Does each symmetric operator have a stability domain?

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Abstract

We show that any symmetric operator H has a dense maximal b-stability domain $\mathcal{D}_s$ (i.e. $H|\mathcal{D}_s \geq bI$, $b \in \mathbf{R}^1$) if and only if H is unbounded from above. This abstract result allows an application to singular perturbed Schrödinger operators which are not semibounded from below, i.e., to the so-called "fall to the center problem". It turns out that for that case the regularization problem is always ill-posed which has as a consequence that there is no unique "right Hamiltonian" for the perturbed system. We give an example of singular perturbed Schrödinger operator for which stability domains are described explicitly.

Mathematics Subject Classification

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singular perturbation, symmetric operator, self-adjoint extension, regularization, Schrödinger operator, fall to the center

Contents

1 Introduction

If a self-adjoint operator H on separable Hilbert space $\mathcal{H}$ should be the Hamiltonian of some quantum system with finitely many particles, then it has to be semi-bounded from below. Otherwise the particles would occupy states with lower and lower energies, which finally leads to the collapse of the system. However, having some semi-bounded self-adjoint operator A and some singular perturbation W of A it can happen that the perturbed operator H,

$$Hf = Af + Wf, \qquad f \in \mathrm{dom}(H) = \mathcal{D} \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W), \tag{1.1}$$

where $\mathcal{D}$ is some linear dense subset of $\mathrm{dom}(A) \cap \mathrm{dom}(W)$, is not essentially self-adjoint and, moreover, it is not semi-bounded from below. If H admits self-adjoint extensions no of them can be regarded as the Hamiltonian of the perturbed quantum system since each extension is not semi-bounded from below.

Example 1.1 Let $A = -d^2/dx^2$ be the Laplace operator on $\mathcal{H} = L^2(\mathbf{R}^1)$ and let W be the multiplication operator induced by

$$W(x) = -\frac{1}{4}\frac{1}{|x|^2}, \qquad x \in \mathbf{R}^1. \tag{1.2}$$

and

$$H_{\frac{1}{\kappa}} f = Af + \frac{1}{\kappa} Wf, \qquad f \in \mathrm{dom}(H_{\frac{1}{\kappa}}) = \mathcal{D} = C_0^\infty(\mathbf{R}^1 \setminus \{0\}), \quad 0 < \kappa < \infty. \tag{1.3}$$

If $\kappa \geq 1$, then the operator $H_{\frac{1}{\kappa}}$ is semi-bounded from below (see [2], [3]) while for $\kappa \in (0,1)$ not: "fall to the center". Since the operator $H_{\frac{1}{\kappa}}$ commutes with the complex conjugation it has equal deficiency indices (von Neumann's theorem, see e.g. [8], Th. X.3). Hence $H_{\frac{1}{\kappa}}$ has several self-adjoint extensions and, moreover, if $\kappa \in (0,1)$, then each of these extensions is not semi-bounded from below. Therefore, in the last case there is no self-adjoint extension which would be a candidate for Hamiltonian.

To find a way out from this situation one can take the view that the domain $\mathrm{dom}(H)$ is too large and, hence, even the symmetric operator H is wrong defined. However, it might be possible that there is a dense domain in $\mathcal{D}_s \subseteq \mathrm{dom}(H)$ in $\mathcal{H}$ such that $H|\mathcal{D}_s$ is semi-bounded.

Definition 1.2 *Let H be a (closed or unclosed) unbounded symmetric operator. A dense linear subset $\mathcal{D}_s \subseteq \mathrm{dom}(H)$ is called a stability domain of H if $H|\mathcal{D}_s$ is semi-bounded from below.*

A stability domain $\mathcal{D}_s$ of H is called a b-stability domain, $b \in \mathbf{R}^1$, if

$$\inf_{f \in \mathcal{B}_1(\mathcal{D}_s)} (Hf, f) \geq b, \tag{1.4}$$

where

$$\mathcal{B}_1(\mathcal{D}_s) = \{f \in \mathcal{D}_s : \|f\| \leq 1\}. \tag{1.5}$$

A b-stability domain $\mathcal{D}_s$ is called maximal if for each b-stability domain $\mathcal{D}'_s$ obeying $\mathcal{D}_s \subseteq \mathcal{D}'_s$ one gets $\mathcal{D}_s = \mathcal{D}'_s$.

In other words the problem of construction a semi-bounded from below Hamiltonian is reduced to the existence of a stability domain of H. However, this leads to a general question: whether each symmetric operator has a stability domain? The answer is, of course, *no*.

Definition 1.3 *Let H be a (closed or unclosed) densely defined unbounded symmetric operator. We say H is unbounded from above if*

$$\lambda^+_1(H) \stackrel{\rm def}{=} \sup_{f\in\mathcal{B}_1(\mathrm{dom}(H))} (Hf, f) = +\infty. \tag{1.6}$$

The operator is called unbounded from below if $(-H)$ is unbounded from above.

If H is unbounded from *below* but semi-bounded from *above*, then evidently H has no stability domain. In this case the existence of a stability domain $\mathcal{D}_s$ of H would imply that H is bounded. So the problem reduces to the question: whether each symmetric operator, which is *unbounded from above*, has a stability domain? The answer to this question is *yes*. The aim of this paper is to prove this.

Note that the problem is trivial if the operator H is unbounded from above but semi-bounded from below. So we face with a real problem if H is unbounded from *below* and *above*.

A partial solution of this problem is obtained in [9]. There it is proven that two (closed or unclosed) densely defined unbounded symmetric operators H_1 and H_2 which satisfy

$$\mathrm{dom}(H_s) = \bigcap_{n=1}^{\infty} \mathrm{dom}(\overline{H^n_s}), \qquad s = 1, 2, \tag{1.7}$$

possess unitarily equivalent densely defined restrictions if and only if both operators are *strongly* unbounded from below or from above.

Definition 1.4 *Let H be a (closed or unclosed) densely defined unbounded symmetric operator such that $H(\mathrm{dom}(H)) \subseteq \mathrm{dom}(H)$ and let $k, n \in \mathbf{N}$, where $\mathbf{N} = \{1, 2, \ldots\}$. We set*

$$\mathcal{B}_n(dom(H)) \stackrel{\rm def}{=} \{f \in \mathrm{dom}(H) : |(H^j f, f)| \leq 1 \qquad for \quad j = 0, \ldots, n-1\} \tag{1.8}$$

and

$$\begin{aligned} \lambda^+_{2k-1}(H) &\stackrel{\rm def}{=} \sup\nolimits_{f\in\mathcal{B}_{2k-1}(\mathrm{dom}(H))} (H^{2k-1}f, f) \\ \lambda^-_{2k-1}(H) &\stackrel{\rm def}{=} \inf\nolimits_{f\in\mathcal{B}_{2k-1}(\mathrm{dom}(H))} (H^{2k-1}f, f). \end{aligned} \tag{1.9}$$

The operator H is called strongly unbounded from above (below) if $\lambda^+_{2k-1}(H) = +\infty$ ($\lambda^-_{2k-1}(H) = -\infty$) for all $k \in \mathbf{N}$.

In application to our situation this means that if the symmetric operator H satisfies conditions $H(\mathrm{dom}(H)) \subseteq \mathrm{dom}(H)$ and

$$\mathrm{dom}(H) = \bigcap_{n=1}^{\infty} \mathrm{dom}(\overline{H^n}) \tag{1.10}$$

and is strongly unbounded from above, then H possesses a stability domain. To see this we set $H_1 = H$ and choose for H_2 a restriction of an arbitrary unbounded non-negative self-adjoint operator K to the domain $\mathcal{D}_\infty \stackrel{\mathrm{def}}{=} \bigcap_{n=1}^{\infty} \mathrm{dom}(K^n)$, i.e. $H_2 = K|\mathcal{D}_\infty$ which is essentially self-adjoint. One can easily prove that H_1 and H_2 obey (1.7) and, moreover, that H_2 is strongly unbounded. Hence H_1 and H_2 have densely defined restrictions which are unitarily equivalent. In particular, this means that H has densely defined non-negative restriction, i.e., H has a stability domain. However, the condition (1.10) and the strong unboundedness are too limited for our purpose.

The paper is organized as follows. In the next section we give necessary and sufficient conditions (Theorem 2.3) that a symmetric operator has a maximal b-stability domain. In Section 3 we discuss applications of the main Theorem 2.3 to the problem of the "right Hamiltonian" for non-positive singular perturbations. The principle observation is a kind of "no-go" Theorem 3.3 which says that if the natural defined perturbed operator is not semi-bounded from below, then there is no *unique* "right Hamiltonian". We demonstrate the above abstract statements by an instructive quantum mechanical example in Section 4. Concluding remarks and discussions are postponed to the last section.

2 Stability

The aim of this section is to prove the assertion made above about the existence of a stability domain for an unbounded from above symmetric operator.

Lemma 2.1 *Let H be a (closed or unclosed) densely defined unbounded symmetric operator which is unbounded from above. Let $\mathcal{F} \subseteq \mathrm{dom}(H)$ be a finite dimensional subspace. Then*

$$\sup_{f \in \mathcal{B}_1(\mathrm{dom}(H) \ominus \mathcal{F})} (Hf, f) = +\infty \tag{2.1}$$

Proof: Let $P^{\mathcal{H}}_{\mathcal{F}}$ be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{F}$. Since $\mathcal{F} \subseteq \mathrm{dom}(H)$ and $\dim(\mathcal{F}) < \infty$ there is a constant $C > 0$ such that $||HP^{\mathcal{H}}_{\mathcal{F}}f|| \leq C||f||$, $f \in \mathcal{H}$. Hence

$$|(H(I - P^{\mathcal{H}}_{\mathcal{F}})f, (I - P^{\mathcal{H}}_{\mathcal{F}})f) - (Hf, f)| \leq 2C||f||^2, \qquad f \in \mathrm{dom}(H). \tag{2.2}$$

Since H is unbounded from above, for each $n \in \mathbf{N}$ there is a $f_n \in \mathrm{dom}(H)$, $||f_n|| \leq 1$, such that $(Hf_n, f_n) \geq n$. We set

$$g_n = (I - P^{\mathcal{H}}_{\mathcal{F}})f_n, \qquad n \in \mathbf{N}. \tag{2.3}$$

Then $g_n \in \mathrm{dom}(H) \ominus \mathcal{F}$, $||g_n|| \leq 1$ and by (2.2) one gets

$$(Hg_n, g_n) = (H(I - P^{\mathcal{H}}_{\mathcal{F}})f_n, (I - P^{\mathcal{H}}_{\mathcal{F}})f_n) \geq n - 2C \tag{2.4}$$

for $n \in \mathbf{N}$ which proves (2.1). •

Next we need some facts from linear algebra.

Lemma 2.2 *Let H be a (closed or unclosed) symmetric operator such that there is a sequence $\{g_l\}^m_{l=1}$, $g_l \in \mathrm{dom}(H)$, of linearly independent non-trivial elements for which one has*

$$(Hg_l, g_l) > 0, \qquad l = 1, \ldots, m. \tag{2.5}$$

Let $\mathcal{G}_m$ be the subspace spanned by $\{g_l\}^m_{l=1}$. Then one has $(Hf, f) > 0$ for any non-trivial element $f \in \mathcal{G}_m$ if and only if the determinants

$$D(g_1, \ldots, g_k) = \det \begin{pmatrix} (Hg_1, g_1) & \cdots & (Hg_k, g_1) \\ (Hg_1, g_2) & \cdots & (Hg_k, g_2) \\ \cdots & \cdots & \cdots \\ (Hg_1, g_k) & \cdots & (Hg_k, g_k) \end{pmatrix} \tag{2.6}$$

obey $D(g_1, \ldots, g_k) > 0$, for each $k = 1, 2, \ldots, m$.

Proof: We set

$$a_{ln} \stackrel{\text{def}}{=} (Hg_n, g_l), \qquad 1 \leq l, n \leq m. \tag{2.7}$$

Let $f = \sum^m_{n=1} c_n g_n$. Obviously one has

$$\begin{aligned}(H\sum_{n=1}^m c_n g_n, \sum_{l=1}^m c_l g_l) \\ = \sum_{n=1}^m \sum_{l=1}^m c_n \overline{c_l} (Hg_n, g_l) = \sum_{n=1}^m \sum_{l=1}^m a_{ln} c_n \overline{c_l}.\end{aligned} \tag{2.8}$$

Let

$$A_m = \|a_{ln}\|_{l,n=1}^{m} = \left\| \begin{array}{ccc} (Hg_1, g_1) & \cdots & (Hg_m, g_1) \\ (Hg_1, g_2) & \cdots & (Hg_m, g_2) \\ \cdots & \cdots & \cdots \\ (Hg_1, g_m) & \cdots & (Hg_m, g_m) \end{array} \right\| \tag{2.9}$$

Note that the matrix A_m is symmetric, i.e. $a_{nl} = \overline{a_{ln}}$, $1 \le l, n \le m$. Furthermore, the kernel of A_m would be non-trivial if $\{g_l\}_{l=1}^{\infty}$ are not linearly independent. Introducing the vector

$$\vec{c} = \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{pmatrix} \tag{2.10}$$

one gets (see (2.7))

$$(Hf, f) = < A_m \vec{c}, \vec{c} >, \tag{2.11}$$

where $< \cdot, \cdot >$ is the usual scalar-product in $\mathbf{C}^m$. Therefore one has $(Hf, f) > 0$ for each $f \in \mathcal{G}_m$, $f \neq 0$, if and only if the matrix A_m is positive. However, by a well-known criterion (Theorem 27,[10]) this is equivalent to the fact the determinants $D(g_1, ..., g_k)$ are positive for each $k = 1, 2, \ldots, m$. •

In the following we use decomposition of determinants into minors. One gets

$$D(g_1, ..., g_{m+1}) = a_{m+1,1}(-1)^{m+2} M(g_2, ..., g_{m+1}) + \ldots + a_{m+1,m+1}(-1)^{2m+2} M(g_1, ..., g_m), \tag{2.12}$$

where

$$a_{m+1,j} = (Hg_j, g_{m+1}), \qquad j = 1, 2, \ldots, m+1. \tag{2.13}$$

The minors $M(g_1, g_2, ..., g_{j-1}, g_{j+1}, ..., g_{m+1})$ are given by

$$M(g_1, ..., g_{j-1}, g_{j+1}, ..., g_{m+1}) = \tag{2.14}$$

$$\det \begin{pmatrix} (Hg_1, g_1) & \cdots & (Hg_{j-1}, g_1) & (Hg_{j+1}, g_1) & \cdots & (Hg_{m+1}, g_1) \\ (Hg_1, g_2) & \cdots & (Hg_{j-1}, g_2) & (Hg_{j+1}, g_2) & \cdots & (Hg_{m+1}, g_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (Hg_1, g_m) & \cdots & (Hg_{j-1}, g_m) & (Hg_{j+1}, g_m) & \cdots & (Hg_{m+1}, g_m) \end{pmatrix}.$$

Note that $D(g_1, ..., g_m) = M(g_1, ..., g_m)$. If $g_1, ..., g_m$ are fixed elements, then there are constants $C_j(g_1, \ldots, g_m)$ such that

$$|M(g_1, ..., g_{j-1}, g_{j+1}, ..., g_{m+1})| \le C_j(g_1, ..., g_m)\|g_{m+1}\|, \qquad j = 1, 2, \ldots, m. \tag{2.15}$$

This comes from the estimate $|(Hg_{m+1}, g_n)| \le \|Hg_n\|\|g_{m+1}\|$, $1 \le n \le m$.

Theorem 2.3 *Let H be a (closed or unclosed) densely defined symmetric operator. The operator H has a maximal b-stability domain, $b \in \mathbf{R}^1$, if and only if H is unbounded from above.*

Proof: Let $b = 0$ and $\{\xi_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathcal{H}$ such that $\xi_n \in \mathrm{dom}(H)$ for each $n = 1, 2, \ldots$. Such basis always exists. On the other hand, by Lemma 2.1 there is a sequence $\{f_m\}_{m=1}^{\infty}$, $f_m \in \mathrm{dom}(H)$, $\|f_m\| \leq 1$, such that

(i) $f_m \perp f_1, \ldots, f_{m-1}$, $\qquad m = 2, 3, \ldots$,

(ii) $f_m \perp \xi_1, \ldots, \xi_{[m]+1}$, $\qquad m = 1, 2, \ldots$,

(iii) $(Hf_m, f_m) > \gamma_m > 0$, $\qquad m = 1, 2, \ldots$,

where $\{\gamma_m\}_{m=1}^{\infty}$ is a sequence of positive numbers such that $\lim_{m\to\infty} \gamma_m = +\infty$, which will be defined in the following, and

$$[m] \stackrel{\mathrm{def}}{=} max\{n \in \mathbf{N}_0 : \frac{n(n+1)}{2} < m\}, \qquad m \in \mathbf{N}, \tag{2.16}$$

where $N_0 = \{0, 1, 2, \ldots\}$ and $\mathbf{N} = \{1, 2, \ldots\}$. We introduce the function $p(\cdot) : \mathbf{N} \longrightarrow \mathbf{N}$,

$$p(m) \stackrel{\mathrm{def}}{=} m - \frac{[m]([m]+1)}{2} \tag{2.17}$$

and the elements

$$g_m \stackrel{\mathrm{def}}{=} f_m + ([m] - p(m) + 2)\xi_{p(m)}, \qquad m \in \mathbf{N}. \tag{2.18}$$

The first members of the sequence $\{g_m\}_{m=1}^{\infty}$ are given by

$$\begin{aligned} g_1 &= f_1 + 1 \cdot \xi_1 \\ g_2 &= f_2 + 2 \cdot \xi_1 \\ g_3 &= f_3 + 1 \cdot \xi_2 \\ g_4 &= f_4 + 3 \cdot \xi_1 \\ g_5 &= f_5 + 2 \cdot \xi_2 \\ g_6 &= f_6 + 1 \cdot \xi_3 \\ g_7 &= f_7 + 4 \cdot \xi_1 \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{2.19}$$

Note that the sequence $\{g_m\}_{m=1}^{\infty}$ consists of linearly independent elements. To see this let us assume that

$$\sum_{l=1}^{L} c_l g_l = 0. \tag{2.20}$$

By the properties (i) and (ii) one has $f_L \perp f_1, \dots, f_{L-1}$ and $f_L \perp \xi_1, \dots, \xi_{[L]+1}$. Therefore

$$0 = \left(\sum_{l=1}^{L} c_l g_l, f_L \right) = c_L. \tag{2.21}$$

Hence $\sum_{l=1}^{L-1} c_l g_l = 0$. Repeating this consideration one gets $c_1 = c_2 = \dots = c_L = 0$.

Let us prove that there is a suitable sequence $\{\gamma_m\}_{m=1}^{\infty}$ such that the sequence $\{g_m\}_{m=1}^{\infty}$ obeys $D(g_1, ..., g_m) > 0$ for each $m = 1, 2, ...$ The existence of $\{\gamma_m\}_{m=1}^{\infty}$ we prove by induction. For $m = 1$ we set

$$\gamma_1 = 3\|H\xi_1\|. \tag{2.22}$$

Then we obtain

$$D(g_1) = (Hg_1, g_1) = (H(f_1 + \xi_1), (f_1 + \xi_1)) \geq (Hf_1, f_1) - 3\|H\xi_1\| > 0. \tag{2.23}$$

Let as assume that there are $\gamma_1, \dots, \gamma_m$ such that the corresponding $g_1, \dots, g_m$ satisfy the condition $D(g_1, \dots, g_l) > 0$ for $l = 1, 2, \dots, m$. We have to show that there is a γ_{m+1} such that $D(g_1, ..., g_m, g_{m+1}) > 0$. We set

$$\begin{aligned} \gamma_{m+1} = ([m+1]+1)(([m+1]+3) \sup_{1 \leq k \leq [m+1]} \|H\xi_k\| \\ + \frac{1 + ([m+1]+1)^2}{D(g_1, ..., g_m)} \sum_{j=1}^{m} \|Hg_j\| C_j(g_1, ..., g_m), \end{aligned} \tag{2.24}$$

where $C_j(g_1, ..., g_m)$ is given by (2.15). Taking into account formula (2.13) one gets

$$\begin{aligned} D(g_1, ..., g_{m+1}) \geq (Hg_{m+1}, g_{m+1}) D(g_1, ..., g_m) \\ - \sum_{j=1}^{m} \|Hg_j\| \|g_{m+1}\| |M(g_1, ..., g_{j-1}, g_{j+1}, ..., g_{m+1})|. \end{aligned} \tag{2.25}$$

Using (2.15) we obtain

$$D(g_1, ..., g_{m+1}) \geq (Hg_{m+1}, g_{m+1}) D(g_1, ..., g_m) - \|g_{m+1}\|^2 \sum_{j=1}^{m} \|Hg_j\| C_j(g_1, ..., g_m). \tag{2.26}$$

Since $D(g_1, ..., g_m) > 0$ by assumption one has $D(g_1, ..., g_{m+1}) > 0$ if

$$\Delta \stackrel{\text{def}}{=} (Hg_{m+1}, g_{m+1}) - \frac{\|g_{m+1}\|^2}{D(g_1, ..., g_m)} \sum_{j=1}^{m} \|Hg_j\| C_j(g_1, ..., g_m) > 0 \tag{2.27}$$

Obviously, from (2.18) we find

$$\|g_{m+1}\|^2 \leq 1 + ([m+1] - p(m+1) + 2)^2 \leq 1 + ([m+1]+1)^2 \tag{2.28}$$

and

$$(Hg_{m+1}, g_{m+1}) \geq (Hf_{m+1}, f_{m+1}) - \\ -2([m+1] - p(m+1) + 2)\|H\xi_{p(m+1)}\| - ([m+1] - p(m+1) + 2)^2\|H\xi_{p(m+1)}\|. \tag{2.29}$$

Hence

$$(Hg_{m+1}, g_{m+1}) \geq (Hf_{m+1}, f_{m+1}) - ([m+1]+1)([m+1]+3) \sup_{1\leq k\leq[m+1]} \|H\xi_k\|. \tag{2.30}$$

Therefore, the estimate (2.27) gets the form:

$$\Delta \geq (Hf_{m+1}, f_{m+1}) - \\ ([m+1]+1)([m+1]+3) \sup_{1\leq k\leq[m+1]} \|H\xi_k\| - \frac{1 + ([m+1]+1)^2}{D(g_1, ..., g_m)} \sum_{j=1}^{m} \|Hg_j\| C_j(g_1, ..., g_m). \tag{2.31}$$

Using definition (2.24) one finally gets

$$\Delta \geq (Hf_{m+1}, f_{m+1}) - \gamma_{m+1}. \tag{2.32}$$

Applying (iii) we immediately obtain $\Delta > 0$ which yields $D(g_1, ..., g_{m+1}) > 0$. So, by Lemma 2.2 the proof of stability follows by induction.

It remains to show that the linear span $\mathcal{G}$ of $\{g_m\}_{m=1}^{\infty}$ is dense in $\mathcal{H}$. To this end we note that

$$\eta_m \stackrel{\text{def}}{=} \frac{1}{[m] - p(m) + 2} f_m + \xi_{p(m)} \in \mathcal{G}, \qquad m = 1, 2, \ldots. \tag{2.33}$$

If $m = \frac{n(n+1)}{2} + k$, $1 \leq k \leq n+1$. Then by definitions (2.16), (2.17) we have $[m] = n$ and $p(m) = k$. Hence $[m] - p(m) + 2 = n - k + 2$. Therefore, (2.33) transforms into

$$\eta_{\frac{n(n-1)}{2}+k} = \frac{1}{n-k+2} f_{\frac{n(n-1)}{2}+k} + \xi_k. \tag{2.34}$$

Fixing k and tending $n \longrightarrow \infty$ one gets $\lim_{n\to\infty} \eta_{\frac{n(n-1)}{2}+k} = \xi_k$. However, by construction $\{\xi_k\}_{k=1}^{\infty}$ forms an orthonormal basis in $\mathcal{H}$. Hence $\mathcal{G}$ is dense in $\mathcal{H}$.

If $b \neq 0$, then we consider instead of H the operator $H_b \stackrel{\text{def}}{=} H - bI$. Applying the first part of the proof we obtain a stability domain $\mathcal{D}_s$ such that $H_b|\mathcal{D}_s \geq 0$. Hence $H|\mathcal{D}_s \geq bI$.

It remains to show that there is always a maximal stability domain. To prove this we introduce the set X_b of all dense domains $\mathcal{D}_s \subseteq \text{dom}(H)$ such that $H|\mathcal{D}_s \geq bI$. With respect to the set inclusion relation the set X_b is partially ordered. Moreover, for every linearly ordered subset Y_b of X_b there is a upper bound $\mathcal{D}_p$. The upper bound is given by $\mathcal{D}_p = \bigcup_{\mathcal{D}_s \in Y_b} \mathcal{D}_s$. Obviously, $\mathcal{D}_p$ is a linear dense subset of $\text{dom}(H)$ such that $H|\mathcal{D}_p \geq bI$. Then by the Zorn's lemma (see e.g. [7],Th. I.2) in X_b there is at least one maximal element $\mathcal{D}_m$, i.e., such that from $\mathcal{D}'_s \supseteq \mathcal{D}_m$, $\mathcal{D}'_s \in X_b$, it follows $\mathcal{D}'_s = \mathcal{D}_m$. •

3 Singular perturbations

Let us relate the above result to our previous papers [5, 6]. There we associated with two self-adjoint operators $A \geq 0$ and $W \leq 0$ having a common dense domain $\mathcal{D} \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W)$ the symmetric operator

$$H_\alpha f = Af + \alpha W f, \qquad \mathrm{dom}(H_\alpha) = \mathcal{D}, \quad \alpha > 0. \tag{3.1}$$

A dense linear subset $\mathcal{D}_s \subseteq \mathcal{D}$ was called a *stability domain of the pair* $\{A, W\}$ if there are constants a, b, $0 < a < 1$, $b \geq 0$, such that the estimate

$$(-Wf, f) \leq a(Af, f) + b(f, f), \qquad f \in \mathcal{D}_s, \tag{3.2}$$

takes place. In general, however, such a stability domain $\mathcal{D}_s \subseteq \mathcal{D}$ might not exist for given constants a, b, $0 < a < 1$, $b \geq 0$. So, the natural question arises whether there is always a stability domain of the pair $\{A, W\}$ with respect to $\mathcal{D}$? Moreover, does such stability domain exist for any constants a, b, $0 < a < 1$, $b \geq 0$? Finally, is there always a *maximal stability domain* for given constants a, b? This means a domain $\mathcal{D}_m$ such that for any other stability domain $\mathcal{D}_s$, $\mathcal{D}_m \subset \mathcal{D}_s \subseteq \mathcal{D}$, for which (3.2) holds, one gets $\mathcal{D}_m = \mathcal{D}_s$.

Theorem 3.1 *Let $\{A \geq 0, W \leq 0\}$ be a pair of self-adjoint operators with common dense domain $\mathcal{D} \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W)$. For each a, $0 < a < 1$, and $b \geq 0$ there is a maximal stability domain $\mathcal{D}_{a,b} \subseteq \mathcal{D}$ of the pair $\{A, W\}$ if and only if for $\alpha = 1/a$ the symmetric operator $H_{1/a}$ given by (3.1) is unbounded from above.*

Proof: Let $H_{1/a}$ be unbounded from above. Applying Theorem 2.3 to the operator $H_{1/a}$ we find $\left(-\frac{b}{a}\right)$-stability domain $\mathcal{D}_{a,b} \subseteq \mathcal{D}$ of $H_{1/a}$. This means that

$$-\frac{b}{a}(f, f) \leq (Af, f) + \frac{1}{a}(Wf, f), \qquad f \in \mathcal{D}_{a,b}, \tag{3.3}$$

which immediately yields that $\mathcal{D}_{a,b}$ is a stability domain of $\{A, W\}$. Moreover, by Theorem 2.3 the domain $\mathcal{D}_{a,b}$ can be chosen maximal for $H_{1/a}$ which implies that $\mathcal{D}_{a,b}$ is a maximal stability domain of $\{A, W\}$ for constants a, b. The converse is obvious. •

Consequently, constants a, b obeying $0 < a < 1$, $b \geq 0$ exist if and only if at least for one $\alpha > 1$ the operator H_α (3.1) is unbounded from above. However, this seems to be a natural assumption from the physical point of view.

To proceed further we have to recall briefly main results of [5, 6]. Assume that $\mathcal{D}$ itself is a stability domain of $\{A, W\}$, i.e. $\mathcal{D}_s = \mathcal{D}$, and that symmetric operators $A_0 \stackrel{\mathrm{def}}{=} A|\mathcal{D}$ and H_α,

$0 < \alpha \leq 1$, are *not* essentially self-adjoint on this stability domain . Hence the problem arises to find out the right Hamiltonian for the perturbed system which is expected to be a self-adjoint extension of H_α.

To obtain this extension one associates with H_α an approximating sequence of self-adjoint operators $\{\tilde{H}_{\alpha,n}\}_{n=1}^{\infty}$ of the form

$$\tilde{H}_{\alpha,n} f = \tilde{A} f + \alpha W_n f, \qquad f \in \mathrm{dom}(\tilde{H}_{\alpha,n}) = \mathrm{dom}(\tilde{A}), \tag{3.4}$$

where $\{W_n\}_{n=1}^{\infty}$ is a regularizing sequence of bounded operators, which converges in the strong resolvent sense to W, and $\tilde{A}$ is a self-adjoint extension of A_0. If the sequence $\{\tilde{H}_{\alpha,n}\}_{n=1}^{\infty}$ tends in the strong resolvent sense to some self-adjoint operator $\tilde{H}_\alpha$ and

$$\sup_n \|W_n f\| < \infty, \qquad f \in \mathcal{D}, \tag{3.5}$$

then indeed $\tilde{H}_\alpha$ is a self-adjoint extension of H_α and it is usually adopted as the right Hamiltonian of the perturbed system. An essential ingredient of this approach is a *maximality* of the Friedrichs extension $\hat{A}$ of A_0 with respect to the perturbation W. This means that if for some semi-bounded self-adjoint extension $\tilde{A}$ we have

$$\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\tilde{\nu}) \subseteq \mathrm{dom}(\sqrt{-W}), \tag{3.6}$$

where $\hat{\nu}$ and $\tilde{\nu}$ are the quadratic forms associated with $\hat{A}$ and $\tilde{A}$, respectively, then $\tilde{A}$ coincides with the Friedrichs extension $\hat{A}$. Note that by (3.2) one always has $\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\sqrt{-W})$.

Under assumptions that $\hat{A}$ is maximal with respect to the perturbation W, that the regularizing sequence $\{W_n\}_{n=1}^{\infty}$ satisfies the condition

$$\lim_{n\to\infty} \sqrt{|W_n| + I} f = \sqrt{-W + I} f, \qquad f \in \mathrm{dom}(\sqrt{-W}), \tag{3.7}$$

and that the approximating sequence $\{\tilde{H}_{\alpha,n}\}_{n=1}^{\infty}$ obeys

$$\sup_n \|\sqrt{|W_n|}(\tilde{H}_{\alpha,n} - z)^{-1}\sqrt{|W_n|}\| < \infty, \qquad \Im\mathrm{m}(z) \neq 0, \tag{3.8}$$

it follows that if convergence of the sequence (3.4) takes place in the strong resolvent sense to some semi-bounded self-adjoint operator, then the limit is an extension of H_α, which is canonical in the sense that it necessarily coincides with the form sum $\hat{H}_\alpha = \hat{A} \dotplus \alpha W$. Conversely, for each semi-bounded self-adjoint extension $\tilde{A}$ of A_0 there is a regularizing sequence $\{W_n\}_{n=1}^{\infty}$ such that the corresponding approximating sequence (3.4) tends to the form sum $\hat{H}_\alpha$ and the condition (3.8) is satisfied. So, either we have convergence to the unique "canonical" form sum $\hat{H}_\alpha$ or we have no convergence at all. That is why the case, when the Friedrichs extension is *maximal* with respect to the perturbation, was called in [6] the *well-posed regularization problem*. We have shown (see e.g. [3, 4]) that the well-posed regularization problem takes place if and only if

$$\mathrm{dom}(\sqrt{-W}) \cap \ker(A_0^* - \eta I) = \{0\} \tag{3.9}$$

for some $\eta < 0$.

The opposite case when $\mathrm{dom}(\sqrt{-W}) \cap \ker(A_0^* - \eta I) \neq \{0\}$ for some $\eta < 0$ and, hence for each $\eta < 0$, was called in [6] the *ill-posed regularization problem.* In this case for each semi-bounded self-adjoint extension $\tilde{A}$ of A_0 obeying $\mathrm{dom}(\tilde{\nu}) \subseteq \mathrm{dom}(\sqrt{-W})$ the approximating sequence (3.4) converges in the strong resolvent sense to the form sum $\tilde{H}_\alpha = \tilde{A} \dotplus \alpha W$ at least for sufficiently small coupling constants α. Moreover, the condition (3.8) is satisfied. Consequently, in the case of ill-posed problem we have as a disadvantage the lost of the *uniqueness* of the approximating procedure. Hence, there are a lot of candidates $\tilde{H}_\alpha$ for physical Hamiltonian of the perturbed system. In other words, the approximating procedure is not helpful to find out the right Hamiltonian since it gives a variety of possible Hamiltonians. To single out among this variety the right Hamiltonian one has to apply additional physical or mathematical arguments, for instance, to restrict the class of allowed semi-bounded self-adjoint extensions $\tilde{A}$ of A_0.

Let us show that the *ill-posed* case realizes if $\mathcal{D}$ itself is not a stability domain of the pair $\{A, W\}$.

Lemma 3.2 *Let $\nu \geq 0$ and $\gamma \geq 0$ be two closed quadratic forms such that* $\mathrm{dom}(\nu) \cap \mathrm{dom}(\gamma)$ *is dense. There is a dense domain $\mathcal{D} \subseteq$* $\mathrm{dom}(\nu) \cap \mathrm{dom}(\gamma)$ *and constants α, a, b, $0 < \alpha \leq 1$, $0 < a < 1$, $b \geq 0$, such that*

$$\alpha\gamma(f, f) \leq a\nu(f, f) + b(f, f), \qquad f \in \mathcal{D}, \tag{3.10}$$

if and only if there is a densely defined closed restriction $\hat{\nu}$ of ν such that $\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\gamma)$.

Proof: If (3.10) is satisfied, then one obviously has that the closure $\hat{\nu}$ of $\nu|\mathcal{D}$ obeys $\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\nu)$ and $\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\gamma)$. Conversely, let us assume that there is a closed restriction $\hat{\nu}$ of ν such that $\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\gamma)$. Let us introduce the scalar product

$$(f, g)_\nu = \nu(f, g) + (f, g), \qquad f, g \in \mathrm{dom}(\nu). \tag{3.11}$$

Since ν is a closed form the domain $\mathrm{dom}(\nu)$ endowed with the scalar product (3.11) forms a Hilbert space $\mathcal{H}_\nu \stackrel{\mathrm{def}}{=} \{\mathrm{dom}(\nu), (\cdot, \cdot)_\nu\}$. The set $\mathrm{dom}(\hat{\nu})$ is a closed subspace of $\mathcal{H}_\nu$. If $\mathrm{dom}(\gamma) \supseteq \mathrm{dom}(\hat{\nu})$, then $\mathrm{dom}(\gamma)$ contains the closed subspace of $\mathrm{dom}(\hat{\nu})$. Applying the closed graph principle (see e.g. [2], III §5.4) one gets a constant $C > 0$ such that

$$\gamma(f, f) \leq C(\hat{\nu}(f, f) + (f, f)), \qquad f \in \mathrm{dom}(\hat{\nu}). \tag{3.12}$$

Setting $\mathcal{D} = \mathrm{dom}(\hat{\nu})$ we find constants α, a, b such that (3.10) holds. •

Theorem 3.3 *Let $\{A \geq 0, W \leq 0\}$ be a pair of self-adjoint operators with common dense domain $\mathcal{D} \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W)$. If $\mathcal{D}$ is not a stability domain of $\{A, W\}$, then for any stability domain $\mathcal{D}_s \subseteq \mathcal{D}$ of $\{A, W\}$ the Friedrichs extension $\hat{A}$ of $A|\mathcal{D}_s$ is not maximal with respect to W.*

Proof: Let us denote closed quadratic forms associated with A and $-W$ and the Friedrichs extension $\hat{A}$ of $A_0 = A|\mathcal{D}_s$ by ν, γ and $\hat{\nu}$, respectively. Since $\mathcal{D}_s \subseteq \mathcal{D}$ is a stability domain of $\{A, W\}$ there are constants a, b, $0 < a < 1$, $b \geq 0$, such that (3.2) is satisfied. By Lemma 3.2 this yields that $\mathrm{dom}(\hat{\nu})$ is a closed subspace of $\mathcal{H}_\nu$ obeying $\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\gamma) = \mathrm{dom}(\sqrt{-W})$. Hence one gets $\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\nu) \cap \mathrm{dom}(\gamma)$.

Note that $\mathcal{D} \subseteq \mathrm{dom}(\hat{\nu})$ is impossible. Otherwise $\mathcal{D}$ would be a stability domain of $\{A, W\}$. Hence there is an element $f \in \mathcal{D}$ such that $f \notin \mathrm{dom}(\hat{\nu})$. Since $f \in \mathrm{dom}(\nu)$ the element f admits an orthogonal decomposition in $\mathcal{H}_\nu$ of the form

$$f = g + h, \qquad g \in \mathrm{dom}(\hat{\nu}), \qquad h \perp \mathrm{dom}(\hat{\nu}). \tag{3.13}$$

Since $f, g \in \mathrm{dom}(\gamma)$ one obviously has $h \in \mathrm{dom}(\gamma)$. Hence $h \in \mathrm{dom}(\nu) \cap \mathrm{dom}(\gamma)$ and $h \perp \mathrm{dom}(\hat{\nu})$ in $\mathcal{H}_\nu$, i.e.

$$\nu(h, f) + (h, f) = 0, \qquad \forall f \in \mathcal{D}_s. \tag{3.14}$$

Since $\mathcal{D}_s \subseteq \mathrm{dom}(A)$ one gets

$$(h, (A + I)f) = (h, (A_0 + I)f) = 0, \qquad \forall f \in \mathcal{D}_s. \tag{3.15}$$

Therefore $h \in \ker(A_0^* + I)$. Since $h \in \mathrm{dom}(\sqrt{-W})$ one gets $\mathrm{dom}(\sqrt{-W}) \cap \ker(A_0^* + I) \neq \{0\}$. Hence $\hat{A}$ is not maximal with respect to W. •

If $\mathcal{D}$ is not a stability domain by Theorem 3.1 there is always a smaller stability domain $\mathcal{D}_s \subseteq \mathcal{D}$ of $\{A, W\}$ provided H_α is unbounded from above at least for one $\alpha > 1$. Hence the situation of [5, 6] can be generated. However, this has the price that the *ill-posed* case inevitably emerges even if the stability domain $\mathcal{D}_s$ of $\{A, W\}$ is maximal for given constants a, b, $0 < a < 1$, $b \geq 0$. In particular this means that approximating method fails to define the right Hamiltonian.

Note that the $\mathcal{D}$ is not a stability domain of $\{A, W\}$ if and only if H_α is unbounded from below for any $\alpha > 1$. Hence the conclusions of Theorem 3.3 are meaningful if for any $\alpha > 1$ the operator H_α is unbounded from below ($\mathcal{D}$ is no stability domain) and at least for one $\alpha > 1$ the operator H_α is unbounded from above (existence of stability domain $\mathcal{D}_s \subseteq \mathcal{D}$).

4 Example

Let us illustrate Theorem 2.3 by Example 1.1 when $\kappa \in (0,1)$: "fall to the center problem". Since we fix $\kappa \in (0,1)$ in the following we omit the κ indices for simplicity, i.e., we set $H \stackrel{\text{def}}{=} H_{\frac{1}{\kappa}}$. As far as the operator H is the direct sum of two symmetric operators $H^{\pm}$ which corresponds to the positive and negative half axis, i.e.

$$(H^{\pm}f)(x) = -\frac{d^2}{dx^2}f(x) - \frac{1}{4\kappa}\frac{1}{x^2}f(x), \qquad f \in \mathrm{dom}(H^{\pm}) = C_0^{\infty}(\mathbf{R}^1_{\pm}), \quad x \in \mathbf{R}^1_{\pm}, \tag{4.1}$$

we can concentrate considerations only on H^+. Since H^+ is unbounded from above, by Theorem 2.3 one gets that for each $b \in \mathbf{R}^1$ there is a maximal b-stability domain $\mathcal{D}_b \subseteq C_0^{\infty}(\mathbf{R}^1_+)$ for H^+, cf. Theorem 3.3. Below we describe this domain in explicit form.

Let us introduce the auxiliary operator

$$\begin{aligned}(H(a)f)(x) &= -\frac{d^2}{dx^2}f(x) - \frac{1}{4\kappa}\frac{1}{(x+a)^2}f(x)\,, \\ \mathrm{dom}(H(a)) &= \{f \in W_2^2([0,\infty)) : f(0) = 0\}, \qquad a > 0.\end{aligned} \tag{4.2}$$

The operator $H(a)$ is self-adjoint and semi-bounded from below for each $a > 0$. Let

$$\lambda(a) \stackrel{\text{def}}{=} \inf \sigma(H(a)), \quad a > 0, \tag{4.3}$$

where by $\sigma(X)$ we denote the spectrum of an operator X. Then the function $\lambda(\cdot)$ is continuous. Since the family of operators $\{H(a)\}_{a>0}$ is non-decreasing, one gets that $\lambda(\cdot)$ is non-decreasing too. A straightforward computation shows that $\lim_{a\to\infty}\lambda(a) = 0$. Since H^+ is unbounded from below one has that $\inf_{a>0}\lambda(a) = -\infty$.

Let $b < 0$. Then we set

$$\mathcal{N}(b) \stackrel{\text{def}}{=} \{a > 0 : \lambda(a) \geq b\}, \quad b < 0. \tag{4.4}$$

Let $a_0 \stackrel{\text{def}}{=} \inf \mathcal{N}(b)$. Then by continuity of $\lambda(\cdot)$ one gets $\lambda(a_0) = b$. Furthermore, $a < a_0$ yields $\lambda(a) < b$.

Further let us introduce the operator

$$\begin{aligned}(K(r)f)(x) &= -\frac{d^2}{dx^2}f(x) - \frac{1}{4\kappa}\frac{1}{(x+r)^2}f(x)\,, \\ \mathrm{dom}(K(r)) &= \{f \in W_2^2([0,1]) : f(0) = f(1) = 0\}, \qquad r > 0.\end{aligned} \tag{4.5}$$

and

$$\mu(r) \stackrel{\text{def}}{=} \inf \sigma(K(r)). \tag{4.6}$$

First of all we note that $\mu(r)$ is continuous and non-decreasing function of $r > 0$ and one has

$$\lim_{r\to\infty} \mu(r) = \inf \sigma(K(\infty)), \tag{4.7}$$

where

$$\begin{aligned}(K(\infty)f)(x) &= -\frac{d^2}{dx^2} f(x), \\ \mathrm{dom}(K(\infty)) &= \{f \in W_2^2([0,1]) : f(0) = f(1) = 0\}.\end{aligned} \tag{4.8}$$

Since $\inf \sigma(K(\infty)) = \pi^2$ one gets

$$\lim_{r\to\infty} \mu(r) = \pi^2 > 0. \tag{4.9}$$

As above one can prove that $\inf_{r>0} \mu(r) = -\infty$.

Next we introduce the set

$$\mathcal{M}(a) = \left\{ r \in (0,\infty) : \mu(r) - \frac{a^2 b}{(1+r)^2} \geq 0 \right\}, \quad a > 0. \tag{4.10}$$

Since $\lim_{r\to 0} \frac{a^2b}{(1+r)^2} = a^2 b < 0$ and $\lim_{r\to\infty} \frac{a^2b}{(1+r)^2} = 0$ the set $\mathcal{M}(a)$ is always not empty.

Let a_0 be chosen as above. For $a = a_0$ we set

$$r_1 \stackrel{\mathrm{def}}{=} \inf \mathcal{M}(a_0). \tag{4.11}$$

By the continuity of $\mu(\cdot)$ one always has

$$\mu(r_1) = \frac{a_0^2 b}{(1+r_1)^2}. \tag{4.12}$$

Let

$$a_1 = a_0 \frac{r_1}{1+r_1} < a_0. \tag{4.13}$$

Now we consider the set $\mathcal{M}(a_1)$. Then we define $r_2 \stackrel{\mathrm{def}}{=} \inf \mathcal{M}(a_1)$ and set

$$a_2 = a_1 \frac{r_2}{1+r_2} < a_1 < a_0. \tag{4.14}$$

Moreover, one has

$$a_2 = a_0 \frac{r_2}{1+r_2} \frac{r_1}{1+r_1}. \tag{4.15}$$

By this manner we work out the sequence of numbers $\{r_n\}_{n=1}^\infty$ which defines the sequence $\{a_n\}_{n=1}^\infty$, where

$$a_n = a_0 \prod_{k=1}^n \frac{r_k}{1+r_k}. \tag{4.16}$$

Let us show that the sequence $\{a_n\}_{n=0}^{\infty}$ tends to zero as $n \longrightarrow \infty$. To this end we note that $r_j \le r_0$, $j = 1, 2, \ldots$, where $r_0 \stackrel{\text{def}}{=} \inf \mathcal{M}(0)$. Since

$$\frac{r_j}{1+r_j} \le \frac{r_0}{1+r_0} < 1, \qquad j = 1, 2, \ldots, \tag{4.17}$$

one gets

$$a_n = a_0 \prod_{k=1}^{n} \frac{r_k}{1+r_k} \le \left(\frac{r_0}{1+r_0}\right)^n a_0, \qquad n = 1, 2, \ldots, \tag{4.18}$$

which immediately implies $\lim_{n\to\infty} a_n = 0$.

We set

$$\Gamma = \bigcup_{n=0}^{\infty} \{a_n\}. \tag{4.19}$$

and define domain $\mathcal{D}_b = \{f \in C_0^\infty(\mathbf{R}_+^1) : f(a_n) = 0, \quad a_n \in \Gamma\}$ which is obviously dense in $L^2(\mathbf{R}_+^1)$.

Let us show first that $\mathcal{D}_b$ is a b-stability domain for H^+. To prove this one has to verify that the operator $H_b^+ = H^+|\mathcal{D}_b$ satisfies

$$H_b^+ \ge bI. \tag{4.20}$$

Let $f \in C_0^\infty(\mathbf{R}_+^1)$ and $\operatorname{supp}(f) \subseteq (a_0, \infty)$. Then we have to prove that

$$\int_{a_0}^{\infty} |f'(x)|^2 dx - \frac{1}{4\kappa}\int_{a_0}^{\infty} \frac{1}{x^2}|f(x)|^2 dx \ge b\int_{a_0}^{\infty} |f(x)|^2 dx. \tag{4.21}$$

Setting $x = t + a_0$ and $g(t) \stackrel{\text{def}}{=} f(t + a_0)$ this is equivalent to verify

$$\int_0^\infty |g'(t)|^2 dt - \frac{1}{4\kappa}\int_0^\infty \frac{1}{(t+a_0)^2}|g(t)|^2 dt \ge b\int_0^\infty |g(t)|^2 dt, \qquad g \in C_0^\infty(\mathbf{R}_+^1). \tag{4.22}$$

However, this is equivalent to $H_{a_0} \ge bI$ which is right due to (4.4).

Let $f \in C_0^\infty(\mathbf{R}_+^1)$ and $\operatorname{supp}(f) \subseteq (a_1, a_0)$. Then as above one has to prove that

$$\int_{a_1}^{a_0} |f'(x)|^2 dx - \frac{1}{4\kappa}\int_{a_1}^{a_0} \frac{1}{x^2}|f(x)|^2 dx \ge b\int_{a_1}^{a_0} |f(x)|^2 dx. \tag{4.23}$$

Using the transformation $x = \frac{a_1}{r_1}(y + r_1)$, where $r_1 = \frac{a_1}{a_0 - a_1}$, we obtain that (4.23) is equivalent to

$$\int_0^1 |h'(y)|^2 dy - \frac{1}{4\kappa}\int_0^1 \frac{1}{(y+r_1)^2}|h(y)|^2 dy \ge \frac{a_1^2 b}{r_1^2}\int_0^1 |h(y)|^2 dy, \quad h \in C_0^\infty([0,1]), \tag{4.24}$$

where $h(y) \stackrel{\text{def}}{=} f(\frac{a_1}{r_1}(y + r_1))$. Since a_0, a_1 and r_1 are related by (4.13) we finally get

$$\int_0^1 |h'(y)|^2 dy - \frac{1}{4\kappa}\int_0^1 \frac{1}{(y+r_1)^2}|h(y)|^2 dy \ge \frac{a_0^2 b}{(1+r_1)^2}\int_0^1 |h(y)|^2 dy, \quad h \in C_0^\infty([0,1]). \tag{4.25}$$

However, (4.25) is equivalent to

$$K(r_1) \geq \frac{a_0^2 b}{(1+r_1)^2} I, \tag{4.26}$$

which immediately follows from definition of r_1 (4.11).

Similarly one gets that

$$\int_{a_n}^{a_{n-1}} |f'(x)|^2 dx - \frac{1}{4\kappa} \int_{a_n}^{a_{n-1}} \frac{1}{x^2} |f(x)|^2 dx \geq b \int_{a_n}^{a_{n-1}} |f(x)|^2 dx, \tag{4.27}$$

for $f \in C_0^\infty(\mathbf{R}_+^1)$, $\mathrm{supp}(f) \subseteq (a_n, a_{n-1})$ and $n = 2, 3, \ldots$, is equivalent to

$$K(r_n) \geq \frac{a_{n-1}^2 b}{(1+r_n)^2} I, \qquad n = 2, 3, \ldots. \tag{4.28}$$

So, in fact the restricted operator H_b^+ is semi-bounded by b and, hence, $\mathcal{D}_b$ is a dense b-stability domain.

Next let us prove that the domain $\mathcal{D}_b$ is maximal. To this end we assume that $\mathcal{D}_b$ is not maximal. Therefore, it exists at least one function $f_0 \in C_0^\infty(\mathbf{R}_+^1)$ such that $f_0 \notin \mathcal{D}_b$ but

$$(H^+ f_0, f_0) \geq b(f_0, f_0) \tag{4.29}$$

and $f_0(a_p) \neq 0$ at least for one $a_p \in \Gamma$. Otherwise $f_0 \in \mathcal{D}_b$. We set

$$\Gamma_0 = \{a_n \in \Gamma : f_0(a_n) = 0\} \subseteq \Gamma. \tag{4.30}$$

Since f_0 belongs to a larger domain $\mathcal{D}' \supseteq \mathcal{D}_b$, by linearity one gets that

$$\mathcal{D}' \supseteq \{f \in C_0^\infty(\mathbf{R}_+^1) : f(a_n) = 0, \quad a_n \in \Gamma_0\}. \tag{4.31}$$

Let $a_0 \notin \Gamma_0$, i.e. $f_0(a_0) \neq 0$. Then $a_l = \sup \Gamma_0 < a_0$, i.e. $l > 0$. Since $\mathcal{D}'$ is a b-stability domain of H^+ this yields that $a_l \in \mathcal{N}(b)$. Therefore, by definition of a_0 one gets $a_l \geq a_0$ which is impossible. Consequently, $a_0 \in \Gamma_0$, i.e. $f_0(a_0) = 0$. Assume now that $a_1 \notin \Gamma_0$. Then $a_l = \sup\{\Gamma_0 \setminus \{a_0\}\} < a_1$, i.e. $l > 1$. Since $\mathcal{D}'$ is a b-stability domain one gets (see (4.23)) that $r_1' = \frac{a_l}{a_0 - a_l} \in \mathcal{M}(a_0)$. Hence $r_1' < r_1$ which contradicts to (4.11). Therefore, $a_1 \in \Gamma_0$, i.e. $f(a_1) = 0$. Proceeding in this way further one verifies that $f_0(a_m) = 0$ for each $m = 0, 1, 2, \ldots$. Hence $f_0 \in \mathcal{D}_b$ or $\mathcal{D}' = \mathcal{D}_b$.

The proof of the maximality shows that $\mathcal{D}_b$ is not unique as a maximal domain. To check this we choose a number $a = a_0' > a_0$ (see (4.4)). After that we find the numbers $a_1', a_2', \ldots$ following the formula (4.16) as above. Assume that $a_1' < a_0 < a_0'$ which can be arranged by the choice of a_0'. Then we set

$$\mathcal{D}_b' = \{f \in C_0^\infty(\mathbf{R}_+^1) : f(a_m') = 0, \quad m = 0, 1, 2, \ldots\}. \tag{4.32}$$

Repeating the above line of reasoning for $a = a_0'$ we obtain that the domain $\mathcal{D}_b'$ is also maximal but $\mathcal{D}_b' \neq \mathcal{D}_b$.

Note that by construction the symmetric b-stable restriction (4.20) has (as one should to anticipate) infinite deficiency indices $\{\infty, \infty\}$ due to the infinite set Γ.

Furthermore, the Example 1.1 allows to describe explicitly the Friedrichs extension $\hat{A}^+$ of $A_0^+ = A|\mathcal{D}_{b,m}$. Denoting by Δ_n the intervals $\Delta_0 = [a_0, \infty)$ and $\Delta_n = [a_n, a_{n-1}]$, $n = 1, 2, \ldots$, one can decompose the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^1_+)$ into

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \qquad \mathcal{H}_n = L^2(\Delta_n). \tag{4.33}$$

Denoting by $\hat{L}_n$, $n = 0, 1, 2, \ldots$, the operators

$$(\hat{L}_0 f)(x) = -\frac{d^2}{dx^2} f(x), \qquad f \in \mathrm{dom}(\hat{L}_0) = \{f \in W_2^2(\Delta_0) : f(a_0) = 0\} \tag{4.34}$$

and

$$(\hat{L}_n f)(x) = -\frac{d^2}{dx^2} f(x), \qquad f \in \mathrm{dom}(\hat{L}_n) = \{\mathrm{dom}(W_2^2(\Delta_n) : f(a_n) = f(a_{n-1}) = 0\}, \tag{4.35}$$

$n = 1, 2, \ldots$, one gets

$$\hat{A}^+ = \bigoplus_{n=0}^{\infty} \hat{L}_n. \tag{4.36}$$

To illustrate Theorem 3.3 we show that the Friedrichs extension $\hat{A}^+$ is not maximal with respect to W. The element h belongs to the deficiency subspace $\mathcal{N}_\eta = \ker(A_0^{+*} - \eta I)$ if and only if h is continuous and admits the representation

$$h(x) = b_0 e^{-\sqrt{-\eta} x}, \qquad x \in \Delta_0, \quad b_0 \in \mathbf{C}, \tag{4.37}$$

and

$$h(x) = b_n e^{-\sqrt{-\eta} x} + c_n e^{\sqrt{-\eta} x}, \qquad , x \in \Delta_n, \quad b_n \in \mathbf{C}, \quad n = 1, 2, \ldots. \tag{4.38}$$

The continuity of h imposes some relations between the coefficients b_n and c_n, $n = 0, 1, 2, \ldots$. Obviously there are elements $h \in \mathcal{N}_\eta$ such that $h(x) = 0$ for $x \in \Delta_n$ for n greater some natural number $N > 1$. For instance, if $N = 2$ the function

$$h(x) = \begin{cases} b_0 e^{-\sqrt{-\eta} x} & x \in \Delta_0 \\ b_0 e^{-\sqrt{-\eta} a_0} \frac{\sinh(\sqrt{-\eta}(x - a_1))}{\sinh(\sqrt{-\eta}(a_0 - a_1))} & x \in \Delta_1 \\ 0 & x \in \Delta_n, \quad n = 2, 3, \ldots \end{cases} \tag{4.39}$$

belongs to $\mathcal{N}_\eta$ and has the property that

$$h \in \mathrm{dom}(\sqrt{-W}), \tag{4.40}$$

which means that

$$\int_0^\infty \frac{1}{x^2}|h(x)|^2 dx = \int_{a_1}^\infty \frac{1}{x^2}|h(x)|^2 dx < +\infty. \tag{4.41}$$

On the other hand, the element $h(x) = be^{-\sqrt{-\eta}x}$, $x \in \mathbf{R}^1_+$, also belongs to $\mathcal{N}_\eta$, but $h \notin \mathrm{dom}(\sqrt{-W})$. Therefore, $\mathrm{dom}(\sqrt{-W}) \cap \ker(A_0^{+*} - \eta I) \neq \{0\}$, i.e., the regularization problem is ill-posed.

Note that by construction (4.36) of $\hat{A}^+$ the spectrum of form sum $H^+ = A^+ \dot{+} \frac{1}{\kappa}W$, $0 < \kappa < 1$, for $b < 0$ has besides the discrete part an absolutely continuous part which results from the infinite interval Δ_0. The situation changes, if the b is chosen non-negative, i.e. $b \geq 0$.

Since the set $\mathcal{N}(b)$ (4.4) is empty for $b \geq 0$ (see e.g. [1]), we proceed as follows. First we choose an arbitrary $a = a_0 > 0$. Then as above we define with this a_0 the sequence $\{a_n\}_{n=1}^\infty$ and we can prove that $\lim_{n\to\infty} a_n = 0$. After that we introduce the set

$$\mathcal{M}^-(a) \stackrel{\text{def}}{=} \{r \in (0,\infty) : \mu(r) - \frac{a^2 b}{r^2} \geq 0\}, \qquad a > 0. \tag{4.42}$$

Since $\lim_{r\to 0} \frac{a^2 b}{r^2} = \infty$ (or zero if $b = 0$) and $\lim_{r\to\infty} \frac{a^2 b}{r^2} = 0$ and (4.9) the set $\mathcal{M}^-(a)$ is always not empty. Furthermore, one has $\inf \mathcal{M}^-(a) > 0$. So we can define

$$r_{-1} = \inf \mathcal{M}^-(a_0) \tag{4.43}$$

and

$$a_{-1} = a_0 \frac{1 + r_{-1}}{r_{-1}} > a_0. \tag{4.44}$$

Then we pass to $r_{-2} = \inf \mathcal{M}^-(a_1)$ and

$$a_{-2} = a_1 \frac{1 + r_{-2}}{r_{-2}} = a_0 \frac{1 + r_{-2}}{r_{-2}} \frac{1 + r_{-1}}{r_{-1}} > a_{-1} > a_0. \tag{4.45}$$

Finally, we obtain a sequence $\{r_{-n}\}_{n=1}^\infty$ which defines the sequence $\{a_{-n}\}_{n=1}^\infty$, where

$$a_{-n} = a_0 \prod_{k=1}^n \frac{1 + r_{-k}}{r_{-k}}. \tag{4.46}$$

Let us show that $\lim_{n\to\infty} a_{-n} = \infty$. By monotonicity $\lim_{n\to\infty} a_{-n} = a_{-\infty}$ exist. Assume that $a_{-\infty} < +\infty$. We set $r_{-\infty} = \inf \mathcal{M}^-(a_{-\infty})$. Obviously we have $r_{-j} \leq r_{-\infty}$ which yields

$$1 < \frac{1 + r_{-\infty}}{r_{-\infty}} \leq \frac{1 + r_{-j}}{r_{-j}}, \qquad j = 1, 2, \dots . \tag{4.47}$$

Since

$$a_{-n} = a_0 \prod_{k=1}^n \frac{1 + r_{-k}}{r_{-k}} \geq a_0 \left(\frac{1 + r_{-\infty}}{r_{-\infty}}\right)^n, \qquad n = 1, 2, \dots, \tag{4.48}$$

one gets $\lim_{n\to\infty} a_{-n} = a_{-\infty} = \infty$. Therefore, the sequence $\{a_{-n}\}_{n=0}^{\infty}$ tends to infinity as $n \longrightarrow \infty$.

Setting

$$\Gamma^- = \bigcup_{n=-\infty}^{\infty} \{a_n\} \tag{4.49}$$

one defines as above $\mathcal{D}_b = \{f \in C_0^\infty(\mathbf{R}_+^1) : f(a_n) = 0, \quad a_n \in \Gamma^-\}$. Similarly to the case $b < 0$ one can show that $\mathcal{D}_b$ is a maximal b-stability domain for H^+. Moreover, starting with $a = a_0' \neq a_0$ one can construct another maximal b-stability domain $\mathcal{D}_b'$ for $b \geq 0$.

In contrast to the case $b < 0$, when $\lim_{n\to\infty} a_n = 0$, it turns out that for $b \geq 0$, when $\lim_{n\to\infty} a_{-n} = +\infty$, the spectrum of the form sum $H^+ = \hat{A} \dotplus \frac{1}{\kappa}W$, $0 < \kappa < 1$, is purely point one, i.e., absolutely continuous spectrum is absent.

5 Conclusions

This paper is initiated by a general problem of definition and construction of the right Hamiltonian for non-positive singular perturbations, see [3, 4, 5, 6]. The theory developed there for a pair of self-adjoint operators $\{A \leq 0, W \geq 0\}$ includes as an essential ingredient a stability domain of the pair $\{A, W\}$, i.e., a dense domain $\mathcal{D}_s \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W)$ such that (3.2) is satisfied.

A priori the existence of $\mathcal{D}_s$ is not evident. Our abstract Theorem 3.1 gives a simple criterion for that: symmetric operator $H_\alpha = A + \alpha W$ with domain $\mathrm{dom}(H) = \mathrm{dom}(A) \cap \mathrm{dom}(W)$ should be unbounded from above at least for one $\alpha > 1$ (Theorem 3.3). From the physical point of view this is a very reasonable condition which is, of course, very often satisfied in physical systems. Naturally this leads to a simple idea to handle non-positive singular perturbations as follows: we choose a stability domain of the pair $\{A, W\}$ and a regularizing sequence for the non-positive singular perturbation W, with this we can construct the approximating Hamiltonian sequence and determine the limit of this sequence which we call the "right Hamiltonian" of the perturbed system.

However, this simple idea is false. The main reason for that is that in general the "right Hamiltonian" is *not* uniquely determined in this way. In order to guarantee uniqueness of the "right Hamiltonian" one has to verify *maximality* of Friedrichs extension $\hat{A}$ of $A|\mathcal{D}_s$ with respect to the singular perturbation W [5, 6]. If one has maximality, then the "right Hamiltonian" for $\{A, W\}$ coincides with the form sum of Friedrichs extension $\hat{A}$ and the perturbation W. Since this case is very satisfactory it was called "well-posed regularization problem". In the opposite case, i.e., if there is no maximality, one gets a variety of "right Hamiltonians". So, the pleasant and important from the physical point of view uniqueness is lost. Consequently, the regularization problem was called *ill-posed*. In [6] it was mentioned that in some sense an ill-posed regularization problem is an incomplete well-posed one and that it can be completed to latter but not in a unique way. Consequently, in order to guarantee the uniqueness of the "right Hamiltonian" it is necessary to introduce besides the regularization method some supplementary physical principles to select it.

Unfortunately the ill-posed case always realizes if the set $\mathrm{dom}(A) \cap \mathrm{dom}(W)$ is not a stability domain of $\{A, W\}$. This means, that the Theorem 2.3, which guarantees the existence of a stability domain $\mathcal{D}_s \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W)$ of $\{A, W\}$, does not save the situation: the ill-posed case necessarily appears due to Theorem 3.3. Hence the regularization method of [3, 4, 6, 5] is unsuitable to find out the "right Hamiltonian" if we have no canonical stability domain from the beginning. In other words, the regularization problem is well-posed for the pair $\{A, W\}$ only if $\mathrm{dom}(A) \cap \mathrm{dom}(W)$ is a stability domain of $\{A, W\}$ and, moreover, the Friedrichs extension $\hat{A}$ of $A|\mathrm{dom}(A) \cap \mathrm{dom}(W)$ is maximal with respect to the singular perturbation W. This is the physical quintessence of the Section 3.

Besides this we hope that Theorem 2.3 has an independent meaning since it gives a criterion when a symmetric operator can be restricted to semi-bounded one. In particular it implies that well-known momentum operator on the real axis can be restricted to a non-negative one.

Finally, in Section 4 we present a solution of the quantum mechanics "fall to the center" problem for the Hamiltonian $H_{\frac{1}{\kappa}}$, $\kappa \in (0,1)$, of Example 1.1. Theorem 2.3 suggests the idea that $H_{\frac{1}{\kappa}}$ is defined on a domain which is too large. Hence one has to restrict $H_{\frac{1}{\kappa}}$ in a suitable manner such that it becomes semi-bounded. In Section 4 it is shown how this can be done. From the construction of the stability domain it follows that the physical interpretation of the way to prevent the fall to the center is to introduce a suitable sequence of point barriers of infinite height which come closer and closer to the center. How to choose the sequence of point barriers is indicated in by Γ (4.19) and Γ^- (4.49). However, it is easy to see that there are several possibilities to do this. Finally, to illustrate our "no-go" Theorem 3.3 we show that the Friedrichs extension $\hat{A}^+$ is not maximal with respect to the perturbation W.

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