# Localization for One Dimensional Long Range Random Hamiltonians

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**Abstract**

We study spectral properties of random Schrödinger operators $h_\omega = h_0 + v_\omega(n)$ on $l^2(Z)$ whose free part $h_0$ is long range. We prove that the spectrum of $h_\omega$ is pure point for typical $\omega$ if the random variables $v_\omega(n)$ have sufficently long tails and if off-diagonal terms of $h_0$ decay as $|i-j|^{-\gamma}$ for some $\gamma > 8$.

# 1 Introduction

In this paper we study spectral properties of random Schrödinger operators

$$h_\omega = h_0 + v_\omega(n) \tag{1.1}$$

on $l^2(Z)$ where $h_0$ is not the usual free Hamiltonian but only a bounded self-adjoint operator with some off-diagonal decay. Throughout, we assume that $v_\omega(n)$ are independent and identically distributed random variables with density $p(x)$. We denote by $\mathcal{V}$ the support of $p$, i.e. the closure of the set $\{x : p(x) \neq 0\}$. We further assume that $h_0$ is translation invariant, namely that there is a function $j : Z \mapsto C$, $j(0) = 0$, such that

$$(\delta_m, h_0\delta_n) = j(n-m), \tag{1.2}$$

for all $n, m \in Z$. We are interested in the case where

$$|j(n)| \leq C(1+|n|)^{-\gamma} \tag{1.3}$$

for some constants $\gamma > 1$ and $C > 0$. Let

$$\hat{j}(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_n j(n) e^{in\varphi}, \qquad \varphi \in [-\pi, \pi].$$

If (1.3) holds, then

$$\sigma(h_0) = \sigma_{ac}(h_0) = [\alpha, \beta],$$

where $\alpha = \min \hat{j}(\varphi)$, $\beta = \max \hat{j}(\varphi)$. The standard argument yields that for almost all $\omega$

$$\sigma(h_\omega) = [\alpha, \beta] + \mathcal{V}.$$

We are interested in the structure of $\sigma(h_\omega)$. It is known that in the large coupling or high energy regime, the spectrum of $h_\omega$ is pure point for typical $\omega$ [AM]. This result holds for the $d$-dimensional analog of (1.1) if $\gamma > d$. In one dimension and under suitable technical conditions, however, one expects localization for all energies. To the best of our knowledge, the only known result in this direction is proven in [SS]: if $\gamma > 4$, then under general conditions $\sigma_{ac}(h_\omega) = \emptyset$ for typical $\omega$. Here we show that under somewhat stronger conditions, the spectrum of $h_\omega$ is pure point for typical $\omega$. In the sequel $|A|$ denotes the Lebesgue measure of the set $A \subset R$. Let

$$\begin{array}{rcl} \nu_0 & \equiv & \sup_{x\in R} |p(x)|, \\ j_0 & \equiv & \sum |j(n)|, \\ \eta_0 & \equiv & \nu_0 - j_0. \end{array} \tag{1.4}$$

If $\eta_0 > 0$, we set $I_\nu \equiv [-\eta_0, \eta_0]$. Note that if $|\mathcal{V}| = \infty$ then $I_\nu = R$. Our first result is:

**Theorem 1.1** *Assume that $\gamma > 8$. If $\eta_0 > 0$, then for a.e. $\omega$*

$$\sigma(h_\omega) \cap I_\nu = \sigma_{pp}(h_\omega).$$

*In particular, if $|\mathcal{V}| = \infty$ then $\sigma(h_\omega) = \sigma_{pp}(h_\omega)$ for a.e. $\omega$.*

**Remark.** We expect that a similar result holds for $\gamma > 2$. For technical reasons, however, we need $\gamma > 8$.

Theorem 1.1 is expected for the following well-known heuristic reason: A quantum particle with energy in the band $I_\nu$ has to travel through an infinite sequence of long and high potential bariers to reach infinity. These barriers are effective in preventing tunneling, and they keep the particle localized within a finite region of space. This intuition is formalized in the following theorem which is our principal technical result.

**Theorem 1.2** *Let $h_0$ be given by (1.2) and let $\mu, \rho > 0$. Assume that (1.3) holds for some $\gamma > 8$. Let $v$ be a potential and $a \geq 2$ an integer. Assume that there exists an integer $N \geq 0$ such that, $\forall n > 0$, the intervals*

$$\pm\, [a^{N+n} + 1, a^{N+n+1} - 1] \tag{1.5}$$

*contain sub-intervals $I_{\pm n}$ of length $|I_{\pm n}| \geq n$ within which*

$$|v(k)| \geq j_0 + \mu + \rho, \qquad k \in I_{\pm n}. \tag{1.6}$$

*Then, for a.e. $E \in (-\mu, \mu)$ with respect to the Lebesgue measure and for all $n \in Z$,*

$$\limsup_{\zeta \downarrow 0} \sum_{m \in Z} \left| (\delta_n, (h - E - i\zeta)^{-1} \delta_m) \right|^2 < \infty, \tag{1.7}$$

*where $h = h_0 + v$.*

**Remark 1.** The proof of Theorem 1.2 is based on a suitable cluster expansion of the resolvent $(h - z)^{-1}$. The basic set of ideas concerning this type of expansion is developed in [KMP], [M], [M1] and [GJMS].

**Remark 2.** An immediate consequence of (1.7) is that $\sigma_{ac}(h) \cap (-\mu, \mu) = \emptyset$.

**Remark 3.** Clearly, Theorem 1.2 implies that (1.7) holds for a.e. $E \in (-\mu - \rho, \mu + \rho)$, and without loss of generality we could set $\rho = 0$ in (1.6). It is, however, notationally convenient to prove Theorem 1.2 with $\rho > 0$ in (1.6).

**Remark 4.** Theorem 1.1 follows directly from Theorem 1.2, the Simon-Wolff theorem, and a simple probabilistic argument which we present in Section 4.

**Remark 5.** The results proven here will be used in [JMP1] to study the propagation properties of surface waves in regions with corrugated boundaries in dimension $d = 2$. For additional information on the theory of surface waves and its relation to spectral theory of long range Hamiltonians, we refer the reader to [JMP], [G] and [JL].

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## 2 Preliminaries

In this section we collect a few technical results which we will use in the sequel.

A *path* $\tau$ connecting $n$ and $m$ is any sequence of sites $\tau = (i_0, i_1, \dots i_k)$ such that $i_0 = n$, $i_k = m$. The *length* of this path is $|\tau| = k$. To the path $\tau$ we associate a sequence of bonds $\tau_b = (b_1, \dots, b_k)$, where

$$b_1 = (i_0, i_1),\ b_1 = (i_1, i_2), \dots, b_k = (i_{k-1}, i_k).$$

We write $s \in \tau$ if $s$ is one of the sites of the path $\tau$, and $b = (s,t) \in \tau_b$ if $b$ is one of its bonds. We use the shorthand $j(b) = j(s-t)$. Let $h = h_0 + v$, where $v$ is arbitrary potential, and

$$\begin{aligned} R(z) &= (h-z)^{-1}, \\ R(n,m;z) &= (\delta_n, R(z)\delta_m). \end{aligned}$$

Recall that $j_0$ is given by (1.4). Let $\delta_{nm}$ be the usual Kronecker symbol.

**Proposition 2.1** *If $Im(z) > j_0$ then*

$$R(n,m;z) = -\frac{\delta_{nm}}{z-v(n)} - \sum_{\tau} \left[\prod_{s\in\tau} \frac{1}{z-v(s)}\right] \cdot \left[\prod_{b\in\tau_b} j(b)\right], \tag{2.8}$$

*where the sum is over all paths connecting $n$ and $m$. For each $\varepsilon > 0$ the series converges uniformly in the half-plane $Im(z) \geq j_0 + \varepsilon$.*

*Proof:* We split the set of paths connecting $n$ and $m$ into the disjoint classes $\mathcal{T}_k$ such that $\tau \in \mathcal{T}_k$ iff $|\tau| = k$. If $\tau \in \mathcal{T}_k$ then

$$\prod_{b\in\tau_b} |j(b)| \leq \left(\sum_n |j(n)|\right)^k = j_0^k,$$

and if $Im(z) > j_0$,

$$\sum_{k>0} \sum_{\tau\in\mathcal{T}_k} \left[\prod_{s\in\tau} \left|(z-v(s))^{-1}\right|\right] \cdot \left[\prod_{b\in\tau_b} |j(b)|\right] \leq \sum_{k>0} \frac{j_0^k}{|Im(z)|^{k+1}} < \infty.$$

Thus, the series (2.8) converges uniformly in the half-plane $Im(z) \geq j_0 + \varepsilon$ for any $\varepsilon > 0$. Since

$$(h_0 + v)R(z) = I + zR(z),$$

we get

$$(h_0\delta_n, R(z)\delta_m) = \delta_{nm} + (z - v(n))R(n, m; z).$$

Expanding $h_0\delta_n$ in the basis $\{\delta_i\}$ we get

$$R(n, m; z) = -\frac{\delta_{nm}}{z - v(n)} + \sum_i \frac{j(n-i)}{z - v(n)} R(i, m; z).$$

Iterating this formula we derive Relation (2.8). □

Proposition 2.1 is known as the path expansion of the resolvent. A similar result holds if the system is restricted to a box. Let $L = [a, b] \subset Z$ be an interval, and let $h_0^L$ be restriction of $h_0$ to $L$ with Dirichlet boundary condition. We recall that $h_0^L$ acts on $l^2(L)$ according to the formula

$$(h_0^L\psi)(n) = \sum_{m \in L} j(n-m)\psi(m), \qquad n \in L.$$

Let $R_L = (h_0^L + v - z)^{-1}$. Then for $n, m \in L$,

$$R_L(n, m; z) = -\frac{\delta_{nm}}{z - v(n)} - \sum_\tau \left[\prod_{s\in\tau} \frac{1}{z - v(s)}\right] \cdot \left[\prod_{b \in \tau_b} j(b)\right], \tag{2.9}$$

where the sum is over all paths which connect $n$ and $m$ and belong to $L$. If $n$ or $m \notin L$, we set

$$R_L(n, m; z) = 0. \tag{2.10}$$

The final ingredients we need are the following two versions of the well-known Kolmogorov inequality. For the proofs we refer the reader to [M1], [GJMS]. The history of the Kolomogorov inequality is discussed in [A]. Recall that $|A|$ stands for the Lebesgue measure of the set $A$.

**Proposition 2.2** *Let $\lambda_1, \ldots, \lambda_n$ be fixed real numbers, and $h_1, \ldots, h_n$ fixed complex numbers. Let $h : R \mapsto C$ be the function defined by*

$$h(\lambda) = \sum_{k=1}^n \frac{h_k}{\lambda - \lambda_k}, \qquad \lambda \in R.$$

*Assume that $\sum |h_k| \leq 1$. Then for any $M > 0$,*

$$|\{\lambda : |h(\lambda)| > 1/M\}| \leq 1/M,$$

$$|\{\lambda : |h'(\lambda)| > 1/M\}| \leq 4(n/M)^{1/2}.$$

# 3 Proof of Theorem 1.2

We first introduce several sequences of intervals which will play an important role in the sequel. Let the $I_n$'s be as in Theorem 1.2. We denote by $a_n$ and $b_n$ the end-points of the intervals $I_n$:

$$I_n \equiv [a_n, b_n], \qquad |n| > 0.$$

Let $M_0 = [a_{-1}, b_1]$. For $n > 0$, we set $M_n = [a_n, b_{n+1}]$, and for $n < 0$, $M_n = [a_{n-1}, b_n]$. We will refer to the intervals $M_n$ as the *main blocks*. Let $\Delta_0 = [b_{-1}, a_1]$. For $n > 0$, we set $\Delta_n = [b_n, a_{n+1}]$, and for $n < 0$, $\Delta_n = [b_{n-1}, a_n]$. In the sequel we will refer to the $I_n$'s as the *black blocks* and to $\Delta_n$'s as the *white blocks*. Note that for $n > 0$,

$$M_n = I_n \cup \Delta_n \cup I_{n+1}. \tag{3.11}$$

A similar relation holds for $n < 0$.

The strategy of our argument is the following. The black blocks are potential barriers (recall (1.6)). Although we do not have any information about the values of the potential within the white blocks, we know that these blocks are not "too long." Our first goal is to write a suitable expansion of the resolvent $(h - z)^{-1}$ in terms of the main blocks $M_n$. We then use the decomposition (3.11) and tunneling estimates to further refine this expansion, and to establish (1.7).

We denote by $h_{M_n}$ the restriction of $h$ to $M_n$ with Dirichlet boundary condition. Let $R_{M_n}(z)$ be the resolvent of $h_{M_n}$ and $R_{M_n}(p, q; z)$ its matrix elements. We first collect some a priori estimates on $R_{M_n}$. Let

$$x_n^{(1)} = a_n,\ x_n^{(2)} = b_n,\ x_n^{(3)} = a_{n+1},\ x_n^{(4)} = b_{n+1}.$$

Recall that $\gamma > 8$. Throughout, we will freely use the convention (2.10).

**Proposition 3.1** *Let $\delta > 0$ be such that $\gamma/4 > 2(1 + \delta)$. Then for every $\varepsilon > 0$ there is a set $\mathcal{M}_\varepsilon \subset R$ such that:*
*1. $|R \setminus \mathcal{M}_\varepsilon| = 0$.*
*2. For each $E \in \mathcal{M}_\varepsilon$ there is a positive integer $n_{E,\varepsilon}$ such that for $|n| \geq n_{E,\varepsilon}$ the following estimates hold:*

$$\max_{i,j} \left| R_{M_n}(x_n^{(i)} + p, x_n^{(j)} + q; E) \right| \leq \varepsilon (1 + |n|)^{1+\delta} (1 + |p|)^{1+\delta} (1 + |q|)^{1+\delta}, \tag{3.12}$$

$$\max_i \sum_{q \in M_n} \left| R_{M_n}(x_n^{(i)} + p, q; E) \right|^2 \leq (|M_n| + 1)(1 + |n|)^{2(1+\delta)} (1 + |p|)^{2(1+\delta)}. \tag{3.13}$$

*Proof:* Let $L_n = |M_n| + 1$. Then

$$R_{M_n}(x_n^{(i)} + p, x_n^{(j)} + q; E) = \sum_{k=1}^{L_n} \frac{\overline{\phi}_k(x_n^{(i)} + p)\phi_k(x_n^{(j)} + q)}{E - E_k},$$

where $\phi_k$ are eigenfunctions and $E_k$ eigenvalues of $h_{M_n}$. Let

$$A_n(p,q) = \Big\{E : \Big|R_{M_n}(x_n^{(i)} + p, x_n^{(j)} + q; E)\Big| >$$

$$\frac{\varepsilon}{16}(1+|n|)^{1+\delta}(1+|p|)^{1+\delta}(1+|q|)^{1+\delta} \text{ for } 1 \le i,j \le 4\Big\}.$$

Note that

$$\sum_{k=1}^{L_n} |\phi_k(x_n^{(i)} + p)| \cdot |\phi_k(x_n^{(j)} + q)| \le \left(\sum_{k=1}^{L_n} |\phi_k(x_n^{(i)} + p)|^2\right)^{1/2} \left(\sum_{k=1}^{L_n} |\phi_k(x_n^{(j)} + q)|^2\right)^{1/2} = 1.$$

It follows from Proposition 2.2 that

$$|A_n(p,q)| \le \frac{1}{\varepsilon}\left[(1+|n|)(1+|p|)(1+|q|)\right]^{-1-\delta}.$$

Since

$$\sum_n \sum_{p,q} |A_n(p,q)| < \infty,$$

it follows from the Borel-Cantelli lemma that there exists a measurable set $\mathcal{M}'_\varepsilon$ such that 1. and 2., (3.12) hold.

We now consider Relation (3.13). Note first that

$$\sum_{q\in M_n} |R_{M_n}(x_n^{(i)} + p, q; E)|^2 = \sum_{k=1}^{L_n} \frac{|\phi_k(x_n^{(i)} + p)|^2}{|E - E_k|^2}$$

It follows again from Proposition 2.2 and the Borel-Cantelli lemma that there exists a measurable set $\mathcal{M}''$ (which does not depend on $\varepsilon$) such that 1. and 2., (3.13) hold. Taking $\mathcal{M}_\varepsilon = \mathcal{M}'_\varepsilon \cap \mathcal{M}''$, we deduce the statement. □

**Remark.** The various parametrizations $x_n^{(i)}$ in the previous lemma are introduced for later convenience.

Proposition 3.1 gives information on the resolvent matrix elements of $R_{M_n}$ starting with a sufficently large index $n$ which depends on the energy. To circumvent some difficulties which arise from this $E$-dependence, we introduce the sets

$$\mathcal{M}_{k,\varepsilon} = \{E : \ E \in (-\mu,\mu) \text{ and } n_{E,\varepsilon} = k\}.$$

Since $R_{M_n}(s,t;E)$ are Lebesgue measurable functions of $E$, the sets $\mathcal{M}_{k,\varepsilon}$ are measurable. Clearly, if $i > k$ then $\mathcal{M}_{k,\varepsilon} \subset \mathcal{M}_{i,\varepsilon}$ Furthermore, it follows from Proposition 3.1 that for each $\varepsilon > 0$, $\cup_{k\ge 0}\mathcal{M}_{k,\varepsilon}$ is of full measure in $(-\mu,\mu)$. We will prove below the following result:

**Theorem 3.2** *Assume that conditions of Theorem 1.2 are satisfied. Then there exists $\varepsilon > 0$ such that*

$$\limsup_{\zeta \to 0} \sum_m |R(n,m;E+i\zeta)|^2 < \infty, \tag{3.14}$$

*for a.e. $E \in \cup_{k\geq 0} \mathcal{M}_{k,\varepsilon}$ and for all $n \in Z$.*

Clearly, Theorem 1.2 is an immediate consequence of Theorem 3.2.

We also need some information on the resolvents $R_{I_n}$ of the black blocks. Recall that $\gamma > 8$. Let $j_0$, $\mu$ and $\rho$ be as in (1.6).

**Proposition 3.3** *For all $E \in (-\mu, \mu)$ and all $p, q \in I_n$,*

$$|R_{I_n}(p,q;E)| \leq C\left[|p-q|^{-\gamma/4} + 1\right].$$

*The constant $C$ depends only on $\rho$ and the function $j$.*

*Proof:* We consider first the case $p \neq q$. Let $\alpha = j_0 + \rho$ and $\ell = |p-q|$. The path expansion of the resolvent $R_{I_n}$ (recall (2.9)) leads to the bound

$$|R_{I_n}(p,q;E)| \leq \frac{1}{\alpha}\sum_{k=1}^{\infty}\frac{1}{\alpha^k}\sum_{\substack{s_1,\dots s_k \in Z\\ s_1+\dots+s_k=\ell}}\prod_{1\leq i\leq k}|j(s_i)|.$$

If

$$h(\varphi) = \sum_m e^{im\varphi}|j(m)|,$$

then

$$\sum_{\substack{s_1,\dots s_k \in Z\\ s_1+\dots+s_k=\ell}}\prod_{1\leq i\leq k}|j(s_i)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\varphi\ell}h(\varphi)^k d\varphi.$$

Since $|h(\varphi)| \leq j_0 < \alpha$, we have that

$$|R_{I_n}(p,q;E)| \leq \frac{1}{2\pi\alpha}\sum_{k=1}^{\infty}\int_{-\pi}^{\pi} e^{-i\varphi\ell}\left(\frac{h(\varphi)}{\alpha}\right)^k d\varphi = \frac{1}{2\pi\alpha}\int_{-\pi}^{\pi} e^{-i\varphi\ell}\frac{h(\varphi)}{\alpha - h(\varphi)}d\varphi.$$

Integrating by parts $[\gamma]-2$ times, we obtain

$$\left|\int_{-\pi}^{\pi} e^{-i\varphi\ell}\frac{h(\varphi)}{\alpha - h(\varphi)}d\varphi\right| = \frac{1}{\ell^{[\gamma]-2}}\left|\int_{-\pi}^{\pi} e^{-i\varphi\ell}f(\varphi)d\varphi\right|.$$

Since $f \in L^1[-\pi,\pi]$ and $[\gamma]-2 > \gamma/4$, the result follows.

If $p = q$, the argument is simpler:

$$R_{I_n}(p,p;E) \leq \frac{1}{\alpha} + \sum_{k=2}^{\infty}\frac{1}{\alpha^{k+1}}\sum_{\substack{s_1,\dots s_k \in Z\\ s_1+\dots+s_k=0}}\prod_{1\leq i\leq k}|j(s_i)|$$

$$\leq \frac{1}{\alpha} + \sum_{k=1}^{\infty} \frac{1}{\alpha^{k+1}} \left( \sum_s |j(s)| \right)^k = \frac{1}{\rho}. \quad \square$$

**Notation.** In the sequel we will denote by the same letter $C$ various constants which depend *only* on $j$ in (1.2) and $\rho$ in (1.6). The values of these constants may change from estimate to estimate.

We are now fully equipped to proceed with the proof of Theorem 3.2. We will establish Relation (3.14) in the case where $n = 0$. Also, we will assume that $a = 2$ in (1.5). A similar argument applies to the other values of $n$ and $a$.

Our first goal is to develop a suitable expansion of the matrix resolvent element $R(0, m; z)$ with respect to $R_{M_k}$. We first assume that $E \in \mathcal{M}_{0,\varepsilon}$. After this special case is established, an argument based on finite rank perturbation theory will yield the statement for a.e. $E \in \mathcal{M}_{k,\varepsilon}$, $k > 0$ (see the discussion at the end of the section). We will also assume that $m \in \Delta_{n_0}$ for some $n_0 > 0$. A similar argument applies if $m < 0$ or $m \in I_{n_0}$.

Let $\tau$ be any path in the expansion (2.8) which connects 0 and $m$, $\tau = (0, n_1, n_2, \ldots, n_k, m)$. To such a path we associate a sequence of bonds $(b_1, \ldots, b_l)$ and a sequence of blocks $(M_{s_1}, \ldots, M_{s_l})$ in the following way. Let $n_{k_1}$ be the first of the $n_l$'s which is not in the block $M_0$. Then let $b_1 = (n_{k_1-1}, n_{k_1})$. We denote the block to which $n_{k_1}$ belongs by $M_{s_1}$. Let $n_{k_2}$ be the first of the $n_l$'s, for $l > k_1$, which is not in $M_{s_1}$, and let $b_2 = (n_{k_2-1}, n_{k_2})$. We denote the block to which $n_{k_2}$ belongs by $M_{s_2}$. If $n_{k_2} \in M_s \cap M_t$ then, by definition, $k_2 = \min\{s,t\}$ if $s, t \geq 0$, and $k_2 = \max\{s,t\}$ if $s, t \leq 0$. We now continue inductively. It is helpful to invoke the following picture. The path $\tau$ starts in the block $M_0$, and wanders for some time within this block. It then leaves $M_0$ and jumps to a different block $M_{s_1}$. In the bond $b_1$ we record the site $n_{k_1-1} \in M_0$ at which the path takes off, and the site $n_{k_1} \in M_{s_1}$ at which it lands. The path now wanders through $M_{s_1}$ and then jumps to $M_{s_2}$, etc. The last bond $b_l = (n_{k_{l-1}}, n_{k_l})$ corresponds to the last entry into the block $M_{s_l} \equiv M_{n_0}$ which contains $m$. Since neighboring blocks intersect, the paths can land at the site which belongs simultaneously to two blocks; in this case, by definition, we say that the path landed in the block which is closer to 0. Clearly, the sequences $\{b_i\}$ and $\{M_{s_i}\}$ are not uniquely determined by the path $\tau$: a great many paths $\tau$ will determine the same sequences of blocks. Note that $\{b_i\}$, however, uniquely determines $\{M_{s_i}\}$. Let $\mathcal{B}$ be the set of all sequences of bonds $\tau_b = \{b_i\}$ obtained in the above way.

Regrouping the elements in the expansion (2.8) and using the formula (2.9) we get

$$R(0, m; z) = \sum_{\tau_b \in \mathcal{B}} R_{M_0}(0, n_{k_1-1}; z) j(n_{k_1-1} - n_{k_1}) R_{M_{s_1}}(n_{k_1}, n_{k_2-1}; z) \ldots$$

$$\ldots R_{M_{s_{l-1}}}(n_{k_{l-1}}, n_{k_l-1}; z) j(n_{k_l-1} - n_{k_l}) R_{M_{n_0}}(n_{k_l}, m; z). \tag{3.15}$$

Let us consider a typical term in this expansion:

$$R_{M_{s_{i-1}}}(n_{k_{i-1}}, n_{k_i-1}; z) j(n_{k_i-1} - n_{k_i}) R_{M_{s_i}}(n_{k_i}, n_{k_{i+1}-1}; z). \tag{3.16}$$

We will assume that $s_{i+1} > s_i > s_{i-1} > s_{i-2} > 0$; one argues similarly in the other cases. We denote by $d(s_{i-1}, s_i)$ the distance between the blocks $M_{s_{i-1}}$ and $M_{s_i}$. Clearly, if $s_i = s_{i-1}+1$, then $d(s_{i-1}, s_i) = 0$. Our goal is to obtain a suitable estimate on

$$R_{M_{s_{i-1}}}(n_{k_{i-1}}, n_{k_i-1}; z) j(n_{k_i-1} - n_{k_i}). \tag{3.17}$$

We have to consider several cases, depending on whether the sites $n_{k_{i-1}}$, $n_{k_i-1}$ belong to white or black blocks. We will consider the following two cases:
1. $n_{k_{i-1}}, n_{k_i-1} \in \Delta_{s_{i-1}}$,
2. $n_{k_{i-1}} \in \Delta_{s_{i-1}}$, $n_{k_i-1} \in I_{s_i}$.
All the other possibilities are treated along the same lines.
*Case 1.* Recall that $\Delta_{s_{i-1}} = [b_{s_i-1}, a_{s_i}]$. It follows from (1.3) that

$$|j(n_{k_i-1} - n_{k_i})| \le C\left[(1+|n_{k_{i-1}} - a_{s_i}|)(1+|a_{s_i} - b_{s_i}|)(1+d(s_{i-1}, s_i))(1+|n_{k_i} - a_{s_{i+1}}|)\right]^{-\gamma/4},$$

where $C$ is a universal constant. The last term in the product, $(1+|n_{k_i} - a_{s_{i+1}}|)^{-\gamma/4}$, is used in estimating the term in the expansion which follows after (3.17); in estimating (3.17) we will make use of the similar term which arises in the estimation of $j(n_{k_{i-2}} - n_{k_i-1})$, namely $(1+|n_{k_{i-1}} - a_{s_i}|)^{-\gamma/4}$. With this convention, we get that for $E \in \mathcal{M}_{0,\varepsilon}$,

$$\left|R_{M_{s_{i-1}}}(n_{k_{i-1}}, n_{k_i-1}; E)\right| \le \varepsilon\left[(1+|s_{i-1}|)(1+|n_{k_{i-1}} - a_{s_i}|)(1+|n_{k_i-1} - b_{s_i}|)\right]^{1+\delta}$$

Since $b_{s_i} - a_{s_i} \ge |s_i|$ and $\gamma/4 > 2(1+\delta)$, we get that

$$|R_{M_{s_i}}(n_{k_{i-1}}, n_{k_i-1}; E)| \le C\varepsilon\left[(1+|s_i|)(1+|n_{k_{i-1}} - a_{s_i}|)(1+|n_{k_i-1} - b_{s_i}|)\right]^{-1-\delta} \times \tag{3.18}$$

$$(1+d(s_i, s_{i+1}))^{-\gamma/4},$$

for some constant $C$. Thus, for $E \in M_{0,\varepsilon}$,

$$\sum_{n_{k_{i-1}}, n_{k_i-1} \in \Delta_{s_i-1}} \left|R_{M_{s_i}}(n_{k_{i-1}}, n_{k_i-1}; E)\right| \cdot |j(n_{k_i-1} - n_{k_i})| \le C\varepsilon(1+|s_i|)^{-1-\delta}(1+d(s_{i-1}, s_i))^{-\gamma/4}. \tag{3.19}$$

*Case 2.* The critical input in deriving estimate (3.18) is that the path jumps over the long black block $I_{s_i}$. If $n_{k_i-1} \in I_{s_i}$ and

$$n_{k_i-1} - a_{s_i} \le 3\,|I_{s_i}|\,/4,$$

then the same argument (with a change of the constant, of course) yields the estimate (3.18). We have choosen the constant $3/4$ for definiteness; any $0 < \delta < 1$ will do as well. Difficulties arise if $n_{k_i-1}$ is close to $b_{s_i}$; in this case, the jump $M_{s_{i-1}} \to M_{s_i}$ could be very short and we cannot use the previous arguments. It is now critical that the black blocks are the potential barriers. For the rest of the argument we assume that

$$n_{k_i-1} - a_{s_i} > 3\,|I_{s_i}|\,/4. \tag{3.20}$$

It follows from (2.9) that

$$R_{M_{s_{i-1}}}(n_{k_{i-1}}, n_{k_i-1}; E) = \sum_{\substack{r\in I_{s_{i-1}}\cup\Delta_{s_{i-1}} \\ t\in I_{s_i}}} R_{M_{s_{i-1}}}(n_{k_{i-1}}, r; E) j(r-t) R_{I_{s_i}}(t, n_{k_i-1}; E).$$

We will make use of the following two elementary estimates:

$$|j(n_{k_i-1} - n_{k_i})| \le C\left[(1+|n_{k_i-1} - a_{s_i}|)(1+d(s_{i-1}, s_i))(1+|n_{k_i} - a_{s_{i+1}}|)\right]^{-\gamma/4} \tag{3.21}$$

$$|j(r-t)| \le C\left[(1+|r-a_{s_i}|)(1+|t-a_{s_i}|)\right]^{-\gamma/4}.$$

It follows from Propositions 3.1 and 3.3 that for $E \in M_{0,\varepsilon}$,

$$\left|R_{M_{s_{i-1}}}(n_{k_{i-1}}, r; E)\right| \le \varepsilon\left[(1+|s_{i-1}|)(1+|n_{k_{i-1}} - a_{s_i}|)(1+|r-a_{s_i}|)\right]^{1+\delta}$$

$$\left|R_{I_{s_i}}(t, n_{k_i-1}; E)\right| \le C(1+|t-n_{k_i-1}|)^{-\gamma/4}.$$

We now again drop the last term in (3.21) and use the similar term arising from $j(n_{k_{i-2}} - n_{k_{i-1}})$. With this convention and arguing as before, we deduce

$$\sum_{\substack{n_{k_{i-1}}\in\Delta_{s_{i-1}} \\ n_{k_i-1}\in I_{s_i}}} \left|R_{M_{s_{i-1}}}(n_{k_{i-1}}, n_{k_i-1}; E)\right| \cdot |j(n_{k_i-1} - n_{k_i})| \le C\varepsilon(1+|s_{i-1}|)^{1+\delta}(1+d(s_{i-1}, s_i))^{-\gamma/4}\mathcal{S}_{s_i},$$

where

$$\mathcal{S}_{s_i} \equiv \sum_{t\in I_{s_i}} \left[(1+|t-a_{s_i}|)(1+|t-n_{k_i-1}|)\right]^{-\gamma/4}.$$

It is easy to show that (recall (3.18))

$$\mathcal{S}_{s_i} \le C(1+|s_i|)^{-2(1+\delta)}.$$

Thus, we have derived an estimate analogous to (3.17): for some constant $C$, which depends only on $\rho$ and the function $j$,

$$\sum_{\substack{n_{k_{i-1}}\in\Delta_{s_{i-1}} \\ n_{k_i-1}\in I_{s_i}}} \left|R_{M_{s_{i-1}}}(n_{k_{i-1}}, n_{k_i-1}; E)\right| \cdot |j(n_{k_i-1} - n_{k_i})| \le C\varepsilon(1+|s_{i-1}|)^{-1-\delta}(1+d(s_{i-1}, s_i))^{-\gamma/4}. \tag{3.22}$$

This concludes the discussion of the Case 2. We emphasize that in all the other cases, a similar argument leads to the estimates analogous to (3.19) and (3.22). Also, a similar argument shows that

$$\sum_{t\in M_0} |R_{M_0}(0,t;E)j(t-n_1)| \le C\varepsilon(1+d(0,s_1))^{-\gamma/4}$$

for a suitable constant $C$. Since the first parameter is fixed to be zero, here we do not need a contribution from $j$ on the left-hand side to compensate for the sum with respect to this parameter.

We now go back to the expansion (3.15). We split the set of bonds $\mathcal{B}$ as $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ ($\mathcal{B}_2$) consists of bonds associated to the paths whose length is $< n_0/2$ ($\ge n_0/2$). Accordingly, we decompose $R(0,m;z)$ as

$$R(0,m;z) = A_1(0,m;z) + A_2(0,m;z).$$

To simplify the notation, we will assume that $n_0$ is even, $n_0 = 2n_0'$. A similar argument applies if $n_0$ is odd. To estimate $A_2$ we do not need the contribution from $d(s_{i-1},s_i)$. From the previous analysis it follows easily that for $E \in \mathcal{M}_{0,\varepsilon}$,

$$\sup_{0<\zeta<1} |A_2(0,m;E+i\zeta)| \le \sum_{j\ge n_0'} \left( C\varepsilon \sum_{k>0} \frac{1}{k^{1+\delta}} \right)^j G(m) \le (C\varepsilon)^{n_0'/2} G(m),$$

where

$$G(m) \equiv \max_i \sum_{t\in M_{n_0}} (1+|x_{n_0}^{(i)} - t|)^{-\gamma/4} \left| R_{M_{n_0}}(t,m;E) \right|.$$

The contribution $(1+|x_{n_0}^{(i)} - t|)^{-\gamma/4}$ is the remaining part of the estimate analogous to (3.18) applied to (3.17) with $i = l$ (recall the convention introduced in the derivation of (3.18)). We have chosen $\varepsilon$ in the above estimates so that

$$(C\varepsilon)^{1/2} \sum_{k>0} k^{-1-\delta} < \frac{1}{3}; \tag{3.23}$$

(recall that $C$ depends only on $\rho$ and the function $j$). We now proceed to estimate $A_1$. Here, we have to make use of the contributions arising from $d(s_{i-1},s_i)$. Let us split $\mathcal{B}_1$ into sets of disjoint bonds such that associated paths have lengths $1,2,\ldots,n_0'-1$. Accordingly, $A_1$ splits as

$$A_1 = \sum_{l=1}^{n_0'-1} A^{(l)}.$$

To estimate $A^{(1)}(0,m;z)$, we note that if the path has length 1, then it has to jump from the block $M_0$ to the block $M_{n_0}$. Thus, it has to go completely over the block $M_{n_0-2}$ whose length is $2^{n_0-2}$, and we arrive at the estimate

$$\sup_{0<\zeta<1} |A^{(1)}(0,m;E+i\zeta)| \le C\varepsilon 2^{-\gamma(n_0-2)/4} G(m).$$

We have absorbed $N$ from (1.6) into the constant $C$. If the path has the length $l$ and $n_0 > 2l$, then it has to jump over at least one block whose index is $n_0 - 2l$, which leads to the estimate

$$\sup_{0<\zeta<1} |A^{(l)}(0, m; E + i\zeta)| \leq (C\varepsilon)^l 2^{-\gamma(n_0-2l)/4} G(m).$$

Since $\gamma/4 > 2$, if we choose $\varepsilon$ such that $C\varepsilon < \frac{1}{12}$, we get the estimate

$$\sup_{0<\zeta<1} |A_1(0, m; E + i\zeta)| \leq 2^{-\gamma n_0/2} G(m).$$

Thus, we summarize: if $\varepsilon$ is sufficiently small, then for all $E \in \mathcal{M}_{0,\varepsilon}$ we have

$$\sup_{0<\zeta<1} |R(0, m, E + i\zeta)|^2 \leq \left((C\varepsilon)^{n_0/4} + 2^{-\gamma n_0/2}\right)^2 G(m)^2. \tag{3.24}$$

We proceed to estimate $G(m)^2$. Note first that

$$G(m)^2 \leq |M_{n_0}| + \max_i \sum_{t \in M_{n_0}} (1 + |x_{n_0}^{(i} - t|)^{-2(1+\delta)} |R_{M_{n_0}}(t, m; E)|^2.$$

It follows from Proposition 3.1 that

$$\sum_{m \in M_{n_0}} G(m)^2 \leq |M_{n_0}|^2 + |M_{n_0}|^2 (1 + |n_0|)^{2(1+\delta)}.$$

Choosing $\varepsilon$ in (3.25) such that $C\varepsilon < 1/32$, we get that for $E \in \mathcal{M}_{0,\varepsilon}$,

$$\sup_{0<\zeta<1} \sum_{m \in M_{n_0}} |R(0, m; E + i\zeta)|^2 \leq C 2^{-\delta n_0}.$$

Thus, if $\varepsilon$ is sufficiently small, then

$$\limsup_{\zeta \to 0} \sum_{m \in Z} |R(0, m; E + i\zeta)|^2 \leq \sum_n \limsup_{\zeta \to 0} \sum_{m \in M_n} |R(0, m; E + i\zeta)|^2$$

$$\leq C \sum_n 2^{-\delta |n|} < \infty.$$

This yields (3.14) if $E \in \mathcal{M}_{0,\varepsilon}$ and $n = 0$. Clearly, a similar argument shows that for $E \in \mathcal{M}_{0,\varepsilon}$, and any $n \in Z$,

$$\limsup_{\zeta \to 0} \sum_m |R(n, m; E + i\zeta)|^2 < \infty. \tag{3.25}$$

Let us now assume that $E \in \mathcal{M}_{k,\varepsilon}$, namely that inequalities (3.12) and (3.13) hold for $|n| \geq k$. We introduce an auxiliary potential $v_L(n)$ by the formula:

$$v_L(n) = v(n), \qquad \text{if } n \in M_s,\ |s| \geq k,$$

$$v_L(n) = L, \qquad \text{if } n \in M_s,\ |s| < k.$$

We denote by $R_L$ the resolvent of the operator $h_L = h_0 + v_L$. Clearly, by choosing $L$ large enough, the estimates (3.12) and (3.13) will hold for $h_L$ and $E \in \mathcal{M}_{k,\varepsilon}$ starting with $n = 0$. Thus, Relation (3.14) holds for $R_L$ and $E \in \mathcal{M}_{k,\varepsilon}$. It follows from the resolvent identity that

$$R(0,m;E+i\zeta) = R_L(0,m;E+i\zeta) + \sum_{i \in M_s, |s|<k} (L - v(i)) R(0,i;E+i\zeta) R_L(i,m;E+i\zeta).$$

Since for a.e. $E \in R$,

$$\lim_{\zeta \to 0} R(0,i;E+i\zeta)$$

exists and is finite, we derive that for a.e. $E \in \mathcal{M}_{k,\varepsilon}$,

$$|R(0,m;E+i\zeta)|^2 \leq C_{k,E} \sum_{i \in M_s, |s|<k} |R_L(i,m;E+i\zeta)|^2.$$

This proves Theorem 3.2, and completes the proof of Theorem 1.2.

# 4 Proof of Theorem 1.1

We denote by $(\Omega, \mathcal{F}, P)$ the probability space associated to the model (1.1) (see, e.g., [CFKS] or [CL]). For $\varepsilon > 0$ we define $\mu_\varepsilon = \eta_0 - \varepsilon$. Note that

$$P\left(\left\{|v_\omega(n)| > \mu_\varepsilon + j_0 + \frac{\varepsilon}{2}\right\}\right) = \delta_\varepsilon > 0.$$

Let $a$ be an integer such that $a\delta_\varepsilon^2 > 1$. Let us consider the sub-intervals of $[a^n, a^{n+1}]$, $[-a^{n+1}, -a^n]$ of the following form:

$$I_n^{(k)} = [a^n + 2(k-1)n + 1, a^n + 2(k-1/2)n + 1], \qquad I_{-n}^{(k)} = -I_n^{(k)},$$

where $1 \leq k \leq [a(a^n - 1)/2n] - 1$ ($[\cdot]$ is the greatest integer part). Clearly, these are mutually disjoint intervals of length $n$. Let

$$A_{n,k_1,k_2} = \left\{\omega : |v_\omega(i)| > \mu_\varepsilon + j_0 + \frac{\varepsilon}{2} \textit{ for all } i \in {I_n}^{k_1} \cup I_{-n}^{k_2}\right\}.$$

The probability of this event is $P(A_n) = \delta_\varepsilon^{2n}$. Let $B_n$ be the event that no $A_{n,k_1,k_2}$ take place, i.e.

$$B_n = \Omega \setminus (\cup_{k_1,k_2} A_{n,k_1,k_2}).$$

Clearly,

$$P(B_n) = (1 - \delta_\varepsilon^{2n})^{[a(a^n-1)/2n]-1}.$$

A simple analysis leads to a (rough) bound

$$P(B_n) = O(2^{-a(\delta_\varepsilon^2 a)^{2n}/2n}).$$

Since $a\delta_\varepsilon^2 > 1$,

$$\sum_n P(B_n) < \infty,$$

and the Borel-Cantelli lemma yields that for typical $\omega$ only finitely many events $B_n$ take place. Thus, for typical $\omega$ there is $N(\omega)$ such that all the conditions of Theorem 1.2 are satisfied. We conclude that for a.e. $\omega \in \Omega$, a.e. $E \in (-\mu_\varepsilon, \mu_\varepsilon)$, and for all $n \in Z$,

$$\limsup_{\zeta \to 0} \sum_m \left|(\delta_n, (h_\omega - E - i\zeta)^{-1}\delta_m)\right|^2 < \infty.$$

It now follows from the Simon-Wolff theorem [SW] (for its various reformulations, see [AM]) that for a.e. $\omega$,

$$\sigma(h_\omega) \cap (-\mu_\varepsilon, \mu_\varepsilon) = \sigma_{pp}(h_\omega) \cap (-\mu_\varepsilon, \mu_\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, Theorem 1.1 follows.

# References


[A] Aizenman M., Localization at Weak Disorder: Some Elementary Bounds. *Rev. Math. Phys.*, **6**, 1163 (1994)

[AM] Aizenman M., Molchanov S., Localization at Large Disorder and at Extreme Energies: An Elementary Derivation. *Commun. Math. Phys.* **157**, 245 (1993)

[CL] Carmona R., Lacroix J., *Spectral Theory of Random Schrödinger Operators.* Birkhauser, Boston 1990.

[CFKS] Cycon H., Froese R., Kirsch W., Simon B., *Schrödinger Operators.* Springer-Verlag, Berlin-Heidelberg 1987

[G] Grinshpun V., Dopovidi Akademii Nauk Ukrainy (Proc. Acad. Sc. Ukraine), **8**, 18, (1992); **9**, 26 (1993)

[GJMS] Gordon Y., Jakšić V., Molchanov S., Simon B., Spectral Properties of Random Schrödinger Operators with Unbounded Potentials. *Commun. Math. Phys.* **157**, 23 (1993)

[JL] Jakšić V., Last Y., in preparation.

[JMP] Jakšić V., Molchanov S., Pastur L., On the Propagation Properties of Surface Waves. IMA Preprint 1316, (1995)

[JMP1] Jakšić V., Molchanov S., Pastur L., in preparation.

[KMP] Kirch W., Molchanov S., Pastur L., One-dimensional Schrödinger Operator with Unbounded Potential. *Func. Anal. Prilozhen.*, **24**, 14 (1990)

[L] Levitov L., Anderson Model for Localization with Long Range Transfer Amplitudes: Delocalization of Vibrational Modes Caused by Their Dipole Interaction. *Ann. New York Acad. Sc.*, **581**, 95 (1990)

[M] Molchanov S., Lectures given at Caltech, Spring 1990.

[M1] Molchanov S., Lectures on Random Media. In *Lectures on Probability*, ed. P. Bernard, Lecture Notes in Mathematics, **1581**, Springer-Verlag, Heidelberg 1994

[SS] Simon B., Spencer T., Trace Class Perturbations and the Absence of Absolutely Continuous Spectrum. *Commun. Math. Phys.* **125**, 113 (1989)

[SW] Simon B., Wolff T., Singular Continuous Spectrum Under Rank One Perturbations and Localization for Random Hamiltonians. *Commun. Pure Appl. Math.* **39**, 75 (1986)