Some examples in one-dimensional "geometric" scattering on manifolds

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Abstract

We consider "geometric" scattering for a Laplace-Beltrami operator on a compact Riemannian manifold inserted between two half-lines. We discuss applicability and correctness of this model. With an example, we show that such scattering problem may exhibit unusual properties: the transition coefficient has a sequence of sharp peaks which become more and more distant at high energy and otherwise turns to zero.

Introduction

In this paper we consider certain boundary value problems, namely, the one-dimensional scattering on two- or three-dimensional compact Riemannian manifolds. The motivation for studying these problems is twofold. The first comes from the paper of J. Avron, P. Exner and Y. Last [2], who dicussed the problem of approximating the scattering properties of δ'- potential by certain graphs. By δ'- potential we mean the boundary condition of the type $u'(+a) = u'(-a)$, $u(+a) - u(-a) = \alpha u'(a)$ imposed on the functions from the domain of the operator at the given point a. We recall that for one-dimensional scattering on the line on a sufficiently rapidly decaying potential, it is typical that the transition coefficient tends to one at high energies. On the other hand, the δ'- potential has in this respect very different properties: the transition coefficient $t(\lambda)$ turns to 0 as $\lambda \to \infty$ at the rate $\lambda^{-1/2}$. This feature leads to important spectral consequences for δ' Wannier-Stark ladders. Namely, the absolutely continuous spectrum in a system of periodic δ' interactions with electric field is void [4] under very general assumptions on the coupling constants. This is a very special phenomena in the sense that one cannot have such an effect if δ' is replaced by some smooth potential with compact support.

Avron, Exner and Last suggested that certain geometric scatterers may possess the properties similar to those of δ'. In [2], they showed that the transition coefficient for an "onion" – N segments of equal length l glued together at two points, to which two half-axes are attached – may approximate the behavior of δ' transition coefficient on an arbitrary large scale of energy, if one adjusts N and l in a suitable way. Eventually, however, as is typical for all graphs of this type [6], the behavior of the transition coefficient is periodic in energy. The example we will consider in this paper shows that one can find some geometric structures for which the transition coefficient will in general decay at infinity, without being periodic. There will be present a sequence of sharp resonances, however – an interesting phenomena on its own.

The second motivation comes from the fact that in some situations, the boundary value problems we consider here are known to approximate, in a certain sense, the corresponding "real" boundary value problems. In our situation it would be a scattering problem for a Laplace-Beltrami operator on the manifold with two thin half-infinite tubes attached to it. We will discuss the known results in more detail in Section 1, when we explain the choice of the boundary conditions at the contact points. We do not expect, however, a straightforward analogy in the case we study here, since by taking a line instead of a tube we ignore the effects due to the transverse modes existing in the tube, which is essential for high-energy limit. It is reasonable, however, to expect that the model problem will approximate the real one on a certain scale of energies related to the size of the channel in a real problem.

The paper is organized as follows: in Section 1, we formally define the Laplace operator on our domain and discuss natural restrictions and choice of the boundary conditions. In Section 2, we compute the transition coefficient in a general situation. Section 3 is devoted to a specific example, when we take a sphere as a manifold.

Setup of the problem

We would like to consider a scattering problem for a Laplacian on the compact smooth Riemannian manifold M inserted between two half-lines R^+ and R^-. We will assume, unless stated otherwise, that M is a C^∞ manifold without boundary. Let x_1, x_2 be the points at which the half-lines join the manifold: $x_1 = R^- \cap M$ and $x_2 = R^+ \cap M$. We denote by Ω the union $R^- \cup R^+ \cup M$. $-\Delta_M$ will stay for the Laplace-Beltrami operator on M and $D_\mp$ for the operators of double differentiation on $R^\mp$ with Dirichlet boundary conditions at x_1 or x_2 correspondingly. To define a Laplace operator on the domain Ω, we proceed in a classic way: first define some symmetric operator given by the differential expression of Laplacian on the set of infinitely differentiable functions vanishing in the neighborhood of the special points x_1, x_2 and then consider its self-adjoint extensions. Namely, on the set of functions

$$C_0^\infty(D, x_1, x_2) = C_0^\infty(R^+) \cup C_0^\infty(R^-) \cup C_0^\infty(M, x_1, x_2)$$

described above, we define an operator Δ_0, which acts on the function f as follows:

$$\Delta_0 f(x) = \begin{cases} -\frac{d^2}{dx^2} f(x), & x \in R^+ \text{ or } \mathrm{R}^-; \\ -\Delta_M f(x), & x \in M. \end{cases}$$

Hence, Δ_0 acts as a double differentiation operator on half-axes and as a Laplace-Beltrami operator on M. The closure of this operator $\overline{\Delta}_0$ is clearly symmetric but not self-adjoint. Among the obvious self-adjoint extensions of Δ_0 is the operator $D_- \oplus -\Delta_M \oplus D_+$, and the operators with other than Dirichlet boundary conditions at x_1, x_2, but still without interaction between half-lines and the manifold. We are of course interested in the Laplacians on

Ω not splitting into a direct sum with summands acting on different geometric components. Therefore, we look for some boundary conditions at the points x_1, x_2 other than leading to orthogonal sums. The range of the applicability of our model is given by the following

Proposition 1.1. *Suppose that* $\dim M = 2$ *or* 3. *Then the closure* $\overline{\Delta}_0$ *is a symmetric operator with deficiency indices* $(4, 4)$. *If* M *has higher dimension, the deficiency indices are* $(2, 2)$.

Proof. The function f from the deficiency subspace $\mathcal{N}_\lambda$ of the operator $-\overline{\Delta}_0$ corresponding to some regular value λ of the spectral paramter is an L^2- function which should satisfy

$$\langle(-\overline{\Delta}_0 - \lambda)g, f\rangle = 0$$

for any $g \in C_0^\infty(\Omega, x_1, x_2)$, where $\langle , \rangle$ stays for the scalar product in $L^2(\Omega)$. Equivalently, $(-\Delta - \overline{\lambda})f$ is a distribution supported on x_1 and x_2. By Δ we mean here a differential expression for the Laplacian. The action of Δ is defined on f in the distributional sense. By well-known facts from distribution theory, it follows that

$$(-\Delta - \overline{\lambda})f = \sum_\alpha (a_{1,\alpha}\delta^{[\alpha]}(x - x_1) + a_{2,\alpha}\delta^{[\alpha]}(x - x_2)), \tag{1}$$

where on the right-hand side we have a sum of some derivatives of δ-functions at x_1 and x_2. In other words, f should be a linear combination of the Green's function of Δ and its derivatives. It is a standard fact that two dimensions of the defect subspace arise from the operator on the half-axes $R^\mp$. One can take as defect elements corresponding to a given regular value of energy λ solutions $\exp(\pm\sqrt{\lambda}x)$ on R^- and R^+, with the sign in the exponent chosen depending on the sign of $\Im(\lambda)$ and x, so that these solutions are correspondingly in $L^2(R^\pm)$. For the manifold M, we consider a scale of the Sobolev spaces associated with the Laplace-Beltrami operator. For the relevant definitions and facts, we refer to [13]. The second Sobolev space $H_2(M)$ coincides with the domain of the Laplace-Beltrami operator. $-\Delta_M - \lambda$ maps $H_2(M)$ onto $L^2(M)$ for any regular value of λ. Furthermore, for such λ,

$-\Delta - \lambda$ maps $L^2(M)$ onto $H_{-2}(M)$, the dual space of $H_2(M)$. Hence, for equation (1) to have a solution in $L^2(M)$, the δ-function and perhaps some of its derivatives should be in $H_{-2}(M)$. It follows from the Sobolev embedding theorems [1] that this is only the case when $n < 4$. Moreover, if $2 \leq n \leq 3$, only the δ-function belongs to $H_{-2}(M)$ and hence the only L^2 solutions of (1) supported on M are the linear combinations of the Green's functions $G_M(x, x_i, \lambda)$, $i = 1, 2$. This proves the lemma. □

Remark. For the sake of simplicity we restricted ourselfves to the case when M has no boundary; but the approach certainly works in many cases for manifolds with boundary, in particular, when the contact points belong to ∂M. For example, when M is a bounded domain with smooth boundary in R^n, we replace everywhere in the above Laplace operator by the Neumann Laplacian and all considerations clearly remain true. The problems of this type have been studied in particular in [9], [3], [10], [8], where one can also find more references. We stress, however, that for the Dirichlet boundary condition the presented approach does not work even for $n = 2$, since there is obviously no $\delta(x)$ functional over the domain of the operator if x is on the boundary and one has to start from $\delta^{[\alpha]}$ functionals in the equation (1). See, for example, [11] for more details on this matter and an approach to the construction of similar models in this case.

For the rest of the paper we assume that $\dim M$ is equal to 2 or 3.

Proposition 1.2. *Fix some negative number λ_0. Every function f from the domain $D(-\Delta_0^*)$ of the adjoint operator $-\Delta_0^*$ may be represented as follows:*

$$f(x) = \begin{cases} u_-^f(x), & x \in R^- \\ a_1^f G_M(x, x_1, \lambda_0) + a_2^f G_M(x, x_2, \lambda_0) + u^f(x), & x \in M \\ u_+^f(x), & x \in R^+. \end{cases} \tag{2}$$

Here $u_\mp^f \in H_2(R^\mp)$ and $u^f \in H_2(M)$.

Proof. Only the representation for $x \in M$ needs comments. From the theorems on self-adjoint extensions of symmetric operators (see, e.g., [12]), it follows that

$$\dim D(-\Delta_0^*)/D(-\Delta_M) = 2$$

(exactly the deficiency indices of $-\overline{\Delta}_0|_M$). On the other hand, every negative number and hence λ_0 is a regular value of the spectral parameter. Therefore, $G_M(x, x_i, \lambda_0)$, $i = 1, 2$, belong to the domain of $D(-\Delta_0^*)$ (and $(-\Delta_0^* - \lambda_0)G_M(x, x_i, \lambda_0) = 0$). These functions are linearly independent over $H_2(M)$ because of the singularities at the different points (x_1 and x_2) and hence they, together with the functions from H_2, span the whole $D(\Delta_0^*)$. □

For a given function $f \in D(-\Delta_0^*)$, let us denote

$$b_i^f = a_i^f G^N(x_1, x_2, \lambda_0) + u^f(x_i), \quad i = 1, 2.$$

We have the following:

Proposition 1.3. *The boundary form of the operator $-\Delta_0^*$ is given by*

$$\langle -\overline{\Delta}_0^* f, g \rangle - \langle f, -\overline{\Delta}_0^* g \rangle =$$
$$-(a_1^f \overline{b}_1^g - b_1^f \overline{a}_1^g) - (u_-^{f'} \overline{u}_-^g - u_-^f \overline{u}_-^{g'})|_{x_1} - (a_2^f \overline{b}_2^g - b_2^f \overline{a}_2^g) + (u_+^{f'} \overline{u}_+^g - u_+^f \overline{u}_+^{g'})|_{x_2}. \quad (3)$$

Proof. The second and fourth terms on the right-hand side of the equation are standard boundary terms of the double differentiation operator on the half-line. The other two terms are obtained by a direct computation using the representation for a function from $D(\Delta_0^*)$ on M and the facts that $(-\Delta_0^* - \lambda_0)G^N(x, x_i, \lambda_0) = 0$ and that $-\Delta_M$ is contained in Δ_0^*. □

The self-adjoint extensions of $-\overline{\Delta}_0$ are given by the subspaces of the eight-dimensional space, spanned by the coefficients entering (3) corresponding to a function f, which nullify the boundary form. A complete description of all possible self-adjoint boundary conditions in such situation may be found, for example, in [5]. We will not be concerned here with studying the properties of all possible boundary conditions. Our main goal in this paper is to try to give some interesting examples, rather than to present a complete study of the model. We will consider the following family of boundary conditions:

$$\begin{cases} a_1^f \;=\; -\beta(u_-^f)'(x_1), & a_2^f \;=\; \beta(u_+^f)'(x_2), \\ \\ u_-^f(x_1) \;=\; \beta b_1^f, & u_+^f(x_2) \;=\; \beta b_2^f. \end{cases} \tag{4}$$

Our family depends on two parameters β and λ_0. It would be reasonable to denote the self-adjoint operator defined by (4) as $\Delta_\Omega(\beta, \lambda_0)$. Since, for our purpose, virtually all essential properties of this family of the operators will turn out to be independent of β and λ_0, we will often omit these indices and talk about Δ_Ω.

Remark. We note that the family of boundary conditions (4) is natural in the following sense. In [8], following the scheme introduced above, the Neumann Laplacian on a "dumbell" domain, composed from two disjoint regions in R^n, $n = 2, 3$, connected by segment, was studied. It is known that in the corresponding "real" problem for the Neumann Laplacian on the dumbell domain Ω_D, composed from two regions Ω_1, Ω_2 connected by a thin channel P_ω, the eigenvalues of $-\Delta_{\Omega_D}^N$ turn to the eigenvalues of the direct sum $-\Delta_{\Omega_1}^N \oplus -\Delta_{P_0}^D \oplus -\Delta_{\Omega_2}^N$ (where $-\Delta_{P_0}^D$ is an operator of the double differentiation on the segement P_0 with Dirichlet boundary conditions) as the width of the channel ω goes to zero. In [7], the first term of the asymptotics in ω for the eigenvalues of $-\Delta_{\Omega_D}^N$ was obtained in the case $n \geq 3$. It was shown in [8] that in the case $n = 3$ the eigenvalues of the operator $-\Delta_\Omega(\beta, \lambda_0)$ defined by (4) have the same first term of the asymptotics in β as β goes to zero as in the real problem if we let $\beta = \sqrt{\omega}$. Hence, the operator $-\Delta_\Omega(\beta, \lambda_0)$ "approximates" in a certain sense the Neumann Laplacian, at least for the small width of the channel.

Now we are done with all the formal preparations and we are ready to study the scattering properties of the system. For the reader's convenience, we summarize the given information about the operator $-\Delta_\Omega$ we consider:

1. $-\Delta_\Omega$ is defined on the functions f which have the representation (2) with the coefficients which satisfy (4).

2. $-\Delta_\Omega$ acts as an operator of double differentiation on R^+ and R^-. On the functions from $H_2(M)$, $-\Delta_\Omega$ acts as a Laplace-Beltrami operator while $(-\Delta_\Omega - \lambda_0)G_\Omega(x, x_i, \lambda_0) = 0$. This defines the action of $-\Delta_\Omega$ completely.

Transition coefficient and transfer matrix: General case

In this section we are going to compute the matrix $L(\lambda)$ connecting the values and derivatives of the function $f(x, \lambda)$ solving the equation $(-\Delta_\Omega - \lambda)f(x, k) = 0$ at the points x_1, x_2 on $R^\mp$. This will help us to compute the transition coefficient in the scattering problem. Let us arrange the eigenvalues of $-\Delta_M$ in increasing order counting multiplicities. We will denote the n-th eigenvalue λ_n and the corresponding eigenfunction $\phi_n(x)$. The following are the key expressions which naturally enter the calculations and contain all the necessary spectral information about the operator $-\Delta_M$:

$$g_i(\lambda) = (G_M(x, x_i, \lambda) - G_M(x, x_i, \lambda_0))|_{x=x_i} = (\lambda - \lambda_0) \sum_{n=1}^{\infty} \frac{|\phi_n(x_i)|^2}{(\lambda_i - \lambda_0)(\lambda_i - \lambda)};$$

$$h(\lambda) = G_M(x_1, x_2, \lambda) = G_M(x_2, x_1, \lambda) = \sum_{n=1}^{\infty} \frac{\phi_n(x_1)\overline{\phi}_n(x_2)}{(\lambda_n - \lambda)}.$$

Henceforth we will assume that λ is different from any of λ_n. For such values of λ, the expressions above are well-defined. The treatment of the eigenvalue frequencies presents purely technical difficulties; we will define the values of the transition coefficient $t(\lambda_n)$ by continuity.

We need the following lemmas:

Lemma 2.1. *Suppose* $\dim M = 2$ *or* 3. *Then for any two regular values of the spectral parameter* λ_1, λ_2, *the function* $q(x, x_i, \lambda_1, \lambda_2) = G_M(x, x_i, \lambda_1) - G_M(x, x_i, \lambda_2)$ *belongs to* $H_2(M)$.

Proof. Certainly the Green's function $G_M(x, x_i, \lambda)$ of the Laplace-Beltrami operator itself is not from $H_2(M)$; it may have stronger singularities at the points x_i. The lemma says that these singularities do not depend on the spectral parameter λ. For the proof, we use the well-known properties of the action of $-\Delta_M$ on the scale of the associated Sobolev spaces (we refer to [13] for these facts). We have

$$G_M(x, x_i, \lambda_1) = (-\Delta - \lambda_1)^{-1}\delta(x - x_i).$$

By the resolvent identity we find that

$$G_M(x, x_i, \lambda_1) - G_M(x, x_i, \lambda_2) = (\lambda_1 - \lambda_2)(-\Delta - \lambda_1)^{-1}(-\Delta - \lambda_2)^{-1}\delta(x - x_i).$$

The function on the right hand side is from $H_2(M)$ since $\delta(x - x_i)$ is from $H_{-2}(M)$ under the conditions of the lemma and the action of the resolvent of $-\Delta_M$ at the regular point increases by two the index of Sobolev space to which the function belongs. □

Lemma 2.2. *Suppose that the function* $f(x, \lambda)$ *satisfies*

$$(-\Delta_\Omega - \lambda)f(x, \lambda) = 0, \tag{5}$$

$\lambda \neq \lambda_n$ *for any* n. *Then* $f(x, \lambda)$ *has the following representation:*

$$f(x, \lambda) = \begin{cases} c_1^- \exp(i\sqrt{\lambda}x) + c_2^- \exp(-i\sqrt{\lambda}x), & x \in R^- \\ a_1^f G_M(x, x_1, \lambda) + a_2^f G_M(x, x_2, \lambda), & x \in M \\ c_1^+ \exp(i\sqrt{\lambda}x) + c_2^+ \exp(-i\sqrt{\lambda}x), & x \in R^+. \end{cases} \tag{6}$$

The coefficients in the formula above should be chosen in order to satisfy the boundary conditions (4).

Proof. The representation on the half-axes is obvious. Since the operator $-\Delta_M$ is a restriction of $-\Delta_0^*$, we have that $(-\Delta_M - \lambda)G_M(x, x_i, \lambda) = 0$, $i = 1, 2$. On the other hand, suppose that some function f from $D(-\Delta_M)$ and hence with representation (2) satisfies equation (5). Consider the function

$$q(x) = f(x) - a_1^f G_M(x, x_1, \lambda) - a_2^f G_M(x, x_2, \lambda).$$

Then $q(x)$ also satisfies equation (5) on M, and by Lemma 2.1 it is from $H_2(M)$. But on the functions from $H_2(M)$, the operator $-\Delta_\Omega$ acts as a Laplace-Beltrami operator $-\Delta_M$. Hence, since $\lambda \neq \lambda_n$ for any n, we must have $q(x) = 0$. $\square$

Now we compute the matrix $L(\lambda)$, using the representation (6) and boundary conditions (4). Substituting (6) into (4) we get:

$$\begin{cases} a_1^f = -\beta(u_-^f)'(x_1); & \beta(a_1^f g_1(\lambda) + a_2^f h(\lambda)) = u_-^f(x_1) \\ a_2 = \beta(u_+^f(x_2); & \beta(a_2 g_2(\lambda) + a_1 h(\lambda)) = u_+^f(x_2). \end{cases}$$

Solving these equations for the matrix $L(\lambda)$ such that

$$\begin{pmatrix} (u_+^f)'(x_2) \\ u_+^f(x_2) \end{pmatrix} = L(\lambda) \begin{pmatrix} (u_-^f)'(x_1) \\ u_-^f(x_1) \end{pmatrix},$$

we find that

$$L(\lambda) = \frac{1}{h(\lambda)} \begin{pmatrix} g_1(\lambda) & \frac{1}{\beta^2} \\ \beta^2(\frac{g_1(\lambda)g_2(\lambda)}{h(\lambda)}) & g_2(\lambda) \end{pmatrix}.$$

To evaluate the transition coefficient $t(\lambda)$, we seek the solutions $f(x,\lambda)$ of equation (5) in the form of scattered waves:

$$f(x,\lambda) = \begin{cases} \exp(i\sqrt{\lambda}x) + r(\lambda)\exp(-i\sqrt{\lambda}x), & x \in R^-; \\ t(\lambda)\exp(i\sqrt{\lambda}x), & x \in R^+. \end{cases}$$

We have the following linear system from which we can determine $t(\lambda)$

$$t(\lambda)\begin{pmatrix} 1 \\ i\sqrt{\lambda} \end{pmatrix} - r(\lambda)L(\lambda)\begin{pmatrix} 1 \\ -i\sqrt{\lambda} \end{pmatrix} = L(\lambda)\begin{pmatrix} 1 \\ i\sqrt{\lambda} \end{pmatrix}.$$

An easy computation gives the result:

$$t(\lambda) = \frac{2i\sqrt{\lambda}h(\lambda)}{-\frac{1}{\beta^2} + \lambda\beta^2(g_1(\lambda)g_2(\lambda) - h(\lambda)^2) + i\sqrt{\lambda}(g_1(\lambda) + g_2(\lambda))}. \tag{7}$$

The sphere example

In this section we will study the transition coefficient in the particular situation when the manifold M is a two-dimensional sphere with radius one. We assume that the half-lines $R^{\mp}$ are joined to M at the opposite points x_1, x_2. We will use the following well-known information about the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on the sphere:

1. The eigenvalues $\lambda_{l,m}$, $m = -l, ..., 0, ..., l$ are equal to $l(l+1)$ with the degeneracy $2l+1$.
2. The corresponding normalized eigenfunctions (spherical harmonics) are

$$\phi_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} P_l^{|m|}(\cos\theta)\exp(im\psi),$$

where $P_l^{|m|}$ are adjoint Lagrange polynomials and (θ, ψ) are the standard coordinates on the sphere.

Let us choose the coordinate system on the sphere so that the points x_1 and x_2 have the coordinates $\theta = 0$ and π respectively. The adjoint Lagrange polynomials have the following well-known properties:

$$P_l^m(\pm 1) = 0, \text{ if m} > 0; \;\; \mathrm{P}_1^0(1) = 1 \text{ and } \mathrm{P}_1^0 = (-1)^{\mathrm{l}}.$$

Hence, all the eigenfunctions which correspond to the index $m \neq 0$ vanish at the points x_1 and x_2 and therefore need not be taken into account. The key expressions $h(\lambda)$ and $g(\lambda, \lambda_0)$ do not depend on the eigenfunctions which vanish at the joint points (we note that in our case because of the symmetry $g_1(\lambda, \lambda_0) = g_2(\lambda, \lambda_0)$; henceforth, we may denote this function by $g(\lambda)$, not showing explicitly dependence on λ_0). Therefore, the "input" spectral information for our particular example is as follows:

$$\lambda_l = l(l+1); \;\; \phi_l(x_1) = \sqrt{\frac{2l+1}{4\pi}} \text{ and } \phi_l(x_2) = \sqrt{\frac{2l+1}{4\pi}}(-1)^l.$$

Remark. We note that due to the symmetry of the problem we can view our construction as a coupling of two half-lines to the singular Sturm-Liouville operator on the segment (which in our case is given by the Lagrange differential expression). Indeed, the Laplace-Beltrami operator for a sphere in parabolic coordinates r, x, ϕ decomposes into a direct sum of the one-dimensional operators H_m

$$H_m = \frac{d}{dx}((1-x^2)\frac{d}{dx}) + \frac{m^2}{1-x^2}$$

acting on the subspaces of the functions of type $f(x)\exp(\mp m\phi)$. When $m > 0$, due to the singularity of the potential these operators are limit point at the end points ∓ 1 and hence may not be coupled to the half-lines. The author is grateful to Professor P. Exner for this remark.

Now we prove several technical lemmas which describe the behavior of the functions h and g at high energies. First, let us write an explicit formula for these functions in the sphere case:

$$h(\lambda) = \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{(2l+1)(-1)^l}{l(l+1)-\lambda} \tag{8}$$

$$g(\lambda) = \frac{1}{4\pi}(\lambda-\lambda_0)\sum_{l=0}^{\infty}\frac{(2l+1)}{(l(l+1)-\lambda)(l(l+1)-\lambda_0)}. \tag{9}$$

Lemma 3.1. *For every real value of λ, we have $g'(\lambda) > |h'(\lambda)| + \frac{1}{20}$.*

Remark. Certainly the constant is far from optimal. We need only the fact that the function g changes faster than h by some fixed constant at any point.

Proof. The proof is straightforward. We compute that

$$g'(\lambda) = \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{2l+1}{|l(l+1)-\lambda|^2};$$

$$h'(\lambda) = \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{(2l+1)(-1)^l}{|l(l+1)-\lambda|^2}.$$

Therefore,

$$g'(\lambda) - |h'(\lambda)| \geq \max_{j=0,1}\frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{(2(2l+j)+1)}{|(2l+j)(2l+j+1)-\lambda|^2}.$$

All terms in the last sum are positive. Estimating the minimum of the sum of the two terms closest to the given λ, we obtain the statement of the lemma.

Lemma 3.2. *For every λ in energy interval $(l(l-1), l(l+1))$, the function $h(\lambda)$ has the following asymptotic behaviour as $l \to \infty$:*

$$h(\lambda) = \frac{1}{4\pi}\left(-\frac{2l-1}{l(l-1)-\lambda} + \frac{2l+1}{l(l+1)-\lambda}\right)(-1)^l + O(1), \tag{10}$$

with $O(1)$ being uniform for all $\lambda \in R^+$.

Proof. We need to estimate the sum (8) on the interval $(l(l-1), l(l+1))$ with two singular terms deleted. The sum (8) on this interval is actually a sum with terms changing signs. If both singular terms in (8) are positive (which is the case when l is even), than both second terms "going from λ" are negative, then comes a pair of positive terms and so on. A simple calculation shows that if we write h as a sum of such a sign-changing series, the absolute value of each summand is greater than that of the next one. Hence, it is enough to estimate the first (non-singular) pair of the summands

$$\frac{1}{4\pi}\left(\left|\frac{2l-3}{\lambda-(l-1)(l-2)}\right| + \left|\frac{2l+3}{(l+1)(l+2)-\lambda}\right|\right)$$

for the values of λ in $(l(l-1), l(l+1))$. The first summand is maximal when $\lambda = l(l-1)$, with maximal value equal to $\frac{2l-3}{2(l-1)}$. The second summand is maximal at $\lambda = l(l+1)$ and hence does not exceed $\frac{2l+3}{2(l+1)}$. Taking into account the factor of $\frac{1}{2\pi}$, we see that the maximal value of the sum on the interval of interest to us is smaller than $\frac{1}{2\pi}$. Hence, we see that

$$\left|h(\lambda) - \frac{1}{4\pi}\left(\frac{2l-1}{-l(l-1)+\lambda} + \frac{2l+1}{l(l+1)-\lambda)}\right)\right| < \frac{1}{2\pi}.$$

This proves the lemma. $\square$

Lemma 3.3. *Let $\lambda \in (l(l-1), l(l+1))$. Then, for l large enough, the following representation for the function $g(\lambda)$ is valid in this interval* :

$$g(\lambda) = \frac{1}{4\pi}\left(\frac{2l-1}{l(l-1)-\lambda} + \frac{2l+1}{l(l+1)-\lambda}\right) - \frac{1}{2\pi}\log l + O(1), \tag{11}$$

with $O(1)$ being uniform for all $\lambda \in R^+$.

Proof. Again, we single out two terms which are singular on the considered interval and estimate the remaining sum. First, note that

$$\frac{(\lambda-\lambda_0)(2l-1)}{(l(l-1)-\lambda_0)(l(l-1)-\lambda)} - \frac{2l-1}{l(l-1)-\lambda} = \frac{2l-1}{l(l-1)-\lambda_0} = O(1/l)$$

so that we can replace the cumbersome singular terms which we have in a formula (9) for g by the ones we use in (11). To see the behavior of the rest of the sum for g, let us split it into two groups:

$$g_-(\lambda, l) = (\lambda-\lambda_0)\sum_{m=0}^{l-2}\frac{2m+1}{(m(m+1)-\lambda_0)(m(m+1)-\lambda)}$$

and

$$g_+(\lambda, l) = (\lambda-\lambda_0)\sum_{m=l+1}^{\infty}\frac{2m+1}{(m(m+1)-\lambda_0)(m(m+1)-\lambda)}\,.$$

The function g_- is a negative part of the summands in a sum (9) for g, while g_+ is a positive part. The following estimate is standard:

$$g_-(\lambda, l) = (\lambda-\lambda_0)\int_1^{l-2}\frac{2x+1}{(x(x+1)-\lambda_0)(x(x+1)-\lambda)}\,dx + O(1). \tag{12}$$

Indeed, we can estimate the series for g_- by the integral from above and below, depending on whether we include or omit several summands which are of order $O(1)$ uniformly in λ. We can further simplify the expression (12) by writng an equality

$$(\lambda-\lambda_0)\int_1^{l-2}\frac{2x+1}{(x(x+1)-\lambda_0)(x(x+1)-\lambda)}\,dx = 2\lambda\int_1^{l-2}\frac{dx}{x(x(x+1)-\lambda)} + O(1),$$

which is easy to verify keeping in mind that λ_0 is a once and for all fixed negative number. The latter integral may already be estimated by easily computable integrals:

$$2l(l-1)\int_1^{l-2}\frac{dx}{x(x(x+1)-l(l+1))}$$

for the lower bound and

$$2l(l+1)\int_1^{l-2}\frac{dx}{x(x+1)-l(l-1))}$$

for the upper. Let us compute, for example, the upper bound integral:

$$\frac{1}{(x(x+1)-l(l-1))}=-\frac{1}{l(l-1)}\frac{1}{x}+\frac{1}{(2l-1)(l-1)}\frac{1}{(x-l+1)}+\frac{1}{(2l-1)l}\frac{1}{(x+l)},$$

and hence

$$2l(l+1)\int_1^{l-2}\frac{dx}{x(x+1)-l(l-1)}=-2\log l-\log l+\log 2l-\log l+O(1).$$

Therefore, the upper bound integral behaves as $-3\log l+O(1)$ for l large. The uniformity of $O(1)$ in λ is easy to check. The computation of the lower bound integral proceeds in a similar way and gives the same asypmtotic behavior. This proves that

$$g_-(\lambda,l)=-3\log l+O(1)$$

as $l\to\infty$ with $O(1)$ uniform in λ. For the function g_+ we get an analogous estimate that

$$g_+(\lambda,l)=2\lambda\int_{l+1}^{\infty}\frac{dx}{x(x(x+1)-\lambda)}+O(1).$$

Again, we can estimate the latter integral from above and below using the assuptions on the interval where λ varies. The two integrals which stay as bounds in this estiamte turn out to have identical asymptotic behavior up to a uniform constant term. Let us for example evaluate the upper bound:

$$2l(l+1)\int_{l+1}^{\infty}\frac{dx}{x(x(x+1)-l(l+1))}.$$

First we find that

$$\frac{1}{x(x(x+1)-l(l+1))}=-\frac{1}{l(l+1)}\frac{1}{x}+\frac{1}{l(2l+1)}\frac{1}{(x-l)}+\frac{1}{(l+1)(2l+1)}\frac{1}{(x+l+1)}.$$

Next we compute that

$$2l(l+1)\int_{l+1}^{\infty}\frac{dx}{x(x(x+1)-l(l+1))}=$$

$$=\lim_{A\to\infty}\left(-2\int_{l+1}^{A}\frac{1}{x}\,dx+\frac{2l(l+1)}{l(2l+1)}\int_{l+1}^{A}\frac{1}{x-l}\,dx+\frac{2l(l+1)}{(l+1)(2l+1)}\int_{l+1}^{A}\frac{1}{x+l+1}\right)=$$

$$=-2(\log A-\log(l+1))+\frac{2(l+1)}{2l+1}\log(A-l)+\frac{2l}{2l+1}(\log(A+l+1)-\log 2(l+1))=$$

$$=\log l+O(1).$$

It is easy to conclude from the calcualtions that the $O(1)$ in the last formula is uniform in λ. Together with the expression for g_-, the last formula concludes the proof of the lemma. One has to add the asymptotic expressions for g_- and g_+ and remember about the $\frac{1}{4\pi}$, which we omitted in the series for $g_\mp$. $\square$

Remark. One can notice that the asymptotic in high energy behavior of the function g, with two terms singular on the given energy omitted, is identical to the behavior of the singularity of the Green's function in space (as a coordinate x turns to the singular point x_0, $G(x,x_0,\lambda_0)=-\frac{1}{2\pi}\log|x-x_0|+O(1)$). Intuitevely, by scaling argument, similar relation may also be true in more general situation.

Now we have all the necessary tools to study the transition coefficient $t(\lambda)$. The formula (7) for $t(\lambda)$ suggests that one can expect the decay of transition coefficient as $\lambda\to\infty$. Indeed, the highest power of λ in the denominator is one, while in the numerator we have only $\sqrt{\lambda}$. However, h and g may have singularities, which might dominate any λ-growth at certain intervals. Also, g^2-h^2 may have zeroes, which will kill the λ-growth for some energies. The composition of all these factors shapes the benavior of $t(\lambda)$.

The following lemma describes the zeroes of the expression $g^2 - h^2$.

Lemma 3.4. *For l sufficiently large in each interval $(l(l-1), l(l+1))$, there exists a unique number μ_l such that $g^2(\mu_l) - h^2(\mu_l) = 0$. The position of μ_l satisfies the condition*

$$l(l+1) - \mu_l = 2l(\log l)^{-1}(1 + o(1)). \tag{13}$$

Proof. $g^2 - h^2 = (g-h)(g+h)$. Using the formulas (10) and (11) it is easy to check that for l large enough, the function $g - (-1)^l h$ is always negative, while $g + (-1)^l h = 0$ has a root μ_l satisfying

$$\frac{4l+2}{l(l+1) - \mu_l} - 2\log l + O(1) = 0\,.$$

But the equation $g + h = 0$, as well as $g - h = 0$, has always a unique, if any, root on the interval $(l(l-1), l(l+1))$ (this follows easily from Lemma 3.1: $g'(\lambda) > |h'(\lambda)| + c$ for every λ). This implies, for l large enough, the uniqueness of the root of $g^2 - h^2$ in the stated interval. From the equation above follows (13). □

Now we are in a position to prove the main result of this section:

Theorem 3.5. *Let $K_\epsilon = R^+ \setminus \cup_{l=2}^{\infty}(\mu_l - \epsilon(l)(\log l)^{-2}, \mu_l + \epsilon(l)(\log l)^{-2})$, where $\epsilon(x)$ is an arbitrary positive monotonously increasing to $+\infty$ function. Then, as $\lambda \to \infty$ on K_ϵ, the transition coefficient turns to zero with the rate $\epsilon(\lambda)^{-1}$. That is, there exists the constant C_ϵ, such that if $\lambda \in K_\epsilon$, then $t(\lambda) \leq C\epsilon(\lambda)^{-1}$.*

On the other hand, $t(\mu_l) = 1 + O((l\log l)^{-1})$, so that for l large enough, $t(\lambda)$ has sharp peaks on each interval $(l(l-1), l(l+1))$.

Remark. It may seem somewhat surprising that the high energy resonances do not lie close to the eigenvalues of Δ_M, but rather are connected with more subtle spectral information.

Proof. Suppose that λ lies in $K_{l,\epsilon} \cap (l(l-1), l(l+1))$. Then

$$|g(\lambda) + (-1)^l h(\lambda)| \geq \frac{1}{2}\epsilon(\lambda) l^{-1}$$

for λ large enough. Indeed,

$$|g(\lambda) + (-1)^l h(\lambda)| = |\frac{2(2l+1)}{\lambda_l - \lambda} - 2\log l + O(1)| \geq \frac{1}{\lambda_l - \lambda}(\log l)(\log l)^{-2}\epsilon(\lambda) \geq \frac{1}{2}\epsilon(\lambda) l^{-1},$$

since $O(1)$ in the formula above is smaller than $\log l$ for l large and the shift from the root μ_l is larger than $\epsilon(\lambda)(\log l)^{-2}$. Suppose now that λ lies to the left from μ_l. Then for large l we have that

$$|t(\lambda)| \leq \frac{-\frac{2l-1}{\lambda_{l-1}-\lambda} - 2\log l}{\frac{1}{8\pi}\epsilon(\lambda)(\sqrt{\lambda} l^{-1})(-\frac{(2l-1)}{\lambda_{l-1}-\lambda} - 2\log l + O(1)) + \frac{2l-1}{\lambda_{l-1}-\lambda} + 2\log l + O(1)}.$$

The $O(1)$ terms are not significant for large l and therefore we obtain that

$$|t(\lambda)| \leq (\frac{1}{2}\epsilon(\lambda) - 1)^{-1}\frac{l}{\sqrt{\lambda}} \leq C(\epsilon(\lambda))^{-1}.$$

The case when λ lies to the right from μ_l is considered in the same way. Omitting not important (up to a different constant in front of inequality) $O(1)$ terms and denoting C some uniform in λ (not necessarily the same) constants, we have that:

$$|t(\lambda)| \leq C\frac{\frac{2l+1}{\lambda_l - \lambda}}{2\log l + \frac{2l+1}{\lambda_l-\lambda} - \sqrt{\lambda}\log l(g(\lambda) + (-1)^l h(\lambda))} \leq$$

$$\leq C\frac{1}{3 - \sqrt{\lambda}\log l(1 - (2\log l + O(1))\frac{\lambda_l - \lambda}{2l+1})} \leq C(\epsilon(\lambda))^{-1}.$$

In the second inequality we used the asymptotics (10) and (11) of the functions h, g and in the last inequality we used that λ is shifted from the root μ_l by more than $\epsilon(\lambda)(\log l)^{-2}$ and the fact that $\sqrt{\lambda} \geq \sqrt{l(l-1)}$ on the interval we consider. This proves the first assertion of the theorem. For the proof of the second claim ("sharp peaks"), we notice that

$$|t(\mu_l)| = \frac{h(\mu_l)}{-\frac{1}{2i\sqrt{\lambda}\beta^2} + g(\mu_l)} = 1 + O((l\log l)^{-1}),$$

since $|h(\mu_l)| = |g(\mu_l)|$ and (13) together with the representations (11) and (12) allow us to compute the order of their common value. $\square$

We note that varying the function $\epsilon(x)$, we get different pictures: if we take ϵ, for example, to be equal to $\log x$, we have peaks in the intervals of size $(\log l)^{-1}$ around μ_l and the decay of $t(\lambda)$ proportional to $(\log(\lambda))^{-1}$ for the rest of the energies. Taking $\epsilon(x)$ equal to $x(\log x)^{-1}$, we get the decay of $t(\lambda)$ on K_ϵ proportional to $(\sqrt{\lambda})^{-1}$, but now the segments we had to exclude are not small: they are of the size $l(\log l)^{-1}$ on the intervals $(l(l-1), l(l+1))$ and hence may be considered "small" only relatively. These two choices characterize to a certain degree the properties of the transition coefficient at high energies: the sharpness of peaks and the regions of fast decay.

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