Correlation spectrum of quenched and annealed equilibrium states for random expanding maps

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Abstract. We show that the integrated transfer operators for positively weighted independent identically distributed smooth expanding systems give rise to annealed equilibrium states for a new variational principle. The unique annealed equilibrium state coincides with the unique annealed Gibbs state. Using work of Ruelle [1990] and Fried [1995] on generalised Fredholm determinants for transfer operators, we prove that the discrete spectrum of the transfer operators coincides with the correlation spectrum of these invariant measures (yielding exponential decay of correlations), and with the poles of an annealed zeta function, defined also for complex weights. A modified integrated transfer operator is introduced, which describes the (relativised) quenched states studied e.g. by Kifer [1992], and conditions (including SRB) ensuring coincidence of quenched and annealed states are given. For small random perturbations we obtain stability results on the quenched and annealed measures and spectra by applying perturbative results of Young and the author [1993].

1. Introduction

The study of equilibrium states for a single map $f : M \to M$ and a positive weight function g on M, i.e., the analysis of f-invariant Borel probability measures μ on M which maximise the expression

$$h_f(\nu) + \int \log g(x)\, \nu(dx) \tag{1.1}$$

(with $h_f(\nu)$ the entropy of (f, ν)) is now a well-developed subject in a variety of settings (see e.g. Ruelle [1989] and references therein). One of the main tools for this is a transfer operator acting on a suitable Banach space of test functions $\varphi : M \to \mathbb{C}$ by

$$\mathcal{L}\varphi(x) = \sum_{fy=x} \varphi(y) g(y)\,. \tag{1.2}$$

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In many cases one constructs the equilibrium state μ by combining maximal eigenfunctions of $\mathcal{L}$ and its dual, and one obtains exponential decay of the corresponding correlation functions

$$C_{\varphi_1\varphi_2}(n) = \int (\varphi_1 \circ f^n(x))\varphi_2(x)\,\mu(dx) - \int \varphi_1(x)\mu(dx) \int \varphi_2(x)\,\mu(dx) \tag{1.3}$$

for suitable φ_1, φ_2 by proving that there is a gap in the spectrum of $\mathcal{L}$. The discrete spectrum of $\mathcal{L}$ can be shown to correspond to the poles of the Fourier transform of C_{φ_1,φ_2} in some strip, these poles are the *resonances* of Ruelle [1987].

A natural generalisation of this problem (see e.g. Ruelle [1995] for an overview) consists in starting from a family of maps f_ξ (or their inverse branches) and positive weights g_ξ for $\xi \in E$, and defining the mixed or generalised transfer operator

$$\mathcal{L}\varphi(x) = \sum_\xi \sum_{f_\xi(y)=x} \varphi(y) g_\xi(y) \tag{1.4}$$

(the sum over ξ being replaced by an integral when the index set E is uncountable). This framework appears naturally when considering (weighted) independent identically distributed (i.i.d.) random compositions of maps f_ξ associated with a probability measure $\theta(d\xi)$ on the index set E, a convenient description of the system being given by the weighted (two-sided) skew product on $M \times E^{\mathbb{Z}}$

$$\tau(x,\omega) = (f_{\omega_0}(x), \sigma\omega)\,, \quad g(x,\omega) = g_{\omega_0}(x)\,, \tag{1.5}$$

with σ the shift on $E^{\mathbb{Z}}$, or its corresponding "one-sided" version τ^+ (see (2.3)).

For weighted random (not necessarily i.i.d.) compositions, equilibrium states for a relativised variational principle (Ledrappier-Walters [1977], see (2.6) below) have been studied, in particular by Kifer [1992]. In the case where the maps f_ξ are expanding, and the weights are given by the Jacobians $g_\xi(x) = 1/|\det D_x f_\xi|$, the integrated transfer operator (1.4) (see (2.16) for a precise formula) gives rise to this *relativised equilibrium state* $\mu^{(q)}$, which is just the SRB measure of the random system, and has been studied in particular by Baladi-Young [1993]. The discrete spectrum of the operator is then also related to integrated correlation functions (2.20) for the random SRB measure.

In more general cases (consider for example a family of one-dimensional repellors f_ξ, each with an invariant Cantor set of Hausdorff dimension α_ξ and escape rate P_ξ, and the weight $g_\xi = 1/|f'_\xi|$), we find that the annealed integrated operator $\widehat{\mathcal{L}}$ (1.4)-(2.16) acting on a suitably "large" space does *not* always give rise to the relativised equilibrium state, but to another τ-invariant measure (see (2.10) below for the corresponding variational principle) which we call the *annealed* equilibrium state $\mu^{(a)}$ (in particular, we solve a conjecture of Ruelle [1995, Section 7], see Section 2.4). Extending the analogy with spin-glasses (see e.g. Mézard et al. [1987], and our random Ising model example in Section 2.4) we rename the relativised equilibrium states *quenched* equilibrium states. Using previous work of Kifer [1992] and Khanin-Kifer [1996] we describe the modified (and less directly accessible, for example in computer simulations) quenched integrated transfer operator $\widehat{\mathcal{M}}$ (see (4.12)) which gives rise to the quenched states.

We are then able to extend the very powerful transfer operator techniques (including perturbative results from Baladi-Young [1993], as well as the analysis of the discrete spectrum in terms of zeta functions or generalised Fredholm determinants of Ruelle [1990]) to both integrated operators $\widehat{\mathcal{L}}$, $\widehat{\mathcal{M}}$, obtaining a good understanding of the ergodic properties of both the annealed and quenched invariant measures $\mu^{(a)}$, $\mu^{(q)}$, including the resonances of their integrated correlation functions. Some of our results also apply to negative, or even complex weights (negative weights appear naturally e.g. in the study of renormalisation, see e.g. Christiansen et al. [1990], Jiang et al. [1992] and references therein) where perturbative results are also desirable.

See Baladi et al. [1995] for a treatment of random correlation functions (as opposed to the integrated correlation functions (2.20)), without any i.i.d. assumption. The Birkhoff cones used there do not seem to be directly applicable to other Ruelle resonances than the first one. We refer to Ruedin [1994], Lanford-Ruedin [1995] for a study of pressure and Gibbs state via similar integrated transfer operators.

In this paper we use mainly three ingredients: We adapt the results of Ruelle [1990] and Fried [1995] to our (one-sided) skew product situation; we transport the two-sided techniques of Kifer [1992] and Khanin-Kifer [1996] to our one-sided skew product; we apply the perturbative results in Baladi-Young [1993] to get stochastic stability. In some sense we are considering a "toy model": our uniform expansion and smoothness assumptions are the strongest possible. We expect and hope that the techniques developed here may be extended to more realistic settings (expanding in average as in Khanin-Kifer [1996], or more generally non uniformly hyperbolic, and/or piecewise smooth). The theory presented here is particularly simple in the i.i.d. setting, but most of it can be extended to more general situations as pointed out to us by David Ruelle (see Appendix B).

The outline of the paper is as follows: In Section 2 we define precisely our model for random compositions of expanding maps, as well as the annealed and quenched equilibrium and Gibbs states. We also state the main results: Theorem 1 (existence and uniqueness of annealed states), Theorem 3 (stochastic stability for annealed and quenched states), Theorem 4 (giving the relationship between the spectrum of the integrated operators and the correlation functions for annealed and quenched states), Theorem 5 (stability of these correlation spectra) and finally Theorem 6 on annealed zeta functions and annealed Fredholm determinants and their stability. Section 3 contains a proof of Theorem 1 based on an analysis of an integrated transfer operator ($\widehat{\mathcal{L}}$, see Proposition 3.1) which also yields Theorem 4, and proofs of the stability results concerning the annealed states in Theorems 3 and 5. Theorem 6 is also proved in Section 3. Finally, Section 4 is devoted to the proofs of the claims in Theorems 3, 4 and 5 on quenched states, using the transfer operator $\widehat{\mathcal{M}}$ (Proposition 4.2).

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2. Definitions and statement of results

2.1. Weighted random composition of expanding maps.

For fixed $r \geq 1$ let M be a compact, connected, C^r Riemann manifold endowed with a Riemann metric d_M. For $\gamma > 1$ let $C^r_\gamma(M, M)$ denote the space of all γ-expanding C^r maps $f : M \to M$ (i.e., maps such that for all $x \in M$, and all $v \in T_x M$, we have $\|D_x f(v)\| > \gamma \|v\|$), endowed with the C^r metric. Finally, let $C^r(M, \mathbb{C})$, respectively $C^r(M, \mathbb{R}^+_*)$ be the space of all complex-valued or positive C^r functions, endowed with the C^r metric d_r or norm $\| \cdot \|_r$. (Many of our results have versions for M a compact metric space and Lipschitz or Hölder smoothness, or replacing the inverse branches of expanding maps by suitable families of contractions.)

Let E be a compact subspace of $C^r_\gamma(M, M) \times C^r(M, \mathbb{C})$, for the C^r metric d_E. Let Ω^+ be the compact space of one-sided sequences $E^{\mathbb{Z}_+}$ endowed with the distance $d_\alpha(\omega, \tilde\omega) = \sum_{k=0}^\infty \alpha^k d_E(\omega_k, \tilde\omega_k)$ for some $0 < \alpha < 1$, and let $\Omega = E^{\mathbb{Z}}$ be the corresponding two-sided space, with an analogous metric also denoted d_α.

Let σ^+ be the one-sided shift to the left on Ω^+, and σ the two-sided shift to the left on Ω. Fix θ a Borel probability on E. The product measure $\Theta^+ = \theta^{\mathbb{Z}_+}$ on Ω^+ is σ^+-invariant and σ^+ is ergodic for Θ^+. For a Borel measure υ on $M \times \Omega^+$, we shall write π_υ for the marginal of υ on Ω^+. We let $\mathcal{P}_\Theta$ denote the space of τ^+-invariant probability measures μ on $M \times \Omega^+$ with $\pi_\mu = \Theta^+$.

If $\omega \in \Omega^+$, write f_ω for the first coordinate of $\omega_0 \in E$, and g_ω for the second coordinate of ω_0. We consider the independent identically distributed compositions

$$f^{(n)}_\omega = f_{(\sigma^+)^n \omega} \circ \cdots \circ f_{\sigma^+ \omega} \circ f_\omega \,, \tag{2.1}$$

weighted by

$$g^{(n)}_\omega = g_{(\sigma^+)^n \omega} \circ f^{(n-1)}_\omega \cdots g_{\sigma^+ \omega} \circ f_\omega \cdot g_\omega \tag{2.2}$$

where $n \geq 1$ and $(f_\omega, g_\omega) = (f_{\omega_0}, g_{\omega_0})$ is chosen in E following the distribution $\theta(d\omega_0)$. In other words, we are iterating the (weighted) one-sided skew-product $\tau^+ : M \times \Omega^+ \to M \times \Omega^+$

$$\tau^+(x, \omega) = (f_{\omega_0}(x), \sigma^+(\omega)) \,, \quad g(x, \omega) = g_{\omega_0}(x) \,. \tag{2.3}$$

The map τ^+ is in general not positively expansive, but for each $\xi \in E$ and each local inverse branch $(f_\xi)^{-1}_i$ of f_ξ the inverse branch

$$(\tau^+)^{-1}_{\xi,i}(x, \omega) = ((f_\xi)^{-1}_i x, \xi \wedge \omega) \tag{2.4}$$

(where $\xi \wedge \omega$, or simply $\xi\omega$, denotes the concatenation of $\xi \in E$ and $\omega \in \Omega^+$) is a $\max(\alpha, 1/\gamma)$ contraction for the metric $d_M \times d_\alpha$. In particular, we shall see that we are in the framework of Ruelle [1990] or Fried [1995].

We call such a system (τ^+, g, θ) (note that the pair (E, θ) contains all the information) a C^r *weighted independent identically distributed (i.i.d.) expanding map.* If all the g_ω are real and positive-valued (respectively nonnegative-valued) the system is called *positively weighted* (respectively nonnegatively weighted).

A special case of a (family of) random i.i.d. expanding maps is obtained by considering *small random perturbations* of (f_0, g_0), for $f_0 \in C^r_\gamma(M, M)$ and $g_0 \in C^r(M, \mathbb{C})$: For each small $\epsilon \geq 0$ we have a probability measure θ_ϵ on some fixed E as above, with

$$\operatorname{supp} \theta_\epsilon \subset B_\epsilon(f_0, g_0)\,, \tag{2.5}$$

where B_ϵ is the ϵ-ball in the d_E metric. (In particular, θ_0 is the Dirac mass at (f_0, g_0).) Many of our results concern this special case.

Remark 2.1. Another model in which our arguments work without modification is given by the following data: let M, γ, r, $C^r_\gamma(M, M)$, $C^r(M, \mathbb{C})$ be as above, let (E, d_E) be a compact metric space endowed with a probability measure θ, set $\Omega^+ = E^{\mathbb{Z}_+}$ endowed with metrics d_α for $0 < \alpha < 1$, and consider

$$f : \Omega^+ \to C^r_\gamma(M, M)\,, \quad g : \Omega^+ \to C^r(M, \mathbb{C})\,,$$

two Lipschitz functions with $f(\omega) = f(\omega_0)$ and $g(\omega) = g(\omega_0)$, which we view as random variables on $(\Omega^+, \theta^{\mathbb{Z}_+})$ (or equivalently (E, θ)). The rest of the setup is as above. This other description is more convenient to describe one-dimensional random Ising models in Section 2.4. Also, it allows generalisations to Lipschitz g on Ω^+ which depend on the full sequence $\omega_0, \omega_1, \ldots$ (but assuming still that $f(\omega) = f(\omega_0)$). In this case, most of our results hold (see the discussion on the operator $\widehat{\mathcal{M}}$ in Section 4). The only differences are that the operator $\mathcal{L}$ does not exist any more (we must work with $\widehat{\mathcal{L}}$, most notably in Theorems 4 and 6, and the maximal eigenfunction $\hat\rho(x, \omega)$ of Proposition 3.1 (2) can depend on ω) and that the definition of the annealed zeta function and determinant (2.24-2.25) must be slightly changed (completing periodically the sequences $\vec\xi$ appearing in $g_{\vec\xi}$).

2.2. Relativised equilibrium and Gibbs states (quenched and annealed).

We assume in this subsection that all weights g_ω are *real and nonnegative.*

Quenched and annealed equilibrium states

Recall that an *equilibrium state for the relativised variational principle* (Ledrappier-Walters [1977], Ruelle [1978 (Sections 6.21-22)], Kifer [1992]), for τ^+, $g(x, \omega) = g_{\omega_0}(x)$, and θ is a Borel probability measure $\mu \in \mathcal{P}_\Theta$ which realises the supremum

$$Q^{(q)}(\log g) = \sup\{h_{\tau^+}(\nu|\Theta^+) + \int \log g(x, \omega)\, \nu(dx, d\omega) \mid \nu \in \mathcal{P}_\Theta\}\,, \tag{2.6}$$

where $h_{\tau^+}(\nu|\Theta^+)$ denotes the relative entropy of ν with respect to its marginal $\pi_\nu = \Theta^+$. We shall apply the formula from Bogenschütz–Crauel [1992]

$$h_{\tau^+}(\nu|\pi_\nu) = \sup_{\mathcal{Q} \text{ finite partition of } M} \lim_{n\to\infty} \frac{1}{n} \int_{\Omega^+} H_{\nu^\omega}\Big(\bigvee_{i=0}^{n-1} (f_\omega^{(n)})^{-1}\mathcal{Q}\Big)\, \pi_\nu(d\omega)\,, \tag{2.7}$$

where we use the essentially unique decomposition

$$\nu(dx, d\omega) = \nu^\omega(dx)\pi_\nu(d\omega)\,, \tag{2.8}$$

and where the entropy $H_\upsilon(\mathcal{Q})$ of a measure υ for finite partition $\mathcal{Q}$ is defined as usual by $-\sum_{Q\in\mathcal{Q}} \upsilon(Q)\log\upsilon(Q)$. We call the relativised equilibrium states defined by (2.6) *quenched (relativised) equilibrium states* for τ^+, g, and θ, and the supremum $Q^{(q)}(\log g)$ the *quenched (relativised) topological pressure* of τ^+, g, and θ.

We now introduce a new type of invariant equilibrium measure. Recall that the *specific entropy per site* $h^\theta(\upsilon)$ of a σ^+ invariant measure υ, relative to the *a priori* measure θ on E is

$$h^\theta(\upsilon) = -\int \log\beta(\xi\omega)\, \upsilon(d(\xi\omega)) = -\int \log\beta(\xi\omega)\ \beta(\xi\omega)\, \theta(d\xi)\, \upsilon(d\omega) \tag{2.9}$$

if $\upsilon(d(\xi\omega))$ is absolutely continuous with respect to $\theta(d\xi)\upsilon(d\omega)$, with Radon-Nikodym derivative $\beta(\xi\omega)$, and otherwise, $h^\theta(\upsilon) = -\infty$. (See Georgii [1988, pp. 317–318], Pinsker [1964, Section 15.2]: the two-sided framework there may be adapted to our one-sided shift σ^+.) Define now an *annealed (relativised) equilibrium state* for τ^+, g and the *a priori* measure θ to be a τ^+-invariant Borel probability measure μ on $M\times\Omega^+$ realising the following supremum:

$$Q^{(a)}(\log g) = \sup\{h_{\tau^+}(\nu|\pi_\nu) + h^\theta(\pi_\nu) + \int \log g(x,\omega)\, \nu(dx,d\omega)\}\,, \tag{2.10}$$

the supremum being over all τ^+-invariant Borel probability measures ν on $M\times\Omega^+$.

We call $Q^{(a)}(\log g)$ the *annealed topological pressure* of τ^+, g, and θ.

Since $h^\theta(\pi_\mu) = h^\theta(\Theta^+) = 0$ if and only if $\mu\in\mathcal{P}_\Theta$ (see e.g. Georgii [1988]), we have $Q^{(a)}(\log g)\ge Q^{(q)}(\log g)$. In some cases equality holds, but not always (see in particular Proposition 2 and Remark 3.4 below).

Quenched and annealed Gibbs states

We introduce first the *random transfer operators* $\mathcal{L}_\xi : C^r(M,\mathbb{C}) \to C^r(M,\mathbb{C})$, defined for $\xi\in E$ by

$$\mathcal{L}_\xi\varphi(x) = \sum_{f_\xi(y)=x} \varphi(y)\, g_\xi(y)\,. \tag{2.11}$$

We also write for $n\ge 1$ and $\omega\in\Omega^+$

$$\mathcal{L}_\omega^n = \mathcal{L}_{\omega_{n-1}}\circ\cdots\circ\mathcal{L}_{\omega_1}\circ\mathcal{L}_{\omega_0}\,. \tag{2.12}$$

Define now a *quenched (relativised) Gibbs state* for τ^+, g, and θ (see Khanin-Kifer [1996], and also Bogenschütz-Gundlach [1995] who used a slightly different but equivalent definition) to be a measure $\mu \in \mathcal{P}_\Theta$ such that the probability measures μ^ω on M arising in the essentially unique decomposition (2.8) satisfy

$$\mu^\omega \text{ is absolutely continuous with respect to } \nu^\omega \,, \tag{2.13}$$

where (Θ^+-almost) each measure ν^ω is a *quenched (relativised) Gibbs measure* for τ^+, g, and θ, i.e., satisfies

$$\int \varphi(x)\,\nu^\omega(dx) = \int \frac{\mathcal{L}_\omega^n \varphi(x)}{\mathcal{L}_\omega^n \mathbf{1}(x)}\,(f_\omega^{(n)})^* \nu^\omega(dx)\,, \tag{2.14}$$

for all continuous $\varphi : M \to \mathbb{C}$, and all $n \geq 1$, where $\mathbf{1}$ denotes the constant function $= 1$ on M. Clearly, the definition of a Gibbs measure is equivalent to requiring that the conditional probabilities $\nu_n^\omega(dy|x)$ of the measure ν^ω conditioned by $f_\omega^{(n)}(y) = x$ be the discrete measures defined on the finite set $(f_\omega^{(n)})^{-1}(x)$ by

$$\int \varphi(y)\nu_n^\omega(dy|x) = \frac{\mathcal{L}_\omega^n \varphi(x,\omega)}{\mathcal{L}_\omega^n \mathbf{1}(x,\omega)}\,. \tag{2.15}$$

Defining the *integrated transfer operator* $\widehat{\mathcal{L}}$ acting on measurable functions $\varphi : M \times \Omega^+ \to \mathbb{C}$ (we write $\varphi(x,\omega) = \varphi_\omega(x)$) by:

$$\widehat{\mathcal{L}}\varphi(x,\omega) = \int_E (\mathcal{L}_\xi \varphi_{\xi\wedge\omega})(x)\,\theta(d\xi)\,, \tag{2.16}$$

we define an *annealed (relativised) Gibbs measure* for τ^+, g and the a priori measure θ to be a Borel probability measure ν on $M \times \Omega^+$ such that for all measurable $\varphi : M \times \Omega^+$ and all $n \geq 0$

$$\int \varphi(x,\omega)\,\nu(dx,d\omega) = \int \frac{\widehat{\mathcal{L}}^n \varphi(x,\omega)}{\widehat{\mathcal{L}}^n \mathbf{1}(x,\omega)}\,((\tau^+)^n)^* \nu(dx,d\omega)\,. \tag{2.17}$$

Again, there is an interpretation in terms of conditional probabilities: a Borel probability measure ν on $M \times \Omega^+$ is an annealed Gibbs measure if for any integer $n \geq 1$ the conditional probability $\nu_n((dy,d\eta)|(x,\omega))$ under the condition $(\tau^+)^n(y,\eta) = (x,\omega)$ (in particular $\eta_{n+j} = \omega_j$ for $j \geq 0$) is equal to the Radon measure

$$\int \varphi(y,\eta)\nu_n(dy,d\eta|(x,\omega)) = \frac{\widehat{\mathcal{L}}^n \varphi(x,\omega)}{\widehat{\mathcal{L}}^n \mathbf{1}(x,\omega)}\,. \tag{2.18}$$

Finally we define an *annealed (relativised) Gibbs state* for τ^+, g and the a priori measure θ to be a Borel probability measure μ on $M \times \Omega^+$ which is τ^+-invariant and absolutely continuous with respect to an annealed Gibbs measure on $M \times \Omega^+$ for τ^+, g, and θ.

2.3 Results.

Let us first recall results due to Kifer, Khanin-Kifer, and Bogenschütz-Gundlach:

Theorem (Unique quenched Gibbs and equilibrium states $\mu^{(q)}$). *A C^r positively weighted i.i.d. expanding map (τ^+, g, θ) admits a unique quenched (relativised) Gibbs state and a unique quenched (relativised) equilibrium state. These two measures coincide.*

For a proof, see Kifer [1992, Theorem A], Khanin-Kifer [1996, Theorem C] and Bogenschütz-Gundlach [1995] (their results are for the two-sided skew product τ in (1.5), but give readily our claim by integration).

Our first main result is:

Theorem 1 (Unique annealed Gibbs and equilibrium states $\mu^{(a)}$). *A C^r positively weighted i.i.d. expanding map (τ^+, g, θ) admits a unique annealed Gibbs state and a unique annealed equilibrium state. These two measures coincide.*

We prove Theorem 1 in Section 3.2. In fact, we also construct there (non necessarily unique) annealed states for nonnegative weights.

In the case of SRB measure the quenched and annealed states are the same:

Proposition 2 (SRB). *For a C^r weighted i.i.d. expanding map (τ^+, g, θ) with $g_\omega(x) = 1/|JacD_x f_\omega|$, the unique annealed equilibrium state and the unique quenched equilibrium state coincide.*

Essentially in the same setting, Kifer [1992] proved that for a C^r weighted i.i.d. expanding map (τ^+, g, θ) with $g_\omega(x) = 1/|\mathrm{Jac}D_x f_\omega|$, the (relativised) equilibrium state is a direct product $\rho \times \Theta^+$, with ρ a measure equivalent with Riemannian volume on M and invariant for the *Markov chain* corresponding to (τ^+, θ), which is defined by the transition probabilities

$$\mathcal{P}(x, A) = \int \chi_A(f_{\omega_0}(x))\, \theta(d\omega_0) \tag{2.19}$$

for $x \in M$ and $A \subset M$ Borel. (The marginal on M of an arbitrary τ^+ invariant measure is not invariant for the Markov chain in general.)

Proposition 2 is a consequence of Remark 3.4 in Section 3. One-dimensional i.i.d. random Ising models give simple examples where the quenched and annealed states differ (see Section 2.4).

We obtain the following stability result in Section 3.4 (the second claim in Theorem 3 was obtained previously by Kifer [1992, Section 4; 1990], see also Bogenschütz [1994]):

Theorem 3 (Stochastic stability for annealed and quenched states). *Let μ_0 be the equilibrium state for $f_0 \in C^r_\gamma(M, M)$ and $\log g_0$ with $g_0 \in C^r(M, \mathbb{R}^+_*)$. Consider a positively weighted small random perturbation of f_0, g_0 given by a family θ_ϵ ($\epsilon \geq 0$).*

(1) *The annealed equilibrium states $\mu_\epsilon^{(a)}$ weakly converge to $\mu_0 \times \delta_{f_0,g_0}^{\mathbb{Z}_+}$, where δ_{f_0,g_0} is the Dirac measure at (f_0, g_0) as $\epsilon \to 0$. The annealed relativised pressure $Q_\epsilon^{(a)}(\log g)$ converges to the topological pressure $P(\log g_0)$ of f_0.*

(2) *The quenched equilibrium states* $\mu_\epsilon^{(q)}$ *weakly converge to* $\mu_0 \times \delta_{f_0,g_0}^{\mathbb{Z}_+}$ *as* $\epsilon \to 0$. *The quenched relativised pressure* $Q_\epsilon^{(q)}(\log g)$ *converges to the topological pressure* $P(\log g_0)$ *of* f_0.

Integrated correlation functions

If μ is a τ^+-invariant probability measure, we define its *integrated* random *correlation function* for $\varphi_1, \varphi_2 \in L^2(\mu)$, and any integer $n \geq 0$ by

$$\begin{aligned} C_{\varphi_1\varphi_2}(n) = \int (\varphi_1 \circ (\tau^+)^n)(x,\omega)\, \varphi_2(x,\omega)\, \mu(dx, d\omega) \\ - \int \varphi_1(x,\omega)\, \mu(dx, d\omega) \int \varphi_2(x,\omega)\, \mu(dx, d\omega)\,. \end{aligned} \tag{2.20}$$

For φ_1, φ_2 in some function class $\mathcal{F}$, we may ask if $|C_{\varphi_1\varphi_2}(n)|$ goes to zero exponentially fast, i.e., if there exists $\tau < 1$ so that for any $\varphi_1, \varphi_2 \in \mathcal{F}$ there is $K(\varphi_1, \varphi_2)$ with $|C_{\varphi_1\varphi_2}(n)| \leq K(\varphi_1, \varphi_2) \cdot \tau^n$ for all n (the smallest such τ is called the (exponential) *rate of decay of correlations* for μ and $\mathcal{F}$). More generally, we can ask if the formal Fourier transform (see Pollicott [1985], Ruelle [1987] for corrresponding objects in a nonrandom setting)

$$\widehat{C}_{\varphi_1\varphi_2}(\eta) = \sum_{n=0}^{\infty} C_{\varphi_1\varphi_2}(n)\, e^{i\eta n} + \sum_{n=1}^{\infty} C_{\varphi_2\varphi_1}(n)\, e^{-i\eta n} \tag{2.21}$$

admits an analytic extension to a strip and a meromorphic extension to a larger domain of the complex plane. Using the notation $\mathcal{L}$ to represent the restriction to $C^r(M, \mathbb{C})$ of the operator $\widehat{\mathcal{L}}$ defined in (2.16), and referring to (4.12) Section 4 for the definition of $\widehat{\mathcal{M}}$ (and to Section 3.1 for the definition of the Banach space $\mathcal{B}(\alpha)$) we have (see Sections 3.4, 3.6 and 4 for proofs):

Theorem 4 (Annealed and quenched integrated correlation spectrum). *Set* $\mathcal{F} = C^r(M, \mathbb{C})$.

(1) *The Fourier transform* $\widehat{C}_{\varphi_1\varphi_2}(\eta)$ *of the integrated correlation function of the annealed equilibrium state* $\mu^{(a)}$ *of a* C^r *positively weighted i.i.d. expanding map* (τ^+, g, θ) *for test functions in* $\mathcal{F}$ *is analytic in a strip* $|\Im(\eta)| \leq \delta^{(a)}$ *for some* $\delta^{(a)} > 0$ *and admits a meromorphic extension to the strip* $|\Im(\eta)| \leq \log \gamma^r$ *where its poles appear at points* η, *where* $\lambda = \exp(-i\eta + Q^{(a)}(\log g))$ *is an eigenvalue of* $\mathcal{L}$ *acting on* $\mathcal{F}$ *with* $\exp(Q^{(a)}(\log g))/\gamma^r < |\lambda| < \exp(Q^{(a)}(\log g))$.

(2) *Let* $\alpha > 1/\gamma$. *The Fourier transform of the integrated correlation function of the quenched equilibrium state* $\mu^{(q)}$ *of a* C^r *positively weighted i.i.d. expanding map* (τ^+, g, θ) *for test functions in* $\mathcal{F}$ *or* $\mathcal{B}(\alpha)$ *is analytic in a strip* $|\Im(\eta)| \leq \delta^{(q)}$ *for some* $\delta^{(q)} > 0$ *and admits a meromorphic extension to the strip* $|\Im(\eta)| \leq \log 1/\alpha$ *where its poles appear at points* η, *where* $\lambda = \exp(-i\eta)$ *is an eigenvalue of* $\widehat{\mathcal{M}}$ *acting on* $\mathcal{B}(\alpha)$ *satisfying* $\alpha < |\lambda| < 1$.

In particular, the rate of decay of $\mu^{(a)}$ for $\mathcal{F} = C^r(M, \mathbb{C})$ coincides with the ratio of the moduli of the "first two eigenvalues" of $\mathcal{L}$ acting on $\mathcal{F}$ and similarly for $\mu^{(q)}$ and $\widehat{\mathcal{M}}$.

Motivated by Theorem 4, and since we will show in Proposition 3.1 (respectively Proposition 4.2) that the essential spectral radius of $\mathcal{L}$ on $\mathcal{F}$ (respectively $\widehat{\mathcal{M}}$ on $\mathcal{B}(\alpha)$ with $\alpha > 1/\gamma$) is not bigger than $\exp(Q^{(a)}(\log g))/\gamma^r$ (respectively α) we call the (discrete) spectrum of $\mathcal{L}$ in the annulus $\exp(Q^{(a)}(\log g))/\gamma^r < |\lambda| < \exp(Q^{(a)}(\log g))$ (respectively of $\widehat{\mathcal{M}}$ in $\alpha < |\lambda| < 1$) the *annealed (respectively quenched) integrated correlation spectrum* of the measure $\mu^{(a)}$ and the function class $\mathcal{F}$ (respectively $\mu^{(q)}$ and $\mathcal{B}(\alpha)$).

Regarding small random perturbations, we shall show in Sections 3.4, 3.6, and 4:

Theorem 5 (Stability of the annealed/quenched integrated correlation spectrum). *Let μ_0 be the equilibrium state for $f_0 \in C^r_\gamma(M, M)$ and $\log g_0$, with $g_0 \in C^r(M, \mathbb{R}^+_*)$, let $P(\log g_0)$ be the corresponding pressure, and let $\tau_0 < 1$ be the rate of decay of correlations for μ_0 and $\mathcal{F} = C^r(M, \mathbb{C})$. Consider a positively weighted small random perturbation of f_0, g_0 given by a family θ_ϵ ($\epsilon \geq 0$).*

1. *The rate of decay $\tau_\epsilon^{(a)}$ of correlations for the annealed equilibrium state $\mu_\epsilon^{(a)}$ of $(\tau^+, g, \theta_\epsilon)$, and test functions in $\mathcal{F}$, satisfies $\limsup_{\epsilon\to 0} \tau_\epsilon^{(a)} \leq \tau_0$. In fact, outside of any disc of radius $e^{P(\log g_0)}/\gamma^r + \delta$, ($\delta > 0$) the integrated correlation spectrum of $\mu_\epsilon^{(a)}$ for $\mathcal{F}$ converges to the correlation spectrum of μ_0 for $\mathcal{F}$.*
2. *The rate of decay $\tau_\epsilon^{(q)}$ of correlations for the quenched equilibrium state $\mu_\epsilon^{(q)}$ of $(\tau^+, g, \theta_\epsilon)$, and test functions in $\mathcal{B}(\alpha)$ ($\alpha > 1/\gamma$), satisfies $\limsup_{\epsilon\to 0} \tau_\epsilon^{(q)} \leq \tau_0$. In fact, outside of any disc of radius $e^{P(\log g_0)}/\gamma + \delta$ ($\delta > 0$) the integrated correlation spectrum of $\mu_\epsilon^{(q)}$ for $\mathcal{B}(\alpha)$ converges to the correlation spectrum of μ_0 for $\mathcal{F}$.*

Annealed zeta functions and annealed generalised Fredholm determinants

First consider the deterministic system f, g (g not necessarily real or positive), and define the formal *zeta function* (see e.g. Ruelle [1989] and references therein) by

$$\zeta(z) = \exp \sum_{m\geq 1} \frac{z^m}{m}\zeta_m \quad \text{where } \zeta_m = \sum_{f^m(x)=x} \prod_{j=0}^{m-1} g(f^j(x)) \,. \tag{2.22}$$

A second formal series, the *generalised Fredholm determinant* (Ruelle [1990]), may be associated with the deterministic system by setting

$$d(z) = \exp - \sum_{m\geq 1} \frac{z^m}{m} d_m \quad \text{where } d_m = \sum_{f^m(x)=x} \frac{\prod_{j=0}^{m-1} g(f^j(x))}{\det(1 - D_x f^{-m})} \,, \tag{2.23}$$

where $D_x f^{-m}$ denotes the derivative of the local inverse branch of f^m associated to the m-periodidic orbit of x.

For a weighted i.i.d. map (τ^+, g, θ), we define the formal *annealed zeta function* by

$$\zeta^{(a)}(z) = \exp \sum_{m\geq 1} \frac{z^m}{m} \zeta^{(a)}_m$$
$$\text{where } \zeta^{(a)}_m = \int_{E^m} \sum_{f^{(m)}_{\vec\xi}(x)=x} \prod_{j=0}^{m-1} g_{\xi_j}(f^{(j)}_{\vec\xi}(x))\, \theta(d\xi_0)\dots\theta(d\xi_{m-1})\,. \tag{2.24}$$

Similarly, we get an *annealed Fredholm determinant* by setting

$$d^{(a)}(z) = \exp - \sum_{m\geq 1} \frac{z^m}{m} d^{(a)}_m$$
$$\text{where } d^{(a)}_m = \int_{E^m} \sum_{f^{(m)}_{\vec\xi}(x)=x} \frac{\prod_{j=0}^{m-1} g_{\xi_j}(f^{(j)}_{\vec\xi}(x))}{\det(1 - D_x f^{(-m)}_{\vec\xi})} \theta(d\xi_0)\dots\theta(d\xi_{m-1})\,. \tag{2.25}$$

The following stability result will be a consequence of the proofs in Ruelle [1990] and the spectral stability obtained in Proposition 3.5 below (see Section 3.5):

Theorem 6 (Annealed zeta functions and Fredholm determinants). *Consider a C^r weighted i.i.d. expanding map (τ^+, g, θ), write $R^{(a)} = \exp Q^{(a)}(\log|g|)$, and let $\mathcal{F} = C^r(M, \mathbb{C})$.*

1. *The annealed zeta function $\zeta^{(a)}(z)$ is analytic in the disc of radius $1/R^{(a)}$ and admits a meromorphic, zero-free, extension to the open disc of radius $\gamma/R^{(a)}$, where its poles are exactly the inverses of the eigenvalues of $\mathcal{L}$ acting on $\mathcal{F}$ of modulus $> R^{(a)}/\gamma$ (including multiplicities).*
2. *The annealed Fredholm determinant $d^{(a)}(z)$ admits an analytic extension to the disc of radius $\gamma^r/R^{(a)}$ where its zeroes are exactly the inverses of the eigenvalues of $\mathcal{L}$ acting on $\mathcal{F}$ of modulus $> R^{(a)}/\gamma^r$ (including multiplicities).*
3. *In the case of a weighted small random perturbation $(\tau^+, g, \theta_\epsilon)$ of (f_0, g_0), writing $R_0 = \exp P(\log|g_0|)$, the functions $(\zeta^{(a)}_\epsilon(z))^{-1}$, respectively $d^{(a)}_\epsilon(z)$ converge to $(\zeta(z))^{-1}$, respectively $d(z)$, as $\epsilon \to 0$ in any compact subset of the disc of radius γ/R_0, respectively γ^r/R_0, in the sense of analytic functions.*

2.4 Two examples.

A conjecture of Ruelle

Our first example is taken from Ruelle [1995, Section 7.4]. Assume that E is countable or finite (with $\theta(\xi) > 0$ for all $\xi \in E$) and that g is nonnegative. Assume also that the spectral radius $R > 0$ of the operator $\mathcal{L}$ acting on $C^r(M, \mathbb{C})$ is the only eigenvalue of modulus R and is simple (this is true for example if g is positive, see Section 3.1). We start by giving a different characterisation of the annealed equilibrium state μ for (τ^+, g)

and θ, assuming further that μ has the property that $h_{\sigma^+}(\pi_\mu) < \infty$. In this case we have on the one hand

$$h_{\tau^+}(\mu|\pi_\mu) = h_{\tau^+}(\mu) - h_{\sigma^+}(\pi_\mu)\,, \tag{2.26}$$

and on the other

$$h_{\sigma^+}(\pi_\mu) = -\int_{\Omega^+} \log \frac{\pi_\mu(\omega)}{\pi_\mu(\sigma^+\omega)}\, \pi_\mu(d\omega)\,, \tag{2.27}$$

where $\pi_\mu(\omega)/\pi_\mu(\sigma^+\omega)$ denotes the Radon-Nikodym derivative (recall that E is countable, π_μ is σ^+-invariant, and use Theorem 4.14 in Walters [1982]). It thus follows from the definition of the specific entropy per site that

$$\begin{aligned} h_{\tau^+}(\mu|\pi_\mu) + h^\theta(\pi_\mu) &+ \int_{M\times\Omega^+} \log g_{\omega_0}(x)\, \mu(dx, d\omega) \\ &= h_{\tau^+}(\mu) + \int_{M\times\Omega^+} \log(\theta(\omega_0) g_{\omega_0}(x))\, \mu(dx, d\omega)\,. \end{aligned} \tag{2.28}$$

In other words, the annealed equilibrium state μ is the (almost ordinary[1]) equilibrium state for τ^+ and the weight $G(x,\omega) = \theta(\omega_0) g_{\omega_0}(x)$ on $M \times \Omega^+$, whenever $h_{\sigma^+}(\pi_\mu) < \infty$. (Note that this finiteness property does *not* always hold for E countable infinite: just consider $g_\xi = 1/|\mathrm{Det}\, Df_\xi|$ so that the annealed equilibrium state satisfies $\pi_\mu = \theta^{\mathbb{Z}_+}$, and take $\theta(n)$ of the order of $1/(n(\log n)^2)$; an example where it does hold would be given by an a priori measure of the order of $1/n^2$.)

To obtain a τ^+-invariant measure, Ruelle starts from ρ, the nonnegative eigenfunction of $\mathcal{L}$ associated to its spectral radius $R \geq 0$ (Ruelle [1989, 1990]), and constructs a measure $\bar\nu$ on $M \times \Omega^+$ by iterating the corresponding eigenfunctional ν of $\mathcal{L}^*$ (noting that ν is a positive measure):

$$\begin{aligned} &\bar\nu(dx, d\omega_0, d\omega_1, \dots) \\ &\quad= \lim_{m\to\infty} \theta(d\omega_0)\dots\theta(d\omega_m) \sum_{i_0} (f_{\omega_0})^{-1}_{i_0} (R^{-1} g_{\omega_0} \cdot (\cdots \sum_{i_m} (f_{\omega_m})^{-1}_{i_m} (R^{-1} g_{\omega_m} \nu(dx))\cdots)\,, \end{aligned} \tag{2.29}$$

where we use the notation $(f_\xi)^{-1}_i$ for the finitely many inverse branches of f_ξ. He then considers the normalisation $\upsilon(dx, d\omega)$ of the τ^+-invariant measure $\rho(x)\bar\nu(dx, d\omega)$ and formulates the

Conjecture (Ruelle [1995, Section 7.4]): *The spectral radius R is the exponential of the topological pressure of $\log G$ for τ^+. The τ^+-invariant probability measure $\upsilon = \rho\bar\nu/\bar\nu(\rho)$ is an equilibrium state for the dynamical system τ^+ on $(M \times \Omega^+)$ and the function $\log G : M \times \Omega^+ \to \mathbb{R} \cup -\infty$.*

It is not difficult to check that $\widehat{\mathcal{L}}^* \bar\nu = R\bar\nu$, so that $\bar\nu = \hat\nu$, the maximal eigenfunctional of $\widehat{\mathcal{L}}^*$ (Proposition 3.1). The invariant measure υ is thus the annealed equilibrium state μ for τ^+, g and θ by Proposition 3.2 below. We have therefore proved the above conjecture,

[1]Restricting to τ^+ invariant measures with finite entropy $h_{\tau^+}(\nu)$ in the variational principle.

under the *assumptions*, nontrivial when E is infinite, that $h_{\sigma^+}(\pi_v) < \infty$ (equivalently $h_{\tau^+}(v) < \infty$, note that the conjecture needs to be reformulated otherwise) and that $E = \{(f_\xi, g_\xi), \xi \in \mathbb{Z}_+\} \subset C^r_\gamma(M,M) \times C^r(M,\mathbb{C})$ is compact for the induced metric.

Ruelle actually works with a countable family of inverse branches instead of expanding maps f_ω defined on the entire space M. However, the assumptions he makes on the support of the corresponding weights ensure that our arguments in Sections 2 and 3 carry through.

One dimensional exponentially decaying random Ising model

Our second example is a one-dimensional Ising model with i.i.d. random external field and coupling constants (see e.g. Ledrappier [1977] for a description in terms of relative variational principle and references). More precisely, we work on the half-lattice $\mathbb{Z}_+$ and consider the full shift f on the metric space $M = \{\pm 1\}^{\mathbb{Z}_+}$ with a metric $d_M(x,y) = \sum_{k\geq 0} |x_k - y_k|/\gamma^k$ for $\gamma > 1$ (a compact set of continuous spins could also be considered). It is more convenient to work in the setup described in Remark 2.1, considering of course Lipschitz (instead of C^r) functions on M. For the weight, we can fix for example some $\beta \geq 0$, consider a probability law $\theta = \theta_1 \times \theta_2$ on a compact square $E = [-A, A]^2 \subset \mathbb{R}^2$ (see Ledrappier [1977] on how to remove the compactness assumption in the nearest neighbour interaction case), and set

$$g_{\omega_0}(x) = \exp(-\beta \cdot (h_{\omega_0} x_0 + J_{\omega_0} x_0 \cdot x_1)), \; x \in M, \tag{2.30}$$

with $\omega_0 = (h_{\omega_0}, J_{\omega_0})$ picked in E with law θ. (At the end of this subsection we explain how to generalise to long-range *exponentially decaying* interactions.) The physical interpretation is that $\log g_\omega(f^k x)$ is the random contribution to the Hamiltonian associated with the k^{th} site of the configuration x (i.e., the sum of the interaction between the k^{th} site and the $(k+j)^{\text{th}}$ sites for $j \geq 0$, as well as the term from the external random field acting on x_k). Note that since the skew product is in fact a direct product here, the marginal on M of an annealed or quenched state will be a shift invariant measure on M.

In other words, if we define by the usual formula the partition function $Z_n(\omega, x) = Z_n(\omega_0, \ldots, \omega_{n-1}, x)$ ($\omega \in \Omega^+$, $x \in M$) of a finite one-sided box $[0, n-1]$ corresponding to the random Hamiltonian with fixed boundary condition $y_{n+j} = x_j$, $j \geq 0$, we find that $Z_n(\omega, x) = \mathcal{L}^{(n)}_\omega \mathbf{1}(x)$. The results of Ledrappier [1977] for finite range interaction and more generally (a slight modification) of Kifer [1992, Theorem 3.2 iii] imply that for $\theta^{\mathbb{Z}_+}$-almost all ω and all $x \in M$

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\omega, x) = Q^{(q)}(\log g). \tag{2.31}$$

Therefore it follows from Proposition 3.2 below that for $\theta^{\mathbb{Z}_+}$-almost all ω and all $x \in M$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log Z_n(\omega, x) &= Q^{(q)}(\log g) \\ &\leq Q^{(a)}(\log g) = \lim_{n\to\infty} \frac{1}{n} \log \int_{E^n} Z_n(\omega, x)\theta(d\omega_0)\cdots\theta(d\omega_{n-1}),\end{aligned} \tag{2.32}$$

with a *strict inequality* in general (for example, in the simplest case (2.30), with $h \equiv 0$ we see that the necessary condition stated in Remark 3.4 below is only satisfied when J takes θ-almost everywhere at most two values). Our definitions of one-sided quenched and annealed Gibbs states are consistent with the standard terminology and we recover in particular by Theorem 4 the folklore theorem of exponential decay of correlations for both states (note that the integrated correlations (2.20) are simply the space-correlation functions of observables in phase space M for the shift-invariant M-marginal). Even when the quenched and annealed states are different, it is not obvious that they have different marginals on M. Since the physically observable measure is this M-marginal, it would be of interest if possible to find conditions ensuring that the quenched and annealed marginal are the same.

We point out also that considering partition functions $Z_n^{(\mathrm{per})}(\omega)$ with periodic boundary conditions $x_{n+j} = x_j$, $j \geq 0$ yields

$$Q^{(a)}(\log g) = \lim_{n\to\infty} \frac{1}{n} \log \int_{E^n} Z_n^{(\mathrm{per})}(\omega)\theta(d\omega_0)\cdots\theta(d\omega_{n-1}) \tag{2.33}$$

(because $\zeta_m^{(a)} = \int_{E^n} Z_n^{(\mathrm{per})}(\omega)\theta(d\omega_0)\cdots\theta(d\omega_{n-1})$ and by Theorem 6 (1) on the annealed zeta function).

We may also consider exponentially decaying long-range interactions such as

$$g_{\omega_0}(x) = \exp\left[-\beta\cdot\left(h_{\omega_0^0}x_0 + \sum_{j=1}^{\infty} J_{\omega_0^j}\frac{x_0\cdot x_j}{\gamma^j}\right)\right],\ x\in M\,, \tag{2.34}$$

with $\omega_0 = (h_{\omega_0^0}, J_{\omega_0^j}, j \geq 1)$ chosen in $E = [-A,A]\times[-A,A]^{\mathbb{Z}_+}$ with law $\theta_1 \times \theta_2^{\mathbb{Z}_+}$ where θ_1 and θ_2 are two probabilities on $[-A,A]$. Since we are in a purely Lipschitz context, there is no need to modify the results of Ruelle [1990] (see beginning of the proof of Proposition 3.1 (2)) and it suffices to check that g is a Lipschitz function on $M \times \Omega^+$ if we endow E with a metric d_α for $\alpha > 1/\gamma$ in order to apply our results.

3. The annealed transfer operators $\widehat{\mathcal{L}}$

3.1 The integrated annealed transfer operators $\widehat{\mathcal{L}}$.

Let $\mathcal{B} = \mathcal{B}(\alpha)$ denote the Banach space of Lipschitz functions $\varphi : \Omega^+ \to C^r(M,\mathbb{C})$ (for the metric d_α on Ω and d_r on $C^r(M,\mathbb{C})$) endowed with the norm $\|\varphi\|_\alpha = \sup_\omega \|\varphi\|_r + \mathrm{Lip}_\omega\varphi$, where $\mathrm{Lip}_\omega\varphi$ denotes the smallest Lipschitz constant. We may view an element of $\mathcal{B}$ as a function on $M\times\Omega^+$ by setting $\varphi(x,\omega) = \varphi(\omega)(x)$ and it is easy to see that the operator $\widehat{\mathcal{L}}$ defined by (2.16) preserves the Banach space $\mathcal{B}$.

We consider the operator $\mathcal{L} = \mathcal{L}_g$ defined by restricting $\widehat{\mathcal{L}}$ (see (2.16)) to measurable functions on M:

$$\mathcal{L}\varphi(x) = \int (\mathcal{L}_\xi\varphi)(x)\theta(d\xi)\,. \tag{3.1}$$

The transfer operator $|\mathcal{L}|$ obtained by replacing g_ξ by $|g_\xi|$ in (3.1) is bounded when acting on the Banach space of bounded functions on M endowed with the supremum norm $\|\varphi\|_\infty$. Denote by $R = R(|g|)$ its spectral radius which satisfies by definition

$$R = \lim_{m\to\infty} (\||\mathcal{L}|^m 1\|_\infty)^{1/m} . \tag{3.2}$$

The basic properties of $\mathcal{L}$ and $\widehat{\mathcal{L}}$ that we shall use are:

Proposition 3.1 (Quasicompacity). *Set $\mathcal{F} = C^r(M, \mathbb{C})$.*

(1) *The spectral radius of $\mathcal{L}$ acting on $\mathcal{F}$ is bounded above by R, its essential spectral radius is bounded above by R/γ^r. If g is nonnegative and $R > 0$, then R is an eigenvalue of $\mathcal{L}$ with a nonnegative eigenfunction $\rho \in \mathcal{F}$. If g is positive then ρ is positive and R is a simple eigenvalue; moreover it is the only eigenvalue of modulus R, and the corresponding eigenfunctional for $\mathcal{L}^*$ is a positive measure ν such that*

$$\lim_{n\to\infty} \sup_{x\in M} \left| \frac{\mathcal{L}^n \varphi(x)}{R^n} - \rho(x)\cdot\nu(\varphi) \right| = 0 \tag{3.3}$$

for all $\varphi \in L^1(\nu)$.

(2) *The essential spectral radius of the operator $\widehat{\mathcal{L}}$ acting on $\mathcal{B}(\alpha)$ is not larger than $R\cdot\max(\alpha, \gamma^{-r})$. The spectra of $\mathcal{L}$ acting on $\mathcal{F}$ and $\widehat{\mathcal{L}}$ acting on $\mathcal{B}(\alpha)$ coincide, including multiplicities in the domain $\{|z| > R\cdot\max(\alpha,\gamma^{-r})\}$. If g is positive, then R is a simple eigenvalue of $\widehat{\mathcal{L}}$ with eigenfunction $\hat\rho$ equal to the eigenfunction ρ of $\mathcal{L}$, the corresponding positive eigenfunctional $\hat\nu$ is a positive measure with marginal ν on M. Also, when g is positive*

$$\lim_{n\to\infty} \sup_{(x,\omega)\in M\times\Omega^+} \left| \frac{\widehat{\mathcal{L}}^n \varphi(x,\omega)}{R^n} - \rho(x)\cdot\hat\nu(\varphi) \right| = 0 \tag{3.4}$$

for all $\varphi \in L^1(\hat\nu)$.

Proof of Proposition 3.1.

(1) The bounds on the spectral and essential spectral radius are proved in Ruelle [1990, Theorem 1.1, Theorem 1.3] (condition (ii) of Ruelle is satisfied up to using a partition of unity). If g is nonnegative, the spectral radius of $\mathcal{L}$ acting on $\mathcal{F}$ is equal to R by (3.2). To prove that there is a corresponding nonnegative eigenfunction, just use the algebra in Ruelle [1989, (4.10–4.12)] (the stronger assumptions of that paper were not used in this particular argument, see also Baladi–Kitaev–Ruelle–Semmes [1995, Proof of Theorem 2.5] for more details). If g is positive then since each f_ω is transitive we may use easy modifications of standard arguments (see e.g. Parry-Pollicott [1990, pp. 23-24]) to show that each nonnegative eigenfunction is positive. To show that there is a nonnegative eigenfunctional which is a positive measure, one may consider as usual the

weight defined by $\bar{g}(y,\omega) = g(y,\omega)\rho(y)/(R\rho(f_\omega(y))$. The constant function 1 is then fixed by the integrated operator $\mathcal{L}_{\bar{g}}$, so that the dual of this operator preserves the compact convex space of Borel probability measures. By Schauder-Tychonoff, $\mathcal{L}_{\bar{g}}^*$ therefore has a fixed point $\bar{\nu}$ and then the normalisation of the measure defined by $\nu = \bar{\nu}/\rho$ is the desired maximal eigenmeasure for $\mathcal{L}$. Since the iterates of $\mathcal{L}_{\bar{g}}$ satisfy a classical Yorke-type inequality (see e.g. Lemma 4.2 in Baladi et al. [1995]) and each f_ω is topologically mixing, the standard convexity argument (see Parry-Pollicott [1990, pp. 25-26]) may be applied to $\mathcal{L}_{\bar{g}}$, yielding $\mathcal{L}_{\bar{g}}^n\varphi \to \nu(\varphi)$ for continuous φ, the result for ν-integrable φ follows from Lusin's theorem. Standard arguments (see Parry-Pollicott [1990, pp. 25-26]) then show that $\mathcal{L}_{\bar{g}}$ restricted to $\{\varphi \in C^r(M) \mid \nu(\varphi) = 0\}$ has spectral radius < 1, so that the spectrum of $\mathcal{L}_{\bar{g}}$ is formed of the simple eigenvalue 1 and a subset of a disc with radius strictly less than 1.

(2) Theorem 1.1 in Ruelle [1990] yields the upper bound $R \cdot \max(\alpha, \gamma^{-1})$ for the essential spectral radius of $\widehat{\mathcal{L}}$ acting on the Banach space of Lipschitz functions on $M \times \Omega^+$ with the metric $d + d_\alpha$. To get the better bound claimed for the space $\mathcal{B}(\alpha)$, we could adapt Ruelle's original computation, but have chosen to follow Fried's [1995] subsequent presentation. Our setting is much simpler than the one considered by Fried, in particular, we are only considering a graph (V, A) with vertex set V reduced to a point $M \times \Omega^+$ so that all arrows A have initial and final vertex equal to $M \times \Omega^+$. The arrows are simply an index set for the contractions which are the inverse branches $\psi_{\xi,i}(x,\omega) = ((f_\xi)_i^{-1}x, \xi\omega)$ (for $\xi \in E$ and i in a finite set depending on ξ) as in (2.4), that we view as being defined on a closed subset $\hat{M}_{\xi,i} \times \Omega^+$ of $M \times \Omega^+$. We say that an n-tuple of local inverse branches of τ_+ is admissible if the corresponding composition $\psi^n_{\vec{\xi},\vec{\imath}} = \psi_{\xi_n,i_n} \circ \cdots \circ \psi_{\xi_1,i_1}$ has a non empty domain of definition $D_{\vec{\xi},\vec{\imath}} \times \Omega^+$ in $M \times \Omega^+$. We need to refine Lemma 1 from Fried [1995], adapting it to our skew-product situation: We claim that there is a constant $C > 0$ so that for all $n \geq 0$, each admissible composition $\psi^n = \psi^n_{\vec{\xi},\vec{\imath}}$ of n local inverse branches of τ^+, and any $(\tilde{x}, \tilde{\omega})$ in the image of $\psi^n(D_{\vec{\xi},\vec{\imath}} \times \Omega^+)$, then if $T = T_{\psi^n}(\tilde{x}, \tilde{\omega})$ denotes the finite rank operator on $\mathcal{B}$ given by $T\varphi =$ the Taylor expansion of order r about $\tilde{x}$ of $\varphi_{\tilde{\omega}}$, we have:

$$\|(\psi^n)^*(I - T)\varphi\|_{\mathcal{B}} \leq C \max(\alpha, \gamma^{-r})^n \|\varphi\|_{\mathcal{B}} \tag{3.5}$$

(see Appendix A for a proof, where we explain the slight differences with Fried's assumptions and bounds). Using (3.5) in place of Lemma 1 in Fried [1995], the proof of Proposition 1 from Fried [1995] combined with the Leibniz-telescoping argument in the proof of Ruelle [1990, Proposition 2.5] (useful to replace the growth rate appearing in Fried [1995] by the better bound R) then yields our claim. (We may ensure Fried's [1995, p. 1064] gap condition by using a suitable partition of unity.)

Applying Theorem 1.1 in Ruelle [1990], we see that the eigenvalues of both $\mathcal{L}$ and $\widehat{\mathcal{L}}$ acting on Lipschitz functions (on M, respectively $M \times \Omega^+$) in the domain

$|z| > R\max(\alpha, \gamma^{-1})$ are exactly (including multiplicities) the inverses of the poles of the zeta function $\zeta^{(a)}(z)$ (2.24). Since any eigenfunction for $\mathcal{L}$ is clearly an eigenfunction for $\widehat{\mathcal{L}}$, the statements on $\hat\rho$ and $\hat\nu$ in the case of a positive weight g follow from the simplicity of the eigenvalue R for $\mathcal{L}$.

Using (3.5) to generalise the main theorem in Fried [1995, Section 3, p. 1067], we obtain a bijection between the spectra of $\mathcal{L}$, $\widehat{\mathcal{L}}$ and the zeroes of the determinant $d^{(a)}(z)$ (2.25) in the bigger domain $|z| > R\max(\alpha, \gamma^{-r})$. (To find the formula for the trace of each finite rank operator $\mathcal{K} = \mathcal{L}_{\psi^n} T$ associated to an admissible composition ψ^n and the corresponding operator $T = T_{\psi^n}(\tilde x, \tilde\omega)$, where we choose $(\tilde x, \tilde\omega)$ to be a fixed point of ψ^n if possible, and otherwise a point in $\psi^n(D_{\vec\xi,\vec\imath} \times \Omega^+) \setminus D_{\vec\xi,\vec\imath} \times \Omega^+$, with $\mathcal{K}$ acting on $\mathcal{B}$, we may compute instead the trace of $\mathcal{K}$ as an operator on functions $\mathcal{B}_{\tilde\omega}$ depending only on the $x \in M$ variable, setting the random argument to be equal to $\tilde\omega$ in the notation above. This is possible because the corresponding projection $\Pi = \Pi_{\tilde\omega} : \mathcal{B} \to \mathcal{B}_{\tilde\omega}$ satisfies $\mathcal{K} = \Pi\mathcal{K}\Pi$ so that $\mathrm{Tr}\ \mathcal{K} = \mathrm{Tr}\ \Pi\mathcal{K}\Pi = \mathrm{Tr}\ \mathcal{K}|_{\mathcal{B}(\tilde\omega)}$ and this last trace is computed as in page 1067 of Fried [1995].) $\square$

3.2 Annealed equilibrium and Gibbs states.

Theorem 1 will be an immediate consequence of Proposition 3.2 and Proposition 3.3:

Proposition 3.2. *Assume that g is positive, and let ρ be the maximal eigenfunction and $\hat\nu$ the maximal eigenmeasure of $\widehat{\mathcal{L}}$ from Proposition 3.1(2). Then the probability measure $\mu = \rho\hat\nu/\hat\nu(\rho)$ is the unique annealed equilibrium state for (τ^+, g, θ). The maximal eigenvalue R of $\widehat{\mathcal{L}}$ is equal to $e^{Q^{(a)}(\log g)}$.*

Proposition 3.3. *Assume that g is positive. The probability measure μ in Proposition 3.2 is the unique annealed Gibbs state for (τ^+, g, θ).*

Proof of Proposition 3.2. To check that the measure μ is τ^+-invariant, consider the following chain of equalities, which holds for any $\varphi \in L^1(\mu)$ (assume that $\hat\nu(\rho) = 1$):

$$\begin{aligned}
\int \varphi(x,\omega)\mu(dx,d\omega) &= \int \varphi_\omega(x)\rho(x)\hat\nu(dx,d\omega) \\
&= \frac{1}{R}\int \varphi_\omega(x) \int \mathcal{L}_\xi(\rho)(x)\theta(d\xi)\hat\nu(dx,d\omega) \\
&= \frac{1}{R}\int\int \mathcal{L}_\xi((\varphi_\omega \circ f_{\xi\omega})\cdot\rho)(x)\theta(d\xi)\hat\nu(dx,d\omega) \\
&= \int \varphi_{\sigma^+\omega}(f_\omega x)\rho(x)\hat\nu(dx,d\omega) = \int \varphi\circ\tau^+(x,\omega)\mu(dx,d\omega)\,.
\end{aligned} \tag{3.6}$$

The basic strategy now is to go to the two-sided situation in order to apply the arguments in Kifer [1992] and Khanin-Kifer [1996]. We will use two other random transfer operators to construct an invariant measure v for the two-sided skew product

τ with same relative entropy as μ. Consider first the random operator $\mathcal{L}'_\xi$, defined by formula (2.11) for the weight

$$g'_\xi(x) = g_\xi(x)\rho(x)/(R \cdot \rho(f_\xi x))$$

(note that $g' \in \mathcal{B}$). The operator $\mathcal{L}'_\xi$ has by definition the property that for all $\varphi \in L^1(\mu)$:

$$\int\int (\mathcal{L}'_\xi \varphi(\xi\omega))(x)\theta(d\xi)\mu(dx, d\omega) = \int \varphi(x, \omega)\mu(dx, d\omega)\,. \tag{3.7}$$

It follows from the definitions that the measure $\pi_\mu(d\xi\omega)$ is equivalent with the product measure $\theta(d\xi)\pi_\mu(d\omega)$ with a density denoted by $\beta(\xi\omega) \in L^1(\pi_\mu)$. In fact, from (3.7) for functions $\varphi(x, \xi\omega)$ independent of x, we obtain by Fubini the explicit formula for π_μ almost all $\xi\omega \in \Omega^+$

$$\beta(\xi\omega) = \int_M \frac{1}{R\rho(x)} \sum_{f_\xi(y)=x} g_\xi(y)\rho(y)\, \mu^\omega(dx)\,, \tag{3.8}$$

(where we use the decomposition (2.8) for μ). It is clear from (3.8) that $\beta(\xi\omega)$ is Θ^+-almost everywhere uniformly bounded and bounded away from zero (combining the (uniform) smoothness and positivity of ρ and g_ξ, together with the fact that the number of inverse branches of the f_ξ is uniformly bounded).

We then define the second modified random operator associated to $g''_\omega = g'_\xi/\beta(\xi\omega)$ by

$$\mathcal{L}''_{\xi\omega}\varphi(x) = \frac{\mathcal{L}'_\xi\varphi(x)}{\beta(\xi\omega)}\,, \tag{3.9}$$

whose dual has the key property that for π_μ-almost all $\xi\omega$:

$$(\mathcal{L}''_{\xi\omega})^*\mu^\omega = \mu^{\xi\omega}\,. \tag{3.10}$$

To prove (3.10), consider an arbitrary $\varphi \in L^1(\mu)$ and write (using (3.7))

$$\begin{aligned}
&\int\int (\mathcal{L}''_{\xi\omega}\varphi(\xi\omega))(x)\, \mu^\omega(dx)\pi_\mu(d\xi\omega) \\
&\quad = \int\int \beta(\xi\omega)^{-1}(\mathcal{L}'_\xi\varphi(\xi\omega))(x)\mu^\omega(dx)\pi_\mu(d\xi\omega) \\
&\quad = \int\int\int (\mathcal{L}'_\xi\varphi(\xi\omega))(x)\theta(d\xi)\mu^\omega(dx)\pi_\mu(d\omega) \\
&\quad = \int\int \varphi(x, \omega)\mu^\omega(dx)\pi_\mu(d\omega) = \int\int \varphi(x, \xi\omega)\mu^{\xi\omega}(dx)\pi_\mu(d\xi\omega)\,.
\end{aligned} \tag{3.11}$$

Consider now two-sided sequences $\omega \in \Omega$ viewing $g''_\omega = g'_\xi/\beta(\xi\omega)$ as a function of ω depending only on the ω_i with $i \geq 0$, and let π_υ denote the natural extension of π_μ to

Ω. Since the family of positive weights g''_ω satisfies the equi-Hölder continuous property of Kifer [1992, (1.7)] we are now in a position to apply Kifer [1992, Proposition 2.5] to the operators $\mathcal{L}''_\omega$. Recall that in this two-sided and not necessarily i.i.d. setting Kifer constructs for π_υ almost all $\omega \in \Omega$ uniquely defined numbers $\lambda^\omega > 0$, probability measures $(\mu'')^\omega$ on M, and positive Hölder functions $h''_\omega : M \to \mathbb{R}$ with $(\mu'')^\omega(h''_\omega) = 1$ and such that

$$\mathcal{L}''_\omega h''_\omega = \lambda^\omega h''_{\sigma\omega}\,, \quad (\mathcal{L}''_\omega)^*(\mu'')^{\sigma\omega} = \lambda^\omega(\mu'')^\omega\,. \tag{3.12}$$

It follows from (3.10) and the uniqueness statement in Kifer that $\lambda^\omega \equiv 1$ and $\mu^\omega = (\mu'')^\omega$.

By construction, the two-sided probability measure

$$\upsilon(dx, d\omega) = h''_\omega(x)\mu^\omega(dx)\,\pi_\upsilon(d\omega) \tag{3.13}$$

is invariant under the two-sided skew product τ, and from Theorem 3.2 in Kifer [1992] it is the unique (quenched) equilibrium state with marginal π_υ for the pair (τ, g'') on $M \times \Omega$. We claim that the relative entropy of (τ, υ) over (σ, π_υ) coincides with the relative entropy of (τ^+, μ) over (σ^+, π_μ). This follows from formula (2.7) applied both in the one-sided and two-sided settings, and the fact that $\sup_x |\log h''_\omega(x)|$ is bounded uniformly π_υ-almost everywhere by Kifer [1992, Proposition 2.5, (2.16)]. (Indeed, this uniform bound implies that there is a positive constant $C > 0$ so that $C \cdot H_{\upsilon^\omega}(\mathcal{Q}) \le H_{\mu^\omega}(\mathcal{Q}) \le H_{\upsilon^\omega}(\mathcal{Q})/C$ for any finite partition $\mathcal{Q}$ and π_υ almost all $\omega \in \Omega$ where we used the decomposition $\upsilon(dx, d\omega) = \upsilon^\omega(dx)\pi_\upsilon(d\omega)$.)

We now show that

$$0 = h_\tau(\upsilon|\pi_\upsilon) + \upsilon(\log g'')\,. \tag{3.14}$$

Equality (3.14) follows from Kifer [1992, Proposition 3.1] which tells us in particular that if we set $g'''_\omega = g''_\omega h''_\omega/(h''_{\sigma\omega} \circ f_\omega)$ then we have for almost all ω

$$0 = \upsilon^\omega(I_{\upsilon^\omega}(\mathcal{B}_M|f_\omega^{-1}\mathcal{B}_M) + \log g'''_\omega)\,, \tag{3.15}$$

where $I_\eta(\mathcal{B}_M|f_\omega^{-1}\mathcal{B}_M)$ denotes the conditional information of the partition $\mathcal{E}$ of M into points with respect to the partition $f_\omega^{-1}\mathcal{E}$ for the probability measure η. (Just integrate (3.15) with respect to π_υ, use the definitions of g''' and υ and the fact that $h_\tau(\upsilon|\pi_\upsilon) = \int \upsilon^\omega(I_{\upsilon^\omega}(\mathcal{B}_M|f_\omega^{-1}\mathcal{B}_M))\pi_\upsilon(d\omega)$.)

Since we have

$$\begin{aligned} h_\tau(\upsilon|\pi_\upsilon) + \upsilon(\log g'') &= h_{\tau^+}(\mu|\pi_\mu) + \mu(\log g'') \\ &= h_{\tau^+}(\mu|\pi_\mu) + \mu(\log g') + h^\theta(\pi_\mu)\,, \end{aligned} \tag{3.16}$$

Equation (3.14) implies by definition of g' that

$$h_{\tau^+}(\mu|\pi_\mu) + \mu(\log g) + h^\theta(\pi_\mu) = \log R\,. \tag{3.17}$$

We now check that $\log R = Q^{(a)}(\log g)$. Let then $\hat\mu(dx, d\omega) = \hat\mu^\omega(dx)\pi_{\hat\mu}(d\omega)$ be a one-sided τ^+-invariant measure on $M \times \Omega^+$ with $h^\theta(\pi_{\hat\mu}) > -\infty$, and let $\hat\beta(\omega)$ be the

Radon-Nikodym derivative of $\pi_{\hat\mu}(d\omega)$ with respect to $\theta(d\omega_0)\pi_{\hat\mu}(d\sigma\omega)$ (note that $\hat\beta$ is $\pi_{\hat\mu}$-almost everywhere nonzero since $1/\hat\beta(\omega)$ is in $L^1(\pi_{\hat\mu})$ because the specific entropy per site is finite). We consider $\hat\upsilon(dx, d\omega) = \hat\upsilon^\omega(dx)\pi_{\hat\upsilon}(d\omega)$ the Ω-natural extension of $(\hat\mu, \tau^+)$, i.e., the unique τ invariant measure $\hat\upsilon$ such that $\hat\upsilon(\varphi) = \hat\mu(\varphi)$ for all $\varphi \in L^1(\hat\mu)$ which depend only on (x and) ω_j for $j \geq 0$. The Ω-natural extension is constructed just like the standard natural extension and has the property that it leaves the relativised entropy invariant. (It is possible although not necessary to show that υ above is the Ω-natural extension of μ.) Note that $\pi_{\hat\upsilon}$ is the (ordinary) natural extension of $\pi_{\hat\mu}$.

Next, let $\hat h_\omega > 0$ be the functions associated to the weight $g'_{\omega_0}(x)/\hat\beta(\omega)$ by Theorem 3.1 in Khanin–Kifer [1996] (note that the corresponding $\hat\lambda^\omega$ are equal to one $\pi_{\hat\upsilon}$ almost everywhere by a uniqueness argument and a computation identical to (3.11)). Set

$$\hat g_\omega(x) = \frac{g'_{\omega_0}(x)\hat h_\omega(x)}{\hat\beta(\omega)(\hat h_{\sigma\omega}\circ f_{\omega_0}(x))}\,. \tag{3.18}$$

By construction, $\sum_{f_\omega y = x} \hat g_\omega(y) = 1$ for any $x \in M$ and $\pi_{\hat\upsilon}$ almost all $\omega \in \Omega$. Applying finally the arguments of Kifer [1992, (3.8)–(3.11)] to $\hat\upsilon$ we get for $\pi_{\hat\upsilon}$ almost every ω

$$\begin{aligned}\hat\upsilon^\omega\big(I_{\hat\upsilon^\omega}(\mathcal{B}_M|f_\omega^{-1}\mathcal{B}_M) + \log\hat g_\omega\big) \leq&\\ \int \sum_{y\in f_\omega^{-1}f_\omega(x)} &\hat g_\omega(y)\,\hat\upsilon^\omega(dx) - 1 = 0\,,\end{aligned} \tag{3.19}$$

Integrating both sides of (3.19) with respect to $\pi_{\hat\upsilon}$ and using the definition of $\hat g$, we get

$$\begin{aligned} &h_\tau(\hat\upsilon|\pi_{\hat\upsilon}) + \hat\upsilon(\log g') + h^\theta(\pi_{\hat\mu})\\ &\qquad = h_{\tau^+}(\hat\mu|\pi_{\hat\mu}) + \hat\mu(\log g') + h^\theta(\pi_{\hat\mu}) \leq 0\,.\end{aligned} \tag{3.20}$$

Since $\hat\mu(\log g') = \hat\mu(\log g) - \log R$, we are done.

It remains to prove uniqueness of the annealed state. Let $\hat\mu(dx, d\omega) = \hat\mu^\omega(dx)\pi_{\hat\mu}(d\omega)$ be a τ^+ invariant probability measure with $h^\theta(\pi_{\hat\mu}) > -\infty$ and such that the inequality in (3.20) is an equality, so that the inequality for the corresponding $\hat\upsilon$ on $M\times\Omega$ in (3.19) is $\pi_{\hat\upsilon}$ almost everywhere an equality, i.e.,

$$\begin{aligned}\hat\upsilon^\omega\big(I_{\hat\upsilon^\omega}(\mathcal{B}_M|f_\omega^{-1}\mathcal{B}_M) + \log\hat g_\omega\big) =&\\ \int \sum_{y\in f_\omega^{-1}f_\omega(x)} &\hat g_\omega\,\hat\upsilon^\omega(dx) - 1 = 0\,.\end{aligned} \tag{3.21}$$

Starting from (3.21), we may proceed exactly as Kifer [1992, (3.12)] obtaining that $\mathcal{L}^*_{\hat g_{\xi\omega}}\hat\upsilon^\omega = \hat\upsilon^{\xi\omega}$ ($\pi_{\hat\upsilon}$ almost everywhere) for $\mathcal{L}_{\hat g_\omega}$ the random operator associated to $\hat g_\omega$. Integration then shows that (3.7) holds with $\hat\mu$ instead of μ:

$$\int \sum_{f_\xi y = x} \frac{g'_\xi(y)}{\hat\beta(\xi\omega)}\hat h_{\xi\omega}(y)\varphi_{\xi\omega}(y)\,\hat\mu^\omega(dx)\,\pi_{\hat\mu}(d\xi\omega) = \int \varphi_\omega(x)\hat h_\omega(x)\,\hat\mu^\omega(dx)\pi_{\hat\mu}(d\omega)\,. \tag{3.22}$$

Since the simplicity of the maximal eigenvalue statement in Proposition 3.1 (2) applies to the dual of the integrated operator $\hat{\mathcal{L}}'$ associated to g'_ξ, we get the claimed equality $\mu = \hat\mu$ from (3.22) by definition of $\hat\beta$. $\square$

Proof of Proposition 3.3. We essentially follow the path laid out by Khanin and Kifer [1996, Section 4], proving first the existence and uniqueness of the annealed Gibbs measure. Observe that any limit point of the probability measures

$$\varphi \mapsto \frac{\widehat{\mathcal{L}}^n \varphi(x,\omega)_n}{\widehat{\mathcal{L}}^n \mathbf{1}(x,\omega)_n}, \quad \varphi \in C^0(M \times \Omega^+, \mathbb{C}), \tag{3.23}$$

as $n \to \infty$ with $(x,\omega)_n \in M \times \Omega^+$ is an annealed Gibbs measure, and that all annealed Gibbs measures are constructed with this procedure. Such a limit point must exist by standard compactness arguments. In fact it follows from the proof of Proposition 3.1 that for all continuous $\varphi : M \times \Omega^+ \to \mathbb{C}$

$$\lim_{n\to\infty} \sup_{(x,\omega),(y,\tilde\omega)} \left| \frac{\widehat{\mathcal{L}}^n \varphi(x,\omega)}{\widehat{\mathcal{L}}^n \mathbf{1}(x,\omega)} - \frac{\widehat{\mathcal{L}}^n \varphi(y,\tilde\omega)}{\widehat{\mathcal{L}}^n \mathbf{1}(y,\tilde\omega)} \right| = \lim_{n\to\infty} \left| \frac{\widehat{\mathcal{L}}^n \varphi(x,\omega)}{\widehat{\mathcal{L}}^n \mathbf{1}(x,\omega)} - \hat\nu(\varphi) \right| = 0\,, \tag{3.24}$$

uniformly in (x,ω), where $\hat\nu$ is defined in Proposition 3.1. (Indeed, the difference $\widehat{\mathcal{L}}^n \varphi(x,\omega)/(R^n \rho(x)) - \hat\nu(\varphi)$ converges to zero uniformly in (x,ω).) In particular, we also get uniqueness of the Gibbs measure, which coincides with $\hat\nu$. Clearly, the annealed equilibrium state $\mu = \rho\hat\nu$ is therefore also an annealed Gibbs state.

To prove that there is no other annealed Gibbs state we note that any such state μ' has a density $\rho' \in L^1(\hat\nu)$ with respect to $\hat\nu$ which satisfies $\widehat{\mathcal{L}}\rho' = R\rho'$ in $L^1(\hat\nu)$ (indeed, we have by assumption $\hat\nu((\varphi\circ\tau^+)\rho') = \hat\nu(\varphi\rho')$ for all $\varphi \in L^1(\hat\nu)$, and we may use $\widehat{\mathcal{L}}\hat\nu = R\hat\nu$). Since $(\widehat{\mathcal{L}}^n\varphi(x))/R^n$ converges to $\rho(x)$ for all continuous φ with $\hat\nu(\varphi) = 1$ and continuous functions are dense in $L^1(\hat\nu)$ by Lusin's Theorem, we get (Theorem VIII.5.1 in Dunford-Schwartz [1988]) that $\frac{1}{n}\sum_{k=0}^{n-1}(\widehat{\mathcal{L}}^k\varphi(x)/R^k)$ converges to $\rho(x)$ for all $\varphi \in L^1(\hat\nu)$ with $\hat\nu(\varphi) = 1$, in particular $\rho' = \rho$ as desired. $\square$

Remark 3.4. The annealed equilibrium state μ is also a quenched equilibrium state if and only if $\pi_\mu = \Theta^+$ (if and only if $Q^{(a)}(\log g) = Q^{(q)}(\log g)$). A *necessary* condition for this is the existence of a probability measure $\nu(dx)$ on M so that $\int_M \mathcal{L}_\xi \mathbf{1}(x)\,\nu(dx)$ is θ-almost everywhere constant (because $\mu \in \mathcal{P}_\Theta$ and $\widehat{\mathcal{L}}^*\mu \in \mathcal{P}_\Theta$, in particular when integrating functions independent of $x \in M$). This constancy condition is violated for example if the number of branches of the f_ξ is constant in ξ, and $g_\xi(x)$ is constant in x but depends (essentially) on ξ. A *sufficient* condition ensuring $\pi_\mu = \Theta^+$ is the existence of a probability $\nu(dx)$ on M and a constant $\lambda > 0$ such that $\mathcal{L}^*_\xi(\nu) = \lambda(\nu)$ for θ almost all $\xi \in E$. By definition of the Jacobian, this property holds with $\lambda = 1$ for $g_\xi = 1/|\det D(f_\xi)|$ and ν Lebesgue measure, proving Proposition 2. In Section 4, Remark 4.3, we mention a weaker sufficient condition.

3.4. Stability of the discrete spectrum and annealed state.

The stability claims in Theorems 3 and 5 will be a consequence of the following proposition and results from Baladi-Young [1993]:

Proposition 3.4. *Consider a weighted small random perturbation of $f_0 \in C^r_\gamma(M,M)$, $g_0 \in C^r(M,\mathbb{C})$ given by a family θ_ϵ ($\epsilon \geq 0$) and write $\widehat{\mathcal{L}}_\epsilon$, $\mathcal{L}_\epsilon$ ($\epsilon \geq 0$) for the corresponding transfer operators acting on $\mathcal{B}(\alpha)$, respectively $\mathcal{F} = C^r(M,\mathbb{C})$. Write $R = \exp P(\log|g_0|)$ as usual.*

(1) *For any fixed $\psi \in \mathcal{B}$, $\varphi \in \mathcal{F}$, $n \geq 1$*

$$\lim_{\epsilon\to 0} \|\widehat{\mathcal{L}}^n_\epsilon \psi - \widehat{\mathcal{L}}^n_0 \psi\|_{\mathcal{B}} = 0\,, \quad \lim_{\epsilon\to 0} \|\mathcal{L}^n_\epsilon \varphi - \mathcal{L}^n_0 \varphi\|_{\mathcal{F}} = 0\,. \tag{3.25}$$

(2) *Let $\bar\gamma < \gamma$ and $\bar\alpha > \alpha$. Then there is a constant $C > 0$ and an integer $N \geq 0$, so that for all $n \geq N$ there is $\epsilon(n)$ such that for all $\epsilon < \epsilon(n)$*

$$\begin{aligned} \|\widehat{\mathcal{L}}^n_\epsilon - \widehat{\mathcal{L}}^n_0\|_{\mathcal{B}} &\leq CR^n \max(\bar\gamma^{-rn}, \bar\alpha^n)\,, \\ \|\mathcal{L}^n_\epsilon - \mathcal{L}^n_0\|_{\mathcal{F}} &\leq CR^n \max(\bar\gamma^{-rn}, \bar\alpha^n)\,. \end{aligned} \tag{3.26}$$

Proof of Proposition 3.4.

(1) By the triangle inequality it suffices to prove the claims for $n = 1$. To do this, use that each θ_ϵ is a probability distribution and observe that

$$\begin{aligned} &\lim_{\epsilon\to 0} \sup_{\xi\omega \in \text{support}\theta_\epsilon^{\mathbb{Z}_+}} \|\mathcal{L}_\xi \psi_{\xi\omega} - \mathcal{L}\psi_{\xi\omega}\|_{\mathcal{F}} = 0\,, \\ &\lim_{\epsilon\to 0} \sup_{\xi\omega \in \text{support}\theta_\epsilon^{\mathbb{Z}_+}} \mathrm{Lip}_\omega D^j(\mathcal{L}_\xi \psi_{\xi\omega} - \mathcal{L}\psi_{\xi\omega}) = 0\,, 0 \leq j \leq r\,, \end{aligned} \tag{3.27}$$

(simply apply the Leibniz formula to each term in the finite sums over inverse branches of the f_ξ and use the definition of a small random perturbation).

(2) The argument follows the lines of the proof of Lemma 5 in Baladi-Young [1993] or Lemma A.1 in Baladi et al. [1995] and is left to the reader. (We may use that $\widehat{\mathcal{L}}^n_\epsilon \varphi(x,\omega)$ can be written as an integral over $\theta_\epsilon(d\xi_1)\cdots\theta_\epsilon(d\xi_n)$ of random operators where the weights g are evaluated at points which depend on x and $\xi_1, \ldots, \xi_n$ but not on ω.) $\square$

Theorem 3 (1) is an immediate consequence of Proposition 3.5:

Proposition 3.5. *Consider a weighted small random perturbation of $f_0 \in C^r_\gamma(M,M)$, $g_0 \in C^r(M,\mathbb{C})$ given by a family θ_ϵ ($\epsilon \geq 0$), and write $\widehat{\mathcal{L}}_\epsilon$, $\mathcal{L}_\epsilon$ ($\epsilon \geq 0$) for the corresponding transfer operators acting on $\mathcal{B}(\alpha)$, respectively $\mathcal{F} = C^r(M,\mathbb{C})$. Write $R = \exp P(\log|g_0|)$ as usual. Let $\bar\gamma < \gamma$ and assume that $\alpha < 1/\bar\gamma^r$. The spectrum of $\mathcal{L}_\epsilon$ and that of $\widehat{\mathcal{L}}_\epsilon$ outside of the disc of radius $R/\bar\gamma^r$ contains only isolated eigenvalues of* *finite multiplicity for small enough ϵ, and both spectra converge to the spectrum of $\mathcal{L}_0$ acting on $\mathcal{F}$ (outside of this disc) as $\epsilon \to 0$. The corresponding generalised eigenspaces of $\mathcal{L}_\epsilon$, respectively $\widehat{\mathcal{L}}_\epsilon$ converge in the $\mathcal{F}$, respectively $\mathcal{B}(\alpha)$, topology to those of $\mathcal{L}_0$, respectively $\widehat{\mathcal{L}}_0$, and the dual eigenspaces converge in the weak topology. In particular, for positive weights, the maximal eigenmeasure $\hat{\nu}_\epsilon$ of $\widehat{\mathcal{L}}_\epsilon$ converges to $\nu_0 \times \delta^{\mathbb{Z}_+}$ with δ the Dirac mass at (f_0, g_0).*

Proof of Proposition 3.5. The stability of the spectrum and the convergence of the eigenfunctions (in particular $\lim_{\epsilon\to 0} \|\rho_\epsilon - \rho_0\|_r = 0$ for positive weights) follows from Lemma 3 in Baladi-Young [1993] applied to the operators $\mathcal{L}_\epsilon$, using the statements in Proposition 3.4 about the operators $\mathcal{L}_\epsilon$. Indeed we get from Baladi-Young [1993] that the spectrum of $\mathcal{L}_\epsilon$ acting on $\mathcal{F}$ and of $\widehat{\mathcal{L}}_\epsilon$ acting on $\mathcal{B}$ outside of the disc of radius $R/\bar{\gamma}^r$ both converge to the spectrum of $\mathcal{L}_0$ acting on $\mathcal{F}$ outside of this disc as ϵ goes to zero. The eigenfunctions converge in the $\mathcal{F}$, respectively $\mathcal{B}$ norm.

To get the weak convergence of the eigenfunctionals it suffices to observe that the bounds for $\widehat{\mathcal{L}}_\epsilon$ in Proposition 3.4 also apply to $\widehat{\mathcal{L}}_\epsilon^*$ by definition of the dual norm. Therefore, Lemma 3 in Baladi-Young [1993] may also be applied to the family $\widehat{\mathcal{L}}_\epsilon^*$, yielding the desired convergence. For the final claim, use $\widehat{\mathcal{L}}_0^*(\nu \times \delta^{\mathbb{Z}_+}) = R(\nu \times \delta^{\mathbb{Z}_+})$ for any maximal eigenmeasure ν of $\mathcal{L}_0$ and the fact that the multiplicity of the maximal eigenvalue is constant for small enough ϵ from Proposition 3.4. □

3.5. The annealed zeta functions.

Theorem 6 will be a consequence of Proposition 3.5 and the following result of Ruelle:

Theorem (Ruelle [1990, Theorem 1.1, Theorem 1.3]). *Consider a C^r (complex) weighted γ-expanding system (τ^+, g, θ), write $\mathcal{L}$ for the corresponding transfer operator acting on $\mathcal{F} = C^r(M, \mathbb{C})$, and let $R = \exp Q^{(a)}(\log|g|)$.*

(1) *The zeta function $\zeta^{(a)}(z)$ is analytic in the disc of radius R^{-1} and admits a zero-free meromorphic extension to the disc of radius $R^{-1}\gamma$ where its poles coincide (including multiplicity) with the inverses of the eigenvalues of modulus larger than R/γ of $\mathcal{L}$ acting on $\mathcal{F}$.*

    More precisely, if E is fixed then for any $\delta > 0$ and $\bar{\gamma} < \gamma$ there is a constant $C(\bar{\gamma}, \delta) > 0$ which does not depend on the probability distribution θ on E, so that if $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of $\mathcal{L}$ of modulus larger than $R/\bar{\gamma}$ then the coefficients a_n in the expansion

$$\sum_{n=0}^{\infty} a_n z^n := \log(\zeta^{(a)}(z) \cdot \prod_{i=1}^{N} (1 - \lambda_i^{-1} z)) \tag{3.28}$$

    satisfy the uniform bounds

$$|a_n| \le C \exp(n(Q^{(a)}(\log|g|) + \delta))/\bar{\gamma}^n\,. \tag{3.29}$$

(2) *The generalised Fredholm determinant $d^{(a)}(z)$ admits an analytic extension to the disc of radius $R^{-1}\gamma^r$ where its zeroes coincide (including multiplicity) with the inverses of the eigenvalues of modulus larger than R/γ^r of $\mathcal{L}$.*

More precisely, if E is fixed then for any $\delta > 0$ and $\bar\gamma < \gamma$ there is a constant $C(\bar\gamma, \delta) > 0$ which does not depend on the probability distribution θ on E, so that if $\lambda_1, \ldots, \lambda_M$ are the eigenvalues of $\mathcal{L}$ of modulus larger than $R/\bar\gamma^r$ then the coefficients b_n in the expansion

$$\sum_{n=0}^{\infty} b_n z^n := \log(d^{(a)}(z)/\prod_{i=1}^{M}(1-\lambda_i^{-1}z)) \tag{3.30}$$

satisfy the uniform bounds

$$|b_n| \le C\exp(n(Q^{(a)}(\log|g|)+\delta))/\bar\gamma^{rn}\,. \tag{3.31}$$

(Ruelle does not state explicitly the θ-uniform bounds (3.29),(3.31) but they are easily obtained from his proofs.)

Proof of Theorem 6. For each fixed $m \ge 1$ we get by definition of a small random perturbation that $\zeta_\epsilon^{(a)}(m)$ converges to $\zeta_0^{(a)}(m)$ and $d_\epsilon^{(a)}(m)$ converges to $d_0^{(a)}(m)$ as $\epsilon \to 0$. Moreover, the eigenvalues $\lambda_{i,\epsilon}$ of $\mathcal{L}_\epsilon$ of modulus larger than $R/\bar\gamma^r$ converge to the corresponding eigenvalues of $\mathcal{L}_0$ by Proposition 3.5. The result is therefore an easy exercise on convergent power series using the uniform bounds in the theorem of Ruelle stated above. □

3.6. Integrated annealed correlation functions.

For $\varphi_1, \varphi_2 \in \mathcal{B}$, and $\mu = \rho\hat\nu$ the annealed equilibrium state of a positively weighted i.i.d. expanding map (τ^+, g, θ), writing R for the spectral radius of $\widehat{\mathcal{L}}$ on $\mathcal{B}$, we get

$$\int(\varphi_1\circ(\tau^+)^n)(x,\omega)\varphi_2(x,\omega)\rho(x,\omega)\hat\nu(dx,d\omega) = \frac{1}{R^n}\int\varphi_1\widehat{\mathcal{L}}^n(\rho\varphi_2)(x,\omega)\hat\nu(dx,d\omega)\,. \tag{3.32}$$

(Just use the fact that $\hat\nu$ is an eigenfunctional for the dual of $\widehat{\mathcal{L}}$ and the eigenvalue R.) If $C_{\varphi_1\varphi_2}(n)$ denotes the correlation function (2.20), it follows formally that

$$\sum_{n\ge 0} e^{in\eta}C_{\varphi_1\varphi_2}(n) = \int\left[\left(1-(\frac{e^{i\eta}}{R})\widehat{\mathcal{L}}\right)^{-1}(\rho\varphi_2)\right](x,\omega)\varphi_1(x,\omega)\hat\nu(dx,d\omega)\,. \tag{3.33}$$

Our results on the spectrum of $\widehat{\mathcal{L}}$ in Proposition 3.1 give the desired meaning to (3.33). This proves Theorem 4 (1) (For $\varphi_1, \varphi_2 \in C^r(M,\mathbb{C})$, we may in fact replace $\widehat{\mathcal{L}}$ by $\mathcal{L}$ and $(\tau^+)^n$ by $f_\omega^{(n)}$ in (3.32) and (3.33).)

Finally, Theorem 5 (1) follows from Proposition 3.5, just as in the proof of Theorem 3.

4. The quenched transfer operator $\widehat{\mathcal{M}}$

In this section, we restrict again to the case of positive weights and we construct a normalised integrated operator $\widehat{\mathcal{M}}$ related to the quenched state. This will be useful to study the quenched correlation spectrum and its stability for small perturbations, in particular to prove Theorem 3 (2), Theorem 4 (2), and Theorem 5 (2). Consider first the two-sided situation (τ, g, θ), viewing g as a function on $M \times \Omega$ depending only on x and ω_0. Using the notations and results from Kifer [1992] recalled in (3.12) above, i.e., uniquely defined positive numbers λ^ω, Borel probability measures ν^ω on M, and functions $h_\omega : M \to \mathbb{R}$ with $\nu^\omega(h_\omega) = 1$, such that $\mathcal{L}_\omega h_\omega = \lambda^\omega h_{\sigma\omega}$, and $\mathcal{L}^*_\omega \nu^{\sigma\omega} = \lambda^\omega \nu^\omega$, we first show:

Proposition 4.1 (Properties of λ^ω and ν^ω).

(1) *Since $\mathcal{L}_\omega = \mathcal{L}_{\omega_0}$, the objects λ^ω and ν^ω only depend on ω_k for $k \geq 0$.*
(2) *The map $\omega \mapsto \log \lambda^\omega$ is Lipschitz from $\Omega^+ \to \mathbb{R}^+$ for the metric d_α for any $\alpha > 1/\gamma$.*

Proof of Proposition 4.1. The first assertion is a consequence of the proof of Lemma 2.2 in Kifer [1992].

To prove the second claim, we use the observations of Kifer [1992, p. 16] that $\lambda^\omega = \nu^{\sigma\omega}(\mathcal{L}_\omega(1))$, and

$$\nu^\omega(\varphi) = \lim_{n\to\infty} \Big(\frac{\mathcal{L}^n_\omega \varphi}{\mathcal{L}^n_\omega 1}\Big) . \tag{4.1}$$

Let $1 < \bar\gamma < \gamma$ and $\alpha = 1/\bar\gamma$, we shall prove that there is a constant $C > 0$ so that for any $\omega, \tilde\omega \in \Omega^+$ we have $|\log \lambda^\omega - \log \lambda^{\tilde\omega}| \leq C d_\alpha(\omega, \tilde\omega)$.

We begin with two purely dynamical remarks. First observe that, by compactness of E, there is $\bar\epsilon(E)$ such that whenever $d_r(f_\omega, f_{\tilde\omega}) < \bar\epsilon$ then f_ω and $f_{\tilde\omega}$ have the same degree, and, moreover, for each $x \in M$ the bijection Ψ between $\{y \mid f_\omega(y) = x\}$ and $\{\tilde y \mid f_{\tilde\omega}(\tilde y) = x\}$ can be chosen in such a way that

$$d_M(y, \Psi(y)) \leq \frac{d_r(f_\omega, f_{\tilde\omega})}{\gamma} . \tag{4.2}$$

Indeed, if $\bar\epsilon$ is small enough we may chose Ψ so that y and $\Psi(y)$ are in the image of the same local inverse branch of f_ω, so that if (4.2) were violated for some y we would have

$$\begin{aligned} 0 = d_M(x,x) &= d_M(f_\omega y, f_{\tilde\omega}(\Psi(y)) \\ &\geq d_M(f_\omega y, f_\omega(\Psi(y)) - d_M(f_\omega(\Psi(y)), f_{\tilde\omega}(\Psi(y)) \\ &> \gamma \cdot \frac{d_r(f_\omega, f_{\tilde\omega})}{\gamma} - d_r(f_\omega, f_{\tilde\omega}) = 0 , \end{aligned} \tag{4.3}$$

a contradiction.

We claim now that for all ω, $\tilde\omega$, all $n \geq 1$, and up to exchanging ω and $\tilde\omega$, there exists for any point $x \in M$ a surjective map

$$\Psi_{n,\omega,\tilde\omega,x} : Y_{n,\omega,x} = \{y \in M \mid f_\omega^{(n)}(y) = x\} \to Y_{n,\tilde\omega,x} = \{\tilde y \in M \mid f_{\tilde\omega}^{(n)}(\tilde y) = x\}\,. \tag{4.4}$$

(If all f_ω have the same degree, then the $\Psi_{n,\omega,\tilde\omega,x}$ are bijections, otherwise, the cardinality of the fibers can be unbounded as $n \to \infty$.) Moreover, fixing $\bar\epsilon(E)$ as above, there is a constant $C > 0$ so that if, in addition $\delta = d_{1/\bar\gamma}(\omega, \tilde\omega) \leq \bar\epsilon$, then there is $n(\delta)$ so that for $n \geq n(\delta)$ and any $y \in Y_{n,\omega,x}$, we have

$$d_M(y, \Psi_{n,\omega,\tilde\omega,x}(y)) < C d_{1/\bar\gamma}(\omega, \tilde\omega)\,. \tag{4.5}$$

To prove (4.5), we first note that since $d_E(\omega_k, \tilde\omega_k) \leq \delta \cdot \bar\gamma^k$ for all $k \geq 0$, there is for any $\delta \leq \bar\epsilon$ an iterate $k_0(\delta) \geq 0$ with $\delta\bar\gamma^{k_0} \leq \bar\epsilon < \delta\bar\gamma^{k_0+1}$ (if $\delta > \bar\epsilon$ we set $k_0 = 0$). For any $n \geq n(\delta) = k_0(\delta)$, and any $x \in M$ consider the finite sets $Y = Y_{n-k_0,\sigma^{k_0}\omega,x}$ and $\tilde Y = Y_{n-k_0,\sigma^{k_0}\tilde\omega,x}$, assuming that $\#Y \geq \#\tilde Y$ (the other case is symmetric) and choose an arbitrary surjection $\Psi : Y \to \tilde Y$. If $\delta > \bar\epsilon$, we are done. Otherwise, we fix an arbitrary pair $(y, \tilde y = \Psi(y)) \in Y \times \tilde Y$. Using the fact that for any $j \geq 1$ and any u, v in M the sets $(f_\omega^{(j)})^{-1}(u)$ and $(f_\omega^{(j)})^{-1}(v)$ are in bijection with the distance between two paired points not larger than $d_M(u,v)/\gamma^j$ (recall that each f_ω is γ-expanding), using the simplified notation $d((f_\omega^{(j)})^{-1}(u), (f_\omega^{(j)})^{-1}(v))$ to represent the maximum distance between two such paired points, and defining $d((f_\omega^{(j)})^{-1}(u), (f_{\tilde\omega}^{(j)})^{-1}(u))$ analogously, we get by applying sucessively (4.2) and recalling the definition of k_0

$$\begin{aligned}
d((f_\omega^{(k_0)})^{-1}(y),&(f_{\tilde\omega}^{(k_0)})^{-1}(\tilde y)) \\
&\leq d((f_\omega^{(k_0)})^{-1}(y), (f_\omega^{(k_0)})^{-1}(\tilde y)) + d((f_\omega^{(k_0)})^{-1}(\tilde y), (f_{\tilde\omega}^{(k_0)})^{-1}(\tilde y)) \\
&\leq \frac{\operatorname{diam} M}{\gamma^{k_0}} + d((f_\omega^{(k_0-1)})^{-1}(f_{\omega_{k_0}})^{-1}(\tilde y), (f_\omega^{(k_0-1)})^{-1}(f_{\tilde\omega_{k_0}})^{-1}(\tilde y)) \\
&\qquad\quad + d((f_\omega^{(k_0-1)})^{-1}(f_{\tilde\omega_{k_0}})^{-1}(\tilde y), (f_{\tilde\omega}^{(k_0-1)})^{-1}(f_{\tilde\omega_{k_0}})^{-1}(\tilde y)) \\
&\leq \cdots \\
&\leq \frac{\bar\gamma^{k_0+1} \cdot \operatorname{diam} M}{\gamma^{k_0}\bar\gamma^{k_0+1}} + \sum_{j=0}^{k_0-1} \frac{\bar\epsilon}{\gamma^{k_0-1-j} \cdot \bar\gamma^j} \\
&\leq \delta \cdot \bar\gamma \cdot \left(\frac{\bar\gamma^{k_0} \cdot \operatorname{diam} M}{\bar\epsilon\gamma^{k_0}} + \frac{1}{1 - (\bar\gamma/\gamma)} \right),
\end{aligned} \tag{4.6}$$

as claimed.

We now write

$$\begin{aligned}
\frac{\lambda^{\omega}}{\lambda^{\tilde\omega}} &= \lim_{n\to\infty} \frac{\mathcal{L}_{\omega}^{n}(1)\cdot\mathcal{L}_{\sigma\tilde\omega}^{n-1}(1)}{\mathcal{L}_{\sigma\omega}^{n-1}(1)\cdot\mathcal{L}_{\tilde\omega}^{n}(1)} \\
&= \lim_{n\to\infty}\sup_x \frac{\sum\limits_{f_{\omega}^{(n)}u=x}\prod\limits_{k=0}^{n-1} g_{\sigma^k(\omega)}(f_{\omega}^{(k)}u)\cdot \sum\limits_{f_{\sigma\tilde\omega}^{(n-1)}s=x}\prod\limits_{k=0}^{n-2} g_{\sigma^{k+1}(\tilde\omega)}(f_{\sigma\tilde\omega}^{(k)}s)}{\sum\limits_{f_{\sigma\omega}^{(n-1)}v=x}\prod\limits_{k=0}^{n-2} g_{\sigma^{k+1}(\omega)}(f_{\sigma\omega}^{(k)}v)\cdot \sum\limits_{f_{\tilde\omega}^{(n)}t=x}\prod\limits_{k=0}^{n-1} g_{\sigma^{k}(\tilde\omega)}(f_{\tilde\omega}^{(k)}t)} \\
&= \lim_{n\to\infty}\sup_x \frac{\sum\limits_{\substack{f_{\sigma\omega}^{(n-1)}y=x\\ f_{\sigma\tilde\omega}^{(n-1)}s=x}}\prod\limits_{k=0}^{n-2} g_{\sigma^{k+1}(\omega)}(f_{\sigma\omega}^{(k)}y)\cdot g_{\sigma^{k+1}(\tilde\omega)}(f_{\sigma\tilde\omega}^{(k)}s)\sum\limits_{f_{\omega_0}u=y} g_{\omega_0}(u)}{\sum\limits_{\substack{f_{\sigma\omega}^{(n-1)}v=x\\ f_{\sigma\tilde\omega}^{(n-1)}r=x}}\prod\limits_{k=0}^{n-2} g_{\sigma^{k+1}(\omega)}(f_{\sigma\omega}^{(k)}v)\cdot g_{\sigma^{k+1}(\tilde\omega)}(f_{\sigma\tilde\omega}^{(k)}r)\sum\limits_{f_{\tilde\omega_0}t=r} g_{\tilde\omega_0}(t)}\,.
\end{aligned} \tag{4.7}$$

If $\delta \le \bar\epsilon$ (the case $\delta > \bar\epsilon$ is simpler since we just need to bound (4.7) uniformly from above and below), we consider the right-hand-side of (4.7) for a fixed $n \ge n(\delta)$ and any $x \in M$, and assume that the surjection $\Psi_{n,\omega,\tilde\omega}$ is as in (4.4) (the other case is similar). It suffices to replace $g_{\omega_0}(u)$ by $g_{\tilde\omega_0}(t)\cdot(g_{\omega_0}(u)/g_{\tilde\omega_0}(t))$ (where $t = \Psi_{n,\omega,\tilde\omega}(u)$) in the numerator, and to use the remark that $d_M(t,u) < d_{1/\bar\gamma}(\omega,\tilde\omega)$, the fact that $d_E(\omega_0,\tilde\omega_0) \le \delta$, as well as the following trivial inequalities for numbers $a_i, c_i > 0$ with $i \in \mathcal{I}$ finite

$$\inf_{i\in\mathcal{I}} c_i \le \frac{\sum_{i\in\mathcal{I}} a_i c_i}{\sum_{i\in\mathcal{I}} a_i} \le \sup_{i\in\mathcal{I}} c_i\,. \quad \square \tag{4.8}$$

Remark 4.1. Equation (4.7) in the proof of Proposition 4.1 also shows that λ^{ω} only depends on ω_0 if f_{ω_0} is independent of ω_0 (as in the random Ising model in Section 2.4). In this case, and whenever $\lambda^{\omega} = \lambda^{\omega_0}$, it is not difficult to check by looking at the proof of Kifer [1992, Proposition 2.5] that h_ω only depends on ω_i for $i < 0$.

Remark 4.2.

(1) Since the Lipschitz constant of $\log\lambda_{\epsilon}^{\omega}$ is uniform in ϵ in the case of a small random perturbation θ_ϵ, we get

$$\lim_{\epsilon\to 0}(\sup_{\omega}\log\lambda_{\epsilon}^{\omega} - \inf_{\omega}\log\lambda_{\epsilon}^{\omega}) = 0\,. \tag{4.9}$$

Since $\int \log\lambda^{\omega}\theta_{\epsilon}^{\mathbb{Z}_+}(d\omega) = Q_{\epsilon}^{(q)}(\log g)$ and we know from Kifer [1992, Section 4] that $Q_{\epsilon}^{(q)}(\log g) \to Q_{0}^{(q)}(\log g)$ as $\epsilon \to 0$ we find

$$\lim_{\epsilon\to 0}\sup_{\omega}|\log\lambda_{\epsilon}^{\omega} - Q^{(q)}(\log g)| = 0\,. \tag{4.10}$$

(2) In the case of a small random perturbation θ_ϵ, we claim that $\log\lambda_\epsilon^\omega$ when viewed as a C^β function of ω (for any $0<\beta<1$) has a Hölder constant which goes to zero as $\epsilon\to 0$. (This will be useful in Proposition 4.3 below.) To see this, let $C>0$ be an upper bound for the Lipschitz constant of $\log\lambda_\epsilon^\omega$ for $\epsilon<\epsilon_0$, and observe that for any fixed ϵ and any ω, $\tilde\omega$, with each ω_i, $\tilde\omega_i$ in the support of θ_ϵ, we have $d_\alpha(\omega,\tilde\omega)\le\epsilon/(1-\alpha)$ so that

$$|\log\lambda_\epsilon^\omega-\log\lambda_\epsilon^{\tilde\omega}|\le\frac{C\epsilon^{1-\beta}}{(1-\alpha)^{1-\beta}}\cdot d_\alpha(\omega,\tilde\omega)^\beta\,. \tag{4.11}$$

We now define the *integrated quenched transfer operator* $\widehat{\mathcal{M}}$ acting on bounded functions $\varphi: M\times\Omega^+\to\mathbb{C}$ by

$$(\widehat{\mathcal{M}}\varphi)(x,\omega_+)=\int\frac{1}{\lambda^{\xi\wedge\omega_+}}(\mathcal{L}_\xi\varphi_{\xi\wedge\omega_+})(x)\,\theta(d\xi)\,. \tag{4.12}$$

Observe that the spectral radius of $\widehat{\mathcal{M}}$ acting on bounded functions is equal to 1.

Remark 4.3. There is in general no corresponding operator $\mathcal{M}$ acting on functions which only depend on the x-variable. Such a definition exists if λ^ω only depends on ω_0 (e.g., if the dynamical system is deterministic but not the weight see Remark 4.1). This is the case in particular when λ^ω is constant (for example in the SRB case where $\lambda^\omega\equiv 1$, or for a constant weight whenever the degree of the f_ξ is constant), then the operator $\widehat{\mathcal{M}}$ is simply $\widehat{\mathcal{L}}$ rescaled by $\lambda=\lambda^\omega$ so that $\mu^{(a)}=\mu^{(q)}$.

Since λ^ω is Lipschitz for d_α ($\alpha>1/\gamma$) by Proposition 4.1 (2) we may consider $\widehat{\mathcal{M}}$ as an operator acting on $\mathcal{B}(\alpha)$ and we find:

Proposition 4.2 (Quasicompacity of $\widehat{\mathcal{M}}$). *The operator $\widehat{\mathcal{M}}$ acting on $\mathcal{B}(\alpha)$ for $\alpha>1/\gamma$ has spectral radius equal to 1 and essential spectral radius strictly smaller than 1. The spectral radius is a simple eigenvalue with a corresponding eigenfunction which coincides with*

$$\tilde\rho_{\omega_0\wedge\omega_+}(x)=\int h_\omega(x)\,\Theta^-(d\omega_-)\,. \tag{4.13}$$

where we use the notations $\Omega=\Omega^-\times E\times\Omega^+$, $\omega=\omega_-\wedge\omega_0\wedge\omega_+$, and Θ^- for the marginal of the measure Θ on Ω^-. The corresponding eigenfunctional for $\widehat{\mathcal{M}}^*$ is the positive measure $\tilde\nu$ on $M\times\Omega^+$ defined by $\tilde\nu(\varphi)=\int\nu^{\omega_+}(\varphi_{\omega_+})\,\Theta^+(d\omega_+)$. The probability measure $\tilde\nu\tilde\rho/\tilde\nu(\tilde\rho)$ is the unique relativised quenched equilibrium state for τ^+,g,θ.

Note in particular that $\tilde\rho(x,\omega)\in\mathcal{B}(\alpha)$.

Proof of Proposition 4.2. The claims about the spectral radius and essential spectral radius follow by the same adaptation of the results of Ruelle [1990] and Fried [1995] as in Proposition 3.1. Since we have assumed that the weight g is positive, $\widehat{\mathcal{M}}$ has a simple maximal fixed point also when acting on $L^1(\bar\nu)$ for the unique probability measure such that $\mathcal{M}^*\bar\nu = \bar\nu$. Note that $\rho(x,\omega) \in L^\infty(\bar\nu) \subset L^1(\bar\nu)$. We have

$$\begin{aligned}(\widehat{\mathcal{M}}\tilde\rho)(x,\omega_+) &= \int \frac{1}{\lambda^{\xi\wedge\omega_+}} \mathcal{L}_\xi \tilde\rho_{\xi\wedge\omega_+}\theta(d\xi) \\ &= \int\int \frac{1}{\lambda^{\xi\wedge\omega_+}} \mathcal{L}_\xi h_{\omega_-\cdot\wedge\xi\wedge\omega_+}(x)\,\Theta^-(d\omega_-)\theta(d\xi) \\ &= \int h_{\omega_-\wedge\xi\cdot\wedge\omega_+}(x)\,\Theta^-(d\omega_-)\theta(d\xi) = \tilde\rho(x,\omega_+)\,.\end{aligned} \tag{4.14}$$

Also, since $\mathcal{L}_\xi^*\nu_{\omega_+} = \lambda^{\xi\wedge\omega}\nu_{\xi\wedge\omega_+}$, we get for $\varphi \in L^1(\tilde\nu)$:

$$\begin{aligned}\widehat{\mathcal{M}}^*\tilde\nu(\varphi) &= \int \nu^{\omega_+}(\widehat{\mathcal{M}}\varphi_{\omega_+})\,\Theta^+(d\omega_+) \\ &= \int \nu^{\omega_+}\Big(\int \frac{1}{\lambda^{\xi\wedge\omega_+}}\mathcal{L}_\xi\varphi_{\xi\wedge\omega_+}\,\theta(d\xi)\Big)\,\Theta^+(d\omega_+) \\ &= \int \nu^{\xi\wedge\omega_+}(\varphi_{\xi\wedge\omega_+})\,\theta(d\xi)\,\Theta^+(d\omega_+) = \tilde\nu(\varphi)\,,\end{aligned} \tag{4.15}$$

so that $\tilde\nu = \bar\nu$. Since $\widehat{\mathcal{M}}$ is just the transfer operator associated to the weight $g_{\omega_0}(x)/\lambda^\omega$, the same computation as (3.6) shows that μ is τ^+-invariant (the fact that the weight now depends on the full sequence ω and that the eigenfunction $\tilde\rho$ depends on ω play no role there). Finally, we can check along the lines of the proof of Proposition 3.2 that the measure $\tilde\nu\tilde\rho/\tilde\nu(\tilde\rho)$ is a one-sided quenched relativised equilibrium state. □

When $\lambda^\omega = \lambda^{\omega_0}$ we have that $\tilde\rho_\omega(x)$ only depends on x by Remark 4.1. In that case, the marginal on M of the quenched state $\mu^{(q)} = \tilde\rho\tilde\nu/\tilde\nu(\tilde\rho)$ is equal to $\tilde\rho\bar\nu$ where $\tilde\rho$ and $\bar\nu$ are the maximal eigenfunctions of $\mathcal{M}$ acting on $C^r(M)$ and its dual $\mathcal{M}^*$.

Finally, we have the following stability result which implies Proposition 3 (2):

Proposition 4.3 (Quenched stability). *Consider a positively weighted small random perturbation of $f_0 \in C^r_\gamma(M,M)$, $g_0 \in C^r(M,\mathbb{R}^+_*)$ given by a family θ_ϵ ($\epsilon \ge 0$), and write $\widehat{\mathcal{M}}_\epsilon$, $\widehat{\mathcal{L}}_0$ ($\epsilon \ge 0$) for the corresponding transfer operators acting on $\mathcal{B}(\alpha)$ ($\alpha > 1/\gamma$). Write $R_0 = \exp P(\log|g_0|)$ as usual. The spectrum of $R_0\widehat{\mathcal{M}}_\epsilon$ acting on $\mathcal{B}(\alpha)$ contains only isolated eigenvalues of finite multiplicity outside of the disc of radius R_0/α for small enough ϵ, and converges to the spectrum of $\widehat{\mathcal{L}}_0$ (outside of this disc) acting on $\mathcal{B}$ as $\epsilon \to 0$. The corresponding generalised eigenspaces of $\widehat{\mathcal{M}}_\epsilon$ converge in the $\mathcal{B}(\alpha)$ topology to those $\widehat{\mathcal{L}}_0$, and the dual eigenspaces converge in the weak topology. In particular the maximal eigenmeasure $\tilde\nu_\epsilon$ converges to $\nu_0 \times \delta^{\mathbb{Z}_+}$ with δ the Dirac mass at (f_0, g_0).*

Proof of Proposition 4.3. We first claim that for any $0 < \beta < 1$ the operators $\widehat{\mathcal{M}}_\epsilon$ ($\epsilon \ge 0$) have the same spectral radius when acting on $\mathcal{B}(\alpha)$ or $\mathcal{B}(\alpha,\beta)$ (where $\mathcal{B}(\alpha,\beta)$ is obtained by replacing Lipschitz by β-Hölder in the definition of $\mathcal{B}(\alpha)$), that their essential spectral radius acting on $\mathcal{B}(\alpha,\beta)$ is not bigger than $1/\alpha^\beta$ and that their eigenvalues of modulus larger than $1/\alpha^\beta$ coincide with those of $\widehat{\mathcal{M}}_\epsilon$ acting on $\mathcal{B}(\alpha)$. (Analogous properties hold for $\widehat{\mathcal{L}}_\epsilon$.) This is obtained again by adapting the results of Ruelle [1990] or Fried [1995].

It therefore suffices to show the claim for $\widehat{\mathcal{M}}_\epsilon$ and $\widehat{\mathcal{L}}_0$ acting on $\mathcal{B}(\alpha,\beta)$ for all $0<\beta<1$. Writing R_ϵ for the spectral radius of $\widehat{\mathcal{L}}_\epsilon$, we have by definition of $\widehat{\mathcal{M}}_\epsilon$ and (4.9),(4.11) that for all $n\ge 1$ and small enough ϵ and all $n\ge 1$

$$\begin{aligned}
&\|R_0^n\widehat{\mathcal{M}}_\epsilon^n-\widehat{\mathcal{L}}_\epsilon^n\|_{\mathcal{B}(\alpha,\beta)}\\
&\quad\le \sup_{(\xi_1\cdots\xi_n\omega)\in \mathrm{support}\theta_\epsilon^{\mathbb{Z}_+}}\left[\frac{R_0^n}{\lambda^{\xi_n\omega}\ldots\lambda^{\xi_1\cdots\xi_n\omega}}-1\right]\|\widehat{\mathcal{L}}_\epsilon^n\|_{\mathcal{B}(\alpha,\beta)}\\
&\quad+\sup_{(\xi_1\cdots\xi_n\omega)\in \mathrm{support}\theta_\epsilon^{\mathbb{Z}_+}}\left[\frac{R_0^n}{\lambda^{\xi_n\omega}\ldots\lambda^{\xi_1\cdots\xi_n\omega}}\cdot\sum_{j=1}^n\left(\frac{\mathrm{H}^\beta{}_\omega(\lambda^{\xi_j\cdots\xi_n\omega})}{\lambda^{\xi_j\cdots\xi_n\omega}}\right)\right]\|\widehat{\mathcal{L}}_\epsilon^n\|_{\mathcal{B}(\alpha,\beta)}\\
&\quad\le c_{n,\epsilon}R_\epsilon^n\,,
\end{aligned}\tag{4.16}$$

with $c_{n,\epsilon}$ a constant tending to zero when $\epsilon\to 0$ for each fixed n, and $\mathrm{H}^\beta{}_\omega(\psi(\omega))$ the β-Hölder constant of a function $\psi:\omega^+\to\mathbb{R}$. It follows from (4.16) and Proposition 3.4 (which also holds for $\widehat{\mathcal{L}}_\epsilon$ when considering $\mathcal{B}(\alpha,\beta)$, up to replacing α by α^β in (3.26)) that the analogue of Proposition 3.4 holds for $\widehat{\mathcal{M}}_\epsilon$, i.e., for any fixed $\psi\in\mathcal{B}(\alpha,\beta)$, and $n\ge 1$

$$\lim_{\epsilon\to 0}\|R_0^n\widehat{\mathcal{M}}_\epsilon^n\psi-\widehat{\mathcal{L}}_0^n\psi\|_{\mathcal{B}(\alpha,\beta)}=0\,,\tag{4.17}$$

and there is a constant $C>0$ and an integer $N\ge 0$, so that for all $n\ge N$ there is $\epsilon(n)$ such that for all $\epsilon<\epsilon(n)$

$$\|R_0^n\widehat{\mathcal{M}}_\epsilon^n-\widehat{\mathcal{L}}_0^n\|_{\mathcal{B}(\alpha,\beta)}\le CR^n\alpha^{n\beta}\,.\tag{4.18}$$

We may thus use Lemma 3 in Baladi-Young [1993] just as in the proof of Proposition 3.5. □

We may define correlation functions associated with the quenched Gibbs state $\mu^{(q)}=\tilde\rho\tilde\nu$ and test functions in $\mathcal{B}(\alpha)$ or $\mathcal{B}(\alpha,\beta)$ ($\alpha>1/\gamma$ and $0<\beta<1$) and proceed as in Section 3.6. The relevant spectrum is that of $\widehat{\mathcal{M}}$ acting on $\mathcal{B}(\alpha)$ and we get Theorem 4 (2) by Proposition 4.2 and Theorem 5 (2) by Proposition 4.3.

Note finally that a quenched zeta function $\zeta^{(q)}$ may be introduced by normalising $\zeta_m^{(a)}$ in the definition in (2.24) through the λ^ω associated with m-periodic sequences ω. The results of Ruelle [1990] apply again, relating the discrete spectrum of $\widehat{\mathcal{M}}$ and the poles of $\zeta^{(q)}(z)$.

Appendix A

Proof of (3.5)**.**

We show here how the proof of Lemma 1 in Fried [1995] can be adapted to our skew-product situation, trying to keep close to the notation there. For $\varphi \in \mathcal{B}(\alpha)$ we define bounded functions $\nu_j(x,\omega)$ $(j = 0, \dots, r, r+1)$ on $M \times \Omega^+$ by setting

$$\nu_j(\varphi)(x,\omega) = \begin{cases} |\varphi(x,\omega)| & j = 0 \\ \|D_x^j \varphi(\cdot,\omega)\| & 1 \le j \le r\,, \\ \mathrm{Lip}_\omega D^{j-(r+1)}\varphi(x,\cdot) & r+1 \le j \le 2r+1\,, \end{cases} \tag{A.1}$$

where $\mathrm{Lip}_\omega \psi(x,\cdot)$, for ψ a complex or matrix-valued function, is the smallest constant $K(x)$ so that

$$\|\psi(x,\tilde\omega) - \psi(x,\bar\omega)\| \le K(x) d_\alpha(\tilde\omega,\bar\omega)\,,$$

for all $\tilde\omega$, $\bar\omega$ in Ω^+ (where $\|\cdot\|$ denotes complex modulus or matrix norm). Just like Fried [1995, p. 1063], we find that for all j there are numbers F_{jk} so that

$$\nu_j\big((\psi^n)^*\varphi(x,\omega)\big) \le \sum_{k=0}^{j} F_{jk}\nu_k(\varphi)(\psi^n(x,\omega))\,, \tag{A.2}$$

with $F_{jj} = (\gamma^{-j})^n$ for $0 \le j \le r$, $F_{j,j} = \alpha^n(\gamma^{-j+(r+1)})^n$, for $r+1 \le j \le 2r+1$ and $F_{j\ell} = 0$ for $\ell = 0 < j$. It remains to estimate

$$\sup_{(x,\omega)\in \mathrm{Im}\,\psi^n} \nu_j(\varphi - T\varphi)(x,\omega) \tag{A.3}$$

for functions φ with $\|\varphi\| \le 1$. Obviously

$$\begin{aligned} \|D^j\varphi(x,\omega) - &D^jT(\tilde x,\tilde\omega)(\varphi)(x,\omega)\| \\ &\le \|D^j\varphi(x,\omega) - D^jT(\tilde x,\omega)(\varphi)\| + \|D^jT(\tilde x,\omega)(\varphi) - D^jT(\tilde x,\tilde\omega)(\varphi)\|\,. \end{aligned} \tag{A.4}$$

To bound the first term in the right-hand-side of (A.4) it is useful to observe that M may be embedded in euclidean space such that for any x, y in M there is a piecewise linear path between them with length bounded by a uniform constant times $d_M(x,y)$, and such that the local inverse branches $(f_\xi)_i^{-1}$ may be extended to an open neighbourhood of M with uniformly bounded derivatives (this is a weakened but sufficient version of assumptions (1)–(3) in Fried [1995, p. 1062]). Therefore the arguments of Fried [1995, p. 1064] yield

$$\sup_{x,\omega\in \mathrm{Im}\psi^n} \nu_j((1-T)\varphi)(x,\omega) \le \gamma^{-n(r-j)} + \alpha^n\,,$$

for $0 \le j \le r$ and

$$\sup_{x\in \mathrm{Im}\psi^n} \nu_j((1-T)\varphi(x,\omega)) \le 1\,,$$

for $r+1 \le j \le 2r+1$ so that the proof of (3.5) may be completed just as the proof of Lemma 1 in Fried [1995]. □

Appendix B

The non i.i.d. case.

We use the setup of Section 2.1, except that we *do not* assume that the σ^+ invariant and mixing probability measure Θ^+ on Ω^+ is a product measure, and indicate how our results could be extended.

Since Θ^+ is σ^+ invariant, its decomposition on $E \times \Omega^+$ takes the special form

$$\Theta^+(d\omega) = \theta^{\sigma^+\omega}(d\omega_0)\Theta^+(d\omega)\,. \tag{B.1}$$

We now *assume* further that the functionals θ^ω are Lipschitz functions of $\omega \in \Omega^+$ for some metric d_α. We then define the annealed integrated operator $\widehat{\mathcal{L}}$ acting on $\mathcal{B}(\alpha)$ by

$$(\widehat{\mathcal{L}}\varphi)_\omega(x) = \int_E (\mathcal{L}_\xi \varphi_{\xi\omega})(x)\theta^\omega(d\xi)\,. \tag{B.2}$$

There is in general no operator $\mathcal{L}$ acting on $\mathcal{F}$ in the non i.i.d. setting, but our main results (quasicompactness, annealed zeta function, stability of spectrum, etc.) should hold as before (see Remark 2.1, note however that neither the quenched nor the annealed one-sided SRB state is a product measure on $M \times \Omega^+$ in general).

The definition of the annealed Gibbs state is unchanged, and the definition of the annealed equilibrium state is (2.10), with the following formula for the specific entropy for site of a σ^+ invariant measure υ with respect to the family θ of a priori measures $\theta^\omega(d\xi)$ on E:

$$h^\theta(\upsilon) = -\int_{\Omega^+} \log \beta(\xi\omega)\, \upsilon(d(\xi\omega)) \tag{B.3}$$

if $\upsilon(d(\xi\omega))$ is absolutely continuous with respect to $\theta^\omega(d\xi)\upsilon(d\omega)$, with Radon-Nikodym derivative $\beta(\xi\omega)$, and otherwise, $h^\theta(\upsilon) = -\infty$. The operator $\widehat{\mathcal{M}}$ can be defined as in (4.12), replacing $\theta(d\xi)$ by $\theta^\omega(d\xi)$ and the results on annealed and quenched states should hold in this more general setting.

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