 20 = 09 = 09h_{j_1}(\lambda)\cdots h_{j_i}(\lambda), = 09 \tag{2.9}$$

for all $j \geq 2$.

Now let:

$$\begin{aligned} = 09\sigma_1 \quad &= 3D \quad 1 \\ = 09\sigma_j \quad &= 3D \quad \sum_{i=3D2}^{j} \sum_{j_1+\ldots+j_i=3Dj} = 09\sigma_{j_1}\cdots\sigma_{j_i} \quad \text{for all } j \geq 2. \end{aligned}$$

Lemma 2.3. *There exists a positive constant c_5 such that:*

$$= 09\sigma_j \leq c_5(3 - 2\sqrt{2})^{1-j}, = 09 \tag{2.10}$$

for all $j \geq 1$.

Proof. The generating function=20 $\sigma(z) = 3D\sum_{i=3D1}^{\infty} \sigma_i z^i$ satisfies the=20 functional equation:

$$\sigma(z) = 3Dz + \frac{\sigma(z)^2}{1 - \sigma(z)},$$

so that:

$$\sigma(z) = 3D\frac{1 + z - \sqrt{1 - 6z + z^2}}{4}$$

is analytic in the disk $|z| < 3 - 2\sqrt{2}$ and bounded and continuous=20 on its closure; (2.10) then follows by Cauchy's=20 estimate.=20 □

Lemma 2.4. *Let $\rho > 0$ be sufficiently small. For all $\lambda \in = 20C(\Lambda, \alpha, \rho)$ and for all $j \geq 1$ one has:*

$$= 09|h_j(\lambda)| \leq \sigma_j r^{j-1} \left(= 09 = 09 = 09 = 09 = 09 = 09\frac{1}{c_2|\lambda - \Lambda|} = 09 = 09 \right)^{\lfloor \frac{j-1}{kq} \rfloor}, = 09 \tag{2.11}$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Proof. We proceed by induction: (2.11) is clearly true if $j = 3D1.=20$ Assume it holds for all $j = 3D1, \dots, n-1$. If $n \le kq$ then from=20 remark 2.5 and lemma 2.2 it follows=20 that:

$$|h_n(\lambda)| \le \frac{1}{c_2|\lambda-\Lambda|} = 09 = 09 = 09 \sum_{i=3D2}^{n} c_1|\lambda-\Lambda| r^{i-1}$$
$$= 09 \sum_{n_1+\ldots+n_i=3Dn} \sigma_{n_1}\cdots\sigma_{n_i} = 09 = 09 = 09 r^{i-(n_1+\ldots+n_i} \le \sigma_n r^{1-n}.$$

If $n > kq$ then we split the previous sum into two parts:=20 $2 \le i \le kq$ and $i > kq$. For the first part we remark that:

$$\lfloor \frac{n_1-1}{kq} \rfloor + \ldots + \lfloor \frac{n_i-1}{kq} \rfloor \le \lfloor \frac{n-i}{kq} \rfloor \le \lfloor \frac{n-1}{kq} \rfloor.$$

The induction assumption applied to the second part gives:

$$= 09 \frac{1}{c_2|\lambda-\Lambda|} \sum_{i=3Dkq+1}^{n} r^{1-i} = 09 = 09 \sum_{n_1+\ldots+n_i=3Dn} \sigma_{n_1}\cdots\sigma_{n_i} = 09 = 09 r^{i-(n_1+\ldots+n_i)} = 09 = 09 \left(= 09 = 09 = 09 \frac{1}{c_2|\lambda-\Lambda|} = 09 = 09 \right)^{\lfloor \frac{n-i}{kq} \rfloor}.$$

Since $i > kq+1$ then=20 $\lfloor \frac{n-i}{kq} \rfloor \le \lfloor \frac{n-1}{kq} \rfloor - 1$, and=20 the proof is complete. $\square$

As lemma 2.4 clearly shows, as $\lambda \to = 20\Lambda$ non tangentially, the radius of convergence of h_λ=20 tends to 0 as $|\lambda-\Lambda|^{1/kq}$. The dynamical reason for=20 this fact is given by the following simple lemma.

Lemma 2.5. *Suppose that $f_2 \ne 0$. Then there exists $\varepsilon_1 > 0$ and=20 $\rho_1 > 0$ such that $f_\lambda^{\circ q}$ has exactly kq fixed point= s=20 in $\mathbb{D}^*_{\varepsilon_1}$ for all $\lambda \in \mathbb{C}$ such that=20 $0 < |\lambda-\Lambda| < \rho_1$. These kq fixed points form k=20 cycles of period q for f_λ.=20*

Proof. Note that 0 is a simple fixed point for f_λ^i,=20 $i = 3D1, \dots, q-1$, whereas it is a fixed point of multiplicity $kq+1$=20 for $f_\lambda^{\circ q}$. The zeros of an analytic function are=20 isolated, and f_λ depends analytically on λ, so=20 there exist $\varepsilon_1 > 0$ and $\rho_1 > 0$ such that=20 $|f_\lambda^i(z) - z| \ge c_5 > 0$ for all z such that=20 $|z| = 3D\varepsilon_1$ or $|z| = 3D2\varepsilon_1$ and for all λ such that=20 $\lambda - \Lambda| < \rho_1$. By the argument principle the number=20 $N(f_\lambda^i, \delta)$ of fixed points of f_λ^i=20 contained in the disk $|z| < \delta$ is given by:

$$N(f_\lambda^i, \delta) = 3D \frac{1}{2\pi i} = 09 \oint_{|z|=3D\delta} \mathrm{d}z = 20 = 09 \frac{(f_\lambda^i)'(z) - 1}{f_\lambda^i(z) - z}.$$

$N(f_\lambda^i, \varepsilon_1)$ and $N(f_\lambda^i, 2\varepsilon_1)$ are=20 continuous functions of λ with integer values, so they are=20 necessarily constant for $|\lambda-\Lambda| < \rho_1$ and $1 \le i = 20 \le q$. Thus $N(f_\lambda^{\circ q}, \varepsilon_1) = 3D = 20 N(f_\lambda^{\circ q}, 2\varepsilon_1) = 3D N(f_\Lambda^{\circ q}, 2\varepsilon_1) == 3D = 20 N(f_\Lambda^{\circ q}, \varepsilon_1) = 3Dkq+1$. Therefore $f_{=a}^{\circ q}$ has=20 exactly kq fixed points in $0 < |z| < \varepsilon_1$ and no other fixed=20 point in the annulus $\varepsilon_1 < |z| < 2\varepsilon_1$. On the other hand one=20 can choose ε_1 and ρ_1 such that $|f_\lambda(z)| \le = 202\varepsilon_1$ if $|z| < \varepsilon_1$ and $|\lambda-\Lambda| < \rho_1$, thus=20 the kq fixed points must form k cycles of period q for=20 f_λ.=20 $\square$

2.4. **Proof of the Main Theorem.**

Proof. Since $\lambda \to \Lambda$ non–tangentially, we can assume=20 that $\lambda \in C(\Lambda, \alpha, \rho)$ and $\rho > 0$, sufficiently=20 small.=20

If $\tilde{h}_j(\lambda)$ denotes the j-th coefficient of the=20 power series expansion of $\tilde{h}_\lambda$, then:

$$|\tilde{h}_J(\lambda)| \leq \sigma_j r^{1-j} = 09(\lambda-\Lambda)^{\frac{j-1}{kq}} \left(= 09 = 09 \frac{1}{c_2|\lambda-\Lambda|} = 09 \right)^{\lfloor \frac{j-1}{kq} \rfloor},$$

and by (2.7) we have the uniform convergence of=20 $\tilde{h}_\lambda$ on a disk around 0 of radius at least=20 $(3-2\sqrt{2})rc_2^{1/kq}$.=20

To compute the limit function $\tilde{h}_\Lambda$, note that:

$$\tilde{f}_\lambda^{\circ q} = 3D = 20\lambda^q z + \sum_{j=3D2}^{kq} f_j^{\circ q}(\lambda)(\lambda-\Lambda)^{\frac{j-1}{kq}} z^j + f_{kq+1}^{\circ q}(\lambda)(\lambda-\Lambda) + O((\lambda-\Lambda)^{1+1/kq}),$$

and by lemma 2.1 and remark 2.5 we have that $f_j^{\circ q}(\lambda) \to 0$ for=20 $j = 3D2, \ldots, kq$ and $f_{kq+1}^{\circ q}(\lambda) \to A \neq= 200$. Since:

$$\tilde{f}_\lambda^{\circ q} \circ \tilde{h}_\lambda = 3D\tilde{h}_\lambda \circ \tilde{f}_\lambda^{\circ q},$$

we easily find that $\tilde{h}_\Lambda$ is the solution of the=20 ordinary differential equation:

$$\tilde{h}_\Lambda(z) = 3Dz\tilde{h}'_\Lambda(z) - = 20\frac{A}{q\Lambda^{q-1}}(\tilde{h}_\Lambda(z))^{kq+1},$$

with initial condition $\tilde{h}_\Lambda(0) = 3D0$, whose solution is=20 just (2.3). ∎

3. The Semi–Standard Map

Now we generalize the results on the scaling at resonances of the=20 linearizations for holomorphic germs of $(\mathbb{C}, 0)$ to the one-parameter=20 family of biholomorphic symplectic diffeomorphisms=20 $(x', y') = 3DF_\varepsilon(x, y)$ of $\mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$=20 [5],[6] given by:

$$= 09 \begin{cases} x' & = 3Dx + y + \varepsilon e^{ix}, \\ = 09 = 09 = 09 = 09y' & = 3Dy + \varepsilon e^{ix}. = 09 \end{cases} = 09 \tag{3.1}$$

Clearly $\mathrm{d}x' \wedge \mathrm{d}y' = 3D\mathrm{d}x \wedge \mathrm{d}y$.=20

We look for an analytic embedding H_λ of a one-dimensional=20 complex pointed disk $\mathbb{D}_r^* = 3D\{z \in \mathbb{C} | 0 < |z| < r\}$ on which=20 the dynamics is given by multiplication by λ: $z \mapsto= 20R_\lambda(z) = 3D\lambda z$; then:

$$= 09F_\varepsilon \circ H_\lambda = 3DH_\lambda \circ R_\lambda. = 09 \tag{3.2}$$

Let:

$$\begin{aligned} = 09z \quad &= 3D \quad -i\varepsilon e^{i\theta}, \\ = 09x \quad &= 3D \quad \theta - i\phi_\lambda(z), = 09 \end{aligned} \tag{3.3}$$

$$= 09y \quad = 3D \quad -i(\phi_\lambda(z) - \phi_\lambda(\lambda^{-1}z)) = 20 = 09 - i\log\lambda.$$

Then:

$$H_\lambda = 3D\left(= 09 = 09 = 09 = 09 = 09 - i\left(= 09 = 09 = 09 = 09 = 09 = 09 = 09\log\frac{iz}{\varepsilon} + \phi_\lambda(z) = 09 = 09 = 09 = 09 = 09\right), = 09 = 09 = 09 = 09 = 09 - i(\phi_\lambda(z) - \phi_\lambda(\lambda^{-1}z)) = 09 = 09 = 09 = 09 = 09 = 09 - i\log\lambda = 09 = 09 = 09\right)$$

and our conjugacy equation becomes:

$$= 09(D_\lambda^2\phi)(z) = 3D\phi_\lambda(\lambda z) - = 20 = 09 = 09 = 09 = 09 = 09 = 09 = 092\phi_\lambda(z) + = 09 = 09 = 09 = 09 = 09 = 09 = 09\phi_\lambda(\lambda^{-1}z) = 3D = 09 = 09 = 09 = 09 = 09 = 09 = 09 - ze^{\phi_\lambda(z)}. = 09 \tag{3.4}$$

If $|\lambda| \neq 1$ then (3.4) has a unique analytic=20 solution such that $\phi_\lambda(0) = 3D0$. When $|\lambda| = 3D1$ there=20 are the same three possibilities we discussed for germs. In the=20 *resonant case* F_ε is in general not even formally=20 linearizable. Our conjugacy equation (3.4) has an=20 analytic solution *if and only iff* $\lambda = 3De^{2\pi i\omega}$ and=20 ω is a *Brjuno number* [16], [17].=20

Remark 3.1. Let=09$H(x, y) = 09 = 3D(e^{ix}, y)$. Then=09$H$=09conjugates F_ε to the biholomorphic diffeomorphism G_ε=09of $\mathbb{C}^2$ defined by:

$$G_\varepsilon(z_1, z_2) = 09 = 3D(z_1e^{i(z_2+\varepsilon=09z_1)}, z_2 + \varepsilon z_1).$$

The=09points $(0, z_2)$ are fixed by=09G_ε and the image $H_\lambda(\mathbb{D}_r^*)$ is associated [18] with=09the=09center manifold passing through the point $(0, -i\log\lambda)$ and tangent at this point to $\{z_2 = 3D - i\log\lambda\}$.

As in the case of germs, ϕ_λ depends holomorphically on=20 $\lambda \in \mathbb{D}$ and it has non-tangential limits at all Brjuno=20 numbers. In what follows we will prove the existence of suitable=20 scalings under which ϕ_λ has non-tangential limits at all=20 resonances. We will compute exactly the limit when $\lambda = 20 \to 1$, and we will give a heuristic argument to calculate=20 the limit at all other roots of unity.=20

3.1. **Existence of Non–Tangential Limits.** We now prove the following theorem.

Theorem 3.1. *Let V be a non-tangential sector with vertex Λ and let:*

$$\tilde{\phi}_\lambda(z) = 3D\phi_\lambda((\lambda - \Lambda)^{2/q}z). \tag{3.5}$$

Then:

$$\tilde{H}_\lambda(z) = 3D\left(-i\left(= 09\log\frac{iz}{\varepsilon} + \tilde{\phi}_\lambda(z)\right), = 09 - i(\tilde{\phi}_\lambda(z) - \tilde{\phi}_\lambda(\lambda^{-1}z)) = 09 - i\log\lambda\right)$$

conjugates $F_{(\lambda-\Lambda)^{2/q}\varepsilon}$ to the rotation=20 R_λ. As $\lambda \to \Lambda$ in V,=20 $\tilde{\phi}_\lambda$ converges uniformly on some small open disk=20 around the 0 to an analytic function $\tilde{\phi}_\Lambda$.=20

Proof. The first statement is obvious.

Expanding ϕ_λ into powers of z:

$$\phi_\lambda(z) = 3D \sum_{j=3D1}^{\infty} \phi_j(\lambda) z^j$$

and substituting into (3.4) one finds:

$$= 09\phi_1(\lambda) = 3D\frac{1}{D_1(\lambda)}$$

and, for all $j \geq 2$:

$$= 09\phi_j(\lambda) = 3D\frac{1}{D_j(\lambda)} \sum_{k=3D1}^{j} = 09 = 09 \sum_{j_1+\ldots+j_k=3Dj-1} = 09 = 09\phi_{j_1}(\lambda)\cdots\phi_{j_k}(\lambda),$$

where $D_j(\lambda) = 3D - (\lambda^{j/2} - \lambda^{-j/2})^2$.

Since $\lambda \to \Lambda$ in V there exists a positive=20 constant c_6 such that:

$$|D_j(\lambda)| \geq= 20 \begin{cases} = 09c_6 & \text{if } j \neq 0 \pmod{q}, \\ = 09c_6|\lambda-\Lambda|^2 & \text{if } j = 3D0 \pmod{q}. \end{cases}$$

It is now easy to prove by induction that:

$$|\tilde{\phi}_j(\lambda)| = 3D = 20|\phi_j(\lambda)|\,|\lambda-\Lambda|^{2j/q} \leq c_7\sigma_j c_6^{-2\lfloor\frac{j}{q}\rfloor} = 09|\lambda-\Lambda|^{2\frac{j}{q}-2\lfloor\frac{j}{q}\rfloor}$$

for some constant $c_7 > 0$, and from this estimate the theorem=20 clearly follows. □

Remark 3.2. Note that $\tilde{\phi}_\lambda$ is actually a function of z^q.=20

Remark 3.3. If $\lambda \to 1$ non-tangetially, one can easily compute=20 the limit $\tilde{\phi}_1$. Note that $\tilde{\phi}_\lambda$ is=20 the solution of:

$$D_\lambda^2\tilde{\phi}_\lambda(z) = 3D = 20 = 09 - (\lambda-1)^2 z e^{\tilde{\phi}_\lambda(z)}.$$

Since:

$$= 09\tilde{\phi}_\lambda(\lambda z) \quad = 3D \quad = 09 = 09\tilde{\phi}_\lambda(z) + = 20 = 09 = 09(\lambda-1)z\tilde{\phi}'_\lambda(z) + = 09 = 09\frac{1}{2}(\lambda-1)^2 z^2\tilde{\phi}''_\lambda(z) + = 09 = 09\ldots,$$

$$= 09\tilde{\phi}_\lambda(\lambda^{-1}z) \quad = 3D \quad = 20 = 09 = 09\tilde{\phi}_\lambda(z) + = 09 = 09\frac{1-\lambda}{\lambda}z\tilde{\phi}'_\lambda(z) + = 09 = 09\frac{1}{2}\frac{(1-\lambda)^2}{\lambda^2} = 09 = 09 = 09z^2\tilde{\phi}''_\lambda(z) + = 09 = 09\ldots,$$

taking the limit $\lambda \to 1$ one finds:

$$= 09z\tilde{\phi}'_1(z) + z^2\tilde{\phi}''_1(z) = 3D = 20 = 09 = 09 - ze^{\tilde{\phi}_1(z)}, = 09 \tag{3.6}$$

with the initial condition $\tilde{\phi}_1(0) = 3D0$. Thus:

$$= 09\tilde{\phi}_1(z) = 3D - 2\log\left(1+\frac{z}{2}\right). = 09 \tag{3.7}$$

Remark 3.4. Replacing εe^{ix} in (3.1) with an entire function f=20 of εe^{ix} such that $f(0) = 3D0$ the conjugacy equation=20 (3.4) becomes:

$$(D^2_\lambda \phi)(z) = 3Df(-ze^{\phi_\lambda(z)})$$

and theorem 3.1 can be immediately proved also in this=20 (slightly) more general case.=20

3.2. **Calculation of the Limits at Resonances.** We would like to compute exactly $\tilde{\phi}_\Lambda$ for all=20 $\Lambda = 3D\exp\left(2\pi i\frac{p}{q}\right)$ with $(p,q) = 3D1$. As a=20 preliminary step toward this result, we give an *heuristic*=20 argument, reminiscent of the renormalization group method, which=20 provides just this. While all the combinatorics is handled rigorously,=20 we can't estimate the error terms that we neglect systematically at=20 each step of the "renormalization". We believe, though, that our=20 result are qualitatively accurate and quite likely exact, and that=20 the method could be turned into a tool for the rigorous analysis of=20 such maps.=20

The basic idea behind the procedure is similar to the one previously=20 used for germs, *i. e.* that the behaviour of the semistandard map around=20 a resonance p/q iterated q times is "roughly the same" as the one=20 of the semistandard map around the "main resonance" 0/1. The same=20 idea was used in [1] to handle the resonance 1/2 for the=20 standard map.=20

We need first the following combinatorial lemma.

Lemma 3.1. *Consider the following recursively defined sequences:*

$$\begin{aligned} a_k^{(0)} &= 3D\mu\delta_{k,0}, \quad \textit{for } \mu \in \mathbb{C}, \\ b_k^{(r+1)} &= 3D\Lambda^k a_k^{(r)}, \qquad (3.8) \\ a_k^{(r)} &= 3Db_{k+1}^{(r)} - 2b_k^{(r)} + b_{k-1}^{(r)}. \end{aligned}$$

Then:
*=09*$a_k^{(r)} = 3D0$ *if* $|k| > r$; *=09* $\sum_{k\in\mathbb{Z}} a_k^{(r)} = 3D0$; *=09* $\sum_{k\in\mathbb{Z}} ka_k^{(r)} = 3D0$; *=09* $a_q^{(q)} = 3Da_{-q}^{(q)} = 3D\mu$, $a_0^{(q)} = 3D - 2\mu$, *an= d in all=20 =09for all other k* $a_k^{(q)} = 3D0$.

3. *Proof.* (1), (2) and (3) are most easily=20 proved by induction. In order to prove (4) we use the=20 generating function:

$$f^{(r)}(z) = 3D\sum_{k\in\mathbb{Z}} a_k^{(r)} z^k.$$

$f^{(r)}$ is a rational function because of (1). We also=20 have:

$$a_k^{(r)} = 3D\Lambda^k \left(= 09 = 09\Lambda a_{k+1}^{(r-1)} - 2a_k^{(r-1)} + \Lambda^{-1} a_{k-1}^{(r-1)} = 09 = 09\right),$$

from which it follows that:

$$f^{(r)}(z) = 3D\sum_{k\in\mathbb{Z}} z^k\Lambda^k \left(= 09 = 09\Lambda a_{k+1}^{(r-1)} - 2a_k^{(r-1)} + \Lambda^{-1}a_{k-1}^{(r-1)} = 09 = 09\right)$$
$$= 09 = 3D\left(\frac{1}{z} - 2 + z\right) f^{(r-1)}(\Lambda z)$$
$$= 09 = 3D\frac{(z-1)^2}{z} f^{(r-1)}(\Lambda z).$$

Now, clearly $f^{(0)}(z) = 3D\mu$. Then:

$$f^{(q)}(z) = 3D\mu\frac{\left(\prod_{k=3D1}^{q-1}(\Lambda^k z - 1)\right)^= 2}{=} 09 = 09 z^q\Lambda^{q(q-1)/2}$$
$$= 09 = 3D\mu\Lambda^{-q(q-1)/2}\frac{(1-z^q)^2}{z^q}$$
$$= 09 = 3D\mu(z^q - 2 + z^{-q}),$$

since Λ is a primitive q-th root of 1: this proves=20 (4). $\square$

Now we proceed with our heuristic argument. We have:

$$= 09D^2_{\lambda^q}\tilde\phi_\lambda(z) = 3D = 20 = 09 = 09\sum_{-q}^{q} a_k^{(q)}\tilde\phi_{\lambda,k},$$

where:

$$= 09\tilde\phi_{\lambda,k} = 3D\tilde\phi_\lambda(z_k),$$

and

$$= 09z_k = 3D\lambda^k z.$$

The coefficients a are the one defined in lemma 3.1,=20 with $\mu = 3D1$. We have choosen our notations so that:

$$D_k^2\tilde\phi_{\lambda,k} = 3D = 20\tilde\phi_{\lambda,k+1} - 2\tilde\phi_{\lambda,k} + \tilde\phi_{\lambda,k-1}.$$

We shall apply $q-1$ times a transformation defined in three steps:

=09write everything in terms of $\tilde\phi_{\lambda,k+1} - = 20 = 09 = 092\tilde\phi_{\lambda,k} + \tilde\phi_{\lambda,k-1}$;

=09 apply the functional equation for $\tilde\phi_\lambda$: =09

$$= 09D^2_\lambda\tilde\phi_\lambda(z) = 3D = 20 = 09 = 09 - (\lambda-\Lambda)^{2/q} z e^{\tilde\phi_\lambda(z)}; = 09$$

=09 linearize: =09

$$= 09z_k e^{\tilde\phi_\lambda(z_k)} \approx = 20 = 09 = 09\Lambda^k z_0 e^{\tilde\phi_{\lambda,0}} = 09 = 09\left(1 + \left(\tilde\phi_{\lambda,k} - \tilde\phi_{\lambda,0} - k\log\frac{\lambda}{= da} = 09 = 09\right)\right). = 09$$

Now we use lemma 3.1 and in particular the fact that:

$$\sum_{k=3D-r}^{r} a_k^{(r)} \tilde{\phi}_{\lambda,k} = 3D = 20 \sum_{k=3D-r+1}^{r-1} b_k^{(r)} (\tilde{\phi}_{\lambda,k+1} - 2\tilde{\phi}_{\lambda,k} + \tilde{\phi}_{\lambda=,k-1}),$$

where the sequences $a^{(r)}$ and $b^{(r)}$ are those defined in the=20 lemma. Note that conditions (2) and (3) of=20 lemma 3.1 are crucial for the following to work.

Then we have:

$$\sum_{k=3D-r}^{r} a_k^{(r)} \tilde{\phi}_{\lambda,k} = 3D = 09 = 09 \sum_{k=3D-r+1}^{r-1} b_k^{(r)} = 09 = 09(\tilde{\phi}_{\lambda,k+1} - = 09 = 092\tilde{\phi}_{\lambda,k} + = 09 = 09\tilde{\phi}_{\lambda,k-1})$$

$$= 09 \approx -(\lambda - \Lambda)^{2/q} z e^{\tilde{\phi}_\lambda} = 09 = 09 \sum_{k=3D-r+1}^{r-1} b_k^{(r)} \Lambda^{-k} = 09 = 09 \left(1 + \left(\tilde{\phi}_{\lambda,k} - = 20 = 09 = 09\tilde{\phi}_{\lambda,0} - = 20 = 09 = 09 k \log \frac{\lambda}{\Lambda}\right)\right)$$

$$= 09 = 3D - (\lambda - \Lambda)^{2/q} z e^{\tilde{\phi}_\lambda} = 09 = 09 \left(\sum_{k=3D-r+1}^{r-1} a_k^{(r-1)} - = 09 \log \frac{\lambda}{\Lambda} \sum_{k=3D-r+1}^{r-1} k a_k^{(r-1)} \right)$$

$$= 09 \quad - (\lambda - \Lambda)^{2/q} z e^{\tilde{\phi}_\lambda} = 09 = 09 \sum_{k=3D-r+1}^{r-1} a_k^{(r-1)} = 09 = 09(\tilde{\phi}_{\lambda,k} - \tilde{\phi}_{\lambda,0})$$

$$= 09 = 3D - (\lambda - \Lambda)^{2/q} z e^{\tilde{\phi}_\lambda} = 09 = 09 \sum_{k=3D-r+1}^{r-1} a_k^{(r-1)} \tilde{\phi}_{\lambda,k},$$

$\blacksquare$

and by keeping iterating this identity $q-1$ times (*i. e.* by setting=20 sequentially $r = 3Dq, \dots, 2$) we obtain:

$$D^2_{\lambda^q} \tilde{\phi}_\lambda = 3D = 20 = 09 \sum_{k=3D-q}^{q} a_k^{(q)} \tilde{\phi}_{\lambda,k}$$

$$\approx = 20 = 09 \left(-(\lambda - \Lambda)^{2/q} z e^{\tilde{\phi}_\lambda(z)}\right)^{q-1=} = 09 \left(a_1^{(1)} \tilde{\phi}_{\lambda,1} = 09 + a_0^{(1)} \tilde{\phi}_{\lambda,0} = 09 + a_{-1}^{(1)} \tilde{\phi}_{\lambda,-1}\right).$$

$\blacksquare$

Since $a_1^{(1)} = 3Da_{-1}^{(1)} = 3D\Lambda$, $a_0^{(1)} = 3D - 2\Lambda$, we= =20 finally have:

$$D^2_{\lambda^q} \tilde{\phi}_\lambda \approx = 09 \left(-(\lambda - \Lambda)^{2/q} z e^{\tilde{\phi}_\lambda(z)}\right)^{q-1=} = 09\Lambda(\tilde{\phi}_{\lambda,1} - = 20 = 092\tilde{\phi}_{\lambda,0} + \tilde{\phi}_{\lambda,-1})$$

$$= 3D\Lambda = 20 = 09(-1)^q (\lambda - \Lambda)^2 z^q e^{q\tilde{\phi}_\lambda(z)}.$$

Taking the limit $\lambda \to \Lambda$ we have:

$$\frac{D^2_{\lambda^q}}{(\lambda - \Lambda)^2} \tilde{\phi}_\lambda = 20 \to q^2 \Lambda^{-2} z \frac{\mathrm{d}\tilde{\phi}_\Lambda}{\mathrm{d}z} + q^2 \Lambda^{-2} z^2 \frac{\mathrm{d}^2 \tilde{\phi}_\Lambda}{\mathrm{d}z^2},$$

so that:

$$= 09\frac{\mathrm{d}}{\mathrm{d}z}\left(= 09 = 09z\frac{\mathrm{d}\tilde{\phi}_\Lambda}{\mathrm{d}z}\right) = 3D = 09 = 09\frac{(-1)^q\Lambda^3}{q^3}z^{q-1}e^{q\tilde{\phi}_\Lambda(z)=}, = 09 \tag{3.9}$$

with the initial condition $\tilde{\phi}_\Lambda(0) = 3D0$. Note that=20 the transformation $\psi_\Lambda(w) =$ $3Dq\tilde{\phi}_\Lambda(z)$ and=20 $w = 3D\frac{(-1)^q\Lambda^3}{q^3}z^q$ applied to (**??**) gives the simpler equation:

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(w\frac{\mathrm{d}\psi_\Lambda}{\mathrm{d}w}\right) = 3De^{\psi_\Lambda},$$

whose solution is $\psi_\Lambda = 3D - 2\log(1 - w/2)$. Therefore we=20 finally have:

$$= 09\tilde{\phi}_\Lambda(z) = 3D = 20 = 09 = 09 - \frac{2}{q}\log\left(1 + \frac{(-1)^{q+1}\Lambda^3}{2q^3}z^q = ht\right), = 09 \tag{3.10}$$

which agrees with (3.7) for $q = 3D1$.

References

3. [1] Berretti, A. and Marmi, S., *Scaling near Resonances and Complex Rotation Numbers for the Standard Map*, Nonlinearity **7** 603 (1994)

[2] de la Llave, R., and Tompaidis, S., *C*omputation of domains of analyticity for some perturbative expansions of mechanics, University of Texas at Austin Preprint (1992)

[3] Billi, L., Todesco, E., and Turchetti, G.,=20 *Singularity Analysis by Padé Approximants of some=20 Holomorphic Maps*, J. Phys. A: Math. and Gen. **27** 6215-6229,=20 1994

[4] Yoccoz, J. C., *Théorème de Siegel, Nombres= =20 de Brjuno et Polômes Quadratiques*, Astérisque **231**=20 3-38, 1995

[5] Greene, J. M., and Percival, I. C.,=20 *Hamiltonian Maps in the Complex Plane*, Physica D **3**=20 530-548, 1981

[6] Percival, I. C., *Chaotic Boundary of a=20 Hamiltonian Map*, Physica D **6** 67-74, 1982

[7] Königs, G., *Recherches sur les integrales=20 de certaines equations fonctionelles*, Ann. Sci. Éc. Norm. Sup.=20 (3ème sér.), 1 Suppl. 1–41, 1884

[8] Poincaré, H., *Œuvres* t. I, p.=20 XXXVI-CXXIX, Gauthier-Villars, Paris 1928-1956

[9] Bjuno, A. D., *Analytic Form of Differential=20 Equations*, Trans. Moscow Math. Soc. **25** 131–288, 1971

[10] Marmi, S., Moussa, P., Yoccoz, J. C., *The=20 Brjuno Functions and their Regularity Properties*, preprint Saclay=20 95/028, 1995

[11] Cremer, H., *Über die Häufigkeit der=20 Nichtzentren*, Math. Ann. **115**, 573-580, 1938

[12] Ecalle, J., *Les Fonctions Résurgentes et leurs= =20 Applications. Tomes I, II, III*, Publications Mathématiques=20 d'Orsay **81-05**, **81-06**, **85-05**, 1981-1985

[13] Voronin, S. M., *Analytical Classification of=20 Germs of Conformal Mappings* $(\mathbb{C},0) \mapsto (\mathbb{C},0)$ *with identity=20 linear part*, Funct. Anal. and its Appl. **15** 1-17, 1981

[14] Beardon, A. F., *Iteration of Rational=20 Functions*, Graduate Text in Mathematics **138**, Springer, 1991

[15] Douady, A., and Hubbard, J. H., *On the Dynamics=20 of Polynomial Like Mappings*, Ann. Sc. E.N.S., 4ème Série,=20 **18** 287-343 (1985); see also Douady, A., *Systèmes=20 Dynamiques Holomorphes*, Séminaire Bourbaki, Exposé 599,=20 Astérisque **105-106**, 39-63 (1983)
[16] Marmi, S., *Critical Functions for Complex=20 Analytic Maps*, Journ. of Physics A: Math. and Gen. **23**=20 3447-3474, 1990
[17] Davie, A. M., *Critical Functions for the=20 Semi–standard and Standard Maps*, Nonlinearity **7** 219-229,=20 1994
[18] Herman, M. R., *Recent Results and Some Open=20 Questions on Siegel's Linearization Theorem of Germs of Complex=20 Analytic Diffeomorphisms of $\mathbb{C}^n$ near a Fixed Point*, Proc. VIII=20 Int. Conf. Math. Phys., Mebkhout and Seneor Eds., World Scientific, 1986

Alberto Berretti, Dipartimento di Matematica, II Università = di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy and I= NFN, Sez. Tor Vergata
E-mail address: `berretti@roma2.infn.it`

Stefano Marmi, Dipartimento di Matematica "U. Dini", Universit= à di Firenze, Viale Morgagni 57a, 50134 Firenze, Italy and INFN, Sez. Firenz= e
E-mail address: `marmi@fi.infn.it`