On the Number of Incipient Spanning Clusters

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Abstract In critical percolation models, in a large cube there will typically be more than one cluster of comparable diameter. In 2D, the probability of $k >> 1$ spanning clusters is of the order $e^{-\alpha\ k^2}$. In dimensions $d > 6$, when $\eta = 0$ the spanning clusters proliferate: for $L \to \infty$ the spanning probability tends to one, and the number of robust spanning clusters grows as L^{d-6}, with $|\mathcal{C}_{max}| \approx L^4$. The rigorous results confirm a generally accepted picture for $d > 6$, but also correct some misconceptions concerning the uniqueness of the dominant cluster. We distinguish between two related concepts: the Incipient Infinite Cluster, which by its construction is unique, and the Incipient Spanning Clusters, which are not. The scaling limits of the ISC exhibit interesting differences between low ($d = 2$) and high dimensions. Notably, in the latter case ($d > 6$?) the double limit: infinite volume and zero lattice spacing, will exhibit both percolation at the critical state, and infinitely many infinite clusters.

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Contents

1. Introduction

1.a Main Results

Some of the essential features of critical phenomena are reflected in the structure of the large correlated clusters observed in the critical state [1, 2, 3, 4]. For percolation, the relevant notion is of connectivity, and the characteristic feature of the criticality is the spontaneous formation of large connected clusters, of "macroscopic" diameter. In this paper we present rigorous results on the number of Incipient Spanning Clusters (ISC), in various dimensions. The term *Spanning* is applied generically to clusters which stretch across a large region, and connect different boundary segments (see Fig. 1, and the more detailed explanation in Section 5). *Incipient* is the word we use for the tenuous structures seen at the critical models, at percolation threshold.

Our goals are: i. clarify the issue of uniqueness, concerning which there has been some confusion, ii. present new bounds on the distribution of the number of ISC in 2D models, and iii. prove that in high dimensions Incipient Spanning Clusters behave in a rather different way than they do in 2D.

More explicitly, we establish the following.

1) In any dimension $d > 1$, at $p = p_c$, with positive probability there is *more than one* spanning cluster in slabs of the form $[0, tL] \times [0, L]^{d-1}$; the proof covers the case of t not too large, but independent of L (Theorem 2, below).

2) For completeness, we add that above the critical point, $p > p_c$, there typically is exactly one spanning cluster (for $L >> 1$) (Theorem 6, Appendix C). Below the critical point there is none.

3) In $2D$ critical models the number of spanning clusters if of finite mean, and its probability distribution satisfies:

$$Prob_{p_c}\left(\begin{array}{l}\text{there are exactly } k \text{ unconnected} \\ \text{spanning clusters in } [0,L]^2\end{array}\right) \begin{array}{ll}\geq & A\ e^{-\alpha\ k^2} \\ \leq & e^{-\alpha'\ k^2}\end{array} \tag{1.1}$$

where k counts clusters connecting the two opposite faces $\{x_1 = 0\}$ and $\{x_1 = L\}$ of $[0, L]^2$, and the bounds hold uniformly in L (Theorem 3).

The power k^2 seen in eq. (1.1) represents $k^{d/(d-1)}$. The analysis suggests a plausible argument for a more general validity of such lower bound in low dimensions, but it is not clear whether the actual rate is not slower for d=3,4,5.

4) In dimensions $d > 6$, assuming $\eta = 0$ (η being the critical exponent seen in eq. (1.4)):

- the spanning probability tends to *one* (Theorem 4), and

- spanning clusters proliferate –

$$Prob_{p_c}\Big(\text{the number of spanning clusters} \geq \ o(1)L^{d-6}\Big) \underset{L\to\infty}{\longrightarrow} \ 1. \qquad (1.2)$$

  Furthermore, at least for clusters whose size (the number of points) is comparable to the maximal ($|\mathcal{C}_{max}|$), L^{d-6} provides the actual rate of growth — not just a bound.

- The size of the maximal cluster is typically

$$|\mathcal{C}_{max}| \approx L^4 \ , \qquad (1.3)$$

  in a sense made explicit in Theorem 5.

The assumption on η refers to the purported law:

$$\tau(x,y) \ \equiv \ Prob_{p_c}\,(x \text{ sites } x \text{ and } y \text{ are connected}) \approx \frac{Const.}{|x-y|^{d-2+\eta}} \quad . \qquad (1.4)$$

It is expected that $\eta = 0$ above the upper–critical dimension $d = 6$ [5, 6]. Rigorous results in this direction were proven by Hara and Slade [7], who establish that a related, but somewhat weaker condition, holds at $d > 6$ in models with sufficiently spread finite–range connections, and at somewhat higher dimensions for the nearest–neighbor percolation model.

The results proven here for $d > 6$ reinforce the generally accepted picture, in which the proliferation of large clusters is related to the breakdown of hyperscaling (Coniglio [1]), and the "fractal" dimension of the large clusters stabilizes at $D = 4$ (Aharony, Gefen and Kapitulnik [8], Alexander et.al. (AGNW) [9]). Our analysis is based on the diagrammatic bounds of Aizenman and Newman [10]. The high effectiveness of the method points to the validity of the perspective offered by AGNW [9] that for $d > 6$ large clusters resemble randomly branched chains.

The fact that the spanning probability tends to 1 has interesting implications for the nature of the continuum limit, mentioned in Section 5.

The prototype for the percolation models discussed here, *and the systems to which we refer by default*, are the nearest-neighbor bond, or site, percolation models on the regular lattice Z^d. However, with minimal clarifications the discussion applies also to percolation models with different short–scale characteristics, including finite range bond percolation, n.n. and next–nearest–neighbor site percolation, and also the continuum random dots models. To attain this robustness, the arguments employed refer mostly to large – scale connection events. The essential features (or at least those assumed throughout) are: independence for sufficiently separated regions [a fixed finite distance], and the vdB-K property which is presented in Section 3.

Following are some less technical comments on the results.

1.b Implications of the Multiplicity of Incipient Spanning Clusters

The assertion that in 2D there is positive probability for more than one Incipient Spanning Cluster will not surprise mathematicians familiar with the theory developed by Russo [11] and Seymour and Welsh [12]. Nevertheless, even the 2D case of the general result (1) is in contradiction with statements which for quite a while have apparently been part of a wide consensus. That consensus was recently challenged, and corrected, by a report of a numerical work [13] (see also [14]). As we attempt to make it plain (and as was already stated in ref. [15]), the case of 2D is really simple. To prove the result for general dimension, we develop what may be regarded as a rigorous real – space renormalization group argument.

The misconception of the uniqueness of the spanning clusters could have been facilitated by a number of factors:

i. *Scaling Theory*
The assumption that there typically is one dominant cluster (in a finite region $[0, L]^d$) helps in the explanation of the *hyperscaling* law, which is found to be obeyed in low dimensions. To re-emphasize that low level multiplicity (up to $L^{o(1)}$) is consistent with hyperscaling, we present a version of the heuristic argument in Appendix A. It is seen there that the relevant condition is only that the number of large clusters grows slower than any power of L.

ii. *Uniqueness of the Infinite Cluster*
One of the results known under rather general assumptions (discreteness and "regularity") is that when there is percolation, the infinite cluster is unique [16, 17, 18] (see however Sect. 5 below). The Uniqueness Theorems were occasionally quoted as an indication that at the percolation threshold, where large–scale clusters are expected, there is a single dominant cluster — referred to as the Incipient Infinite Cluster (IIC). Here, the convenient and otherwise appealing terminology might have suggested an incorrect deduction.

iii. *Terminology*
The term Incipient Infinite Cluster is too encompassing. The dominant cluster(s) in a large volume can be viewed from two perspectives, for which it may be better to have separate terminology. We propose to keep the term IIC for the large clusters as seen from the perspective of their sites (which presumably agrees with the conditioned process studied by Kesten [21] in the context of 2D [21] models). The same clusters, as viewed

on the bulk (macroscopic) scale may be referred to as the Incipient Spanning Cluster(s). The distinction becomes more clear as one contrasts the microscopic and the macroscopic view of the system, and contemplates the relevant mathematical descriptions of the *scaling limit* — the limiting situation in which the ratio of the two scales diverges. A more detailed discussion of these issues, is presented in Section 5, which can be read in advance of the more technical Sections. Included in Section 5 are three possible definitions of the IIC, which still ought to be proven equivalent. For each of them it can be shown that if the percolation density vanishes at p_c then the IIC process exhibits a unique infinite cluster. The ISC in general are not unique (Section 2).

Red — Blue bonds
The multiplicity of ISC means also that the important notion of "red bonds" (Stanley [19, 2], Coniglio [20]) needs some care. This notion is best explained in the situation in which a macroscopic piece of material is placed between two conducting plates which are at different electric potentials, and the bonds represent conducting elements. The "red bonds" are often interchangeably described as: i. bonds which are essential for the connection, i.e., are unavoidable for every path connecting the opposite boundary plates, or as ii. bonds which carry the full current. (Other elements in this classification are: "blue" - the backbone bonds, and "yellow" - the dangling ends.) As we now appreciate, these two definitions do not coincide.

The non-uniqueness of the spanning cluster has the consequence that frequently the random configuration will not contain any bond which is essential for the connection (though there will be bonds essential for their specific spanning clusters). Interestingly, this does not mean that no bond carries the full current. If there are large differences in the total resistance of the distinct spanning clusters, than the bulk of the current may still be carried by just one of them. Its essential bonds meet the condition in the second definition of "red bonds". However, as we see now, this terminology involves more than just the topology of the cluster.

Despite the above complication, it is still desirable to have an elementary geometric concept expressing the fact that in typical configurations there are bonds whose removal (or blockage) will result in a change of the available connected routes which is visible on a large scale R. For this purpose one may define a bond b to be *essential on scale* R ($>>$ lattice scale) if there are two realized paths, of end–to–end distance R emanating from the opposite ends of the bond, which are not linked by any other path which stays within the distance R from b.

Further discussion of the results is found in Section 5.

2. Multiple ISC occur in Critical Models in all Dimensions $d > 1$

Theorem 1 *(Incipient Spanning Clusters occur) For any dimension $d > 1$, there is a (decreasing) function $h(t)$, which for $0 \le t < 1/2$ is strictly positive, such that at the critical point for every $0 < L < \infty$:*

$$R_{L,t} \;\equiv Prob_{p_c} \left(\begin{array}{l} \text{the slab } S_{L,t} \equiv [0, tL] \times [0, L]^{(d-1)} \text{ is traversed} \\ \text{in the direction of the 1-st coordinate} \end{array} \right) \;\ge\; h(t) \;. \qquad (2.1)$$

Furthermore, $h(\cdot)$ satisfies

$$h(t) \;\ge\; \begin{array}{ll} 1 - Ae^{-const.\ t^{-(d-1)}} & \text{as } t \searrow 0 \\ C_d\ (1/2 - t)^{(d-1)/2} & \text{as } t \nearrow 1/2 \end{array} \;. \qquad (2.2)$$

The proof is based on arguments which may be regarded as standard. We place it therefore in the Appendix (B).

Theorem 2 *(There can be more than one ISC) For any dimension $d > 1$, there is a function $g(t)$, strictly positive for $0 \le t < t_o$, with which for every finite size L:*

$$D_L(t, p_c) \equiv Prob_{p_c} \left(\begin{array}{c} \text{there is more than one} \\ \text{spanning cluster in } S_{L,t} \end{array} \right) \;\ge\; g(t) \;. \qquad (2.3)$$

Remarks: 1) It should be appreciated that the bound in eq. (2.3) is satisfied only at $p = p_c$. Otherwise:

$$\lim_{L \to \infty} D_L(t, p\ [\neq p_c]) = 0 \qquad (2.4)$$

(for any $t > 0$). For $p < p_c$ eq. (2.4) is true because there are no spanning clusters (the connectivity function decays exponentially), and for $p > p_c$ it holds because there is only one spanning cluster (see Appendix C).

2) We expect the restriction to small t, for the positivity of $h(t)$ in Theorem 1 and $g(t)$ in Theorem 2 not to belong there, since if the spanning probabilities are uniformly positive for some aspect ratio ($t_o > 0$) then that should also be true for any other value of t. However we do not have a proof of that implication. For two dimensions, such a

statement is derived in the works of Russo [11] and Seymour and Welsh [12] (conveniently summarized in [22]). The result provides a very versatile tool for the study of 2D critical models (see below).

The proofs of Theorem 1 and Theorem 2 (for $d \geq 3$) have the flavor of a real - space renormalization argument. By considering suitably chosen local events, determined in blocks of scale L, it is shown that for any p:

i. If $R_{L,t}$ is too small than the connectivity function decays exponentially, and hence $p < p_c$.

ii. If $[1 - R_{L/3,3t}] + D_{L,t}$ is too small then there is percolation, and $p > p_c$.
For $p = p_c$ the term $[1 - R_{L/3,3t}]$ vanishes as $t \searrow 0$, and hence $D_{L,t}$ cannot be too small.

The first argument is recapitulated in Appendix B. The second is given below.

Proof of Theorem 2: As is explained in the next section, for 2D the RSW theory [11, 12] implies that eq. (2.3) holds with a function $g(t)$ which is positive for all $t > 0$.

To prove Theorem 2 for $d \geq 3$ we shall show that there is a constant $b < \infty$ (based on 2D considerations, and thus independent of d) such that for every $p \leq p_c$

$$(1 - R_{L/3,3t}) + D_{L,t} \ \geq \ b \tag{2.5}$$

for any $L < \infty$. Since $R_{L,t} \geq h(t)$, that would establish the claim, with

$$g(t) = b - [1 - h(3t)] \tag{2.6}$$

where $\lim_{t \searrow 0}[1 - h(t)] = 0$, by Theorem 1.

The following argument proves that if (for a suitable choice of the constant b) eq. (2.5) fails for some L than there is percolation in the slab $[0, tL] \times [0, L]^{d-3} \times R^2$. Let us partition the slab into a planar collection of translates of the d-dimensional rectangular cell $[0, tL] \times [0, L]^{d-3} \times [0, L/3]^2$. In a given configuration of the percolation model, a cell is declared *regular* if the following two conditions are met:
i. the cell contains a spanning cluster (in the first direction),
ii. the block made by joining 3×3 cells, with the given one at the center, is traversed by exactly one spanning cluster.

Notice that if within the 2D array (of d–dimensional regions) there is chain of neighboring *regular cells*, then the spanning clusters of these cells belong to a common connected cluster.

The last observation implies that there is percolation in the slab unless any finite region is encircled by a *-connected loop of cells which are not *regular* in our terminology. We estimate the probability of such events by a Peierls–type argument.

While irregularity is not independent for adjacent cell, it is independent for cells with a gap of 2 in one of the two direction. Thus, it is convenient to estimate the probability of the existence of an encircling loop by focusing on sub-chains of minimal steps with such gap. There are 20 possibilities for each step in the chain. The Peierls estimate, suitably–modified, implies that a sufficient condition for percolation of the regular cells is

$$Prob(\text{ cell is irregular }) \cdot 20 \ < \ 1 \tag{2.7}$$

We bound the probability of a cell to be irregular by adding the probabilities of the two possible failures. The result is that there is percolation unless eq. (2.5) holds, with $b = 0.05$. (That may not strike one as a very large number, but still it is positive uniformly in L).

Since the probabilities $R_{L,t}$ and $D_{L,t}$ are continuous in p (pertaining to regions of fixed finite size), if eq. (2.5) fails at p then it does so also for $p-\epsilon$ for $\epsilon > 0$ small enough, and thus the conclusion is that either $p > p_c$, or eq. (2.5) holds.

3. ISC in Two Dimensions

3.a A Two-dimensional Construction

The situation in two dimensions is most amenable to qualitative analysis. The proof that there can be any number (n) of spanning clusters, with probability which does not vanish as $L \to \infty$, is elementary: partition the rectangular region into $(2n-1)$ parallel strips, and note that the event occurs under the scenario depicted in Figure 2.a. Based on the product of probabilities of the $(2n-1)$ independent events, one gets:

$$\begin{aligned} K_L(n) \;\; &\equiv \;\; Prob_{p_c}\begin{pmatrix} \text{there are at least } n \text{ distinct} \\ \text{spanning clusters in } [0,L]^2 \end{pmatrix} \;\geq \\ &\geq \;\; Prob_{p_c}\begin{pmatrix} \text{the strip } [0, L/(2n-1)] \times [0,L] \text{ is traversed in the} \\ \text{long direction by a spanning cluster [dual cluster]} \end{pmatrix}^{2n-1} \\ &\geq \;\; [h(2n-1)]^{2n-1} \; > 0 \end{aligned} \tag{3.1}$$

where the concluding step is by the aforementioned theorem of Russo [11] and Seymour and Welsh [12] that $h(t) > 0$ for all t.

We view the above argument as part of the rich legacy of the RSW theory. This term refers to a versatile method for construction which in combination with other insights has been employed for a variety of far less elementary results: non-percolation at p_c [23], sharpness of the phase transition [24], support for the scaling theory [25] (all the above for 2D models), some rigorous real space renormalization arguments [26] and some extension to higher dimensions [26].

Our main goal in this section is to present large–deviation estimates for the probability that the number of clusters spanning $[0,L]^2$ exceeds (or equals) a large number n.

3.b Bounds for the Number of ISC in 2D

It is easy to be mislead about the large deviation asymptotics. By an application of the van den Berg – Kesten inequality

$$K_L(n) \;\leq\; [R_{L,1}]^n \;\leq\; e^{-|\ln h(1)| \; n} \;, \tag{3.2}$$

with $h(1) = 1/2$ in the self–dual case. Numerical results seem consistent with exponential decay in n (of the form $e^{-Const.\; n}$) [27]. Nevertheless the decay rate is different (faster).

Remark: The inequality of van den Berg and Kesten [28] states that for independent percolation (or, generally, systems of independent variables) the probability of the *dis-*

joint occurrence of two, or more, events (e.g., the existence of two separate connecting paths) is dominated by the product of their probabilities. The statement is not obvious since *disjoint occurrence* does not refer to a-priori specified separate regions.

The faster–than–exponential decay is caused by the mutual exclusion, which limits the clusters to reduced regions. (For the lower bound seen in eq. (3.1) the clusters are produced in disjoint strips).

Theorem 3 *In planar percolation models, at $p = p_c$ the probability there are at least n separate spanning clusters behaves as*

$$K_L(n) \begin{cases} \geq & A\ e^{-\alpha\ n^2} \\ \leq & \quad e^{-\alpha'\ n^2} \end{cases} \tag{3.3}$$

(where the bounds involve different positive constants).

Remark: $Prob$(exactly n crossings) behaves just as $Prob$(at least n crossings) $\equiv K_L(n)$, since the former is $K_L(n) - K_L(n+1)$, and (by the vdB-K inequality) $K_L(n+1) \leq K_L(1) \cdot K_L(n)$ ($= K_L(n)/2$) .

Proof: The lower bound in eq. (3.3) is obtained through the construction discussed above, supplementing eq. (3.1) with the known fact that $h(t) \geq [\approx] e^{-\alpha t}$ (proven by constructing a chain of 1×2 brick crossing events; as seen in Figure 2.b [11, 12, 23, 26]).

The upper bound is slightly more involved. Let us first ask how many distinct spanning clusters will be seen in a tall strip $L_h \times L_v$ ($L_v > L_h$). It is natural to expect that the number, call it N_{L_v, L_h} will typically be of the order of L_v/L_h, since, roughly, there is a finite number of distinct crossings in each of the squares comprising the tall stack. This motivates the following bound, which is proven below using Lemma 1,

$$Prob_{p_c}\left(N_{L_h, L_v} \geq u \frac{L_v}{L_h} \right) \ \leq \ e^{-f(u) \times L_v / L_h} \tag{3.4}$$

with $f(u) > 0$ for u large enough.

For a moment, let us assume eq. (3.4). To estimate $K_L(n)$, we partition the $L \times L$ square into n/u vertical strips, so that the height/base ratio satisfies $n = uL/L_h$. If the square is spanned by at least n disconnected clusters, then that is also the case for each of the vertical strips. (Though the converse need not be true.) These being independent events, the probability is bounded by:

$$\begin{aligned} K_L(n) \quad & \leq \quad \left[e^{-f(u) \times \frac{n}{u}} \right]^{n/u} \\ & \leq \quad e^{-\frac{f(u)}{u^2}\ n^2} \ , \end{aligned} \tag{3.5}$$

which concludes the proof of eq. (3.3).

To prove the large – deviations estimate eq. (3.4) we first derive the following related statement.

Lemma 1

$$\mathbf{E}_{p_c}\left(e^{t\ N_{L_h,L_v}}\right) \ \leq\ e^{\phi(t)\frac{L_v}{L_h}} \tag{3.6}$$

with $\phi(t) < \infty$ *at least for* t *small enough. [A - posteriori, we'll know it to be finite for all* t*.]* $\mathbf{E}(--)$ *represents here expectation value.*

Proof: To avoid more complicated analysis, we shall take advantage of a convenient shortcut. For L given, let us count the clusters connecting the left and right faces of the square, allowing the paths to meander beyond the square into the entire vertical strip of width L containing it. Let $\tilde{N}_L$ be the (random) number of distinct connecting clusters in the strip with those characteristics. We claim that there is a function $\phi(t)$, finite for t small enough, such that:

$$\begin{aligned}\mathbf{E}_c(e^{t\ \tilde{N}_L}) &= 1 + t\int_0^\infty e^{tx}\ Prob(\tilde{N}_L \geq x)\ dx \\ &\leq e^{\phi(t)}\ .\end{aligned} \tag{3.7}$$

The claim is based on two observations: i. $Prob(\tilde{N}_L \geq k) \leq Prob(\tilde{N}_L \geq 1)^k$ (based on the vdB–K inequality), and ii. $Prob(\tilde{N}_L \geq 1)$ is uniformly < 1 (a simple construction using the RSW theory).

A moment of reflection shows that the random variable N_{L_h,L_v} is dominated, in the stochastic sense, by the sum of L_v/L_h independent copies of $\tilde{N}_{L_h}$ [that is the short-cut.] To see this, it is convenient to order the different spanning clusters from the bottom up, and consider the distribution of the increments associated with the stack of the $L_h \times L_h$ squares. Compared with the open case in which the clusters are free to use the entire vertical strip, the conditioned process is restricted by an excluded volume.

The relevant aspect of the stochastic–domination relation is:

$$\begin{aligned}\mathbf{E}\left(e^{t\ N_{L_h,L_v}}\right) &\leq \mathbf{E}\left(e^{t\ \tilde{N}_{L_h}}\right)^{L_v/L_h} \\ &\leq e^{\phi(t)\frac{L_v}{L_h}}\ ,\end{aligned} \tag{3.8}$$

as claimed above.

The remaining path from eq. (3.8) to eq. (3.4) involves the Chebyshev estimate:

$$Prob\left(N_{L_h,L_v} \geq u\ \frac{L_v}{L_h}\right) \ \leq\ \mathbf{E}(e^{t\ N_{L_h,L_v}})\ e^{-t\ u\ L_v/L_h}\ \leq\ e^{-[tu-\phi(t)]L_v/L_h}\ , \tag{3.9}$$

which shows that eq. (3.4) holds with $f(u)$ the Legendre transform:

$$f(u)\ = \sup_t [tu - \phi(t)]\ \ . \tag{3.10}$$

Since there are values of t for which $\phi(t) < \infty$, $f(u) > 0$ for u large enough.

Let us end this section with a question.

Q: It is a standard conjecture that $\kappa(n)\ =\ \lim_{L\to\infty} K_L(n)$ exists. Can one prove the existence of the limit

$$\lim_{n\to\infty} \lim_{L\to\infty} \frac{1}{n^2} \log K_L(n)\ ? \tag{3.11}$$

(With proper interpretation, this may be easier to prove than than existence of $\kappa(n)$.) Better yet — can one evaluate the limit ? Conformal field theory [29] could perhaps help here.

4. Above the Upper Critical Dimension

4.a The Relevant Condition

We shall now prove that in contrast with the situation in 2D, above the upper critical dimension spanning clusters are sure to occur, and they proliferate in the way described in the introduction. A convenient sufficiency condition is that:

$$d > 6 \tag{4.1}$$

and the connectivity function defined by eq. (1.4) (at $p = p_c$) satisfies:

$$\tau(x,y)\ \lesseqgtr\ Const./\ |x-y|^{(d-2+\eta)} \tag{4.2}$$

with

$$\eta = 0\,. \tag{4.3}$$

In this article, the relation ($\lesseqgtr$) seen in eq. (4.2) means that

$$\tau(x,y)\ \ \begin{cases} \leq & C\ /\ |x-y|^{(d-2+\eta)} \\ \geq & C'/\ \ |x-y|^{(d-2+\eta)} \end{cases} \tag{4.4}$$

with a pair of possibly distinct constants, $0 < C' \leq C < \infty$.

It is generally expected that $d > 6 \Longrightarrow \eta = 0$. As mentioned in the introduction, there are mathematical results offering partial support to this claim, however some gaps remain [7] (reviewed in [30]). The statement which is established in ref. [7] for high dimensional models is slightly weaker than eq. (4.2) and eq. (4.3), but sufficient for the "triangle condition" of ref. [10] ("∇–diagram"$(p_c) < \infty$). A strategy which was successful in the derivation of some other results about the critical behavior in high dimensions was to base the analysis on such a weaker assumption (proving "∇-condition" $\Longrightarrow \gamma = 1$ [10], $\beta = 1$ and $\delta = 2$ [31]). We do not pursue that track here.

4.b The Number (L^{d-6}) and Size (L^4) of the Spanning Clusters

A convenient geometry for the study of spanning clusters is that of the annular region $A_{L,s} = \Lambda_L \backslash \Lambda_{sL}$, with $\Lambda_L = [-L, L]^d$, and $0 < s < 1$. The region is enclosed by the outer layer $B_L = \Lambda_{3/2L} \backslash \Lambda_L$, and its hollow inner core is $D_L = \Lambda_{sL}$. The number of clusters spanning $A_{L,s}$ can be presented as

$$S_L = \text{ the number of connected clusters of } \Lambda_{3/2L}, \text{ intersecting both } B_L \text{ and } D_L \ , \tag{4.5}$$

Our main result is:

Theorem 4 *In any dimension $d > 6$, if eq. (4.2) holds with $\eta = 0$, then*

$$Prob_{p_c}\left(S_L \ \geq \ o(1) L^{(d-6)}\right) \ \underset{L\to\infty}{\longrightarrow} \ 1 \ . \tag{4.6}$$

in the sense that eq. (4.6) holds for any function of L, denoted here $o(1)$, which tends to 0 as $L \to \infty$.

The proliferation of the spanning clusters bears some relation to the fact that they are "fractal" objects. It is often stated that their dimension reaches $D = 4$ and it remains at this value, as the lattice dimension is increased over the upper critical dimension $d = 6$ ([8, 9, 1]). The following statement offers some rigorous support to this claim.

Theorem 5 *In $d > 6$ dimensions, assuming the power–low behavior (eq. (4.2)) with $\eta = 0$,*

$$Prob_{p_c}\left(|\mathcal{C}_{max}| \begin{array}{cc} \leq & c \ \log L \ \cdot L^4 \\ \geq & o(1) \ L^4 \end{array} \right) \ \underset{L\to\infty}{\longrightarrow} \ 1 \tag{4.7}$$

where $|\mathcal{C}_{max}| = \max\{|\mathcal{C}| : \mathcal{C}$ *a connected cluster in* $\Lambda_L\}$*, c is a (sufficiently large) finite constant, and $o(1)$ is to be interpreted as in Theorem 4.*

In the derivation of these results we make use of the diagrammatic bounds of ref. [10]. The most elementary of these is the tree–diagram bound, which directly yields:

Lemma 2 *In any dimension, assuming the power-low behavior eq. (4.2), for any $k \geq 2$*

$$\mathbf{E}_{p_c}\left(\sum_{\mathcal{C}\subset\Lambda_L} |\mathcal{C}|^k\right) \;\leq\; \mathbf{E}_{p_c}\left(|\mathcal{C}_{max}|^k\right) \;\leq\; k!\; C_d^k \cdot L^{d-6+3\eta} \cdot L^{(4-2\eta)k} \tag{4.8}$$

where $\mathcal{C}$ refers to the connected clusters.

Somewhat less immediate is the *lower bound* seen in the following intermediate statement.

Lemma 3 *In any dimension $d > 6$, if eq. (4.2) holds with $\eta = 0$, then for $k = 2, 3$*

$$Prob_{p_c}\left(\sum |\mathcal{C}|^k \begin{array}{cc} \leq & 1/o(1) \\ \geq & o(1) \end{array} \times\; L^{(d-6)}\; L^{k\cdot 4}\right) \;\underset{L\to\infty}{\longrightarrow}\; 1 \tag{4.9}$$

where the sum is over the connected clusters $\mathcal{C} \subset \Lambda_{3/2L}$.

Furthermore, for $k = 2$ the lower bound holds also when the sum is restricted to the spanning clusters ($\sum^{(sp)} |\mathcal{C}|^2$), as defined in eq. (4.5).

Before deriving the two Lemmas (in the next subsection), let us go over the deductions of the Theorems (4 and 5).

To estimate S from below, we use the fact that in each realization,

$$S_L \;\geq\; \frac{\left(\sum^{(sp)} |\mathcal{C}|^2\right)^3}{\left(\sum^{(sp)} |\mathcal{C}|^3\right)^2} \tag{4.10}$$

where the sums $\sum^{(sp)}$ are over the set of the spanning connected clusters, whose number is S_L. The above is implied by the Hölder inequality: $\sum |\mathcal{C}|^2 \leq \left(\sum |\mathcal{C}|^3\right)^{2/3} \; \left(\sum 1\right)^{1/3}$. Readers may note that the bound is dimensionally well balanced. Upon the substitution of the typical values of the two sums, as provided by Lemma 3, one obtains Theorem 4.

Similarly,

$$|\mathcal{C}|_{max} \;\geq\; \frac{\sum |\mathcal{C}|^3}{\sum |\mathcal{C}|^2} \tag{4.11}$$

provides the lower bound, $|\mathcal{C}_{max}| \geq o(1)L^4$, claimed in Thm. 5.

We deduce that $|\mathcal{C}_{max}|$ is typically not much larger than L^4, from the rate of growth of the moments described in lemma 2, combined with the Chebyshev inequality:

$$Prob_{p_c}\left(|\mathcal{C}_{max}| \geq \alpha(L)\cdot L^{\lambda}\right) \;\leq\; \frac{\mathbf{E}\left(|\mathcal{C}_{max}|^k\right)}{\alpha(L)^k \cdot L^{k\lambda}} \;. \tag{4.12}$$

A comparison with Lemma 2 shows that a natural choice is $\lambda = (4-2\eta)$ and $\alpha(L) = a\log L$. Optimizing over k (chosen so that $k \approx \alpha(L)/C_d$), one learns that

$$Prob_{p_c}\left(|\mathcal{C}_{max}| \leq a\;\log L \cdot L^{(4-2\eta)}\right) \underset{L\to\infty}{\longrightarrow} \;1\;, \tag{4.13}$$

provided $a > C_d(d-6+3\eta)$. This proves the last part of (eq. (4.7)), in a somewhat more general form.

4.c Derivation of the Moment Bounds

We start from the elementary but important identity

$$\begin{aligned}
\mathbf{E}\left(\sum{}^{[(sp)]}\,|\mathcal{C}|^k\right) \quad &= \quad \mathbf{E}\left(\sum_{\mathcal{C}}{}^{[(sp)]}\sum_{x_1,\ldots,x_k} \mathrm{I}[\mathrm{x_1}\in\mathcal{C}]\;\ldots\mathrm{I}[\mathrm{x_k}\in\mathcal{C}]\right)\\
&= \quad \mathbf{E}\left(\sum_{x_1,\ldots,x_k} \mathrm{I}[\;x_1,\ldots,x_k \text{ belong to a common [spanning] cluster}]\right)\\
&= \quad \sum_{x_1,\ldots,x_k} \tau^{[(sp)]}(x_1,\ldots,x_k)
\end{aligned} \tag{4.14}$$

where I[–] is an indicator function and $\tau_k^{[(sp)]}(x_1,...,x_k)$ is the probability that the k points are all connected, with the optional restriction to spanning clusters.

Next, the argument will employ a number of diagrammatic bounds on the connectivity functions, following the approach presented in ref. [10]. The technique has been further simplified through the development of the van den Berg – Kesten inequality [28] mentioned in Section 3.

Proof of Lemma 2 (following ref. [10]): The tree–diagram bound [10] states that $\tau_k(x_1,\ldots,x_k)$ is dominated by the sum of products of the two point function $\tau(u,v)$, arranged in the form of tree diagrams with the external vertices $x_1,...,x_k$. Its intuitive explanation is that in order for the given k sites to be connected there has to be a connecting tree. The vdB-K inequality [28] (or the alternative argument which was originally used in ref.[10]) permits to bound the probability by the sum over the corresponding tree diagrams. (The bound is not totally obvious since it involves sum over tree diagrams, and not tree graphs, i.e., all trees embedded in the lattice. The latter sum is much too large.)

For $k = 3$ the inequality is:

$$\tau_3(x_1, x_2, x_3) \ \leq \ \sum_u \tau(x_1, u)\ \tau(x_2, u)\ \tau(x_3, u)\ . \tag{4.15}$$

For general $k \geq 3$ there are $(2k-5)!! \leq 2^k k!$ tree graphs for each set of the external vertices, and each diagram has $(2k-2)$ vertices (external + internal) and $(2k-3)$ lines. The sum yields:

$$\mathbf{E}\left(\sum\ |\mathcal{C}|^k\right) \ \leq \ k!\ C_d^k \cdot L^{d(2k-2)}/L^{(4-2\eta)k} \ = \ k!\ C_d^k \cdot L^{d-6+3\eta} \cdot L^{(4-2\eta)k}\ . \tag{4.16}$$

That implies Lemma 2, since $\mathbf{E}\left(|\mathcal{C}_{max}|^k\right) \leq \mathbf{E}\left(\sum\ |\mathcal{C}|^k\right)$.

To prove Lemma 3 we need a slightly more delicate estimate, of the difference between two comparable terms (which under the right conditions would form a small reminder). Following is that auxiliary result.

Lemma 4 *For independent percolation on any graph,*

$$\begin{aligned} 0 \ \leq \ & Prob\begin{pmatrix} \{x_1, x_2\} \text{ and } \{y_1, y_2\} \\ \text{are pairwise connected} \end{pmatrix} - \tau(x_1, x_2) \times \tau(y_1, y_2) \leq \\ \leq \ & \sum_{u,v} \tau(x_1, u)\ \tau(y_1, u)\ \tau(v, u)\ \tau(v, x_2)\ \tau(v, y_2) \ + \ [y_1 \leftrightarrow y_2 \text{ permutation }] \end{aligned} \tag{4.17}$$

and, for $\tau_3(...)$:

$$\begin{aligned} & Prob\begin{pmatrix} x_1, x_2, x_3 \text{ are all connected} \\ \text{and so are } y_1, y_2, y_3 \end{pmatrix} - \tau_3(x_1, x_2, x_3)\ \tau_3(y_1, y_2, y_3) \ \leq \\ \leq \ & \sum_{u,v} \tau(x_1, u)\ \tau_3(u, x_2, x_3)\ \tau(v, u)\ \tau(v, y_1)\ \tau_3(v, y_2, y_3) \ + \\ & \quad + [\text{ permutations }]\ . \end{aligned} \tag{4.18}$$

(as depicted in Figure 4).

Proof of Lemma 4: The positivity is obtained by the standard monotonicity argument (the Harris – FKG inequality [32, 33]): the probability of the simultaneous connection of the two pairs is greater that the product of probabilities. In the opposite direction, the probability of disjoint connections is smaller than the product which is subtracted (vdB-K inequality). Thus, we need to focus on the remaining situation: each of the k-tuples is interconnected, but not disjointly. In each such configuration there is a tree subgraph connecting all the $2k$ vertices. The tree is bound to have a link whose removal will split it into two subgraphs, of equal numbers of sites. Let $\{u, v\}$ be the vertices of that link. There is a further constraint that this link cannot separate the $x-$sites from the y-sites. Finally, we note that for each such tree, there is a collection of clusters which are connected disjointly, as indicated in Figure 3 and Figure 4. The stated claim follows by an application of the afore mentioned vdB–K inequality [28]. (The result directly generalizes to higher values of k.)

Proof of Lemma 3 The upper bounds in eq. (4.9) are a direct consequence of eq. (4.16) (via the Chebyshev inequality). It remains only to prove that typically the sums $\sum^{[(sp)]}|\mathcal{C}|^k$ are not below the claimed level.

Let us consider first the case $k = 2$. Starting from the $|\mathcal{C}|^2$ version of the identity eq. (4.14), it is easy to see that:

$$\sum{}^{(sp)} |\mathcal{C}|^2 \ \geq \ 2 \sum_{x \in B_L,\ y \in D_L} \mathrm{I}[\ x \text{ and } y \text{ are connected }\ ] \ := \ \mathrm{K}\ , \tag{4.19}$$

where K is defined by the sum. The mean value of K is easily determined:

$$\begin{aligned} \mathbf{E}(K) \quad &= \quad \sum_{x \in B_L,\ y \in D_L} \tau(x, y) \\ &\lesssim \ \text{Const. } L^{2d}/L^{d-2+\eta} = \text{Const. } L^{d+2-\eta}. \end{aligned} \tag{4.20}$$

The corresponding expressions for the second moment, and for the variance, are:

$$\mathbf{E}(K^2) \ = \sum_{\substack{x_1, x_2 \in B_L \\ y_1, y_2 \in D_L}} Prob\,(x_1 \text{ is connected to } y_1, \text{ and } x_2 \text{ is connected to } y_2\ ) \tag{4.21}$$

and

$$\begin{aligned} Var(K) \ &= \ \mathbf{E}(K^2) - \mathbf{E}(K)^2 \ = \\ &= \sum_{\substack{x_1, x_2 \in B_L \\ y_1, y_2 \in D_L}} \left[Prob \left(\begin{array}{c} \{x_1, y_1\} \text{ and } \{x_2, y_2\} \\ \text{are pairwise connected} \end{array} \right) - \ \tau(x_1, y_1) \times \tau(x_2, y_2) \right] . \end{aligned} \tag{4.22}$$

Using eq. (4.18) of Lemma 4, we obtain the following estimate:

$$\begin{aligned}
\mathbf{E}\left|\frac{K}{\mathbf{E}(K)}-1\right|^2 &= \frac{\mathbf{E}(K^2)-|\mathbf{E}(K)|^2}{|\mathbf{E}(K)|^2} \\
&\leq \text{Const. } L^{2d}/L^{3(d-2+\eta)} = \frac{\text{Const.}}{L^{d-6+3\eta}} . \qquad (4.23)
\end{aligned}$$

Under the right conditions, this implies implies that only very seldom will K differ by a significant factor from the mean $\mathbf{E}(K)$.

We can proceed only under the assumption that $d - 6 + 3\eta > 0$, for otherwise the last bound still allows the typical values of $K/\mathbf{E}(K)$ to be arbitrarily small. That however is ruled out if the right side of eq. (4.23) is $o(1)$, in which case there is a constant $b > 0$ (determined through eq. (4.20)) for which

$$Prob\Big(K \;\geq\; b\; L^{(d+2-\eta)}\Big) \underset{L\to\infty}{\longrightarrow} 1 . \qquad (4.24)$$

Since $\sum^{(sp)} |\mathcal{C}|^2 \geq K$ that implies the lower bound claimed in eq. (4.9), for $k = 2$.

The analysis of the case $k = 3$ is very similar, with K replaced by $\sum |\mathcal{C}|^3$ and eq. (4.18) by eq. (4.19). The notable fact is that in both cases the "truncation" results diagrammatically in the addition of *two sites* and *three lines*, which translates to the multiplicative factor $Const.\ L^{2d}/L^{3(d-2+\eta)} = Const./\ L^{d-6+3\eta}$.

Remarks: *1.*Purists may note that the above proof, that spanning is certain, actually requires only the weaker condition:

$$d - 6 + 3\eta \geq 0 , \qquad (4.25)$$

under which the conclusion is that the number of spanning clusters is typically greater than $o(1)L^{d-6+3\eta}$. This can be turned around to say that in dimensions in which the spanning probability does not tend to 1: $\eta \leq \frac{6-d}{3}$. The importance of this improvement is dimmed by the fact that according to numerical estimates $\eta < 0$ for $d = 3, 4, 5$.

2. It is natural at this point to enquire about an upper bound on the number of spanning clusters. Comparing Lemma 3 with Theorem 5, we see that for counting clusters whose volume is comparable to the maximal the lower bound we got is sharp up to $\times L^{o(1)}$. However our estimates will not detect the presence of a large number (larger that L^{d-6}, but not too large) of sufficiently thin spanning clusters.

5. The IIC, the ISC, and the Scaling Limit

5.a The Microscopic and the Macroscopic Perspectives

In this section we place the above results in the context of the interplay between the different scales on which the system can be viewed.

While percolation models are often presented on an infinite lattice, many of the interesting questions relate to finite, though large, systems. There are therefore at least two scales of interest (we focus on two but it will also be interesting to bring in a third, intermediate scale, which could allow a better mathematical expression of the renormalization group approach):

i. The *microscopic scale,* for which the unit length (a) is the lattice spacing, or the size of the dots in the random dot model. This is the scale on which the elementary connections are defined.

ii. The *macroscopic scale,* for which the unit length could be the system's width (L). This is the scale of the clusters on which we focused. Of particular interest are the clusters which connect two different boundary faces. We generically refer to those as spanning clusters (in different geometries, see Fig. 1).

The situation in which $L >> a$ appears rather differently from the two perspectives.

On the microscopic scale the entire system appears vast, and it is mathematically advantageous to consider it infinite, and homogeneous (in either a deterministic or a stochastic sense). The limit $L/a \to \infty$ is described by an infinite percolation model, possibly on a lattice. The IIC is naturally seen from that perspective.

On the macroscopic scale, the situation $L/a >> 1$ is expressed through the *scaling limit* $a \to 0$. This limit lends itself naturally to the discussion of the higher symmetry, including possibly conformal invariance, which seems to emerge on the large scales in the critical regime.

The random objects visible on the macroscopic scale are the spanning clusters and other clusters of linear extent comparable with the system's size (longer that sL for any fixed $0 < s < 1$). In the scaling limit the clusters have the appearance of fractal objects. Although, it might be noted, the standard description of fractals does not capture the relevant information on the realized connections, and a new formalism is required. (A proposal is presented in [15] and work in progress.) It is also of interest to take the double limit: $a \to 0$ and $L \to \infty$, which is needed if one wants to ask about percolation in the scaling limit.

The Incipient Infinite Cluster, defined along the lines of Kesten [21], can be regarded as the mathematical description of the large clusters on the microscopic scale, viewed from the perspective of their typical sites. One may guess that the following three algorithms for the construction of a probability measure of the percolation model in which a chosen site is constrained to be connected to infinity, will have common limits. The infinite cluster seen there is the IIC, and the limiting measure will provide its probabilistic description.

a. Generate the probability distribution for a random environment by a two step procedure. First pick a typical random configuration, and then center it relative to one of the sites on the large clusters, sampling with equal weights over all the sites connected to the boundary of $[0, L]^d$ – excluding those too close to it, e.g., closer than sL with some fixed $0 < s < 1$.

b. Take the conditional distribution conditioned on the origin being connected distance L away, and let $L \to \infty$.

c. Raise p over p_c, condition on the origin being in the infinite cluster (which now exists), and let $p \searrow p_c$.

For 2D, Kesten [21] proved the convergence and equality of the limits *b.* and *c.* Presumably, the method should cover also *a.* (which seems to be the most efficient for simulations). While the result was not yet extended to higher dimensions, it is expected to be true for general $d > 1$. Regardless of that, under the assumption $P_\infty(p_c) = 0$ one can prove that in each of the constructions, the limiting measure will exhibit exactly one infinite cluster (for $p = p_c$). This uniqueness, however, is not the consequence of any of the uniqueness results cited above, since the measure(s) are neither regular nor translation invariant.

The ISC on the other hand are in general non-unique, as is proven in Section 2. One may be puzzled by the disappearance of the different Incipient Spanning Clusters in the construction of the IIC. The answer lies in the observation that the infinite lattice captures only the microscopic scale, and represents only a vanishingly small fraction of the bulk. In a sense seen precisely in the theory of measurability in product spaces, the standard infinite system (e.g., the lattice) is just the collection of all regions which remain at fixed microscopic–scale distances from the point represented by the origin. The different ISC are at larger distances apart.

Regarding the scaling limits of the Spanning Clusters (not just at p_c), it is natural to expect that for any $0 < p < 1$ one would find just one of the three possibilities, which would be equally valid, with probability which tends to one, for any bulk shape of the type seen in Fig. 1.

i. No clusters attain size visible in the scaling limit.

ii. There is a unique macroscopic connected cluster filling the region densely (in the macroscopic–scale sense).

iii. Multiple clusters with macroscopic–scale diameters will be seen in any rectangular region (regardless of its aspect ratio).

Existing results permit one to easily deduce:

$$\begin{aligned} p < p_c &\implies \text{alternative i.} \\ p > p_c &\implies \text{alternative ii. (see Appendix C)} \\ \text{for 2D: } p = p_c &\implies \text{alternative iii. (see Section 3)} \end{aligned} \tag{5.1}$$

and in Section 2 we established:

$$\text{any } d > 1: \quad p = p_c \implies \text{a restricted alternative iii.} \tag{5.2}$$

The restriction is to rectangular regions with a conveniently small aspect ratio. Since the discussion refers to the scaling limit, it seems most natural to expect the restriction to be irrelevant in any dimension, though that is not proven here. Such a result will amount to a significant extension of the theory of Russo [11] and Seymour and Welsh [12], which is based on intrinsically 2D arguments.

5.b Distinction between Type I and Type II Models

We shall not discuss here the interesting question of what mathematical object will provide a suitable description of the structure emerging in the continuum limit. Let us, however, point out that a distinction ought to be made between two classes of critical models. We characterize them as follows.

- *Type I models:* The function

$$\limsup_{L\to\infty} Prob_{p_c}\left(\begin{array}{l} \text{the set } [-sL, sL]^d \text{ is connected} \\ \text{to the boundary of } [-L, L]^d \end{array} \right) = \tilde{h}(s) \tag{5.3}$$

  is strictly less than one; in which case $\lim_{s\to\infty} \tilde{h}(s) = 0$.

- *Type II models*: For any tubular regime T,

$$\lim_{s\to\infty} Prob_{p_c}\left(\begin{array}{l} \text{the tube } T_L \text{ is traversed} \\ \text{by a connected cluster} \end{array} \right) = 1 \ , \tag{5.4}$$

  where T_L is the blow–up of the tube T by the factor L.

(*Question:* is all else ruled out.)

When viewed on the macroscopic scale, models of Type I exhibit many clusters of still visible size, but none of them is infinite. The scaling limit of ISC in such models can be formulated along the lines discussed in ref. [15] (and work in progress).

In Type II models, the clusters visible on the macroscopic scale are qualitatively different than in Type I. Since for any two open sets of macroscopic size the probability of connection tends to 1 (as $a \to 0$), the scaling limit, when properly defined, ought to exhibit percolation at the threshold.

The high–dimensional percolation models discussed in Section 4 are of Type II. Their continuum limit has still to be fully formulated (initial steps were already taken [34, 35, 36]). The results discussed in Section 4 allow one to foresee that the limit will exhibit two features to which we are not accustomed in lattice percolation models:

- Percolation at the critical point.
- Infinitely many infinite clusters.

The latter may surprise one familiar with the general uniqueness Theorem of Burton and Kean [18]. However, it should be appreciated that the BK result requires discreteness and regularity, on the corresponding scale, which are lost in the continuum limit.

A. The Relation between Proliferation of ISC and Hyperscaling

The proliferation of the Incipient Spanning Clusters is related to the breakdown of "hyperscaling". In order to clarify this relation, we recapitulate here the heuristic basis of the scaling and hyperscaling relations. Different variants of the argument can be found in the literature [1, 8, 3]. The one given below is cast in terms of quantities which are studied rigorously in this work. However, unlike in the rest of this paper, in this section we do not present rigorous results. (Rigorous results on scaling relations exist for $d = 2$ [25] and $d > d_{u.c.}$ – as mentioned in Sect.4)

The scaling relations of some of the critical exponents can be explained by the following picture. The first–line assumption, related to the self–similarity, is that the relevant quantities scale by power laws. Next:

1. For critical models, there are about $L^{\#}$ clusters in Λ_L with diameters of order L, and their volumes (defined on the U-V/lattice scale) are of the order of L^D. If exceptions occur, it is assumed that they do not affect the cluster statistics by more than $\times L^{o(1)}$. The quantities we shall look at are

$$\mathbf{E}\left(\sum_{\mathcal{C}\in\Lambda_L} |\mathcal{C}|^k \ \mathrm{I}[\mathrm{diam}(\mathcal{C}) \geq \mathrm{wL}] \right) \tag{A.1}$$

   with $w < 1$, fixed as $L \to \infty$, and $k = 1, 2$.

2. For $p < p_c$ there is a characteristic length

$$\xi(p) \ \approx \ (p_c - p)^{-\nu_-} \tag{A.2}$$

   such that: *i.* Only a fraction (say 1/3, or even up to $(1 - L^{-o(1)})$) of the mean value of the cluster size $|\mathcal{C}(0)|$ is from distances greater than ξ.
   ii. Within the distance ξ, the two point function $\tau_p(0, x)$ is comparable (in the sense of bounded ratios) with its value at the critical point, $\tau_{p_c}(0, x)$.

   Note that the mean cluster size and the two point function are in the direct relation:

$$\sum_x \tau_p(o, x) = \mathbf{E}_p(|\mathcal{C}(0)|) \ \left(\approx (p_c - p)^{-\gamma}\right) \ . \tag{A.3}$$

3. For $p > p_c$ the density of the infinite cluster, $M(p)$, scales as

$$M(p) \ \approx \ (p - p_c)^{\beta} \tag{A.4}$$

   and there is a characteristic length $\xi(p) \ \approx \ (p - p_c)^{\nu_+}$ such that $M(p)$ is of the order of the volume-fraction of Λ_ξ which at $p = p_c$ belongs to clusters of diameters comparable with ξ (i.e., Incipient Spanning Clusters on scale ξ).

Let us include in the list the assumption:

4.
$$\nu_+ = \nu_- \ , \tag{A.5}$$

which is not essential for the picture presented here of hyperscaling, but seems to incorporate empirical observations, and fits well in the standard scaling ansatz.

We shall compare now different ways of evaluating two quantities. First, let us look at

$$\sum_{\substack{\mathcal{C} \text{ c.c. in } \Lambda_L \\ \text{diam}\mathcal{C} \geq wL}} |\mathcal{C}| \ = \ \sum_{x\in\Lambda} \mathrm{I}[\mathrm{diam}\mathcal{C}(\mathrm{x}) \geq \mathrm{wL}] \ . \tag{A.6}$$

Under the assumption *3.* the mean value of the expression on the right scales as $L^d L^{\beta/\nu_+}$. On the other hand, assuming *1.*, the expression on the left scales as $L^{\#} L^D$. Thus:

$$\# + D \ = \ d + \frac{\beta}{\nu_+} \tag{A.7}$$

Next, consider

$$\sum_{\mathcal{C} \text{ c.c. in } \Lambda_L} |\mathcal{C}|^2 \ = \ \sum_{x,y\in\Lambda_L} \mathrm{I}[\ x \text{ and } y \text{ are connected }] \ = \ \sum_{\mathrm{x}\in\Lambda_\mathrm{L}} |\mathcal{C}(\mathrm{x}) \cap \Lambda| \tag{A.8}$$

Each of the expressions suggests a different method for evaluating the mean value. Using: *1.* for the leftmost, the definition of η (eq. (1.4)) for the next, and the assumption *2.* for the rightmost, one is led to:

$$L^{\#} L^{2D} \ = \ L^{2d}/L^{d-2+\eta} \ = \ L^d L^{\gamma/\nu_-}. \tag{A.9}$$

i.e.,

$$\# + 2D = d + 2 - \eta \ = \ d + \gamma/\nu_- \ . \tag{A.10}$$

A convenient rearrangement of the information (equations (A.7) and (A.10), incorporating *4.*) is:

$$\boxed{2 - \eta = \gamma/\nu} \qquad \text{(scaling relation)} \tag{A.11}$$

$$\boxed{D = \tfrac{\beta+\gamma}{\nu}} \qquad \text{(dimension of the incipient spanning clusters)} \tag{A.12}$$

$$\boxed{\# = d - \tfrac{2\beta+\gamma}{\nu}} \qquad \text{(ISC proliferation exponent)} \tag{A.13}$$

For independent percolation the proliferation exponent # vanishes in 2D, and apparently also in all dimensions $d \leq 6$. In such situations we get one more equation, which exhibits d along the standard exponents, and hence is called "hyperscaling":

$$\boxed{\# = 0 \ , \text{ or } \ d - \tfrac{2\beta+\gamma}{\nu} = 0} \qquad \text{(hyperscaling relation)} \tag{A.14}$$

(That brings it to four equations for six quantities.)

Thus, the *breakdown of hyperscaling* is intimately related with the proliferation of the Incipient Spanning Clusters. However, its validity does not require uniqueness — just that the number of "macroscopic" clusters be typically smaller than any power of L, or have a finite limit in a probabilistic sense.

Remarks: *1.* It is easy to incorporate in this picture other exponents, which were not listed above. To make the list less incomplete, let us mention:

$$\begin{aligned} Prob_{p_c}\left(|\mathcal{C}(0)| \geq n\right) &\approx n^{-(1/\delta)} & \delta &= \tfrac{D\nu}{\beta} \\ \frac{\mathrm{card}\{c\subset\Lambda_L \,:\, \mathrm{diam}\ c \,\geq\, wL\}}{|\Lambda_L|} &\approx \ (p_c - p)^{2-\alpha} & 2-\alpha &= \nu(d - \#) \\ & & &(= 2\beta + \gamma\ )\ . \end{aligned} \tag{A.15}$$

where we gave the definition (for α not quite the standard one), and a scaling relation derived along the above lines.

2. The above discussion is relevant also for the Ising and Potts spin models, since the heuristic arguments described here make equal sense in the broader context of the Fortuin – Kasteleyn [37] random–cluster models. The resulting scaling relations are the same in term of the recognizable characteristic exponents, except that $d_{u.c.}$ is different and the significance of # is lost if one is not aware of the geometric structure behind the spin correlations. (For Ising spin systems, hyperscaling has also another connotation: its breakdown implies that the scaling limit is a Gaussian random field [38, 39]).

B. Existence of Spanning Clusters in Critical Models

In this Appendix, we prove Theorem 1, of Section 2. A key role is played by the following estimate. Both may be assumed to be known to experts.

Claim: In dimension $d > 1$, for all $s \leq 1/3$:

$$Q_{L,s} \equiv Prob_{p_c} \left(\begin{array}{l} \text{the set } [-sL, sL]^d \text{ is connected} \\ \text{to the boundary of } [-L, L]^d \end{array} \right) \geq C_d \; s^{(d-1)/2} \; . \tag{B.1}$$

Proof of the Claim: By monotonicity, we may assume that $1/s$ is an integer. We shall show that if eq. (B.1) (with C_d to be specified shortly) fails for some L , then the connectivity function decays exponentially (at distances $>> L$). That is well known to be in contradiction with the condition $p = p_c$ [40, 10].

To make the deduction, partition the lattice into cubic blocks of (linear) size sL. For each self-avoiding path linking a site x with y, let us associate a sequence of sL blocks, which are at distances approximately L apart, by the following algorithm. The zero-th block is the one containing x. Next is the block which the path reaches when it hits the boundary of the cube of size $(4+s)L$ concentric with the first block, and so on: once a stopping point and a block are selected, we center on the sL block a large cube of size $(4+s)L$, and let the next stopping point be the exist site, and the next block be the corresponding element of the lattice cubic partition. Notice that for each block in this sequence, other than the end points, the event seen in eq. (B.1) occurs twice (at the end points once) and disjointly so. The van den Berg - Kesten inequality [vdBK], which is described above, permits to deduce that the probability that x and y are connected by a self–avoiding path which corresponds to given sequence of $(k+1)$ blocks, is $\leq Q_{L,s}^{2K}$. Summing over the possibilities we get:

$$\begin{aligned} \tau(x,y) \quad &\equiv \quad Prob_{p_c} (\; x \text{ and } y \text{ are connected }) \\ &\leq \sum_{k \geq |x-y|/(2L)} \left[2d \; 5^{d-1} \; \frac{1}{s^{d-1}} \; Q_{L,s}^2 \right]^K \leq Const. \; e^{-\mu|x-y|} \; , \end{aligned} \tag{B.2}$$

with $\mu > 0$ if $2d \; 5^{d-1} \; \frac{1}{s^{d-1}} \; Q_{L,s}^2 < 1$. Since $\mu = 0$ at the critical point ([40, 10]), we deduce that eq. (B.1) holds, with $C_d = \sqrt{2d} 5^{(d-1)/2}$.

Proof of Theorem 1: If the probability that that the slab $S_{L,t}$ is traversed is very small, then so will be $Q_{L/2,s}$ for $s = 1/2 - t$. To quantify that, let us consider two concentric cubes, $[-L/2, L/2]^d$ and $[-sL/2, sL/2]^d$. If none of the $2d$ similar slabs which enclose the smaller cube is spanned (in the short direction) then that inner cube is not connected to the outer's boundary. Since the $2d$ events are positively correlated ([32, 33]), the probability of their simultaneous occurrence is greater than the product. Thus:

$$\left[1 - Q_{L/2,s} \right]^{2d} \; \geq \; 1 - R_{L,t} \; . \tag{B.3}$$

The combination of eq. (B.3) with eq. (B.1) yields the case $t \nearrow 1/2$ of eq. (2.2).

The crossing of narrow slabs, with $t \searrow 0$, can be estimated by cutting the slab into $(3/t)^{d-1}$ smaller ones with the aspect ratio close to 1/3. *Their* spanning probability is uniformly positive if $p \geq p_c$ (as shown by the previous discussion). Using the independence of the events, one obtains the rest of eq. (2.2).

C. Uniqueness for Supercritical Models

In this appendix we supplement Theorem 2, by proving that only for $p = p_c$ would there be positive probability for observing *more than one* spanning cluster in arbitrarily large systems. We shall use the proven fact that in $d > 2$ dimensions p_c is the limit of the quadrant–percolation thresholds for slabs of finite width [42]. This is an important technical statement, which makes a variety of 2D arguments applicable to supercritical models in dimensions $d > 2$ [26]. (In particular, it permits to prove that the spanning probability itself tends to 1, as $L \to \infty$.)

Theorem 6 *For any $p \neq p_c$, and $t > 0$,*

$$D_L(t,p) \equiv Prob_p \left(\begin{array}{l} \textit{there is more than one spanning} \\ \textit{cluster in } S_{L,t}(\equiv [0,tL] \times [-L,L]^{(d-1)}) \end{array} \right) \underset{L\to\infty}{\longrightarrow} 0 \tag{C.1}$$

Proof: For any $p < p_c$ the two point function decays exponentially [43, 44], and that easily implies: $D_L(t,p) \leq Const.\ L^{2d} e^{-\mu L} \to 0$ (for $L \to \infty$). The assertion which still requires a proof is that for $p > p_c$ (and $d > 2$) the probability of there being more than one spanning cluster tends to 0.

Let us define, for pairs of sites on the left boundary ($\{x_1 = 0\}$) of the semi-infinite cylinder $[0,\infty) \times [0,L]^{d-1}$:

$$G_k(x,y) = Prob_p \left(\begin{array}{l} x \text{ and } y \text{ are in distinct spanning clusters} \\ \text{of } [0,k] \times [0,L]^{d-1} \end{array} \right) . \tag{C.2}$$

The events seen in eq. (C.2) are obviously decreasing in k, and thus the ratio $G_{k+w}(x,y)/G_k(x,y)$ represents a conditional probability. For $p > p_c$, let $w(p)$ be the smallest slab width for which there is percolation in the quadrants $[0,w]^{d-2} \times [0,\infty)^2$ ($w(p) < \infty$ [41, 42]). The above conditional probability, of the $(k+W)$-th event conditioned on the k-th event, is uniformly smaller that one, since there is a uniformly positive probability that the two distinct clusters will be joined by a path within the added slab. (The reason is explained more explicitly in Section 4 of ref. [26], in the context of a rather similar argument.) Thus

$$G_k(x,y) \ \leq G_{k-w}(x,y)\ e^{-\alpha} \ \ldots \ \leq A\ e^{-\alpha k/w} \tag{C.3}$$

with some $\alpha > 0$. Consequently:

$$\begin{aligned} Prob_p \left(\begin{array}{c} \text{there is more than one} \\ \text{spanning cluster in } \Lambda_L \end{array} \right) \quad &\leq \quad \sum_{\substack{x,y\,\in\,\partial[0,2L]^d \\ x_1,y_1=0}} G_{2L}(x,y) \\ &\leq \quad A\ (2L)^{2d} e^{-\alpha 2L/w} \underset{L\to\infty}{\longrightarrow} \ 0\ . \end{aligned} \tag{C.4}$$

Remark: We see that the probability of there being more than one spanning cluster is exponentially small, in L, for p both above and below p_c, but not for $p = p_c$ (eq. (2.3)). This quantity yields a natural and meaningful characteristic length scale $\tilde{\xi}(p)$, and could perhaps offer a useful tool for a more thorough analysis of the the scaling relations.

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Figure 1: The geometric setups used in the discussion of the Incipient Spanning Clusters.

Figure 2: The elementary mechanism for multiple spanning clusters in 2D critical models: a) strips traversed by connecting paths alternate with strips traversed by separating dual paths (which avoid the realized connections), b) long connected paths are formed through the intersection of elementary crossing events.

Figure 3: The inequality used for estimating the variance of $\sum |\mathcal{C}|^2$. The truncation adds in the diagram two vertices and three lines. The result is an extra factor of $Const./L^{d-6+3\eta}$, which above the upper critical dimension tends to 0 for large separations.

Figure 4: The inequality used in the bound on $\mathrm{Var}(\sum |\mathcal{C}|^3)$. Again, the truncation adds to the graph two extra sites and three extra lines and thus results in the multiplicative factor $Const./L^{d-6+3\eta}$, as in Fig. 3.