The scaling limit of lattice trees in high dimensions

Eric Derbez and Gordon Slade
Department of Mathematics and Statistics
McMaster University
Hamilton, ON, Canada L8S 4K1
slade@mcmaster.ca

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Abstract

We prove that above eight dimensions the scaling limit of sufficiently spread-out lattice trees is the variant of super-Brownian motion known as integrated super-Brownian excursion (ISE), as conjectured by Aldous. The same is true for nearest-neighbour lattice trees in sufficiently high dimensions. The proof uses the lace expansion.

1 Introduction

1.1 The scaling limit

Lattice trees arise in polymer physics as a model of branched polymers and in statistical mechanics as an example exhibiting the general features of critical phenomena. A lattice tree in the d-dimensional integer lattice $\mathbb{Z}^d$ is a finite connected set of lattice bonds containing no cycles. Thus any two sites in a lattice tree are connected by a unique path in the tree. For the nearest-neighbour model, the bonds are nearest-neighbour bonds $\{x, y\}$, $x, y \in \mathbb{Z}^d$, $|x-y| = 1$ (Euclidean distance), but we will also consider "spread-out" lattice trees constructed from bonds $\{x, y\}$ with $x \neq y$ and $0 \leq |x^{(j)} - y^{(j)}| \leq L$ for $j = 1, \ldots, d$. Here $x^{(j)}$ denotes the j^{th} component of $x \in \mathbb{Z}^d$, and L is a parameter which will later be taken large. We associate the uniform probability measure to the set of all n-bond lattice trees which contain the origin.

We are interested in the existence of a scaling limit for lattice trees. This involves taking a continuum limit of lattice trees, in which the size of the trees increases simultaneously with a shrinking of the lattice spacing, in such a way as to produce a random fractal. The nature of the scaling limit is believed to depend in an essential way on the

Figure 1: A 2-dimensional lattice tree with 5000 vertices, created with the algorithm of [?].

spatial dimension, but nothing close to existence of the scaling limit has been proven in low spatial dimensions. The corresponding problem for simple random walk has the well-known solution that when space is scaled down by a factor $n^{1/2}$, as the length n of the walk goes to infinity, there is convergence to Brownian motion in any dimension. For self-avoiding walks, it has been shown using the lace expansion that the scaling limit is also Brownian motion in dimensions $d \geq 5$ [?, ?, ?]. The same is believed to be true for $d = 4$ with a logarithmic adjustment to the spatial scaling, but in dimensions 2 and 3 a different limit, currently not understood, is expected.

Here we prove that under certain assumptions the scaling limit of lattice trees is ISE (integrated super-Brownian excursion) for $d > 8$. To be precise about the assumptions, the scaling limit has been shown to be ISE for the spread-out model if $d > 8$ and L is sufficiently large, and for the nearest-neighbour model if d is sufficiently large. The hypothesis of universality implies that the scaling limit should be the same for spread-out and nearest-neighbour lattice trees, and assuming this, our results provide evidence that the scaling limit of nearest-neighbour lattice trees is ISE for $d > 8$. That the scaling limit of lattice trees should be ISE for $d > 8$ was conjectured by Aldous, who has emphasized the role of ISE as a model for the random distribution of mass [?]. In particular, Aldous has shown that ISE arises in various situations where random trees are randomly embedded into $\mathbb{R}^d$ [?, ?, ?]. ISE is super-Brownian motion conditioned to have total mass 1 and is closely connected to the super-processes intensively studied in the probability literature.

It is typical of statistical mechanical models that there is an upper critical dimension above which a model's scaling properties cease to depend on the dimension and become identical with those of a simpler so-called mean-field model. For lattice trees, the fact that ISE occurs as the scaling limit for $d > 8$ adds to the already considerable evidence that the upper critical dimension is 8 [?, ?, ?, ?, ?].

The results of this paper were announced and discussed in [?], and extend the results of [?]. An overview of our method, and a discussion of a relation between critical exponents and the occurrence of ISE as a scaling limit, is given in [?]. We expect that ISE is relevant also for lattice animals and for percolation, above their upper critical dimensions. Since lattice animals are believed to have the same scaling properties as lattice trees, in all dimensions, it is natural to conjecture that ISE is the scaling limit of lattice animals for $d > 8$. In addition, as is described in more detail in [?], Hara and Slade have conjectured that ISE arises also as a scaling limit for percolation models at the critical point, for $d > 6$. More precisely, the conjecture is that for $d > 6$, at the critical point the connected cluster of the origin, conditioned to contain n sites, yields ISE in the limit $n \to \infty$ when space is scaled down by a factor $n^{1/4}$. Hara and Slade are currently investigating whether our method can be combined with the method of [?] to prove the conjecture. The methods of [?, ?] could possibly serve as a starting point to study related questions for oriented percolation. Further discussion of the scaling limit for percolation at the critical point can be found in [?].

This paper is organized as follows. Sections 1.2 and 1.3 provide introductory material concerning ISE and lattice trees. Section 1.4 contains precise statements of results showing that the scaling limit of high-dimensional lattice trees is ISE. The proof is reduced to a statement about generating functions in Section 2. A method of fractional derivatives is outlined in Section 3, which will be used to analyze the generating functions. The relevant fractional derivatives are estimated using the lace expansion, which is reviewed and extended in Section 4. We take this opportunity to make minor corrections to [?, ?], in Section **??**. The main results for the two-point function, the three-point function, and higher-point functions are then proved respectively in Sections **??**, **??** and **??**.

1.2 ISE

ISE can be considered as an abstract continuous random tree embedded in $\mathbb{R}^d$, rooted at the origin and having total mass 1 [?]. It is designed in such a way that if $0, x_1, \ldots, x_{m-1}$ are points in $\mathbb{R}^d$ contained in ISE then there is an underlying tree structure with branch points $b_1, \ldots, b_{m-2} \in \mathbb{R}^d$ and Brownian motion paths connecting the branch points and the points $0, x_1, \ldots, x_{m-1}$ according to an abstract skeleton (minimal spanning subtree); see Figure 2. There are $(2m-5)!!$ distinct "shapes" for the skeleton. See [?, (5.96)] for a proof of this elementary fact; here $N!!$ is defined recursively for $N = -1, 1, 3, 5, 7, 9, \ldots$ by $(-1)!! = 1$ and $N!! = N(N-2)!!$, $N \geq 1$. The shapes for $m = 2, 3, 4$ are illustrated in Figure 3. The joint probability density function for the skeleton shape, the durations $t_1, \ldots, t_{2m-3}$ of each of the Brownian motion paths and the positions of points and branch points is given by the explicit formula

$$\left(\sum_{i=1}^{2m-3} t_i\right) e^{-(\sum_{i=1}^{2m-3} t_i)^2/2} \prod_{i=1}^{2m-3} p_{t_i}(y_i), \tag{1.1}$$

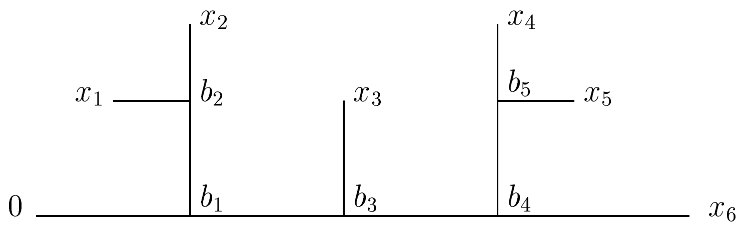


Figure 2: The branch points $b_1, \ldots, b_5$ and abstract skeleton for a realization of ISE containing the sites $0, x_1, \ldots, x_6$.

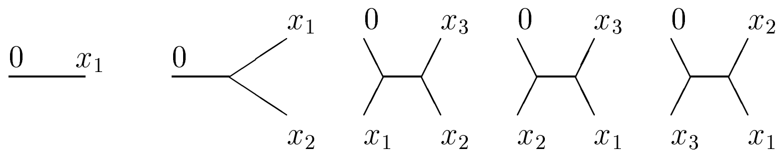


Figure 3: The unique shapes for $m = 2, 3$ and the three shapes for $m = 4$, joining points $0, x_1, \ldots, x_{m-1}$.

where $y_i \in \mathbb{R}^d$ are the vector displacements along the skeleton paths, and $p_t(y)$ is the Brownian transition function

$$p_t(y) = \frac{1}{(2\pi t)^{d/2}} e^{-y^2/2t}. \tag{1.2}$$

In Figure 2, the vector displacements (in $\mathbb{R}^d$) along the skeleton paths are $y_1 = b_1$, $y_2 = b_2 - b_1$, $y_3 = x_1 - b_2$, $y_4 = x_2 - b_2$, and so on. The ordering of the labelling of the displacements is fixed according to some convention, for each skeleton shape σ. The density (1.1) is discussed in [**?**, **?**, **?**]; see also [**?**].

The joint probability density function for the skeleton shape and the positions of the points and branch points, with the time variables integrated out, is given by

$$A^{(m)}(\sigma; y_1, \ldots, y_{2m-3}) = \int_0^\infty dt_1 \cdots \int_0^\infty dt_{2m-3} \left(\sum_{i=1}^{2m-3} t_i \right) e^{-(\sum_{i=1}^{2m-3} t_i)^2/2} \prod_{i=1}^{2m-3} p_{t_i}(y_i). \tag{1.3}$$

The right side is independent of the shape σ and depends only on the displacements y_i. This is indeed a probability measure, because integrating over the y_i's simply removes the product over Brownian transition functions, and the remaining integral over the t_i's equals $1/(2m-5)!!$, the reciprocal of the number of shapes. If we leave the x_i's fixed and integrate out the positions of the branch points and sum over all the $(2m-5)!!$ possible shapes, the result is a measure $P^{(m)}$ on $\mathbb{R}^{d(m-1)}$. The measures $P^{(m)}$, for $m = 2, 3, 4, \ldots$, represent the joint probability densities for ISE to contain the sites $0, x_1, \ldots, x_{m-1}$. In

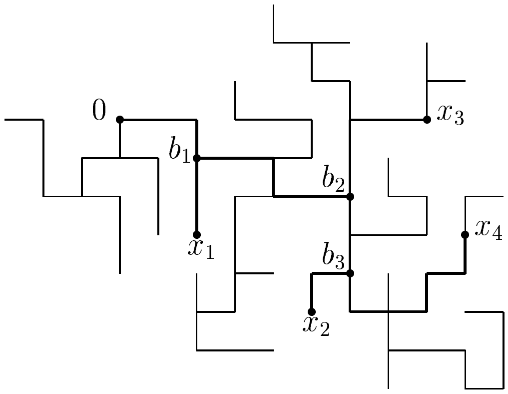


Figure 4: A lattice tree containing the sites $0, x_1, x_2, x_3, x_4$ with its corresponding skeleton and branch points b_1, b_2, b_3. The vector displacements along the skeleton paths are $y_1 = b_1$, $y_2 = x_1 - b_1$, $y_3 = b_2 - b_1$, $y_4 = x_3 - b_2$, and so on. The ordering of the labelling of the displacements is fixed according to some convention, for each skeleton shape σ.

the simplest case $m = 2$, $P^{(2)}(x)$ represents the probability density function for a point chosen randomly from the distribution of ISE.

Explicitly, for $m = 2$,

$$A^{(2)}(x) = P^{(2)}(x) = (2\pi)^{-d/2} \int_0^\infty t^{1-d/2} e^{-t^2/2} e^{-x^2/2t} dt \tag{1.4}$$

and

$$\hat{A}^{(2)}(k) = \int_0^\infty t e^{-t^2/2} e^{-k^2 t/2}\, dt, \tag{1.5}$$

where our convention for the Fourier transform of a function $f : \mathbb{R}^{dn} \to \mathbb{C}$ is

$$\hat{f}(k_1, \ldots, k_n) = \int_{\mathbb{R}^{dn}} f(y_1, \ldots, y_n) e^{ik_1 \cdot y_1 + \cdots + ik_n \cdot y_n}\, d^d y_1 \cdots d^d y_n, \quad k_i \in \mathbb{R}^d. \tag{1.6}$$

The integral (1.5) can be written in terms of the parabolic cylinder function D_{-2} as $\hat{A}^{(2)}(k) = e^{k^4/16} D_{-2}(k^2/2)$ [**?**, 3.462.1]. For general $m \geq 2$,

$$\hat{A}^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) = \int_0^\infty dt_1 \cdots \int_0^\infty dt_{2m-3} \left(\sum_{i=1}^{2m-3} t_i \right) e^{-(\sum_{i=1}^{2m-3} t_i)^2/2} e^{-\sum_{i=1}^{2m-3} k_i^2 t_i/2}. \tag{1.7}$$

1.3 Lattice trees

In this section, we introduce some notation and recall some previous results concerning high-dimensional lattice trees.

Let $t_n^{(1)}$ denote the number of n-bond lattice trees containing the origin, with $t_0^{(1)} = 1$. For $m \geq 2$, let $t_n^{(m)}(\sigma; \vec{y}, \vec{s})$ be the number of n-bond lattice trees with skeleton shape σ and skeleton displacements $y_1, \dots, y_{2m-3}$ as in Figure 4, with the skeleton path corresponding to y_i consisting of s_i steps ($i = 1, \dots, 2m-3$). We also define

$$t_n^{(m)}(\sigma; \vec{y}) = \sum_{\vec{s}} t_n^{(m)}(\sigma; \vec{y}, \vec{s}), \tag{1.8}$$

$$t_n^{(m)}(\vec{y}) = \sum_{\sigma} t_n^{(m)}(\sigma; \vec{y}). \tag{1.9}$$

We will make use of Fourier transforms with respect to the $\vec{y}$ variables, for example,

$$\hat{t}_n^{(m)}(\sigma; \vec{k}) = \sum_{\vec{y}} t_n^{(m)}(\sigma; \vec{y}) e^{i(k_1 \cdot y_1 + \cdots + k_{2m-3} \cdot y_{2m-3})}, \quad k_i \in [-\pi, \pi]^d. \tag{1.10}$$

Note that for $m \geq 2$,

$$\hat{t}_n^{(m)}(\vec{0}) = \sum_{\sigma} \sum_{\vec{y}} t_n^{(m)}(\sigma; \vec{y}) = (n+1)^{m-1} t_n^{(1)}. \tag{1.11}$$

To see this, perform the sums over σ and $\vec{y}$ by first fixing the values of $x_1, \dots, x_{m-1}$ and then summing over all shapes and branch points compatible with $x_1, \dots, x_{m-1}$ as in Figure 4. This leaves the sum over $x_1, \dots, x_{m-1}$ of the number of n-bond lattice trees containing the origin and $x_1, \dots, x_{m-1}$. Then (1.11) follows from the fact that an n-bond lattice tree contains $n+1$ sites.

In [?, ?], some critical exponents for lattice trees were proven to exist and to assume their mean-field values when $d > 8$. More precisely, the results were obtained for the nearest-neighbour model when $d \geq d_0$ for some undetermined dimension $d_0 > 8$, and for spread-out trees when $d > 8$ and L is sufficiently large depending on d. We will refer to these restrictions on the dimension and L as the "high-dimension condition."

In particular, it was shown in [?] that under the high-dimension condition there are positive constants B_2 and z_c (depending on d and L; in the notation of [?], $B_2 = A\pi^{1/2}$) such that

$$t_n^{(1)} \sim B_2 \pi^{-1/2} z_c^{-n} n^{-3/2}, \quad \text{as } n \to \infty. \tag{1.12}$$

In terms of the critical exponent θ occurring in the conjectured asymptotic relation $t_n^{(1)} \sim \text{const.} z_c^{-n} n^{1-\theta}$ and depending, in general, on the dimension, this says that $\theta = \frac{5}{2}$ under the high-dimension condition. The bounds $c_1 n^{-c_2 \log n} z_c^{-n} \leq t_n^{(1)} \leq c_3 n^{1/d} z_c^{-n}$, proved respectively in [?] and [?] and believed not to be sharp, are the best general bounds known at present for $t_n^{(1)}$. With (1.12), (1.11) gives

$$\hat{t}_n^{(m)}(\vec{0}) \sim B_2 \pi^{-1/2} z_c^{-n} n^{m-5/2}. \tag{1.13}$$

A second critical exponent involves R_n, the average radius of gyration of n-bond trees. The squared average radius of gyration is defined by

$$R_n^2 = \frac{1}{t_n^{(1)}} \sum_{T:|T|=n, T \ni 0} R(T)^2, \tag{1.14}$$

where

$$R(T)^2 = \frac{1}{|T|+1}\sum_{x\in T}(x-\bar{x}_T)^2 \tag{1.15}$$

is the squared radius of gyration of T. Here we write $|T|$ to denote the number of bonds in a lattice tree T, $\bar{x}_T = (|T|+1)^{-1}\sum_{x\in T} x$ to denote the centre of mass of T (considered as a set of unit masses at the *sites* of T), and we say that $x \in T$ if x is an element of a bond in T. Equivalently,

$$R_n^2 = \frac{1}{2\hat{t}_n^{(2)}(0)}\sum_x |x|^2 t_n^{(2)}(x). \tag{1.16}$$

It is believed that there is a critical exponent ν such that $R_n \sim Dn^\nu$ as $n \to \infty$, but very little has been proved rigorously about this in general dimensions.

Under the high-dimension condition, it is proved in [**?**] that

$$R_n \sim Dn^{1/4}, \tag{1.17}$$

so that $\nu = \frac{1}{4}$. The amplitude D of (1.17) is a positive constant which depends on d, and for the spread-out model, also on L. Asymptotically, for fixed d, D behaves like a multiple of L as $L \to \infty$. For later use, we define

$$D_1 = 2^{3/4} d^{-1/2} \pi^{-1/4} D. \tag{1.18}$$

The fact that $\nu = \frac{1}{4}$ under the high-dimension condition can be interpreted as saying that the mass n of a tree grows on average like the fourth power of its radius, suggesting a 4-dimensional nature for high-dimensional lattice trees. This compares well with the fact that ISE has Hausdorff dimension 4 (as can be shown to follow from [**?**, Theorem 1.4]), and also permits the upper critical dimension 8 to be interpreted as the dimension above which two 4-dimensional objects generically do not intersect.

1.4 The results

Define

$$p_n^{(m)}(\sigma; \vec{y}) = \frac{t_n^{(m)}(\sigma;\vec{y})}{\hat{t}_n^{(m)}(\vec{0})}, \tag{1.19}$$

which is the probability that an n-bond lattice tree containing the origin has a skeleton of shape σ mediating displacements $\vec{y}$. The following theorem shows that this distribution has the corresponding ISE distribution as its scaling limit, under the high-dimension condition.

Theorem 1.1 *Let $m \geq 2$ and $k_i \in \mathbb{R}^d$ ($i = 1, \ldots, 2m-3$). For nearest-neighbour trees in sufficiently high dimensions $d \geq d_0$, and for sufficiently spread-out trees above eight dimensions,*

$$\lim_{n\to\infty} \hat{p}_n^{(m)}(\sigma; \vec{k} D_1^{-1} n^{-1/4}) = \hat{A}^{(m)}(\sigma; \vec{k}),$$

where D_1 is given by (1.18).

To prove Theorem 1.1, we will obtain the asymptotic behaviour, as $n \to \infty$, of the Fourier transform of the numerator of (1.19). The asymptotic behaviour of the denominator is given already by (1.13).

For a more refined statement than Theorem 1.1, we wish to see the integrand

$$\hat{a}^{(m)}(\sigma; \vec{k}, \vec{t}) \equiv \left(\sum_{i=1}^{2m-3} t_i \right) e^{-(\sum_{i=1}^{2m-3} t_i)^2/2} e^{-\sum_{i=1}^{2m-3} k_i^2 t_i/2} \tag{1.20}$$

of the integral representation (1.7) of $\hat{A}^{(m)}(\sigma; \vec{k})$ as corresponding to Brownian motion paths arising from the scaling limit of the skeleton. For this, we denote by

$$p_n^{(m)}(\sigma; \vec{y}, \vec{s}) = \frac{t_n^{(m)}(\sigma; \vec{y}, \vec{s})}{\hat{t}_n^{(m)}(\vec{0})} \tag{1.21}$$

the probability that an n-bond lattice tree containing the origin has a skeleton of shape σ mediating displacements $\vec{y}$ with skeleton paths of respective lengths $s_1, \ldots, s_{2m-3}$. The following theorem shows the skeleton paths converging to Brownian motions, with the weight factor appearing in (1.20), for $m = 2$ and $m = 3$.

Theorem 1.2 *Let $m = 2$ or $m = 3$, $k_i \in \mathbb{R}^d$, and $t_i \in [0, \infty)$ ($i = 1, \ldots, 2m-3$). For nearest-neighbour trees in sufficiently high dimensions $d \geq d_0$, and for sufficiently spread-out trees above eight dimensions, there is a constant T_1 (defined below in (4.20)) depending on d and L such that*

$$\lim_{n\to\infty} (T_1 n^{1/2})^{2m-3} \hat{p}_n^{(m)}(\sigma; \vec{k} D_1^{-1} n^{-1/4}, \vec{t} T_1 n^{1/2}) = \hat{a}^{(m)}(\sigma; \vec{k}, \vec{t}). \tag{1.22}$$

(As an argument of $\hat{p}_n^{(m)}$, $t_i T_1 n^{1/2}$ is to be interpreted as its integer part $\lfloor t_i T_1 n^{1/2} \rfloor$.)

We believe that Theorem 1.2 holds for all $m \geq 2$, but we encounter technical difficulties for $m \geq 4$ and have proved the theorem only for $m = 2$ and $m = 3$; we comment at the end of of Section **??** on what goes wrong for $m \geq 4$. It would be of interest to prove Theorem 1.2 for general m, and also to investigate tightness to obtain a stronger statement of convergence to ISE.

The factor $(T_1 n^{1/2})^{2m-3}$ on the left side of (1.22) has a natural interpretation. In fact, writing $t_i = s_i/(T_1 n^{1/2})$ in the right side of (1.22), and then multiplying by $(T_1 n^{1/2})^{-(2m-3)}$ and summing over the s_i, gives a Riemann sum approximation to (1.7). Theorem 1.2 indicates that, at least for $m = 2$ and $m = 3$, skeleton paths with length of order $\sqrt{n}$ are typical, and that the skeleton paths converge to Brownian motion paths in the scaling limit. It is natural that the skeleton paths should typically be of length $\sqrt{n} = (n^{1/4})^2$, since distance is scaled as $n^{1/4}$.

2 Generating functions

The proofs of Theorems 1.1 and 1.2 use generating functions and contour integration, with the generating functions controlled using the lace expansion. In this section, we reduce the proofs of Theorems 1.1 and 1.2 to corresponding statements about generating functions. General remarks concerning the relationship between generating functions and ISE can be found in [**?**].

To define the generating functions, for $m \geq 2$ let

$$G^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{y}) = \sum_{n=0}^{\infty} \sum_{\vec{s}} t^{(m)}_n(\sigma;\vec{y},\vec{s})\zeta^s z^n, \quad \zeta_j, z \in \mathbb{C}, \tag{2.1}$$

where $\zeta^s = \zeta_1^{s_1} \cdots \zeta_{2m-3}^{s_{2m-3}}$. We adopt the convention that $\vec{\zeta}$ is dropped from the notation if $\zeta_j = 1$ for all j. If $|\zeta_j| \leq 1$ for all j, then

$$\left| \sum_{\vec{y}} G^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{y}) \right| \leq \sum_{n=0}^{\infty} \hat{t}^{(m)}_n(\sigma;\vec{0})|z|^n. \tag{2.2}$$

In view of (1.11) and the general fact [**?**] that $\lim_{n\to\infty}(t^{(1)}_n)^{1/n} \equiv z_c^{-1} \in (0,\infty)$, it follows from the above bound that the Fourier transform of (2.1) exists for $|\zeta_j| \leq 1$, $|z| < z_c$.

In [**?**], it was shown that

$$\hat{G}^{(2)}_z(0) = \frac{B_2}{\sqrt{1 - z/z_c}} + E(z), \tag{2.3}$$

with B_2 as in (1.12), and $E(z)$ an error term which obeys the bounds $|E(z)| \leq O(|1 - z/z_c|^{\epsilon - 1/2})$ and $|E'(z)| \leq O(|1 - z/z_c|^{\epsilon-3/2})$ uniformly in the disk $|z| \leq z_c$. Here ϵ is a positive constant, whose value may change from line to line in the sequel. The constant B_2 can be expressed in terms of the constants B, B_1 and b occuring respectively in [**?**, (2.21)], [**?**, (2.14)] and [**?**, Lemma 2.1(iii)] as

$$B_2 = \frac{B}{\sqrt{z_c}} = \frac{1}{z_c^{3/2}\Omega B_1} = \frac{1}{z_c \Omega}\left[\frac{1}{2(1+b)}\right]^{1/2}, \tag{2.4}$$

where Ω denotes the coordination number of the lattice ($\Omega = 2d$ for the nearest neighbour model and $\Omega = (2L+1)^d - 1$ for the spread-out model.)

We shall generalize (2.3) to variable $m, \vec{k}, \vec{\zeta}$. For this, we define $E^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{k})$ by

$$\hat{G}^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{k}) = B_2 2^{2m-5/2} \prod_{j=1}^{2m-3} \frac{1}{D_1^2 k_j^2 + 2^{3/2}(1 - z/z_c)^{1/2} + 2T_1(1-\zeta_j)} + E^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{k}), \tag{2.5}$$

where T_1 is the constant occurring in Theorem 1.2, to be defined below in (4.20). Equation (2.5) reduces to (2.3), if we set $m = 2$, $\vec{k} = \vec{0}$, $\vec{\zeta} = \vec{1}$. It will be shown in Theorem 2.1

and the proofs of Theorems 1.1 and 1.2 below that $E^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{k})$ is an error term, for all $m \geq 2$ when $\vec{\zeta} = \vec{1}$ and for $m = 2, 3$ for general $\vec{\zeta}$. Explicit bounds on $E^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{k})$ will be given in Sections **??**, **??** and **??**. With this in mind, we make three remarks.

Remarks. 1. For $m = 2$ and $\zeta = 1$, the main term in (2.5) is proportional to $(D_1^2 k^2 + 2^{3/2}(1 - z/z_c)^{1/2})^{-1}$. This corresponds to the critical exponent values $\eta = 0$ and $\gamma = 3 - \theta = \frac{1}{2}$.

2. Let $m \geq 3$ and $\vec{\zeta} = \vec{1}$. If we define $v = 1/(2B_2^2)$, the main term in (2.5) can be written as

$$v^{m-2} \prod_{j=1}^{2m-3} \frac{B_2 2^{3/2}}{D_1^2 k_j^2 + 2^{3/2}(1 - z/z_c)^{1/2}}. \tag{2.6}$$

The quantity occurring in the product is the leading contribution to the two-point function, and the renormalized vertex factor v accounts for the fact that, due to the self-avoidance interactions inherent in a lattice tree, the m-point function is not a product of independent two-point functions. However, apart from this renormalized vertex factor, the m-point function does behave, to leading order, as a product of two-point functions. A conjecture for percolation theory, in the spirit of (2.6), is stated in [**?**, Section 4.1].

3. Equation (2.5), with $\vec{k} = \vec{0}$ and $\vec{\zeta} = \vec{1}$, is consistent with the following formal calculation. Define $\mathcal{D}f(z) = \frac{d}{dz}(zf(z))$. In view of (1.11), $\hat{G}^{(m)}_z(\vec{0}) = \mathcal{D}^{m-2}\hat{G}^{(2)}_z(0)$. If we ignore the error term in (2.3), then as $z \to z_c$ we obtain

$$\hat{G}^{(m)}_z(\vec{0}) \sim B_2(2m-5)!!2^{-(m-2)}(1 - z/z_c)^{-(2m-3)/2}. \tag{2.7}$$

The factor $(2m-5)!!$ is absent in (2.5) because the shape is fixed there.

We define coefficients $e^{(m)}_n(\sigma;\vec{k},\vec{s})$ and $e^{(m)}_n(\sigma;\vec{k})$ by

$$E^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{k}) = \sum_{n=0}^{\infty}\sum_{\vec{s}} e^{(m)}_n(\sigma;\vec{k},\vec{s})\zeta^s z^n \qquad |z| < z_c, \quad |\zeta_j| \leq 1 \tag{2.8}$$

and

$$E^{(m)}_{z,\vec{1}}(\sigma;\vec{k}) = \sum_{n=0}^{\infty} e^{(m)}_n(\sigma;\vec{k}) z^n \qquad |z| < z_c. \tag{2.9}$$

To abbreviate the notation, we write

$$\kappa = kD_1^{-1}n^{-1/4}. \tag{2.10}$$

Theorem 2.1 *Let $m \geq 2$ and $k_i \in \mathbb{R}^d$ $(i = 1, \ldots, 2m-3)$. For nearest-neighbour trees in sufficiently high dimensions $d \geq d_0$, and for sufficiently spread-out trees above eight dimensions, there is an $\epsilon > 0$ such that*

$$|e^{(m)}_n(\sigma;\vec{\kappa})| \leq const. z_c^{-n} n^{-\epsilon+m-5/2}. \tag{2.11}$$

Let $m = 2$ or $m = 3$, $k_i \in \mathbb{R}^d$, and $t_i \in [0, \infty)$ $(i = 1, \ldots, 2m-3)$. For nearest-neighbour trees in sufficiently high dimensions $d \geq d_0$, and for sufficiently spread-out trees above eight dimensions, there is an $\epsilon > 0$ such that

$$|e_n^{(m)}(\sigma; \vec{\kappa}, \vec{t}T_1 n^{1/2})| \leq const. z_c^{-n} n^{-1-\epsilon}. \tag{2.12}$$

The value of ϵ may vary from line to line in the sequel. Before moving on to the proof of Theorem 2.1, we first show that it allows for proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $c_n^{(m)}(\sigma; \vec{k})$ be defined by

$$\prod_{j=1}^{2m-3} \frac{1}{D_1^2 k_j^2 + 2^{3/2}(1 - z/z_c)^{1/2}} = \sum_{n=0}^{\infty} c_n^{(m)}(\sigma; \vec{k}) z^n, \quad |z| < z_c. \tag{2.13}$$

By (2.5) with $\vec{\zeta} = \vec{1}$,

$$\hat{t}_n^{(m)}(\sigma; \vec{\kappa}) = B_2 2^{2m-5/2} c_n^{(m)}(\sigma; \vec{\kappa}) + e_n^{(m)}(\sigma; \vec{\kappa}). \tag{2.14}$$

In view of (2.11) and (1.13), it suffices to show that

$$c_n^{(m)}(\sigma; \vec{\kappa}) \sim n^{m-5/2} z_c^{-n} 2^{-(2m-5/2)} \pi^{-1/2} \hat{A}^{(m)}(\sigma; \vec{k}) \quad \text{as } n \to \infty. \tag{2.15}$$

The remainder of the proof is concerned with establishing (2.15).

By (2.13),

$$c_n^{(m)}(\sigma; \vec{\kappa}) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} \prod_{j=1}^{2m-3} \frac{1}{k_j^2 n^{-1/2} + 2^{3/2}(1 - z/z_c)^{1/2}}, \tag{2.16}$$

where the integral is taken around a small circle centred at the origin. For simplicity, for the remainder of the proof we suppose that $k_j \neq 0$ for all j. The case where one or more of the k_j is zero can be treated using similar methods, together with a limiting argument. We make the change of variables $w = n(z/z_c - 1)$, and then deform the contour to the branch cut $[0, \infty)$ in the w-plane. The resulting contour goes from right to left below the branch cut and from left to right above the cut. We then apply the identity

$$\frac{1}{k_j^2 + 2^{3/2}(-w)^{1/2}} = \frac{1}{2} \int_0^{\infty} dt_j \, e^{-k_j^2 t_j/2} e^{-2^{1/2}(-w)^{1/2} t_j}. \tag{2.17}$$

Taking into account the correct branch of the square root on either side of the branch cut, and applying Fubini's theorem, this gives

$$\begin{aligned} c_n^{(m)}(\sigma; \vec{\kappa}) &= n^{m-5/2} z_c^{-n} 2^{-(2m-3)} \int_0^{\infty} dt_1 \cdots \int_0^{\infty} dt_{2m-3} \, e^{-(\sum_{j=1}^{2m-3} k_j^2 t_j)/2} \\ &\quad \times \frac{1}{\pi} \int_0^{\infty} \frac{dw}{(1 + w/n)^{n+1}} \sin((\textstyle\sum_{j=1}^{2m-3} t_j)\sqrt{2w}). \end{aligned} \tag{2.18}$$

Since $(1+\frac{w}{n})^{n+1} \geq 1+\frac{(n+1)n}{2}(\frac{w}{n})^2 \geq 1+\frac{w^2}{2}$ for all $n \geq 1$, the dominated convergence theorem can be applied to conclude that as $n \to \infty$ the w-integral converges to

$$\int_0^\infty dw\, e^{-w} \sin((\textstyle\sum_{j=1}^{2m-3} t_j)\sqrt{2w}) = \left(\frac{\pi}{2}\right)^{1/2} (\textstyle\sum_{j=1}^{2m-3} t_j) e^{-(\sum_{j=1}^{2m-3} t_j)^2/2}. \tag{2.19}$$

Therefore, as required,

$$c_n^{(m)}(\sigma;\vec{\kappa}) \sim n^{m-5/2} z_c^{-n} 2^{-(2m-5/2)} \pi^{-1/2} \hat{A}^{(m)}(\sigma;\vec{k}). \tag{2.20}$$

□

Proof of Theorem 1.2. In view of (2.12), recalling (1.13) and taking into account the factor $(T_1 n^{1/2})^{2m-3}$ in (1.22), it suffices to show that for $\vec{k}$ replaced by $\vec{k} D_1^{-1} n^{-1/4}$, the coefficient of $\zeta^{tT_1\sqrt{n}} z^n$ in the first term on the right side of (2.5) is asymptotic to

$$\frac{B_2}{T_1^{2m-3}} \frac{1}{\sqrt{\pi} n z_c^n} \hat{a}^{(m)}(\sigma;\vec{k},\vec{t}). \tag{2.21}$$

Since each factor in the product in (2.5) is the sum of a geometric series in ζ_j, the desired coefficient is equal to

$$B_2 2^{2m-5/2} \frac{1}{(2T_1)^{2m-3}} \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} \prod_{j=1}^{2m-3} \left(1 + \frac{k_j^2}{2T_1\sqrt{n}} + \frac{\sqrt{2}}{T_1}(1-z/z_c)^{1/2}\right)^{-(T_1 t_j \sqrt{n}+1)}, \tag{2.22}$$

where the integration is performed around a small circle centred at the origin. Making the change of variables $w = n(z/z_c - 1)$ and deforming the contour of integration to the branch cut $[0,\infty)$ in the w-plane, the above is equal to

$$\frac{B_2}{T_1^{2m-3}} \frac{\sqrt{2}}{\pi n z_c^n} \mathrm{Im} \int_0^\infty \frac{dw}{(1+w/n)^{n+1}} \prod_{j=1}^{2m-3} \left(1 + \frac{k_j^2}{2T_1\sqrt{n}} - \frac{i\sqrt{2w}}{T_1\sqrt{n}}\right)^{-(T_1 t_j \sqrt{n}+1)}. \tag{2.23}$$

The integrand here is dominated by $(1+\frac{w^2}{2})^{-1} \cdot 1$, so by the dominated convergence theorem, this is asymptotic to

$$\frac{B_2}{T_1^{2m-3}} \frac{\sqrt{2}}{\pi n z_c^n} e^{-(\sum_{j=1}^{2m-3} k_j^2 t_j)/2} \int_0^\infty dw\, e^{-w} \sin(\sqrt{2w}(\textstyle\sum_{j=1}^{2m-3} t_j)). \tag{2.24}$$

By (2.19), this is equals

$$\frac{B_2}{T_1^{2m-3}} \frac{1}{\sqrt{\pi} n z_c^n} \hat{a}^{(m)}(\sigma;\vec{k},\vec{t}). \tag{2.25}$$

□

3 Fractional derivatives

Our remaining task is to prove Theorem 2.1. For this, we shall obtain explicit formulas for the error term $E^{(m)}_{z,\vec{\zeta}}(\sigma; k)$ and use contour integration to obtain the bounds (2.11) and (2.12) on its coefficients of z^n and $z^n \zeta^s$. These bounds result from corresponding bounds on $E^{(m)}_{z,\vec{\zeta}}(\sigma; k)$ itself. The purpose of this section is to review and extend the fractional derivative method of [**?**] (see also [**?**, Section 6.3]) that will be used to carry out this procedure.

Our reliance on this method to obtain bounds on coefficients of a power series from bounds on the series itself is something of a limitation, and it would be of interest to find a method of proving Theorems 1.1 and 1.2 that works directly with the sequence $t^{(m)}_n(\sigma; \vec{y}, \vec{s})$ rather than with generating functions. Such an approach has been used for weakly self-avoiding walks in [**?**, **?**].

Given a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\epsilon \geq 0$, we define the fractional derivative

$$\delta_z^{\epsilon} f(z) = \sum_{n=0}^{\infty} n^{\epsilon} a_n z^n. \tag{3.1}$$

Note that for ϵ equal to a positive integer, δ_z^{ϵ} does not give the usual derivative. Allowing ϵ to take on arbitrary negative values defines a relative of the antiderivative, as follows. For $\alpha > 0$ we define

$$\delta_z^{-\alpha} f(z) = \sum_{n=1}^{\infty} n^{-\alpha} a_n z^n. \tag{3.2}$$

Both of the above quantities will be finite at least strictly within the circle of convergence of $f(z)$. We also define

$$\|f(z)\| = \sum_{n=0}^{\infty} |a_n||z|^n. \tag{3.3}$$

We require the following lemma, quoted from [**?**, Lemma 6.3.1].

Lemma 3.1 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Then for any z with $|z| < R$, and for any $\alpha > 0$,*

$$\delta_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} [f(ze^{-\lambda^{1/\alpha}}) - f(0)] d\lambda. \tag{3.4}$$

We will also use the following lemma, which is a restatement of [**?**, Lemma 6.3.2]. This lemma provides an error estimate analogous to the error estimate in Taylor's theorem. In applications of the lemma, R will be the radius of convergence of f. The norms appearing in the lemma are given, in view of (3.3), by $\|\delta_z^{\epsilon} f(R)\| = \sum_{n=0}^{\infty} n^{\epsilon} |a_n| R^n$ and $\|\delta_z^{1+\epsilon} f(R)\| = \sum_{n=1}^{\infty} n^{1+\epsilon} |a_n| R^n$.

Lemma 3.2 *Let $\epsilon \in (0,1)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $R > 0$. If $\|\delta_z^{\epsilon} f(R)\| < \infty$ (so in particular $f(z)$ converges for $|z| \leq R$), then for any z with $|z| \leq R$,*

$$|f(z) - f(R)| \leq 2^{1-\epsilon} \|\delta_z^{\epsilon} f(R)\| |1 - z/R|^{\epsilon}. \tag{3.5}$$

If $\|\delta_z^{1+\epsilon} f(R)\| < \infty$ (so in particular $f'(z) = \sum_{n=1}^\infty n a_n z^{n-1}$ converges for $|z| \leq R$), then for any z with $|z| \leq R$,

$$|f(z) - f(R) - f'(R)(z-R)| \leq \frac{2^{1-\epsilon}}{1+\epsilon} \|\delta_z^{1+\epsilon} f(R)\| |1 - z/R|^{1+\epsilon}. \tag{3.6}$$

The next lemma will be our main tool in converting bounds on the error term $E^{(m)}_{z,\vec{\zeta}}(\sigma;k)$, which will be obtained using the lace expansion, into the bounds on its coefficients stated in Theorem 2.1. The intuition behind the lemma is that if a power series $f(z) = \sum_{n=0}^\infty a_n z^n$ has radius of convergence $R > 0$ and if $|f(z)|$ is bounded above by a multiple of $|R - z|^{-b}$ on the disk of radius R, with $b \geq 1$, then a_n should be not much worse than order $R^{-n} n^{b-1}$. The lemma provides an extension of [**?**, Lemma 6.3.3]. To abbreviate the notion, we introduce

$$U_p(w) = |1-w|^{-b_p} (1-|w|)^{-\bar{b}_p}. \tag{3.7}$$

Lemma 3.3 *(i) Let $f(z) = \sum_{n=0}^\infty a_n z^n$ have radius of convergence at least R, where $R > 0$. Suppose that $|f(z)| \leq const. U_0(z/R)$ for $|z| < R$, for some $b_0 \geq 1$, $\bar{b}_0 \geq 0$. Then $|a_n| \leq const. R^{-n} n^\alpha$, for any $\alpha > b_0 + \bar{b}_0 - 1$, with the constant independent of n.*

(ii) Let j be a positive integer. Suppose that $|\frac{d^j}{dz^j} f(z)| \leq const. U_0(z/R)$ for $|z| < R$, for some $b_0 \geq 1$, $\bar{b}_0 \geq 0$. Then $|a_n| \leq const. R^{-n} n^{\alpha - j}$ for any $\alpha > b_0 + \bar{b}_0 - 1$.

(iii) Suppose that $f(z, \zeta_1, \ldots, \zeta_q) = \sum_{n,s_1,\cdots,s_q=0}^\infty a_{n,s_1,\cdots,s_q} \zeta^{s_1} \cdots \zeta^{s_q} z^n$ converges absolutely if $|z| < R$ and $|\zeta_p| \leq 1$ ($p = 1, \ldots, q$). Suppose in addition that in this region, $|f(z,\zeta_1,\ldots,\zeta_q)| \leq const. U_0(z/R) \prod_{p=1}^q U_p(\zeta_p)$, where $b_p \geq 1$, $\bar{b}_p \geq 0$ ($p = 0, 1, \ldots, q$). Then $|a_{n,s_1,\cdots,s_q}| \leq const. R^{-n} n^{\alpha_0} \prod_{p=1}^q s_p^{\alpha_p}$ for any $\alpha_p > b_p + \bar{b}_p - 1$ ($p = 0, 1, \ldots, q$).

Proof. (i) Let $\alpha > b_0 + \bar{b}_0 - 1$. By hypothesis, this implies $\alpha > 0$. Since $n^{-\alpha} a_n$ is the coefficient of z^n in the fractional antiderivative $\delta_z^{-\alpha} f(z)$,

$$n^{-\alpha} a_n = \frac{1}{2\pi i} \oint \delta_z^{-\alpha} f(z) \frac{dz}{z^{n+1}}, \tag{3.8}$$

where the integral is around a circle of radius $r < R$ centred at the origin. By Lemma 3.1,

$$n^{-\alpha} |a_n| \leq \frac{1}{2\pi \Gamma(\alpha+1)} r^{-n} \int_{-\pi}^{\pi} d\theta \int_0^\infty d\lambda |f(re^{i\theta} e^{-\lambda^{1/\alpha}}) - f(0)|. \tag{3.9}$$

Since $f(z) - f(0) = O(z)$ for z near zero, the contribution to the integral with respect to λ due to $\lambda \in [1, \infty)$ is finite. Using the assumed bound on f, we thus have

$$n^{-\alpha} |a_n| \leq \text{const.} r^{-n} \left[1 + \int_{-\pi}^{\pi} d\theta \int_0^1 d\lambda \, U_0(rR^{-1} e^{i\theta} e^{-\lambda^{1/\alpha}}) \right]. \tag{3.10}$$

Replacing R on the right side by r gives an upper bound. Taking the limit $r \to R$ in the upper bound then gives

$$n^{-\alpha} |a_n| \leq \text{const.} R^{-n} \left[1 + \int_{-\pi}^{\pi} d\theta \int_0^1 d\lambda \, U_0(e^{i\theta} e^{-\lambda^{1/\alpha}}) \right]. \tag{3.11}$$

It suffices to show that the integral on the right side is finite, and for this it suffices to show (dropping subscripts 0 from b_0 and $\bar{b}_0$) that

$$\int_0^1 d\theta \int_0^1 d\lambda |1 - e^{i\theta} e^{-\lambda^{1/\alpha}}|^{-b} (1 - e^{-\lambda^{1/\alpha}})^{-\bar{b}} < \infty \tag{3.12}$$

for $\alpha > b + \bar{b} - 1$. For this, we make the substitution $u = \lambda^{1/\alpha}$ and estimate the absolute value, to obtain an upper bound

$$\text{const.} \int_0^1 du u^{\alpha-1} (1 - e^{-u})^{-\bar{b}} \int_0^1 d\theta [(1 - e^{-u})^2 + e^{-2u}\theta^2]^{-b/2} \tag{3.13}$$

for (3.12). The change of variables $\theta_1 = \theta e^{-u}(1 - e^{-u})^{-1}$ in (3.13) gives

$$\text{const.} \int_0^1 du u^{\alpha-1} e^{u} (1 - e^{-u})^{-b-\bar{b}+1} \int_0^{e^{-u}(1-e^{-u})^{-1}} d\theta_1 [1 + \theta_1^2]^{-b/2}. \tag{3.14}$$

The θ_1-integral is bounded uniformly in u if $b > 1$, while if $b = 1$ it is finite for u near 1 and $O(|\log u|)$ for u near 0. Hence for $b \geq 1$, (3.14) is bounded above by a multiple of

$$\int_0^1 du\, u^{\alpha-b-\bar{b}} |\log u|, \tag{3.15}$$

which is finite for $\alpha > b + \bar{b} - 1$.

(ii) This is an immediate consequence of (i).

(iii) The proof involves showing that part (i) of the lemma can be iterated, and we demonstrate this for a single iteration with the case $p = 1$. The general case is similar.

Let $p = 1$ and write ζ in place of ζ_1. The proof of part (i), up to (3.11), can be carried out with $a_n = a_n(\zeta) = \sum_{s=0}^{\infty} a_{n,s}\zeta^s$ to give

$$n^{-\alpha_0} R^n |a_n| \leq \text{const.} \left[1 + U_1(\zeta) \int_{-\pi}^{\pi} d\theta \int_0^1 d\lambda\, U_0(e^{i\theta} e^{-\lambda^{1/\alpha}}) \right]. \tag{3.16}$$

The constant in (3.16) is independent of both ζ and n. Since $1 \leq 2^{b_1} U_1(\zeta)$, for $|\zeta| \leq 1$, (3.12) gives

$$n^{-\alpha_0} R^n |a_n| \leq \text{const.} U_1(\zeta). \tag{3.17}$$

Writing $b_{n,s} = n^{-\alpha_0} R^n a_{n,s}$ and $b_n(\zeta) = \sum_{s=0}^{\infty} b_{n,s}\zeta^s$, this means $|b_n(\zeta)| \leq \text{const.} U_1(\zeta)$, and part (i) can be applied to give the desired result. □

The hypothesis $b_0 \geq 1$ in Lemma 3.3(i) cannot be discarded. For example, let $f(z) = \sum_{n=1}^{\infty} n^{-2} z^{2^n}$. Then $f(z)$ is finite for $|z| \leq 1$, so in particular $|f(z)| \leq \text{const.} |1 - z|^{-b}$ for any $b \in [0, 1)$. However $a_N = (\log_2 N)^{-2}$ for $N = 2^n$, so $a_n \neq O(n^\alpha)$ for $\alpha \in (-1, 0)$. The conclusion of Lemma 3.3 gives only *positive* power laws as upper bounds. Thus, for example, to use the lemma for a function which is bounded by $O(|1 - z|^{-1/2})$ for $|z| \leq 1$ (ideally corresponding to coefficients with power law $n^{-1/2}$), we must show that its derivative is bounded above by $O(|1 - z|^{-3/2})$ (yielding power law $n^{1/2+0}$ for the derivative and hence $n^{-1/2+0}$ for the original function). This need to take derivatives will make our calculations more laborious. Specifically, to prove (2.11) will require one derivative for $m = 2$ (none are necessary for $m \geq 3$), and (2.12) requires two derivatives for any m.

4 Lace expansion bounds

Theorem 2.1 will be proved using Lemma 3.3, and to employ the latter we require bounds on the error term $E^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{k})$ of (2.5). These bounds will be obtained using the lace expansion.

This will require some modifications and extensions of the lace expansion methods developed at length for lattice trees in [**?**, **?**] (see also [**?**, Section 5.5.1] and [**?**]). In particular, we extend the lace expansion methods to the context where there is a skeleton activity ζ. Because the extensions required do not involve the development of new methodology, we will be brief with the proofs. All results requiring the lace expansion that will be used in later sections are stated here.

Throughout this section, we assume familiarity with the methods of [**?**, **?**] and will refer frequently to these references for calculations and notation. For simplicity, we work in the remainder of the paper with the spread-out model and assume that $d > 8$ and L is sufficiently large. Similar results can be proven for the nearest-neighbour model in sufficiently high dimensions.

4.1 The lace expansion with backbone activity

With $\zeta = 1$, [**?**, (3.8)] gives

$$\hat{G}^{(2)}_z(k) = \frac{\hat{h}_z(k)}{\hat{F}_z(k)}, \quad \hat{F}_z(k) = 1 - z\Omega\hat{D}(k)\hat{h}_z(k), \tag{4.1}$$

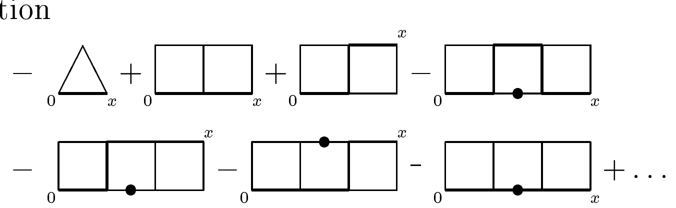
where we are using our convention that when $\zeta = 1$ it is omitted from the notation, $\Omega = (2L+1)^d - 1$ is the coordination number of the lattice (we will abuse notation by also writing Ω for the set of all $x \in \mathbb{Z}^d$ with $x \neq 0$ and $0 \leq |x^{(j)}| \leq L$, $j = 1, \ldots, d$),

$$\hat{D}(k) = \frac{1}{\Omega}\sum_{x\in\Omega} e^{ik\cdot x}, \quad \hat{h}_z(k) = g_z + \hat{\Pi}_z(k), \quad g_z = \sum_{n=0}^{\infty} t^{(1)}_n z^n, \tag{4.2}$$

and in the notation of [**?**, Section 2.1.1],

$$\Pi_z(x) = \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} z^{|\omega|}\left[\prod_{i=0}^{|\omega|}\sum_{R_i\ni\omega(i)} z^{|R_i|}\right] J[0,|\omega|]. \tag{4.3}$$

The above expression for $\Pi_z(x)$ is produced using the lace expansion, and leads to the diagrammatic representation

$\Pi_z(x) =$  (4.4)

The diagrams on the right side represent terms which are given by series that are absolutely bounded by the corresponding Feynman diagram with propagator $G^{(2)}_{|z|}(x)$. This is discussed at length in [?, Section 2.2.1] (see also [?, Section 5.5.1]). The diagrams are convergent when $z = z_c$ for $d > 8$, even if an additional vertex is added to one of the lines. They have an ultra-violet cutoff due to the lattice, and $z = z_c$ corresponds to massless propagators. The heavy lines in the diagrams play no special role for $\zeta = 1$, but will be discussed below for general ζ.

As discussed in [?, Section 2.1], $\hat{F}_{z_c}(0) = 0$. Therefore $\hat{h}_{z_c}(0) = 1/(z_c\Omega)$.

By (1.11),

$$\frac{d}{dz} z g_z = \hat{G}^{(2)}_z(0) \equiv \chi(z), \tag{4.5}$$

where the last equality defines $\chi(z)$. By [?, Lemma 2.1(iii)],

$$\frac{d}{dz} z\hat{\Pi}_z(k) = [z\Omega\chi(z) + 1]\hat{\Psi}_z(k). \tag{4.6}$$

Properties of $\hat{\Psi}_z(k)$ from [?] will be generalized below. The function $\hat{\Psi}_z(k)$ is represented by a sum of diagrams which are convergent when $z = z_c$ for $d > 8$, but not for $d \leq 8$. The diagrams include the diagrams obtained by adding an additional vertex to the diagrams in $\hat{\Pi}_z(k)$, in addition to others.

For $\zeta \neq 1$, the derivation of (4.1) and (4.6) is identical, except for the fact that (in x-space) there is a factor of ζ associated to each bond in the backbone path joining 0 to x. Keeping track of this for the two-point function leads to

$$\hat{G}^{(2)}_{z,\zeta}(k) = \frac{\hat{h}_{z,\zeta}(k)}{\hat{F}_{z,\zeta}(k)}, \quad \hat{F}_{z,\zeta}(k) = 1 - \zeta z\Omega\hat{D}(k)\hat{h}_{z,\zeta}(k), \tag{4.7}$$

where

$$\hat{h}_{z,\zeta}(k) = g_z + \hat{\Pi}_{z,\zeta}(k) \tag{4.8}$$

and

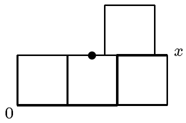
$$\Pi_{z,\zeta}(x) = \sum_{\substack{\omega : 0 \to x \\ |\omega| \geq 1}} \zeta^{|\omega|} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J[0, |\omega|]. \tag{4.9}$$

Thus the only change for $\zeta \neq 1$ occurs in the new factor ζ appearing explicitly in $\hat{F}_{z,\zeta}(k)$ and in the factor $\zeta^{|\omega|}$ in $\hat{\Pi}_{z,\zeta}(k)$. The heavy lines of (4.4) represent the backbone paths carrying factors ζ.

Keeping track of this extra factor $\zeta^{|\omega|}$ throughout the calculations of [?, Section 4.1] gives

$$\frac{d}{dz} z\hat{\Pi}_{z,\zeta}(k) = [z\Omega\chi(z) + 1]\hat{\Psi}_{z,\zeta}(k). \tag{4.10}$$

The quantity $\hat{\Psi}_{z,\zeta}(k)$ is defined in the same way as its $\zeta = 1$ counterpart, except that a factor $\zeta^{|\omega|}$ must be inserted into [?, (4.17)] and [?, (4.24)]. An example of a diagram contributing to $\Psi_{z,\zeta}(x)$, with the backbone carrying ζ drawn with a heavy line, is



For a double power series $f(z,\zeta)=\sum_{n,s} a_{n,s} z^n \zeta^s$, we define

$$\|f(z,\zeta)\| = \sum_{n,s} |a_{n,s}||z|^n|\zeta|^s. \tag{4.11}$$

Basic bounds on the lace expansion are provided by the following lemma.

Lemma 4.1 *(i) There is a constant K such that for large L, $k \in [-\pi,\pi]^d$, $|z| \le z_c$ and $|\zeta| \le 1$,*

$$\|\hat{\Pi}_{z,\zeta}(k)\| \le KL^{1-d}, \quad \|\nabla_k^2 \hat{\Pi}_{z,\zeta}(k)\| \le KL^{3-d}, \quad \|\hat{h}_{z,\zeta}(k)\| \le K, \quad \|\hat{\Psi}_{z,\zeta}(k)\| \le K. \tag{4.12}$$

The critical point z_c is bounded above and below by a constant multiple of Ω^{-1}. In addition, $\hat{h}_{z_c,1}(0)$ is bounded away from zero, and $\lim_{L\to\infty} \hat{\Psi}_{z,\zeta}(k) = 0$ uniformly in z,ζ,k as above.
(ii) There is an $\epsilon > 0$ (depending on $d > 8$) such that the following are bounded uniformly in $k \in [-\pi,\pi]^d$, $|z| \le z_c$, $|\zeta| \le 1$ and large L:

$$\|\delta_z^a \hat{h}_{z,\zeta}(k)\| \ (0 \le a < \tfrac{1}{2}), \quad L^{-2}\|\delta_z^\epsilon \nabla_k^2 \hat{h}_{z,\zeta}(0)\|, \quad L^{-2}\|\delta_z^\epsilon \nabla_k^2 \hat{F}_{z,\zeta}(0)\|, \tag{4.13}$$

$$\|\delta_\zeta^{1+\epsilon} \hat{h}_{z,\zeta}(k)\|, \quad \|\delta_\zeta^{1+\epsilon} \hat{F}_{z,\zeta}(k)\|, \quad \|\delta_z^\epsilon \tfrac{d}{d\zeta} \hat{F}_{z,\zeta}(k)\|. \tag{4.14}$$

(iii) The constants

$$B_3 = -\frac{1}{2d}\nabla_k^2 \hat{F}_{z_c}(0), \quad B_4 = \hat{h}_{z_c}(0) B_2^{-1}, \quad B_5 = -\frac{d}{d\zeta}\hat{F}_{z_c,1}(0) \tag{4.15}$$

are finite and positive, with B_4 and B_5 bounded away from zero and infinity uniformly in large L, and B_3 asymptotic to a multiple of L^2 as $L \to \infty$.

Proof. (i) It suffices to prove the statements for $\zeta = 1$, and the latter follow readily from the bounds of [**?**, **?**]. (More is known about z_c; see [**?**].)
(ii) The bounds (4.13) follow from [**?**, Lemma 2.1] for $\zeta = 1$, and this implies the bounds for all $|\zeta| \le 1$.

For the first bound of (4.14), the ζ-dependence resides in $\hat{\Pi}_{z,\zeta}(k)$. Writing $\hat{\Pi}_{z,\zeta}(k) = \sum_{n,s} \pi_{n,s}(k) z^n \zeta^s$, taking $1+\epsilon$ ζ-derivatives introduces a factor $s^{1+\epsilon}$, which is bounded by sn^ϵ. The factor s can be replaced by a sum over backbone sites, which corresponds diagrammatically to adding a vertex along the backbone. This gives diagrams which are included in $\hat{\Psi}_{z,\zeta}(k)$. The factor n^ϵ corresponds to taking ϵ z-derivatives, and the result remains bounded, since $\|\delta_z^\epsilon \hat{\Psi}_{z_c}(0)\|$ is shown to be finite in the proof of [**?**, Lemma 2.1(iii)]. The other bounds in (4.14) are similar.

(iii) The statements for B_3 follow by direct calculation of the derivative and (4.12). The statement for B_4 follows from (2.4). The statement for B_5 follows since, by definition, $B_5 = 1 + z_c \Omega \frac{d}{d\zeta} \hat{\Pi}_{z_c,1}(0)$, and the derivative on the right side goes to zero as $L \to \infty$ for the same reason $\hat{\Psi}_{z,\zeta}(k)$ does in (i). $\square$

Lemma 4.1(ii) can be used in conjunction with Lemma 3.2, to conclude for example that $\hat{h}_{z,\zeta}(k) - \hat{h}_{z_c,\zeta}(k) = O(|1-z/z_c|^a)$. These two lemmas can also be combined with the elementary facts that $|1-\cos t| \leq O(t^\epsilon)$ and $|\cos t - 1 + t^2/2| \leq O(t^{2+\epsilon})$ for small positive ϵ, and that $|x| \leq n$ if x is a site in an n-bond lattice tree, to conclude for example that

$$\hat{h}_{z_c}(k) - \hat{h}_{z_c}(0) = O(k^\epsilon) \tag{4.16}$$

and

$$\hat{F}_z(k) - \hat{F}_z(0) + \frac{k^2}{2d} \nabla_k^2 \hat{F}_z(0) = O(L^2 k^{2+\epsilon}). \tag{4.17}$$

Explicitly, the latter estimate follows by writing the left side (using symmetry) as

$$\sum_x F_z(x) \left[\cos(k \cdot x) - 1 + \frac{(k \cdot x)^2}{2} \right], \tag{4.18}$$

bounding $|x|^{2+\epsilon}$ by $n^\epsilon |x|^2$, and using the last bound of (4.13) together with Lemma 3.2.

Keeping track of the various constants entering into the definition of the amplitude D in [?, Section 2.3], and using (1.18), we have

$$D_1^2 = 2^{3/2} d^{-1} \pi^{-1/2} D^2 = 2^{3/2} B_3 B_4^{-1}. \tag{4.19}$$

By Lemma 4.1(iii), D_1 grows linearly in L for large L (to leading order), as is to be expected in view of its role as a scaling constant in Theorems 1.1 and 1.2. Also, we can now define the constant T_1 of Theorem 1.2, by

$$T_1 = 2^{1/2} B_5 B_4^{-1}. \tag{4.20}$$

The constant b occurring in (2.4) is given, according to [?, Lemma 2.1(iii)], by

$$b = z_c \Omega \hat{\Psi}_{z_c,1}(0). \tag{4.21}$$

To prove the next lemma, we require the lower bound

$$|\hat{F}_{z,\zeta}(k)| \geq \text{const.} \left(k^2 + |1-\zeta| + |1 - z/z_c|^{1/2} \right) \tag{4.22}$$

of Lemma **??** below (whose proof requires only Lemma 4.1), which is valid uniformly in $|z| \leq z_c$, $|\zeta| \leq 1$, and sufficiently small $k^2 \geq 0$.

Lemma 4.2 *There is an $\epsilon > 0$ (depending on $d > 8$) such that the following bounds hold uniformly in $k \in [-\pi, \pi]^d$, $|z| \leq z_c$, $|\zeta| \leq 1$:*

$$\|\delta_z^\epsilon \hat{\Psi}_{z,\zeta}(k)\| \leq K, \quad \|\delta_\zeta^\epsilon \hat{\Psi}_{z,\zeta}(k)\| \leq K, \tag{4.23}$$

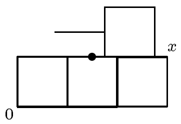
$$\left|\frac{d}{d\zeta}\hat{\Psi}_{z,\zeta}(k)\right| \leq O((1-|\zeta|)^{\epsilon-1}), \tag{4.24}$$

$$\left|\frac{d}{dz}\hat{\Psi}_{z,\zeta}(k)\right| \leq O(|1-z/z_c|^{-1/2}(1-|z|/z_c)^{\epsilon-1/2}), \tag{4.25}$$

$$\left|\frac{d^2}{dz^2}\hat{\Psi}_{z,\zeta}(k)\right| \leq O(|1-z/z_c|^{-1}(1-|z|/z_c)^{\epsilon-1}). \tag{4.26}$$

Proof. The first bound of (4.23) was obtained in the proof of [**?**, Lemma 2.1(iii)] for $\zeta = 1$, and this implies the corresponding bound for general ζ. The second bound of (4.23) then follows, by bounding s^ϵ by n^ϵ as in the proof of (4.14). For (4.24), we argue as in the proof of Lemma 4.1(ii) that taking a ζ-derivative amounts to adding a vertex (along the backbone) to the diagrams representing $\hat{\Psi}_{z,\zeta}(k)$. This yields diagrams with the same power counting as the pentagon; see [**?**, Section 4.3.2] for more details. These diagrams can be bounded above by Feynman diagrams with propagator $\hat{G}_{z_c,|\zeta|}(k)$. (Independent propagators arise via overcounting to remove interactions between diagram lines, and this can be done only for nonnegative activities, hence the absolute values.) This propagator is bounded above by $O([k^2+(1-|\zeta|)]^{-1})$, by (4.22), and the desired result then follows by power counting as in [**?**, Section 4.3.2]. To estimate Feynman diagrams as the propagator's mass goes to zero, we use the power counting bounds of [**?**, Theorem 2], as adapted in [**?**, Appendix A].

The bounds (4.25) and (4.26) are more delicate, due to the presence of $|1-z/z_c|$ and not merely $(1-|z|/z_c)$, and require further application of the lace expansion. We begin with (4.25). Writing $\hat{\Psi}_{z,\zeta}(k) = \sum_{n,s=0}^{\infty} \hat{\psi}_{n,s}(k)\zeta^s z^n$, the quantity $\psi_{n,s}(x)$ counts tree-like geometric objects containing the origin and x and consisting of n bonds including a "backbone" with s steps. These geometric objects are not trees, due to the various self-intersections required by the definition of Ψ. Differentiation with respect to z replaces the factor z^n by nz^{n-1}, and the factor of n arising here can be written as a sum over sites (to be accurate, there are $n+1$ sites, but this is unimportant). Diagrammatically, this corresponds to adding a vertex to a line in $\Psi_{z,\zeta}(x)$ and a new line emanating from that vertex and terminating at a new vertex of degree 1. An example is the following:



(4.27)

Taking absolute values, overcounting by ignoring avoidance constraints between different lines in the diagram, and bounding $|\zeta|$ by 1, leads to a bound by $\chi(|z|) = O((1-|z|/z_c)^{-1/2})$ times a diagram with degree of divergence equal to that of the pentagon and propagator bounded above by $O([k^2+(1-|z|/z_c)^{1/2}]^{-1})$. This gives an overall bound $(1-|z|/z_c)^{\epsilon-1}$, which is insufficient for our needs because it fails to provide the factor $|1-z/z_c|^{-1/2}$. Such factors are important, because of the restriction $b_0 \geq 1$ in Lemma 3.3(i).

To recover the desired factor $|1 - z/z_c|^{-1/2}$ in the upper bound, we proceed in the same fashion used to derive (4.6). This involves performing a lace expansion to remove the interaction between the new line in (4.27) and the remainder of the diagram. The result is a formula of the form

$$\frac{d}{dz} z \hat{\Psi}_{z,\zeta}(k) = \hat{\Phi}^{(1)}_{z,\zeta}(k)[z\Omega\chi(z) + 1], \tag{4.28}$$

where $\hat{\Phi}^{(1)}$ is bounded by a sum of diagrams having the same power counting as the pentagon. The desired factor $|1 - z/z_c|^{-1/2}$ then arises as an upper bound on $\chi(z)$, while $\hat{\Phi}^{(1)}$ is bounded by $O((1 - |z|/z_c)^{\epsilon - 1/2})$.

The situation is similar for (4.26), making use of an additional lace expansion to obtain an identity

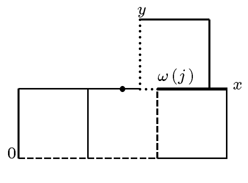
$$\frac{d}{dz} z \hat{\Phi}^{(1)}_{z,\zeta}(k) = \hat{\Phi}^{(2)}_{z,\zeta}(k)[z\Omega\chi(z) + 1], \tag{4.29}$$

with $\hat{\Phi}^{(2)}$ bounded by a sum of diagrams having the same power counting as the hexagon. $\square$

4.2 The three-point function

In this section, we introduce a generalization $\hat{\Psi}_{z,\vec{\zeta}}(\vec{k})$ of the function $\hat{\Psi}_{z,\zeta}(k)$ occurring in (4.10). Here $\vec{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$ with each $|\zeta_j| \leq 1$, and $\vec{k} = (k_1, k_2, k_3)$ with each $k_j \in [-\pi, \pi]^d$. The generalization involves a modification of the definition of $\hat{\Psi}_z(k)$ given in [?, Section 4.1], as follows.

In [?, (4.17)], we multiply the right side by a factor $e^{ik_1 \cdot y} \zeta_1^j e^{ik_2 \cdot (x-y)} \zeta_2^{|\omega| - j}$, where j is such that $\omega(j) = y$. Summing this over x and y gives a contribution to $\hat{\Psi}_{z,\vec{\zeta}}(\vec{k})$ corresponding to [?, (4.17)]. In [?, (4.28)], we modify the right side by changing $\mathcal{F}$ to include a factor $e^{ik_1 \cdot \omega(j)} \zeta_1^j e^{ik_2 \cdot (x - \omega(j))} \zeta_2^{|\omega| - j} e^{ik_3 \cdot (y - \omega(j))} \zeta_3^{|\omega'|}$ in the right side of [?, (4.24)]. This gives the same diagrams as for $\hat{\Psi}_z(k)$, except that now the diagrams contain factors as above. For a specific example arising as in [?, (4.24)], in the diagrammatic contribution



to $\hat{\Psi}_{z,\vec{\zeta}}(\vec{k})$, the dashed line carries a factor $e^{ik_1 \cdot \omega(j)} \zeta_1^j$, the heavy line carries a factor $e^{ik_2 \cdot (x - \omega(j))} \zeta_2^{|\omega| - j}$, and the dotted line carries a factor $e^{ik_3 \cdot (y - \omega(j))} \zeta_3^{|\omega'|}$. Here ω' is the skeleton path joining $\omega(j)$ to y. The method of proof of Lemma 4.2 can then be used to prove the following lemma.

Lemma 4.3 *The bounds of Lemma 4.2 hold, uniformly in $k \in [-\pi, \pi]^d$, $|z| \leq z_c$, $|\zeta_j| \leq 1$, if $\hat{\Psi}_{z,\zeta}(k)$ is replaced by $\hat{\Psi}_{z,\vec{\zeta}}(\vec{k})$ and if ζ is replaced by ζ_j in ζ-derivatives and in the upper bound of (4.24).*

The function $\hat{\Psi}_{z,\vec{\zeta}}(\vec{k})$ arises as a kind of renormalized vertex in the following lemma.

Lemma 4.4 *The three-point function obeys the identity*

$$\begin{aligned}\hat{G}^{(3)}_{z,\vec{\zeta}}(\vec{k}) &= \left[\hat{G}^{(2)}_{z,\zeta_3}(k_3) + \left(1+\zeta_3 z\Omega\hat{D}(k_3)\hat{G}^{(2)}_{z,\zeta_3}(k_3)\right)\hat{\Psi}_{z,\vec{\zeta}}(\vec{k})\right] \\ &\quad \times \prod_{p=1}^{2}\left(\zeta_p z\Omega\hat{D}(k_p)\hat{G}^{(2)}_{z,\zeta_p}(k_p)+1\right). \end{aligned} \tag{4.30}$$

Proof. By definition,

$$\begin{aligned}\hat{G}^{(3)}_{z,\vec{\zeta}}(\vec{k}) &= \sum_{y_1,y_2,y_3}\sum_{\omega:0\to y_1\to y_1+y_2} z^{|\omega|}\sum_{j=0}^{|\omega|}\delta_{\omega(j),y_1}\left(\prod_{i:i\neq j}\sum_{R_i\ni\omega(i)} z^{|R_i|}\right) \\ &\quad\times\sum_{R_j\ni y_1,y_1+y_3} z^{|R_j|}K[0,|\omega|]e^{i(k_1\cdot y_1+k_2\cdot y_2+k_3\cdot y_3)}\zeta_1^{j}\zeta_2^{|\omega|-j}\zeta_3^{|\omega_3|}, \end{aligned}\tag{4.31}$$

where $\omega_3 : y_1 \to y_1 + y_3$ is the backbone in R_j joining y_1 to $y_1 + y_3$, and $K[0, |\omega|]$ is defined in [**?**, (2.3)] and serves to keep the ribs R_l from intersecting one another. By a simple modification of the proof of [**?**, Lemma 5.2.5] to account for the different notion of graph connectivity used in [**?**], it follows that

$$K[0,|\omega|] = \sum_{I:I\ni j} K[0,I_1-1]J[I_1,I_2]K[I_2+1,|\omega|], \tag{4.32}$$

where the sum if over intervals $I = [I_1, I_2]$ of integers such that $0 \le I_1 \le j \le I_2 \le |\omega|$. Here $K[a, b] = 1$ if $a \ge b$, $J[a, a] = 1$, and the degenerate case $I = [j, j]$ is included as a term in the sum.

We insert (4.32) in (4.31). The term with $I = [0, |\omega|]$ and $|\omega| \ge 1$ is the same as the expression for $\frac{d}{dz}z\hat{\Pi}_z(k)$ given in [**?**, (4.5)], except (4.31) is more detailed due to its dependence on $\vec{k},\vec{\zeta}$. Carrying out the analysis of [**?**, Section 4.1] while taking these additional variables into account, this term is equal to

$$\left(1+\zeta_3 z\Omega\hat{D}(k_3)\hat{G}^{(2)}_{z,\zeta_3}(k_3)\right)\hat{\Psi}_{z,\vec{\zeta}}(\vec{k}). \tag{4.33}$$

The factors $\prod_{p=1}^{2}\left(\zeta_p z\Omega\hat{D}(k_p)\hat{G}^{(2)}_{z,\zeta_p}(k_p)+1\right)$ arise from the factors $K[0, I_1-1]$ and $K[I_2+1, |\omega|]$, with the unit terms arising when $I_1 = 0$ or $I_2 = |\omega|$. The term $\hat{G}^{(2)}_{z,\zeta_3}(k_3)$ in (4.30) arises from the case $I = [j, j]$. □

Define

$$V(z,\vec{\zeta},\vec{k}) = \left(1+\zeta_3 z\Omega\hat{D}(k_3)\Psi_{z,\vec{\zeta}}(\vec{k})\right)\prod_{p=1}^{2}\left(\zeta_p z\Omega\hat{D}(k_p)\right) \tag{4.34}$$

and, recalling (4.21) and (2.4),

$$v = V(z_c,\vec{1},\vec{0}) = (1+b)(z_c\Omega)^2 = \frac{1}{2B_2^2}. \tag{4.35}$$

Abbreviating the notation, (4.30) can then be written

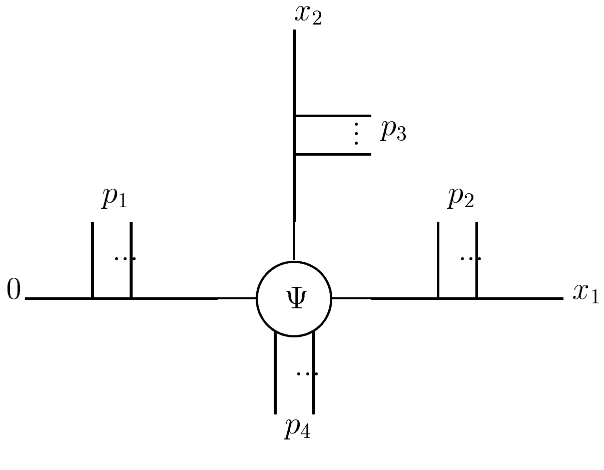
$$\begin{aligned}G^{(3)} &= VG_1G_2G_3 + \Psi(\zeta_2 z\Omega D_2G_2+1)(\zeta_1 z\Omega D_1G_1+1) \\ &\quad + [G_3 + (1+\zeta_3 z\Omega D_3G_3)\Psi]\,(\zeta_1 z\Omega D_1G_1+\zeta_2 z\Omega D_2G_2+1). \end{aligned}\tag{4.36}$$

4.3 The m-point function, $m \geq 4$

This section will be used only in Section **??**, where the case $m \geq 4$ of Theorem 1.1 is proved, so we restrict attention to the case of $\vec{\zeta} = \vec{1}$ and $m \geq 4$.

Consider the m-point function $G_z^{(m)}(\sigma; \vec{y})$ for lattice trees containing the "external" vertices $0, x_1, \ldots, x_{m-1}$, as in Figure 4. Focussing on the three points $0, x_1, x_2$, we apply the proof of Lemma 4.4 to obtain an identity like (4.30), except now in each term in the expansion the geometric objects involved must contain the additional points $x_3, \ldots, x_{m-1}$. In the skeleton for a lattice tree containing $0, x_1, \ldots, x_{m-1}$, we decompose (4.31) according to the possibilities for how the skeleton paths from $x_3, \ldots, x_{m-1}$ connect to the backbone path ω. Given an interval I, let p_1, $\tilde{p}_3$, p_2 respectively be the number of these $m-3$ points which connect to ω in the intervals $[0, I_1 - 1]$, $[I_1, I_2]$, $[I_2+1, |\omega|]$. (If $I_1 = 0$ we take $p_1 = 0$ and if $I_2 = |\omega|$ we take $p_2 = 0$.) Then G_1 and G_2 in (4.30) get replaced by $G_1^{(p_1+2)}$ and $G_2^{(p_2+2)}$, with a suitable allocation of momenta k_j. Of the $\tilde{p}_3$ points connected to $[I_1, I_2]$, let p_3 be the number connected to the portion of the skeleton for $0, x_1, x_2$ which enters into G_3 and let p_4 be the number connected to the portion of the skeleton which enters into Ψ.

Schematically, this can be represented as



although this figure does not show the topological richness of possible connections of $x_3, \ldots, x_{m-1}$ to the backbone. This gives an equation like (4.36) for $G^{(m)}$, with each G_j replaced by $G_j^{(p_j+2)}$ and Ψ replaced by $\Psi^{(p_4)}$ on the right side. Here $\Psi^{(p_4)}$ denotes Ψ with p_4 additional external lines emanating, and we will also write $V^{(p)}$ to denote V with Ψ replaced by $\Psi^{(p)}$. Specification of a shape σ for $G^{(m)}$ imposes corresponding shape restrictions on the $G^{p_j+2)}$, as well as on the possible values for the p_j. Explicitly, (4.36) becomes

$$\begin{aligned} G^{(m)} &= \sum_{p_j \geq 0 : p_1+p_2+p_3+p_4 = m-3} \Big[G^{(p_1+2)} G^{(p_2+2)} G^{(p_3+2)} V^{(p_4)} \\ &\quad + \Psi^{(p_4)} (z\Omega D_2 G_2^{(p_2+2)} + 1)(z\Omega D_1 G_1^{(p_1+2)} + 1) \end{aligned}$$