Stability of Travelling Waves for a Damped Hyperbolic Equation

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Abstract. We consider a nonlinear damped hyperbolic equation in $\mathbf{R}^n$, $1 \leq n \leq 4$, depending on a positive parameter ϵ. If we set $\epsilon = 0$, this equation reduces to the well-known Kolmogorov-Petrovski-Piskunov equation. We remark that, after a change of variables, this hyperbolic equation has the same family of one-dimensional travelling waves as the KPP equation. Using various energy functionals, we show that, if $\epsilon > 0$, these fronts are locally stable under perturbations in appropriate weighted Sobolev spaces. Moreover, the decay rate in time of the perturbed solutions towards the front of minimal speed $c = 2$ is shown to be polynomial. In the one-dimensional case, if $\epsilon < 1/4$, we can apply a Maximum Principle for hyperbolic equations and prove a global stability result. We also prove that the decay rate of the perturbated solutions towards the fronts is polynomial, for all $c > 2$.

1. Introduction

We consider the damped hyperbolic equation

$$\epsilon u_{tt}(\xi,t) + u_t(\xi,t) \;=\; \Delta_\xi u(\xi,t) + f(u(\xi,t)) \;, \tag{1.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{R}^n$, $t \in \mathbf{R}$, $\epsilon > 0$, and $f : \mathbf{R} \to \mathbf{R}$ is a nonlinear map. In the one-dimensional case $n = 1$, equations of the form (1.1) arise as mathematical models describing various natural phenomena, like the propagation of voltage along a nonlinear transmission line, or the random motion of one-celled organisms [DO]. Here we consider the multidimensional case also, and we are interested in situations where the parabolic equation obtained by taking the limit $\epsilon \to 0$ in (1.1) has a continuous family of travelling waves (or fronts) propagating into the unstable state $u \equiv 0$. Sufficient conditions on the nonlinearity f for such a situation to occur are discussed for example in Aronson and Weinberger [AW]. For convenience, we restrict ourselves to the typical example of the Kolmogorov-Petrovsky-Piskunov equation (KPP), which corresponds to $f(u) = u - u^2$, but more general nonlinearities can be treated by the same methods.

Existence of travelling wave solutions to damped hyperbolic equations has been proved by Hadeler [Ha] in a general context. In our case, this is simply done by setting

$$u(\xi,t) \;=\; g(\sqrt{1+\epsilon c^2}\,\xi_1 - ct) \;, \tag{1.2}$$

and inserting into (1.1). One obtains for g the differential equation

$$g''(x) + cg'(x) + g(x) - g(x)^2 \;=\; 0 \;. \tag{1.3}$$

It is well-known [AW] that, for all $c \geq 2$, this equation has a front-like solution $g(x)$ satisfying $g'(x) < 0$ for all $x \in \mathbf{R}$, $\lim_{x\to-\infty} g(x) = 1$, $\lim_{x\to+\infty} g(x) = 0$, and $g(x)$ is unique up to a translation in the variable x. Therefore, for all $\epsilon > 0$, Eq.(1.1) has a continuous family of travelling waves of the form (1.2) indexed by the parameter $c \geq 2$. It should be noted that the speed of such a wave is no longer c, but $c/\sqrt{1+\epsilon c^2}$, a quantity which is bounded by $1/\sqrt{\epsilon}$ as $c \to \infty$. This is of course related to the finite propagation speed property of equation (1.1).

The stability of travelling waves for KPP and similar nonlinear parabolic equations has been intensively studied over many years. Early results have been obtained using comparison theorems based on the Maximum Principle, see [KPP], [Fi], [AW]. Combined with probabilistic techniques, these methods give a very detailed description of the basin of attraction of the wave [Bn]. In parallel, a local stability analysis of the front in suitable weighted spaces has been initiated by Sattinger [Sa] and continued recently by Kirchgässner [Ki], Kapitula [Ka], Bricmont and Kupiainen [BK], Gallay [Ga], Eckmann and Wayne [EW], using functional-analytic techniques, renormalization group methods, or energy functionals. In particular, the decay rate in time of the perturbations in the critical case $c = 2$ has been investigated [Ki], [BK], [Ga]. Similar results have also been obtained for higher dimensional equations [MJ], and for systems of parabolic equations [KR], [RK].

The aim of this paper is to extend part of the stability results available for the KPP equation to the hyperbolic equation (1.1). In particular, using energy functionals, we shall show that the travelling waves (1.2) are locally stable in appropriate function spaces for all $c \geq 2$ and all $\epsilon > 0$. Moreover, using the Maximum Principle for hyperbolic equations, we shall prove a global stability result in the one-dimensional case, provided ϵ is sufficiently small. Finally, a decay rate as $t \to +\infty$ of the perturbations will be obtained if $c = 2$, or if $n = 1$ and $\epsilon < 1/4$.

We now proceed to state our results in a more precise way. Given $\epsilon > 0$, $c \geq 2$, we go to a moving frame using the change of variables

$$u(\xi, t) \;=\; v(\sqrt{1+\epsilon c^2}\xi_1 - ct, \xi_2, \ldots, \xi_n, t) \equiv v(x, y, t) \;, \tag{1.4}$$

where $x = \sqrt{1+\epsilon c^2}\xi_1 - ct$ and, if $n > 1$, $y = (\xi_2, \ldots, \xi_n)$. The equation for v is

$$\epsilon v_{tt} + v_t - 2\epsilon c v_{xt} \;=\; v_{xx} + \Delta_y v + c v_x + v - v^2 \;, \tag{1.5}$$

and by construction $v(x, y, t) = g(x)$ is a stationary solution of (1.5). As in the parabolic case, this solution can only be stable if we restrict ourselves to perturbations which decay to zero sufficiently fast as $x \to +\infty$. To achieve this decay, we look for solutions of the form $v(x, y, t) = g(x) + a(x)w(x, y, t)$, where $a(x) = e^{-\gamma x}$ for some $\gamma > 0$ which will be fixed later. Then $w(x, y, t)$ satisfies the equation

$$\epsilon w_{tt} + (1+2\epsilon c\gamma)w_t - 2\epsilon c w_{xt} \;=\; w_{xx} + \Delta_y w + (c-2\gamma)w_x + (1-c\gamma+\gamma^2-2g)w - aw^2 \;. \tag{1.6}$$

Since Eq.(1.6) is of second order in time, we shall rewrite it in the usual way as a first order system for the pair (w, w_t), and study the stability of the origin $(w, w_t) = (0, 0)$ for this system in a space Z^1_ϵ which we now describe.

Function spaces. For all $j \in \mathbf{N}$, we denote by $H^j = H^j(\mathbf{R}^n)$ the usual Sobolev space of order j over $\mathbf{R}^n$, with $H^0(\mathbf{R}^n) = L^2(\mathbf{R}^n)$. Similarly, we denote by $H^j_a =$

$H_a^j(\mathbf{R}^n)$ the weighted Sobolev space defined by the norm $\|w\|_{H_a^j} = \|aw\|_{H^j}$. We also set $L_a^2 = H_a^0$. We write X^j for the intersection $H^j \cap H_a^j$ equipped with the norm $\|w\|_{X^j}^2 = \|w\|_{H^j}^2 + \|w\|_{H_a^j}^2$, and Z_ϵ^j for the product $X^j \times X^{j-1}$ equipped with the (ϵ-dependent) norm

$$\|(w_1, w_2)\|_{Z_\epsilon^j}^2 = \|w_1\|_{X^j}^2 + \epsilon\|w_2\|_{X^{j-1}}^2 . \tag{1.7}$$

Finally, we define $Y_\epsilon = H^1 \times L^2$ and $Y_{\epsilon a} = H_a^1 \times L_a^2$, equipped with the ($\epsilon$-dependent) norms

$$\|(w_1, w_2)\|_{Y_\epsilon}^2 = \|w_1\|_{H^1}^2 + \epsilon\|w_2\|_{L^2}^2 , \quad \|(w_1, w_2)\|_{Y_{\epsilon a}} = \|(aw_1, aw_2)\|_{Y_\epsilon} .$$

Remark that $Z_\epsilon^1 \equiv Y_\epsilon \cap Y_{\epsilon a}$.

It follows from these definitions that $(w, w_t) \in Z_\epsilon^1$ if and only if $(aw, aw_t)(1+e^{\gamma x}) \in H^1 \times L^2$. Therefore, our perturbation space $\{(aw, aw_t) \,|\, (w, w_t) \in Z_\epsilon^1\}$ depends on γ and becomes smaller when γ is increased. On the other hand, using a direct calculation in Fourier space, it is not difficult to verify that the origin in (1.6) is linearly stable in Z_ϵ^1 if and only if $1 - c\gamma + \gamma^2 \le 0$. In fact, this condition can be read off from the coefficient of w in (1.6). As a consequence, the biggest perturbation space in which we can hope for stability of the wave is obtained by taking

$$\gamma = \frac{c}{2} - \sqrt{\frac{c^2}{4} - 1} . \tag{1.8}$$

Note that this value corresponds to the exponential decay rate of $g(x)$ as $x \to +\infty$, since $g(x) \sim e^{-\gamma x}$ if $c > 2$ and $g(x) \sim xe^{-x}$ if $c = 2$ [AW]. In the sequel, we shall always assume that (1.8) holds, so that (1.6) becomes

$$\epsilon w_{tt} + (1 + 2\epsilon c\gamma)w_t - 2\epsilon c w_{xt} = w_{xx} + \Delta_y w + \sqrt{c^2 - 4}\, w_x - 2gw - aw^2 . \tag{1.9}$$

Furthermore, we shall assume without loss of generality that $g(0) = 1 - \sigma$ for some $\sigma \le 1/8$, and that $g(x) \ge 2a(x)/3$ for all $x \ge 0$. This can always be achieved by replacing $g(x)$ by $g(x - x_0)$ for some sufficiently large $x_0 > 0$.

Remark. As in the parabolic case, one can show that the origin in (1.6) is exponentially stable in Z_ϵ^1 if $c > 2$ and $1 - c\gamma + \gamma^2 < 0$. The fastest decay rate is obtained for the value

$$\hat{\gamma}(\epsilon) = \frac{c}{2}\sqrt{\frac{1 + 4\epsilon}{1 + \epsilon c^2}} . \tag{1.10}$$

Since these results are rather straightforward to prove, we shall focus here on the marginal choice (1.8) for which no exponential decay is expected.

Using these definitions, we can state our first result, which shows that the travelling waves are locally stable.

Theorem 1.1. *Assume that $n \le 4$, and let $\epsilon_0 > 0$, $c \ge 2$. Then there exist constants $\delta_0 > 0$ and $K_0 \ge 1$ such that, for all $0 < \epsilon \le \epsilon_0$, the following holds : for all $(\varphi_0, \varphi_1) \in Z^1_\epsilon$ such that $\|(\varphi_0, \varphi_1)\|_{Z^1_\epsilon} \le \delta_0$, there exists a unique solution $(w, w_t) \in C^0(\mathbf{R}_+, Z^1_\epsilon)$ of (1.9) with initial data $(w(0), w_t(0)) = (\varphi_0, \varphi_1)$. Moreover, one has*

$$\|(w(t), w_t(t))\|_{Z^1_\epsilon} \;\le\; K_0 \|(\varphi_0, \varphi_1)\|_{Z^1_\epsilon} \;, \tag{1.11}$$

for all $t \ge 0$, and

$$\lim_{t\to+\infty} \left(\|\nabla w(t)\|_{X^0} + \|w_t(t)\|_{X^0} + \|w(t)\|_{L^2_a}\right) \;=\; 0 \;. \tag{1.12}$$

In addition, if $c = 2$, one has

$$\lim_{t\to+\infty} \sqrt{t}\, \left(\|\nabla w(t)\|_{X^0} + \|w_t(t)\|_{X^0} + \|w(t)\|_{L^2_a}\right) \;=\; 0 \;. \tag{1.13}$$

Remarks.

1.) By a solution of (1.9), we always mean a *mild* solution, that is a solution of the integral equation associated with (1.9), see the proof of Proposition 2.1 below. In general, such solutions satisfy (1.9) in a distributional sense only, see Lions [Li], Section 1.1. Remark that w_{tt} or w_{xx} belong to $C^0(\mathbf{R}_+, X^{-1})$ only, but $w_{tt} - 2\epsilon c w_{xt} - w_{xx} - \Delta_y w \in C^0(\mathbf{R}_+, X^0)$ by (1.9). Moreover, if $(\varphi_0, \varphi_1) \in Z^2_\epsilon$, then the solution (w, w_t) belongs to $C^1(\mathbf{R}_+, Z^1_\epsilon) \cap C^0(\mathbf{R}_+, Z^2_\epsilon)$ and satisfies (1.9) in a classical sense. In (1.11) and in the sequel, we use the short notation $w(t)$ for $w(\cdot, \cdot, t)$, when no confusion is possible.

2.) The restriction $n \le 4$ arises because we control the nonlinearity aw^2 in (1.9) in the energy space Z^1_ϵ, using the Sobolev embedding of $H^1(\mathbf{R}^n)$ into $L^4(\mathbf{R}^n)$. More generally, if $f(u)$ in (1.1) is a polynomial of degree $p > 1$, we assume that $n \le 2p/(p-1)$. This bound could be improved up to $2(p+1)/(p-1)$ using the more sophisticated $L_p - L_{p'}$ estimates of Strichartz [St], [Br].

3.) If $n \ge 3$, it follows from (1.11), (1.12) and the Sobolev embedding theorem that

$$\lim_{t\to+\infty} \left(\|w(t)\|_{L^q} + \|aw(t)\|_{L^q}\right) \;=\; 0 \;, \quad 2 < q \le \frac{2n}{n-2} \;. \tag{1.14}$$

If $n = 2$, then (1.14) is valid for all $q > 2$ and even for $q = \infty$ if $n = 1$. In the case $c = 2$, (1.11) and (1.13) imply that

$$\lim_{t\to+\infty} \Big(t^\eta \|w(t)\|_{L^q} + t^{1/2}\|aw(t)\|_{L^q} \Big) \;=\; 0 \;, \qquad \eta = \frac{n(q-2)}{4q} \;,$$

for the same values of q.

Theorem 1.1 is a local result in the sense that the size of the basin of attraction of the wave, in particular its dependence on the parameter $\epsilon > 0$, is not specified. However, in the parabolic limit $\epsilon \to 0$, it is known [KR] that the travelling fronts are stable with respect to large positive perturbations, and a similar phenomenon is expected to hold for (1.5) if ϵ is sufficiently small. To investigate this, we restrict ourselves for convenience to one space dimension, and we apply the Maximum Principle for hyperbolic equations, which is briefly recalled in Appendix A. Our second result reads:

Theorem 1.2. *Assume that $n = 1$, and let $\epsilon_0 > 0$, $c \ge 2$, $d \in (0,1]$. Then for any $0 < \epsilon \le \epsilon_0$ and for any constant $K > 0$ such that*

$$1 - 4\epsilon(d+K) \;\ge\; 0 \;, \tag{1.15}$$

there exists a constant $K^* = K^*(\epsilon_0, c, d, K) > 0$ such that the following holds: for any $(\varphi_0, \varphi_1) \in Z^1_\epsilon$ satisfying $\|(\varphi_0,\varphi_1)\|_{Z^1_\epsilon} \le K^*$ and, for (almost) every $x \in \mathbf{R}$,

$$\varphi_0(x) \;\ge\; -(1-d)a(x)^{-1}g(x) \;, \tag{1.16}$$

$$\epsilon\varphi_1(x) \;\ge\; \epsilon c\varphi_0'(x) - \frac{1}{2}(\varphi_0 + (1-d)a^{-1}g)(x) + \epsilon c(-\gamma\varphi_0 + (1-d)a^{-1}g')(x) \;, \tag{1.17}$$

there exists a unique solution $(w, w_t) \in C^0(\mathbf{R}_+, Z^1_\epsilon)$ of (1.9) with initial data (φ_0, φ_1). Moreover, one has

$$\|(w(t), w_t(t))\|_{Z^1_\epsilon} \;\le\; K \;, \quad w(x,t) \;\ge\; -(1-d)a^{-1}(x)g(x) \;, \tag{1.18}$$

for all $x \in \mathbf{R}$, $t \in \mathbf{R}_+$, and (1.12), (1.13) hold.

Remarks.

1) The proof will show that the constant K^* can be chosen so as to satisfy the equation $K_6K^*(1+K^*)^{1/2} = K/2$ for some $K_6 = K_6(\epsilon_0, c, d) \ge 1$. Therefore, if ϵ is small, K (hence K^*) can be chosen very big by (1.15), and Theorem 1.2 shows in this case that

the travelling wave is stable with respect to large perturbations, provided they satisfy the "positivity conditions" (1.16), (1.17). Conversely, if ϵ is large, then K (hence K^*) has to be very small, and Theorem 1.2 reduces to a local stability result similar to Theorem 1.1.

2) The conditions (1.16), (1.17) appear when applying the Maximum Principle to the equation (1.5), see Appendix A. The first one simply says that $v(x,0) \geq dg(x)$ for all $x \in \mathbf{R}$. The condition on the derivative is not very restrictive if ϵ is small, and disappears in the limit $\epsilon \to 0$.

The previous results are incomplete in the sense that they fail to give a decay rate for the perturbations when $c > 2$. Also, it would be very natural to have at least a global existence result if $d = 0$. Indeed, it is known that, if $0 \leq v(x,0) \leq g(x)$, the solution $v(x,t)$ of the parabolic equation (1.5) with $\epsilon = 0$ exists for all times and satisfies $0 \leq v(x,t) \leq g(x)$. In terms of the variable w, this corresponds to $-a^{-1}(x)g(x) \leq w(x,t) \leq 0$. A similar property is expected to hold for the hyperbolic equation (1.9) if ϵ is sufficiently small.

A partial answer to these two questions can be given when $\epsilon \leq 1/4$. Indeed, in this case the Maximum Principle allows us to compare the solution $w(x,t)$ of (1.9) with solutions of *linear* equations, whose initial data are the "positive" and "negative" parts $(\varphi_0^\pm, \varphi_1^\pm)$ of (φ_0, φ_1), in the sense of Appendix A. They are given by

$$\begin{aligned}
&\varphi_0^+(x) = \sup(0, \varphi_0(x))\,,\\
&\varphi_1^+(x) = c(\varphi_0^+)'(x) - (\frac{1}{2\epsilon} + c\gamma)\varphi_0^+(x) + \sup(0, (\varphi_1 - c\varphi_0' + (\frac{1}{2\epsilon} + c\gamma)\varphi_0)(x))\,,
\end{aligned} \tag{1.19}$$

and

$$\begin{aligned}
&\varphi_0^-(x) = \inf(0, \varphi_0(x))\,,\\
&\varphi_1^-(x) = c(\varphi_0^-)'(x) - (\frac{1}{2\epsilon} + c\gamma)\varphi_0^-(x) + \inf(0, (\varphi_1 - c\varphi_0' + (\frac{1}{2\epsilon} + c\gamma)\varphi_0)(x))\,.
\end{aligned} \tag{1.20}$$

Remark that $(\varphi_0^\pm, \varphi_1^\pm)$ belong to Z_ϵ^1 and that $\varphi_i = \varphi_i^+ + \varphi_i^-$ for $i = 0, 1$. Although the norms of $\varphi_1^\pm$ seem to depend strongly on ϵ, it is not the case actually: the reader may check that $|\varphi_1^\pm(x)| \leq |\varphi_1(x)| + c|\varphi_0'(x)|$ a.e. in $\mathbf{R}$.

With these definitions, we can state the last result:

Theorem 1.3. *Assume that $n = 1$, and let $c \geq 2$, $d \in [0,1]$. Then for any $0 < \epsilon \leq 1/4$ and for any nonnegative constant K satisfying*

$$1 - 4\epsilon(1 + K) \geq 0\,,$$

there exists a nonnegative constant $\tilde{K} = \tilde{K}(c, K)$ *such that the following holds: for any* $(\varphi_0, \varphi_1) \in Z^1_\epsilon$ *satisfying (1.16), (1.17) and*

$$\inf\left(\|(\varphi_0, \varphi_1)\|_{Z^1_\epsilon}\,,\ \|(\varphi_0^+, \varphi_1^+)\|_{Z^1_\epsilon}\right) \le \tilde{K}\ ,$$

where $(\varphi_0^+, \varphi_1^+)$ *is given by (1.19), there exists a unique solution* $(w, w_t) \in C^0(\mathbf{R}_+, Z^1_\epsilon)$ *of (1.9) with initial data* (φ_0, φ_1)*. Moreover, one has*

$$-(1-d)g(x) \le a(x)w(x,t) \le K\ ,$$

for all $x \in \mathbf{R}$*,* $t \in \mathbf{R}_+$*. Finally, if* $d > 0$ *and* $\epsilon < 1/4$*, one has*

$$\lim_{t\to+\infty} t^{1/4}\left(\|w(t)\|_{L^\infty} + \|(w(t), w_t(t))\|_{Y_{\epsilon a}}\right) = 0\ .$$

Remark. The constant $\tilde{K}$ is given by $\tilde{K} = K/N$, where $N = N(c)$ is a positive constant. Note that the case $K = 0$ is non trivial: it corresponds to nonpositive initial data.

An outline of the paper is as follows. In Section 2 we introduce energy functionals which allow us to derive *a priori* estimates for the solutions $w(x,t)$ of (1.9) under the assumption that either $\|w(t)\|_{X^0}$ is sufficiently small or $w(x,t)$ satisfies the lower bound in (1.18) on some time interval. Using these energy estimates, we prove Theorem 1.1 in Section 3. Section 4 is devoted to the one-dimensional case $n = 1$. Combining the Maximum Principle with the estimates of Section 2, we derive Theorem 1.2. Furthermore, when $\epsilon \le 1/4$, we obtain linear bounds for the solutions of (1.9) which allow us to prove Theorem 1.3. In Section 5, we consider the limiting parabolic equation (1.1) when $\epsilon = 0$. Noting that all the estimates made in Section 2 are uniform in ϵ when $0 < \epsilon \le \epsilon_0$, and using the Maximum Principle for parabolic equations, we obtain analogues of Theorem 1.1 and Theorem 1.3. Thereby we recover some known stability results for the KPP equation. Finally, in Appendix A, we recall the Maximum Principle for hyperbolic equations [PW] in a version adapted to our purposes.

2. Energy Estimates

In this section, we derive some *a priori* estimates for the solutions of (1.9) which will be needed in the proofs of Theorem 1.1 and Theorem 1.2. We begin with a standard local existence result.

Proposition 2.1. *Let $\epsilon > 0$, $c \geq 2$, and let $(\varphi_0, \varphi_1) \in Z^1_\epsilon$. Then there exists a time $T = T(\epsilon, c, \varphi_0, \varphi_1) > 0$ such that (1.9) has a unique solution $(w, w_t) \in C^0([0,T], Z^1_\epsilon)$ satisfying $(w(0), w_t(0)) = (\varphi_0, \varphi_1)$.*

Remark. In fact, the proof gives a lower bound on the existence time which depends only on ϵ, c and $\|(\varphi_0, \varphi_1)\|_{Z^1_\epsilon}$. Moreover, the energy estimates below will show that this time is independent of ϵ if $\epsilon \in (0, \epsilon_0]$.

Proof. Setting $W = (w, w_t)^{\mathrm{t}}$ (where $^{\mathrm{t}}$ denotes the transposition), we rewrite (1.9) into the "abstract form"

$$\dot{W} \;=\; AW + F(W) \;, \tag{2.1}$$

where A is the linear operator

$$A \;=\; \begin{pmatrix} 0 & 1 \\ \epsilon^{-1}(\partial_x^2 + \Delta_y + \sqrt{c^2-4}\,\partial_x - 2g) & -\epsilon^{-1} + 2c(\partial_x - \gamma) \end{pmatrix} \;,$$

and $F(W) = (0, -\epsilon^{-1} a w^2)^{\mathrm{t}}$. It is not difficult to show that the operator A, defined on the domain $D(A) = Z^2_\epsilon$, is the generator of a C^0-semigroup [Pa] of bounded linear operators in Z^1_ϵ. Indeed, A can be written as the sum of a bounded operator (depending on $x \in \mathbf{R}$ through the function g) and an unbounded operator with constant coefficients, for which the property of being a generator can be verified by a direct calculation (using Fourier transforms). Therefore, it follows from a classical stability theorem ([Pa], Theorem 3.1.1) that A is the generator of a C^0-semigroup e^{At} in Z^1_ϵ.

On the other hand, it is easy to verify that $F : Z^1_\epsilon \to Z^1_\epsilon$ is a C^1 map. Indeed, if $w \in X^1 = H^1 \cap H^1_a$, then by the Sobolev embedding theorem $\|w\|^2_{L^4} \leq K_S \|w\|^2_{H^1}$ and $\|aw\|^2_{L^4} \leq K_S \|w\|^2_{H^1_a}$ for some $K_S > 0$. Therefore, $\|aw^2\|^2_{L^2} \leq \|aw\|^2_{L^4} \|w\|^2_{L^4} \leq K_S^2 \|w\|^2_{H^1} \|w\|^2_{H^1_a}$ and $\|aw^2\|^2_{L^2_a} = \|aw\|^4_{L^4} \leq K_S^2 \|w\|^4_{H^1_a}$. Combining these inequalities, we find that $\|aw^2\|_{X^0} \leq K_S \|w\|^2_{X^1}$, which proves that F maps Z^1_ϵ into itself. Since F is quadratic, the differentiability follows by the same estimates.

In view of these properties, a standard result in semigroup theory ([Pa], Theorem 6.1.4) implies that, for all $\Phi = (\varphi_0, \varphi_1) \in Z^1_\epsilon$, the integral equation

$$W(t) \;=\; e^{At}\Phi + \int_0^t e^{A(t-\tau)} F(W(\tau))\, d\tau \ ,$$

has a unique solution $W \in C^0([0,T], Z^1_\epsilon)$ for some $T > 0$. This is what we call a (mild) solution of (2.1), hence of (1.9). Moreover, if $\Phi \in Z^2_\epsilon$, then $W \in C^1([0,T], Z^1_\epsilon) \cap C^0([0,T], Z^2_\epsilon)$ and satisfies (2.1) in a classical sense ([Pa], Theorem 6.1.5). $\square$

In the sequel, we fix $\epsilon_0 > 0$, $c \geq 2$, and for some $\epsilon \in (0, \epsilon_0]$ we assume that we are given a solution $W = (w, w_t)$ of (1.9) (in the sense of Proposition 2.1) defined on some time interval $[0, T]$ and satisfying *one* of the following two assumptions:

Hypothesis H1:

$$\sup\{\|w(t)\|_{X^0} \,|\, t \in [0,T]\} \leq \delta \text{ for some sufficiently small } \delta > 0,$$

Hypothesis H2:

$$w(x,y,t) \geq -(1-d)a(x)^{-1}g(x) \text{ a.e.}(x,y),\ \forall t \in [0,T], \text{ for some } d \in (0,1].$$

These two cases are adapted to the purposes of the proofs of Theorem 1.1 and Theorem 1.2 respectively. To be specific, we assume in the first case that $\delta \leq 1/(8K_S)$, where K_S is the constant of the Sobolev embedding of H^1 into L^4 (like in the proof of Proposition 2.1).

Under these assumptions, we shall study two families of energy functionals: *unweighted* and *weighted* ones, which control the size of the solution $w(x,y,t)$ in the spaces Y_ϵ and $Y_{\epsilon a}$ respectively. We shall derive differential inequalities for these functionals, which will show that the solution $w(x,y,t)$ is bounded uniformly in time by a quantity depending only on the initial data.

2.1. Unweighted Functionals

Given $w(x,y,t)$ as above, we define

$$\begin{aligned} E_0(t) \;&=\; \int_{\mathbf{R}^n} \left(\frac{\epsilon}{2} w_t^2 + \frac{1}{2}|\nabla_z w|^2 + g w^2 + \frac{1}{3} a w^3 \right) dz \ , \\ E_1(t) \;&=\; \int_{\mathbf{R}^n} \left(\frac{1 + 2\epsilon c\gamma}{2} w^2 + \epsilon w w_t \right) dz \ , \qquad (2.2) \\ E_2(t) \;&=\; \alpha E_0(t) + E_1(t) \ , \end{aligned}$$

where $\alpha = \max(2\epsilon, 1/(2c^2))$. Here and in the sequel, we set $z = (x,y) \in \mathbf{R}^n$ and $dz = dxdy$.

Lemma 2.2. *Assume that H1 or H2 holds. Then*

$$E_0(t) \;\ge\; \int_{\mathbf{R}^n} \left(\frac{\epsilon}{2}w_t^2 + \frac{1}{4}|\nabla_z w|^2 + \frac{1}{2}gw^2\right) dz \;, \tag{2.3}$$

for all $t \in [0,T]$. Moreover, $E_0 \in C^1([0,T])$ and

$$\dot{E}_0(t) \;\le\; (c^2-4)E_0(t) \;. \tag{2.4}$$

Proof. We first control the cubic term in (2.2). Using the Cauchy-Schwarz and Sobolev inequalities, we have

$$\begin{aligned}
\left|\int_{x\le 0} aw^3\,dz\right| &\le \left(\int_{x\le 0} a^2w^2\,dz\right)^{1/2}\left(\int_{x\le 0} w^4\,dz\right)^{1/2} \\
&\le K_S\|w\|_{X^0}\int_{x\le 0}\left(w^2 + |\nabla_z w|^2\right)\,dz \\
&\le 2K_S\|w\|_{X^0}\int_{x\le 0}\left(gw^2 + \frac{1}{2}|\nabla_z w|^2\right)\,dz \;,
\end{aligned} \tag{2.5}$$

since $g(x) \ge 1/2$ for $x \le 0$. Similarly,

$$\begin{aligned}
\left|\int_{x\ge 0} aw^3\,dz\right| &\le \left(\int_{x\ge 0} w^2\,dz\right)^{1/2}\left(\int_{x\ge 0} a^2w^4\,dz\right)^{1/2} \\
&\le K_S\|w\|_{X^0}\int_{x\ge 0}\left(aw^2 + |\nabla_z(a^{1/2}w)|^2\right)\,dz \;.
\end{aligned}$$

The integral of $|\nabla_z(a^{1/2}w)|^{1/2}$ is equal to

$$\int_{x\ge 0}\left(a|\nabla_z w|^2 + \frac{\gamma^2}{4}aw^2 - \gamma aww_x\right)\,dz \;\le\; \int_{x\ge 0}\left(\frac{3}{4}\gamma^2aw^2 + \frac{3}{2}a|\nabla_z w|^2\right)\,dz \;.$$

Since $\gamma^2 \le 1$ and $2a(x)/3 \le g(x) \le 1$ for $x \ge 0$, we thus have

$$\begin{aligned}
\left|\int_{x\ge 0} aw^3\,dz\right| &\le K_S\|w\|_{X^0}\int_{x\ge 0}\left(2aw^2 + \frac{3}{2}a|\nabla_z w|^2\right)\,dz \\
&\le 3K_S\|w\|_{X^0}\int_{x\ge 0}\left(gw^2 + \frac{1}{2}|\nabla_z w|^2\right)\,dz \;.
\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we conclude

$$\left| \int_{\mathbf{R}^n} aw^3 \, dz \right| \leq 3K_S \|w\|_{X^0} \int_{\mathbf{R}^n} \left(gw^2 + \frac{1}{2}|\nabla_z w|^2 \right) dz \ . \tag{2.7}$$

If H1 holds, one has $3K_S\|w\|_{X^0} \leq 3K_S\delta \leq 1/2$, and (2.3) follows immediately. If H2 holds, then $aw^3 \geq -gw^2$ a.e.(x,y) and (2.3) is obvious.

To prove (2.4), we first assume that $(\varphi_0, \varphi_1) = (w(0), w_t(0)) \in Z_\epsilon^2$. In this case, one has $(w, w_t) \in C^1([0,T], Z_\epsilon^1)$, so that $E_0 \in C^1([0,T])$, and a direct calculation shows that

$$\dot{E}_0(t) = -(1 + 2\epsilon c\gamma) \int_{\mathbf{R}^n} w_t^2 \, dz + \sqrt{c^2 - 4} \int_{\mathbf{R}^n} w_x w_t \, dz \ . \tag{2.8}$$

In the general case where $(\varphi_0, \varphi_1) \in Z_\epsilon^1$, we use the fact that the solution $(w, w_t) \in Z_\epsilon^1$ depends continuously on the initial data (φ_0, φ_1), uniformly in $t \in [0,T]$. Therefore, if $G(t)$ denotes the right-hand side of (2.8), we see that (for fixed t) both $E_0(t)$ and $E_0(0) + \int_0^t G(s)\, ds$ are continuous functions of $(\varphi_0, \varphi_1) \in Z_\epsilon^1$. Since they coincide on a dense subset (namely, Z_ϵ^2), they must be equal everywhere. This proves that $E_0 \in C^1([0,T])$ and satisfies (2.8). Finally, since

$$\sqrt{c^2 - 4} \left| \int_{\mathbf{R}^n} w_x w_t \, dz \right| \leq \int_{\mathbf{R}^n} \left(w_t^2 + \frac{c^2 - 4}{4}|\nabla_z w|^2 \right) dz \ ,$$

we see that (2.4) follows from (2.3) and (2.8). □

Lemma 2.3. *Assume that H1 or H2 holds. Then there exist constants $K_1, K_2 > 0$ depending only on ϵ_0, c such that*

$$K_1 \|W(t)\|_{Y_\epsilon}^2 \leq E_2(t) \leq K_2 \|W(t)\|_{Y_\epsilon}^2 (1 + \|w(t)\|_{X^0}) \ , \tag{2.9}$$

for all $t \in [0,T]$. Moreover, $E_2(t) \geq \alpha E_0(t)/2$, $E_2 \in C^1([0,T])$ and

$$\dot{E}_2(t) \leq -\frac{1}{2} E_0(t) \ . \tag{2.10}$$

Remark. We recall that $W = (w, w_t)$. The fact that K_1, K_2 are independent of ϵ will be very important in Section 5, where the limiting case $\epsilon = 0$ is considered. Note that the standard choice $\alpha = 2\epsilon$ in (2.2) would lead to a constant K_1 of order ϵ, see the proof below.

Proof. Since $\alpha \ge 2\epsilon$, we have

$$|\epsilon w w_t| \;\le\; \frac{1}{3}w^2 + \frac{3}{4}\epsilon^2 w_t^2 \;\le\; \frac{1}{3}w^2 + \frac{3}{8}\alpha\epsilon w_t^2 \ . \tag{2.11}$$

Therefore, using (2.3), we find

$$E_2(t) \;\ge\; \int_{\mathbf{R}^n} \left(\frac{\alpha\epsilon}{8}w_t^2 + \frac{\alpha}{4}|\nabla_z w|^2 + \frac{\alpha}{2}g w^2 + \frac{1}{6}w^2\right) dz \ .$$

Furthermore, using (2.7), we obtain

$$\begin{aligned} E_2(t) \;&\le\; \int_{\mathbf{R}^n} \left(\alpha\epsilon w_t^2 + \frac{\alpha}{2}|\nabla_z w|^2 + \alpha g w^2 + (1+\epsilon c\gamma)w^2 + \frac{\alpha}{3}a w^3\right) dz \\ &\le\; (1 + K_S\|w\|_{X^0}) \int_{\mathbf{R}^n} \left(\alpha\epsilon w_t^2 + \frac{\alpha}{2}|\nabla_z w|^2 + \alpha g w^2 + (1+\epsilon c\gamma)w^2\right) dz \ . \end{aligned}$$

Since $0 \le g \le 1$, $\epsilon \le \epsilon_0$, and $1/(2c^2) \le \alpha \le \max(2\epsilon_0, 1/(2c^2))$, we arrive at (2.9), with K_1, K_2 independent of ϵ. Similarly, since $|\epsilon w w_t| \le w^2/2 + \alpha\epsilon w_t^2/4$, it follows from (2.2), (2.3) that

$$E_1(t) \;\ge\; -\frac{\alpha\epsilon}{4}\int_{\mathbf{R}^n} w_t^2\, dz \;\ge\; -\frac{\alpha}{2}E_0(t) \ ,$$

hence $E_2(t) \ge \alpha E_0(t)/2$.

To prove (2.10), we proceed along the same lines as in the preceding lemma. Using a direct calculation and a density argument, we show that

$$\dot E_1(t) \;=\; \int_{\mathbf{R}^n} \left(\epsilon w_t^2 - 2\epsilon w_x w_t - |\nabla_z w|^2 - 2g w^2 - a w^3\right) dz \ ,$$

hence

$$\begin{aligned} \dot E_2(t) \;=\; &(\epsilon - \alpha(1+2\epsilon c\gamma)) \int_{\mathbf{R}^n} w_t^2\, dz + (\alpha\sqrt{c^2-4} - 2\epsilon c) \int_{\mathbf{R}^n} w_x w_t\, dz \\ &- \int_{\mathbf{R}^n} \left(|\nabla_z w|^2 + 2g w^2 + a w^3\right) dz \ . \end{aligned}$$

If $\alpha = 2\epsilon$, then $\alpha\sqrt{c^2-4} - 2\epsilon c = -4\epsilon\gamma$, and

$$4\epsilon\gamma|w_x w_t| \;\le\; 4\epsilon^2 c\gamma w_t^2 + \frac{\gamma}{c}w_x^2 \;\le\; 2\epsilon\alpha c\gamma w_t^2 + \frac{1}{2}|\nabla_z w|^2 \ ,$$

since $\gamma/c \le 1/2$ by (1.8). If $\alpha = 1/(2c^2) \ge 2\epsilon$, then $|\alpha\sqrt{c^2-4} - 2\epsilon c| \le \alpha c$ and

$$\alpha c|w_x w_t| \;\le\; \frac{\alpha}{4}w_t^2 + \alpha c^2 w_x^2 \;=\; \frac{\alpha}{4}w_t^2 + \frac{1}{2}|\nabla_z w|^2 \ .$$

In both cases, we find

$$\dot{E}_2(t) \;\leq\; -\int_{\mathbf{R}^n} \left(\frac{\epsilon}{2}w_t^2 + \frac{1}{2}|\nabla_z w|^2 + 2gw^2 + aw^3\right)\, dz.$$

If H1 holds, then by (2.2), (2.7)

$$\begin{aligned}
\dot{E}_2(t) + \frac{1}{2}E_0(t) \;&\leq\; -\int_{\mathbf{R}^n} \left(\frac{1}{4}|\nabla_z w|^2 + \frac{3}{2}gw^2 + \frac{5}{6}aw^3\right)\, dz \\
&\leq\; -(1 - 5K_S\|w\|_{X^0}) \int_{\mathbf{R}^n} \left(\frac{1}{4}|\nabla_z w|^2 + \frac{1}{2}gw^2\right)\, dz \;\leq\; 0\ .
\end{aligned}$$

If H2 holds, then we simply have

$$\dot{E}_2(t) + E_0(t) \;\leq\; -\int_{\mathbf{R}^n} \left(gw^2 + \frac{2}{3}aw^3\right)\, dz \;\leq\; -\frac{1}{3}\int_{\mathbf{R}^n} gw^2\, dz \;\leq\; 0\ .$$

In both cases, we obtain (2.10). $\square$

Remark. Up to now, we did not use the fact that $d > 0$. Therefore, Lemma 2.2 and Lemma 2.3 are still valid if H2 holds with $d = 0$, and the constants K_1, K_2 are independent of d.

2.2. Weighted Functionals

Under the same assumptions as above, we define the weighted functionals

$$\begin{aligned}
F_0(t) \;&=\; \int_{\mathbf{R}^n} \left(\frac{\epsilon}{2}a^2 w_t^2 + \frac{1}{2}a^2|\nabla_z w|^2 + a^2 gw^2 + \frac{1}{3}a^3 w^3\right)\, dz\ , \\
F_1(t) \;&=\; \int_{\mathbf{R}^n} \left(\frac{1 - 2\epsilon c\gamma}{2}a^2 w^2 + \epsilon a^2 w w_t\right)\, dz\ , \\
F_2(t) \;&=\; \hat{\alpha}F_0(t) + F_1(t) + \beta E_0(t)\ ,
\end{aligned} \tag{2.12}$$

where $\hat{\alpha} = \max(2\epsilon, d/(2c^2))$ and $\beta = 3\hat{\alpha}$. In the case where H1 holds, we set $d = 1$, so that $\hat{\alpha} = \alpha$.

Remark. The additional term $\beta E_0(t)$ in (2.12) guarantees that $F_2(t) \geq 0$. However, if ϵ is sufficiently small, then $\hat{\alpha}F_0(t) + F_1(t)$ is already positive, so we may set $\beta = 0$. This possibility will be used in Section 4.3 below.

Lemma 2.4. *Assume that H1 or H2 holds. Then there exist constants $K_3, K_4, K_5 > 0$ such that*

$$K_3 \|W(t)\|^2_{Y_{\epsilon a}} \;\leq\; F_2(t) \;\leq\; K_4 \|W(t)\|^2_{Y_{\epsilon a}} (1 + \|w(t)\|_{X^0}) + \beta E_0(t) \;, \tag{2.13}$$

for all $t \in [0,T]$. Moreover, $F_2 \in C^1([0,T])$ and satisfies

$$\dot{F}_2(t) + \kappa F_2(t) \;\leq\; K_5 E_0(t) \;, \tag{2.14}$$

where $\kappa = d/(8(1+\hat{\alpha}))$.

Remark. Here and in the sequel, $K_3, K_4, \ldots$ denote positive constants depending only on ϵ_0, c and, if H2 holds, on $d > 0$.

Proof. Using the identity

$$\int_{\mathbf{R}^n} a^2 |\nabla_z w|^2 \, dz \;=\; \int_{\mathbf{R}^n} \left(|\nabla_z (aw)|^2 + \gamma^2 a^2 w^2 \right) \, dz \;, \tag{2.15}$$

together with the relation $1 - c\gamma + \gamma^2 = 0$, we write $F_2(t)$ as

$$\begin{aligned} F_2(t) \;=\; \int_{\mathbf{R}^n} \Big(& \frac{\hat{\alpha}\epsilon}{2} a^2 w_t^2 + \frac{\hat{\alpha}}{2} |\nabla_z (aw)|^2 + c\gamma(\hat{\alpha}/2 - \epsilon) a^2 w^2 + \frac{\hat{\alpha}}{2}(2g-1) a^2 w^2 \\ & + \; \frac{\hat{\alpha}}{3} a^3 w^3 + \frac{1}{2} a^2 w^2 + \epsilon a^2 w w_t \Big) \, dz + \beta E_0(t) \;. \end{aligned} \tag{2.16}$$

To prove (2.13), we first note that $c\gamma(\hat{\alpha}/2 - \epsilon) \leq c\gamma\hat{\alpha}/2 \leq \hat{\alpha}$ and $g \leq 1$. Using (2.11), we thus find

$$F_2(t) \;\leq\; \int_{\mathbf{R}^n} \left(\hat{\alpha}\epsilon a^2 w_t^2 + \frac{\hat{\alpha}}{2} |\nabla_z (aw)|^2 + (1+\hat{\alpha}) a^2 w^2 + \frac{\hat{\alpha}}{3} a^3 w^3 \right) \, dz + \beta E_0(t) \;. \tag{2.17}$$

Furthermore, in analogy with (2.5), we have

$$\left| \int_{\mathbf{R}^n} a^3 w^3 \, dz \right| \;\leq\; K_S \|w\|_{X^0} \int_{\mathbf{R}^n} \left(a^2 w^2 + |\nabla_z (aw)|^2 \right) \, dz \;. \tag{2.18}$$

Therefore, combining (2.17) and (2.18), we easily obtain the upper bound in (2.13).

To prove the lower bound, we first use (2.11) and the fact that $\hat{\alpha} \geq 2\epsilon$. We find

$$\begin{aligned} F_2(t) \;\geq\; & \int_{\mathbf{R}^n} \left(\frac{\hat{\alpha}\epsilon}{8} a^2 w_t^2 + \frac{\hat{\alpha}}{2} |\nabla_z (aw)|^2 + \frac{\hat{\alpha}}{2}(2g-1) a^2 w^2 + \frac{\hat{\alpha}}{3} a^3 w^3 + \frac{1}{6} a^2 w^2 \right) \, dz \\ & + \; \beta E_0(t) \;. \end{aligned} \tag{2.19}$$

If H1 holds, we apply (2.18). Since $K_S\|w\|_{X^0} \leq 3/4$, we obtain

$$F_2(t) \geq \int_{\mathbf{R}^n} \left(\frac{\hat{\alpha}}{8} a^2 w_t^2 + \frac{\hat{\alpha}}{4} |\nabla_z(aw)|^2 + \hat{\alpha}(g-3/4)a^2w^2 + \frac{1}{6}a^2w^2 \right) dz + \beta E_0(t) \ .$$

If $x \leq 0$, then $g(x)-3/4 \geq 1-\sigma-3/4 \geq 0$, since $\sigma \leq 1/8$. If $x \geq 0$, then $g(x)-3/4 \geq -1$ and $a(x)^2 \leq a(x) \leq 3g(x)/2$, so that

$$\int_{x\geq 0} a^2 w^2 \, dz \ \leq \ \frac{3}{2} \int_{x \geq 0} g w^2 \, dz \ \leq \ 3E_0(t) \ , \tag{2.20}$$

by (2.3). Therefore, since $\beta = 3\hat{\alpha}$, we have

$$F_2(t) \ \geq \int_{\mathbf{R}^n} \left(\frac{\hat{\alpha}\epsilon}{8} a^2 w_t^2 + \frac{\hat{\alpha}}{4} |\nabla_z(aw)|^2 + \frac{1}{6} a^2 w^2 \right) dz \ . \tag{2.21}$$

If H2 holds, we observe that

$$\hat{\alpha}(g-1/2)a^2w^2 + \frac{\hat{\alpha}}{3} a^3 w^3 \ \geq \ \hat{\alpha}(2g/3 - 1/2)a^2 w^2 \quad \text{a.e.}(x,y) \ .$$

Again, we have $2g(x)/3 - 1/2 \geq (1-4\sigma)/6 \geq 0$ if $x \leq 0$, and $2g(x)/3 - 1/2 \geq -1$ if $x \geq 0$. Therefore, using (2.19) and proceeding as above, we again arrive at (2.21). This proves the lower bound in (2.13).

To prove (2.14), we proceed along the same lines as in the preceding lemmas. Using a direct calculation and a density argument, we first show that

$$\begin{aligned} \dot{F}_2(t) \ &= \ (\epsilon - \hat{\alpha}) \int_{\mathbf{R}^n} a^2 w_t^2 \, dz + c(\hat{\alpha} - 2\epsilon) \int_{\mathbf{R}^n} a^2 w_x w_t \, dz \\ &\quad - \int_{\mathbf{R}^n} \left(|\nabla_z(aw)|^2 + (2g-1)a^2w^2 + a^3w^3 \right) dz + \beta \dot{E}_0(t) \ . \end{aligned} \tag{2.22}$$

If $\hat{\alpha} = d/(2c^2) > 2\epsilon$, then

$$|c(\hat{\alpha} - 2\epsilon) w_x w_t| \ \leq \ c\hat{\alpha}|w_x w_t| \ \leq \ \frac{\hat{\alpha}}{4} w_t^2 + \frac{d}{2} |\nabla_z w|^2 \ .$$

Therefore, using (2.15) and the fact that $\gamma^2 \leq 1$, we find

$$\dot{F}_2(t) \ \leq \ - \int_{\mathbf{R}^n} \left(\frac{\epsilon}{2} a^2 w_t^2 + \frac{1}{2} |\nabla_z(aw)|^2 + (2g - 1 - d/2)a^2 w^2 + a^3 w^3 \right) dz + \beta \dot{E}_0(t) \ . \tag{2.23}$$

If $\hat{\alpha} = 2\epsilon$, then (2.23) follows immediately from (2.22).

We now combine (2.17) and (2.23). Using (2.4) and the fact that $\kappa\hat{\alpha} \le 1/2$, we easily find

$$\begin{aligned} \dot{F}_2(t) + \kappa F_2(t) \;\le\; &- \int_{\mathbf{R}^n} \Bigl(\frac{1}{4}|\nabla_z(aw)|^2 + (2g-1-d/2-\kappa(1+\hat{\alpha}))a^2w^2\Bigr)\,dz \\ &- (1-\kappa\hat{\alpha}/3)\int_{\mathbf{R}^n} a^3w^3\,dz + \hat{\beta}E_0(t)\;, \end{aligned} \tag{2.24}$$

with $\hat{\beta} = \beta(\kappa + c^2 - 4)$. If H1 holds, we use (2.18) and obtain

$$\dot{F}_2(t) + \kappa F_2(t) \;\le\; -\int_{\mathbf{R}^n} (2g - 3/2 - \kappa(1+\hat{\alpha}) - K_S\|w\|_{X^0})a^2w^2\,dz + \hat{\beta}E_0(t)\;.$$

If $x \le 0$, then $2g(x) - 3/2 - \kappa(1+\hat{\alpha}) - K_S\|w\|_{X^0} \ge 1/2 - 2\sigma - \kappa(1+\hat{\alpha}) - K_S\delta \ge 0$, by assumptions on σ, δ, κ. If $x \ge 0$, the same quantity is bounded from below by -2. Therefore, using (2.20), we find $\dot{F}_2 + \kappa F_2 \le (6+\hat{\beta})E_0$, which is (2.14).

If H2 holds, we infer from (2.24)

$$\dot{F}_2(t) + \kappa F_2(t) \;\le\; -\int_{\mathbf{R}^n} ((1+d)g - 1 - d/2 - \kappa(1+\hat{\alpha}))a^2w^2\,dz + \hat{\beta}E_0(t)\;.$$

Since $g(x) \to 1$ as $x \to -\infty$, there exists $x_d \le 0$ such that $g(x) \ge 1 - d/8$ for all $x \le x_d$. Therefore, if $x \le x_d$, one has $(1+d)g - 1 - d/2 - \kappa(1+\hat{\alpha}) \ge (3d - d^2)/8 - \kappa(1+\hat{\alpha}) \ge 0$ by assumptions on d, κ. If $x \ge x_d$, the same quantity is bounded from below by -2, and

$$\int_{x \ge x_d} a^2w^2\,dz \;\le\; \frac{9}{4}\,e^{-2\gamma x_d}\int_{x\ge x_d} gw^2\,dz \;\le\; \frac{9}{2}\,e^{-2\gamma x_d}E_0(t)\;,$$

since $3g(x) \ge 3g(x - x_d) \ge 2a(x - x_d) = 2e^{\gamma x_d}a(x)$ for all $x \ge x_d$. Combining these inequalities, we find $\dot{F}_2 + \kappa F_2 \le (9e^{-2\gamma x_d} + \hat{\beta})E_0$, which is the desired result. □

Corollary 2.5. *Assume that H1 or H2 holds. Then there exists a constant $K_6 \ge 1$ such that*

$$\|W(t)\|_{Z^1_\epsilon} \;\le\; K_6\|W(0)\|_{Z^1_\epsilon}(1 + \|w(0)\|_{X^0})^{1/2}\;, \tag{2.25}$$

for all $t \in [0,T]$.

Proof. According to Lemma 2.3, we have

$$\|W(t)\|^2_{Y_\epsilon} \;\le\; \frac{K_2}{K_1}\|W(0)\|^2_{Y_\epsilon}(1 + \|w(0)\|_{X^0})\;, \tag{2.26}$$

for all $t \in [0,T]$, since E_2 is a decreasing function of t. On the other hand, it follows from (2.14) and Lemma 2.3 that $\dot{F}_2(t) + \kappa F_2(t) \le \hat{K}_5 E_2(t)$, where $\hat{K}_5 = 2K_5/\alpha$. Integrating this inequality, we find

$$F_2(t) \;\le\; e^{-\kappa t} F_2(0) + \hat{K}_5 \int_0^t e^{-\kappa(t-\tau)} E_2(\tau)\, d\tau \;\le\; F_2(0) + \frac{\hat{K}_5}{\kappa} E_2(0)\ ,$$

hence

$$\|W(t)\|^2_{Y_{\epsilon a}} \;\le\; \frac{1}{K_3}\Bigl(K_4 \|W(0)\|^2_{Y_{\epsilon a}} + (6 + \hat{K}_5/\kappa) K_2 \|W(0)\|^2_{Y_\epsilon}\Bigr)\,(1 + \|w(0)\|_{X^0})\ , \quad (2.27)$$

by (2.9), (2.13). Combining (2.26) and (2.27), the result follows. □

If H2 holds with $d = 0$, it is no longer possible to bound $W(t)$ uniformly in time as in Corollary 2.5, but the energy estimates above still imply that $\|W(t)\|_{Z^1_\epsilon}$ cannot grow faster than an exponential. This result will be useful in Section 4.

Corollary 2.6. *Assume that H2 holds with $d = 0$. Then there exist constants $\rho > 0$ and $K_7 \ge 1$ such that*

$$\|W(t)\|_{Z^1_\epsilon} \;\le\; K_7 (1 + e^{\rho t}) \|W(0)\|_{Z^1_\epsilon} (1 + \|w(0)\|_{X^0})^{1/2}\ , \quad (2.28)$$

for all $t \in [0,T]$.

Proof. We recall that Lemma 2.2 and Lemma 2.3 still hold if $d = 0$, see the remark at the end of Section 2.1. Furthermore, if we define $F_2(t)$ by (2.12) with $\hat{\alpha} = \alpha = \max(2\epsilon, 1/(2c^2))$ and $\beta = 3\alpha$, then it is easily verified that (2.13) is still valid. However, (2.23) has to be replaced by

$$\dot{F}_2(t) \;\le\; -\int_{\mathbf{R}^n} \Bigl(\frac{\epsilon}{2} a^2 w_t^2 + \frac{1}{2} |\nabla_z (aw)|^2 + (g - 3/2) a^2 w^2\Bigr)\, dz + \beta \dot{E}_0(t)\ .$$

Therefore, using (2.4), (2.21), we obtain

$$\dot{F}_2(t) \;\le\; 9 F_2(t) + \beta (c^2 - 4) E_0(t)\ ,$$

which replaces (2.14). Integrating this inequality and proceeding as in the proof of Corollary 2.5, we obtain (2.28), with $\rho = 9/2$. □

3. Local Stability

In this section, we prove Theorem 1.1 using the energy estimates of Section 2.

Proof of Theorem 1.1. Let δ_0, K_0 be defined by the relations

$$K_6\delta_0(1+\delta_0)^{1/2} \;=\; \delta/2\;, \quad K_0 \;=\; K_6(1+\delta_0)^{1/2}\;,$$

where $K_6 \ge 1$ is given in Corollary 2.5 and $\delta = 1/(8K_S)$ in the assumption H1, Section 2. Then, for all $\Phi = (\varphi_0, \varphi_1) \in Z^1_\epsilon$ such that $\|\Phi\|_{Z^1_\epsilon} \le \delta_0$, Eq.(1.9) has a unique global solution $W(t) = (w(t), w_t(t)) \in Z^1_\epsilon$ satisfying $W(0) = \Phi$. Indeed, in view of the local existence result (Proposition 2.1), it is sufficient to show that $\|W(t)\|_{Z^1_\epsilon} < \delta$ whenever $W(t)$ exists. Assume on the contrary that there exists a time $T > 0$ such that $\|W(T)\|_{Z^1_\epsilon} = \delta$ and $\|W(t)\|_{Z^1_\epsilon} < \delta$ for all $t \in [0,T)$. Then $\|w(t)\|_{X^0} \le \|W(t)\|_{Z^1_\epsilon} \le \delta$ for all $t \in [0,T]$, so that H1 holds on $[0,T]$. By Corollary 2.5, it follows that $\|W(T)\|_{Z^1_\epsilon} \le K_6\delta_0(1+\delta_0)^{1/2} = \delta/2$, which is a contradiction. Therefore, $W(t)$ exists for all times and the assumption H1 is always satisfied. By Corollary 2.5 again, we conclude that $\|W(t)\|_{Z^1_\epsilon} \le K_0\|\Phi\|_{Z^1_\epsilon}$ for all $t \ge 0$, which proves (1.11).

To prove (1.12), (1.13), we use the differential inequalities satisfied by the functionals E_0, E_2, F_2 defined in Section 2. The following arguments are standard (see for example [EW]) and will be reproduced here for the sake of completeness. First, since E_2 is a positive, decreasing function of t by Lemma 2.3, $E_2(t)$ converges to a nonnegative limit as $t \to +\infty$. By (2.4), (2.10), so does $E_0 + 2(c^2-4)E_2$. Therefore, $E_0(t)$ converges as $t \to +\infty$, and since $E_0(t) \ge 0$ it follows from (2.10) that the integral

$$\int_0^{+\infty} E_0(\tau)\,d\tau \;\le\; 2(E_2(0) - E_2(+\infty))$$

is finite. Thus $E_0(t)$ converges to zero as $t \to +\infty$. Moreover, integrating the differential inequality (2.14), we find

$$\begin{aligned} F_2(t) \;&\le\; e^{-\kappa t}F_2(0) + K_5\int_0^t e^{-\kappa(t-\tau)}E_0(\tau)\,d\tau \\ &\le\; e^{-\kappa t}F_2(0) + K_5\left(e^{-\kappa t/2}\int_0^{t/2} E_0(\tau)\,d\tau + \int_{t/2}^t e^{-\kappa(t-\tau)}E_0(\tau)\,d\tau\right) \\ &\le\; e^{-\kappa t/2}(F_2(0) + 2K_5E_2(0)) + \frac{K_5}{\kappa}\sup_{\tau\in[t/2,t]} E_0(\tau)\;, \end{aligned} \tag{3.1}$$

hence $F_2(t)$ converges to zero as $t \to +\infty$. Therefore, using the lower bounds in (2.3), (2.13), we obtain (1.12).

In the case where $c = 2$, E_0 itself is a decreasing function of t by (2.4), hence $tE_0(t) \leq 2\int_{t/2}^{t} E_0(\tau)d\tau$. Thus $tE_0(t)$ converges to zero as $t \to +\infty$, and by (3.1) the same is true for $tF_2(t)$. Therefore, using again (2.3), (2.13), we obtain (1.13). This concludes the proof of Theorem 1.1. □

4. Global Stability Results in the One-Dimensional Case

Throughout this section, we assume that $n = 1$. First, we prove Theorem 1.2 using the results of Section 2 and the Maximum Principle for hyperbolic equations. Then, we study in more details the case $\epsilon \leq 1/4$; we give linear upper and lower bounds for the solutions of (1.9). Finally, using these linear bounds, we prove Theorem 1.3.

4.1. Global Stability in the General Case

We first show that the assumption H2 (see Section 2) holds if the solution $w(x,t)$ of (1.9) is bounded from above and if the initial data satisfy (1.16), (1.17).

Proposition 4.1. *Let $\epsilon > 0$, $d \in [0,1]$ and let K be a nonnegative constant such that*

$$1 - 4\epsilon(d+K) \geq 0 \; . \tag{4.1}$$

For some $T > 0$, assume that $(w, w_t) \in C^0([0,T], Z_\epsilon^1)$ is a solution of (1.9) with initial data (φ_0, φ_1) satisfying (1.16), (1.17), namely

$$\varphi_0(x) \;\geq\; -(1-d)a^{-1}(x)g(x) \; , \tag{4.2}$$

$$\epsilon\varphi_1(x) \;\geq\; \epsilon c\varphi_0'(x) - \frac{1}{2}\big(\varphi_0 + (1-d)a^{-1}g\big)(x) + \epsilon c\big(-\gamma\varphi_0 + (1-d)a^{-1}g'\big)(x) \; , \tag{4.3}$$

for (almost) every $x \in \mathbf{R}$. Suppose moreover that

$$a(x)w(x,t) \leq K \; , \quad \forall\, (x,t) \in \mathbf{R} \times [0,T] \; . \tag{4.4}$$

Then

$$w(x,t) \geq -(1-d)a^{-1}(x)g(x) \; , \quad \forall\, (x,t) \in \mathbf{R} \times [0,T] \; . \tag{4.5}$$

Proof. We recall that the inequality (4.5) is equivalent to $v(x,t) \geq dg(x)$, where $v(x,t)$ is the solution of (1.5). Also, we remark that $dg - v$ belongs to the space $C^0([0,T], H^1_{loc}(\mathbf{R})) \cap C^1([0,T], L^2_{loc}(\mathbf{R}))$ and satisfies

$$\tilde{L}(dg - v) \equiv \tilde{\mathcal{L}}(dg - v) + \tilde{h}(x,t)(dg - v) = g^2 d(1-d) ,$$

where $\tilde{\mathcal{L}}(v) = v_{xx} + 2\epsilon c v_{xt} - \epsilon v_{tt} + c v_x - v_t$ and $\tilde{h}(x,t) = 1 - v(x,t) - dg(x)$. Therefore, to prove (4.5), we are led to apply the Maximum Principle (Theorem A.1, Appendix A) to the function $dg - v$ and to the operator $\tilde{L}$. The condition (A.1) is obviously satisfied. Due to (4.4), the condition (A.2) holds, i.e.,

$$\tilde{h}(x,t) \geq 1 - (1+d)g(x) - K \geq -K - d .$$

This estimate and (4.1) imply that (A.3) also holds. Moreover, since $0 \leq d \leq 1$, we have $\tilde{L}(dg - v)(x,t) \geq 0$ a.e.$(x,t) \in \mathbf{R} \times [0,T]$, which is (A.4). Finally, the conditions (A.5) and (A.6) required on $dg - v$ are nothing else but the hypotheses (4.2) and (4.3). Therefore, it follows from Theorem A.1 that $(dg - v)(x,t) \leq 0$, that is, $w(x,t) \geq -(1-d)a(x)^{-1}g(x)$, for all $(x,t) \in \mathbf{R} \times [0,T]$. □

Using Proposition 4.1 and Corollary 2.5, we now prove the first global stability result.

Proof of Theorem 1.2. The proof is very similar to the one of Theorem 1.1 given in Section 3. Let μ be a real number, $0 < \mu < 1$. We define K^* by the relation

$$K_6 K^* (1 + K^*)^{1/2} = (1 - \mu) K ,$$

where $K_6 \geq 1$ has been introduced in Corollary 2.5. According to Proposition 2.1, there exist a time $T > 0$ and a unique solution $W(t) = (w, w_t) \in C^0([0,T), Z^1_\epsilon)$ of (1.9) with initial data (φ_0, φ_1) such that $\|W(t)\|_{Z^1_\epsilon} < K$ for all $t \in [0,T)$ and, if $T < \infty$, $W(t) \in C^0([0,T], Z^1_\epsilon)$ and $\|W(T)\|_{Z^1_\epsilon} = K$. We show by contradiction that $T = \infty$. If $T < \infty$, we have

$$a(x)w(x,t) \leq \|aw(t)\|_{L^\infty} \leq \|aw(t)\|_{H^1} \leq \|W(t)\|_{Z^1_\epsilon} \leq K ,$$

for all $t \in [0,T]$. Thus, by Proposition 4.1, $w(x,t) \geq -(1-d)a(x)^{-1}g(x)$ for all $(x,t) \in \mathbf{R} \times [0,T]$, i.e. the assumption H2 of Section 2 holds on $[0,T]$ (we recall that $d > 0$ here). By Corollary 2.5, it follows that $\|W(T)\|_{Z^1_\epsilon} \leq K_6(1+K^*)^{1/2}K^* = (1-\mu)K$, which is a contradiction. Therefore $T = \infty$, and the inequalities (1.18) hold for all times. The properties (1.12), (1.13) are proved like in Section 3. □

4.2. Linear Bounds in the Case $\epsilon \leq 1/4$

From now on, we assume that $\epsilon \leq 1/4$. In this case, the range of application of the Maximum Principle is much wider, and we can show that the solution $w(x,t)$ of (1.9) is bounded from above and from below by solutions of suitable linear equations. These linear bounds will be crucial for the proof of Theorem 1.3. Before stating the results, we introduce some additional notation.

For all $d \in [-1,1]$, we denote by $S_d(t) \in \mathcal{L}(Z^1_\epsilon, Z^1_\epsilon)$ the linear group associated with the equation $(L_d w)(x,t) = 0$, where

$$L_d w \;=\; -\epsilon w_{tt} - (1+2\epsilon c\gamma)w_t + 2\epsilon c w_{xt} + w_{xx} + \sqrt{c^2-4}\, w_x - (1+d)gw \;. \tag{4.6}$$

For $(\varphi_0, \varphi_1) \in Z^1_\epsilon$, we set

$$S_d(t)(\varphi_0,\varphi_1) \;=\; (\tilde{w}_d(t), \tilde{w}_{dt}(t)) \;. \tag{4.7}$$

In (1.19), (1.20), we have defined the positive and negative parts $(\varphi_0^\pm, \varphi_1^\pm)$ of (φ_0, φ_1). In analogy with (4.7), we set

$$S_d(t)(\varphi_0^\pm,\varphi_1^\pm) \;=\; (\tilde{w}_d^\pm(t), \tilde{w}_{dt}^\pm(t)) \;. \tag{4.8}$$

We now show the existence of a linear upper bound.

Lemma 4.2. *Let $\epsilon \leq 1/4$. For any $(\varphi_0,\varphi_1) \in Z^1_\epsilon$, the solution $(w, w_t) \in C^0([0,T], Z^1_\epsilon)$ of (1.9), with initial data (φ_0,φ_1), satisfies for any $d \in [-1,1]$,*

$$w(x,t) \;\leq\; \tilde{w}_1(x,t) \;\leq\; \tilde{w}_d^+(x,t) \;, \quad \forall\, (x,t) \in \mathbf{R} \times [0,T] \;, \tag{4.9}$$

where $\tilde{w}_1$, $\tilde{w}_d^+$ have been defined in (4.7) and (4.8) respectively.

Proof. We first prove the inequality $w(x,t) \leq \tilde{w}_1(x,t)$. The function $w - \tilde{w}_1$ satisfies the equation $L_1(w-\tilde{w}_1) = aw^2 \geq 0$, where L_1 has been defined in (4.6). Thus, we can apply the Maximum Principle to the function $w - \tilde{w}_1$ and to the operator L_1. Indeed the conditions (A.1), (A.2), (A.4) are satisfied, and, since the initial data for $w, \tilde{w}$ coincide, (A.5) and (A.6) obviously hold. Since $-2g(x) \geq -2$, the condition (A.3) with $\underline{h} = -2$ becomes $(1-4\epsilon)(\epsilon + \epsilon^2 c^2) \geq 0$, which is satisfied because $\epsilon \leq 1/4$. Therefore, Theorem A.1. implies that $w(x,t) - \tilde{w}_1(x,t) \leq 0$ for all $(x,t) \in \mathbf{R} \times [0,T]$.

We next show that $\tilde{w}_d^+(x,t) \geq 0$. Since $L_d(-\tilde{w}_d^+) = 0$, we can apply the Maximum Principle to the function $-\tilde{w}_d^+$ and to the operator L_d. In view of the first part of the proof, the conditions (A.1) to (A.4) hold. Due to the choice of $(\varphi_0^+, \varphi_1^+)$ made in (1.19), the hypotheses (A.5) and (A.6) are also satisfied. Therefore, by Theorem A.1, $\tilde{w}_d^+(x,t) \geq 0$ for all $(x,t) \in \mathbf{R} \times \mathbf{R}_+$.

Finally, we show that $\tilde{w}_1(x,t) \leq \tilde{w}_d^+(x,t)$ for all $d \in [-1,1]$, by applying the Maximum Principle to the function $\tilde{w}_1 - \tilde{w}_d^+$ and to the operator L_1. As we have already remarked, the hypotheses (A.1), (A.2), (A.3) are satisfied. The condition (A.4) holds, since $L_1(\tilde{w}_1 - \tilde{w}_d^+) = (1-d)g\tilde{w}_d^+$ and $\tilde{w}_d^+ \geq 0$. The choice of $(\varphi_0^+, \varphi_1^+)$ in (1.19) also implies that (A.5) and (A.6) hold. Hence, by Theorem A.1, $\tilde{w}_1(x,t) \leq \tilde{w}_d^+(x,t)$ for all $(x,t) \in \mathbf{R} \times [0,T]$. $\square$

In a similar way, we obtain linear lower bounds for $w(x,t)$.

Lemma 4.3. *Let $\epsilon \leq 1/4$, $d \in [0,1]$, and let K be a nonnegative constant such that*

$$1 - 4\epsilon(1+K) \geq 0\ . \tag{4.10}$$

For some $T > 0$, assume that $(w, w_t) \in C^0([0,T], Z_\epsilon^1)$ is a solution of (1.9) with initial data $(\varphi_0, \varphi_1) \in Z_\epsilon^1$ satisfying (4.2) and (4.3). Suppose moreover that (4.4) holds. Then

$$\tilde{w}_{-1}^-(x,t) \leq \tilde{w}_d^-(x,t) \leq w(x,t)\ , \quad \forall\, (x,t) \in \mathbf{R} \times [0,T]\ , \tag{4.11}$$

where $\tilde{w}_{-1}^-$, $\tilde{w}_d^-$ have been defined in (4.8).

Proof. As in the proof of Lemma 4.2, we show that $\tilde{w}_d^-(x,t) \leq 0$ by applying the Maximum Principle to the function $\tilde{w}_d^-$ and to the operator L_d. To show that $\tilde{w}_{-1}^-(x,t) \leq \tilde{w}_d^-(x,t)$, we apply the Maximum Principle to the function $\tilde{w}_{-1}^- - \tilde{w}_d^-$ and to the operator L_{-1}. Since $L_{-1}(\tilde{w}_{-1}^- - \tilde{w}_d^-) = -(1+d)g\tilde{w}_d^-$ and $\tilde{w}_d^- \leq 0$, the hypothesis (A.4) holds. The other conditions are obvious or have been verified in the proof of Lemma 4.2.

It remains to prove that $w(x,t) \geq \tilde{w}_d^-(x,t)$. We again apply the Maximum Principle, but now to the function $\tilde{w}_d^- - w$ and to the operator $L_1^* = L_{-1} + h^*$, where $h^*(x,t) = -(2g(x) + a(x)w(x,t))$. Since $h^*(x,t) \geq -2-K$, the condition (A.3) becomes $(1 - 4\epsilon(1+K))(\epsilon + \epsilon^2 c^2) \geq 0$, which is nothing but (4.10). Moreover, we have $L_1^*(\tilde{w}_d^- - w) = -\tilde{w}_d^-(aw + (1-d)g) \geq 0$, since $\tilde{w}_d^-(x,t) \leq 0$ and $a(x)w(x,t) \geq -(1-d)g(x)$ by Proposition 4.1. Thus (A.4) holds, and due to the choice of $(\varphi_0^-, \varphi_1^-)$ in (1.20) the conditions (A.5) and (A.6) are also satisfied. Therefore $\tilde{w}_d^-(x,t) \le w(x,t)$ by Theorem A.1, and Lemma 4.3 is proved. $\square$

Since $\tilde{w}_1(x,t)$ is a solution of the linear equation $L_1 w = 0$, it is easy to bound it in terms of the initial data (φ_0, φ_1). We have the following result:

Lemma 4.4. *Let $\epsilon \le 1/4$. There exists a constant $N = N(c) \ge 1$ such that*

$$\|S_1(t)\|_{\mathcal{L}(Z_\epsilon^1, Z_\epsilon^1)} \le N \;, \quad \forall\, t \in \mathbf{R}_+ \;. \tag{4.12}$$

Proof. All we need is repeating the energy estimates of Section 2 for the linear equation obtained by dropping the last term $-aw^2$ in (1.9). The functionals $E_0(t), E_2(t), F_2(t)$ are then replaced by the quadratic expressions

$$\begin{aligned} \tilde{E}_0(t) \;&=\; \int_{\mathbf{R}} \left(\frac{\epsilon}{2} w_t^2 + \frac{1}{2}|w_x|^2 + g w^2 \right) dx \;, \quad \tilde{E}_2(t) \;=\; \alpha \tilde{E}_0(t) + E_1(t) \;, \\ \tilde{F}_2(t) \;&=\; \alpha \int_{\mathbf{R}} \left(\frac{\epsilon}{2} a^2 w_t^2 + \frac{1}{2} a^2 |w_x|^2 + a^2 g w^2 \right) dx + F_1(t) + \beta \tilde{E}_0(t) \;, \end{aligned}$$

where $\alpha, \beta, E_1(t), F_1(t)$ are defined in (2.2), (2.12). Of course, the assumptions H1, H2 are not needed anymore, since they were used to control the cubic terms in $E_2(t), F_2(t)$. Following exactly the lines of the proof of Lemma 2.2, Lemma 2.3, Lemma 2.4 (with obvious simplifications), we arrive at Corollary 2.5, which reduces in this case to $\|W(t)\|_{Z_\epsilon^1} \le N\|W(0)\|_{Z_\epsilon^1}$ for some $N(c) \ge 1$. This proves (4.12). $\square$

4.3. Global Stability and Decay in the Case $\epsilon \le 1/4$

Using the linear bounds of the previous paragraph, we are now able to improve the global stability results. Theorem 1.3 will be a direct consequence of the following two propositions:

Proposition 4.5. *Let $\epsilon \le 1/4$, $d \in [0,1]$ and K be a nonnegative constant, such that (4.10) holds. For any $(\varphi_0, \varphi_1) \in Z_\epsilon^1$ satisfying (4.2), (4.3) and*

$$\inf(\|(\varphi_0, \varphi_1)\|_{Z_\epsilon^1}, \|(\varphi_0^+, \varphi_1^+)\|_{Z_\epsilon^1}) \le \frac{K}{N} \;, \tag{4.13}$$

where $(\varphi_0^+, \varphi_1^+)$ is defined in (1.19) and N in Lemma 4.4, there exists a unique global solution $(w, w_t) \in C^0(\mathbf{R}_+, Z_\epsilon^1)$ of (1.9) with initial data (φ_0, φ_1). Moreover, we have

$$-(1-d)g(x) \le a(x)w(x,t) \le K \;, \quad \forall\, (x,t) \in \mathbf{R} \times \mathbf{R}_+ \;, \tag{4.14}$$

and

$$\tilde{w}_{-1}^{-}(x,t) \le w(x,t) \le \tilde{w}_1(x,t) \le \tilde{w}_{-1}^{+}(x,t) , \quad \forall (x,t) \in \mathbf{R} \times \mathbf{R}_+ . \qquad (4.15)$$

In addition, if $d > 0$, the properties (1.12) and (1.13) hold.

Remark. The case $K = 0$ is non trivial, because it corresponds to $(\varphi_0^+, \varphi_1^+) = 0$, i.e.

$$\varphi_0(x) \le 0 , \quad \varphi_1(x) \le c\varphi_0'(x) - (\frac{1}{2\epsilon} + c\gamma)\varphi_0(x) .$$

In this case, (4.14) shows that $w(x,t)$ stays nonpositive for all times.

Proof. According to Proposition 2.1, there exist a maximal time $T > 0$ and a solution $(w, w_t) \in C^0([0,T), Z_\epsilon^1)$ of (1.9) with initial data (φ_0, φ_1) such that either $T = \infty$, or $T < \infty$. In the latter case, there exists a sequence of positive times t_n, $t_n < T$, such that $t_n \to T$ as $n \to +\infty$ and

$$\|(w(t_n), w_t(t_n))\|_{Z_\epsilon^1} \to +\infty , \qquad (4.16)$$

as $n \to +\infty$. By Lemma 4.2 and Lemma 4.4, we have

$$\begin{aligned} a(x)w(x,t) &\le \inf(a(x)\tilde{w}_1(x,t), a(x)\tilde{w}_1^+(x,t)) \le \inf(\|\tilde{w}_1(t)\|_{X^1}, \|\tilde{w}_1^+(t)\|_{X^1}) \\ &\le N \inf(\|(\varphi_0, \varphi_1)\|_{Z_\epsilon^1}, \|(\varphi_0^+, \varphi_1^+)\|_{Z_\epsilon^1}) \le K , \end{aligned}$$

for all $(x,t) \in \mathbf{R} \times [0,T)$. Therefore, by Proposition 4.1,

$$w(x,t) \ge -(1-d)a(x)^{-1}g(x) , \quad \forall (x,t) \in \mathbf{R} \times [0,T) ,$$

which allows us to apply Corollary 2.6. It follows that

$$\|(w(t), w_t(t))\|_{Z_\epsilon^1} \le K_7(1 + e^{\rho T})\|(\varphi_0, \varphi_1)\|_{Z_\epsilon^1}(1 + \|\varphi_0\|_{X^0})^{1/2} , \quad \forall t \in [0,T) , \quad (4.17)$$

which contradicts (4.16). Thus $T = \infty$. The estimates (4.14), (4.15) are direct consequences of Proposition 4.1, Lemma 4.2 and Lemma 4.3. If $d > 0$, the properties (1.12), (1.13) are obtained like in the proof of Theorem 1.1 in Section 3. $\square$

Remark. If $d = 0$, we can still show, by arguing as in the proof of Theorem 1.1 in Section 3, that

$$\lim_{t \to +\infty} (\|w_x(t)\|_{L^2} + \|w_t(t)\|_{L^2}) = 0 ,$$

and that this quantity is $\mathcal{O}(t^{-1/4})$ if $c = 2$. However, since (2.14) no longer holds, we cannot show that $\|(w, w_t)\|_{Y_{\epsilon a}}$ converges to zero in this case.

Proposition 4.6. *Under the assumptions of Proposition 4.5, the solution $(w, w_t) \in C^0(\mathbf{R}_+, Z^1_\epsilon)$ of (1.9) with initial data (φ_0, φ_1) satisfies*

$$\lim_{t\to+\infty} t^{1/4} \|w(t)\|_{L^\infty} \;=\; 0 \;. \tag{4.18}$$

If, in addition, $d > 0$ and $\epsilon < 1/4$, then

$$\lim_{t\to+\infty} t^{1/4} \left(\|w(t)\|_{L^\infty} + \|(w(t), w_t(t))\|_{Y_{\epsilon a}}\right) \;=\; 0 \;. \tag{4.19}$$

Proof. We first prove (4.18). From (4.15), it follows that

$$\|w(t)\|_{L^\infty} \;\le\; \sup(\|\tilde{w}^+_{-1}(t)\|_{L^\infty}, \|\tilde{w}^-_{-1}(t)\|_{L^\infty}) \;, \quad \forall t \in \mathbf{R}_+ \;. \tag{4.20}$$

Therefore, we need only show that (4.18) holds for any solution $\tilde{w} \in C^0(\mathbf{R}_+, X^1) \cap C^1(\mathbf{R}_+, X^0)$ of the linear equation with constant coefficients $L_{-1}\tilde{w} = 0$. Again, this can be done using the energy estimates of Section 2. Indeed, setting $\tilde{w}(x,t) = \omega(x + \nu t, t)$, where $\nu = \sqrt{c^2-4}/(1 + 2\epsilon c\gamma)$, we see that $\omega(x,t)$ satisfies

$$\epsilon\omega_{tt} + (1 + 2\epsilon c\gamma)\omega_t - 2B\omega_{xt} \;=\; A\omega_{xx} \;, \tag{4.21}$$

where $B > 0$ and $A = (1 + \epsilon c^2 + \epsilon c\sqrt{c^2-4})/(1 + 2\epsilon c\gamma) > 0$. Remark that the coefficient of ω_x vanishes in (4.21), like for the equation (1.9) in the case $c = 2$. Therefore, setting

$$E_0(t) \;=\; \int_{\mathbf{R}} \left(\frac{\epsilon}{2}\omega_t^2 + \frac{1}{2}A|\omega_x|^2\right) \,dx \;,$$

and modifying accordingly the definitions of $E_1(t)$ and $E_2(t)$, we show like in Section 2 that $\dot{E}_0(t) \le 0$ and that (2.9), (2.10) hold. Then arguing like in Section 3, we find that $\|\omega(t)\|_{H^1} \le C_0\|\omega(0)\|_{H^1}$, and $\lim_{t\to+\infty} t^{1/2}\|\omega_x(t)\|_{L^2} = 0$. Since $\|\omega(t)\|_{L^\infty} \le \|\omega(t)\|^{1/2}_{L^2}\|\omega_x(t)\|^{1/2}_{L^2}$, we see that $\lim_{t\to+\infty} t^{1/4}\|\omega(t)\|_{L^\infty} = 0$, which together with (4.20) proves (4.18).

To prove (4.19), we recall that, if $\epsilon < 1/4$, we can define $F_2(t)$ by (2.12) with $\beta = 0$. Indeed, if $\epsilon = 1/4 - \delta$ for some $\delta > 0$, it is easy to verify that, under the assumption H2,

$$F_2(t) \;\ge\; \delta K_3(c)\|W(t)\|^2_{Y_{\epsilon a}} \;, \tag{4.22}$$

for some constant $K_3(c) > 0$. Proceeding again like in the proof of Lemma 2.4, we show that

$$\dot{F}_2(t) + \kappa F_2(t) \;\leq\; C_1 \int_{x \geq x_d} a^2(x) w^2(x,t)\,dx \;\leq\; C_2 \|w(t)\|_{L^\infty}^2 \;, \tag{4.23}$$

where $C_2 = (C_1/2\gamma)e^{-2\gamma x_d}$. Integrating this differential inequality and using (4.18) and (4.22), we obtain (4.19). The proof of Proposition 4.6, hence of Theorem 1.3, is complete. □

Remark. Since L_{-1} is a linear operator with constant coefficients, it is possible to obtain explicit expressions for the solutions of the equation $L_{-1}\tilde{w} = 0$ in terms of the initial data, see for example [Sm], chap. VII.2.5. Therefore, (4.18) could also be proved by a direct (but cumbersome) calculation.

5. The Limiting Case $\epsilon = 0$

5.1. Local Stability

If we set $\epsilon = 0$ in (1.1), we obtain the well-known parabolic KPP equation, the travelling wave solutions $g(x)$ of which are given by (1.2) for $c \geq 2$. To study their stability, we proceed like in the Introduction. First, using the change of variables (1.4), we arrive at (1.5) with $\epsilon = 0$. Then, we look for solutions of the form $v(x,y,t) = g(x) + a(x)w(x,y,t)$, where $a(x) = e^{-\gamma x}$, and we are led to study the stability of the solution $w = 0$ of the parabolic equation (1.6) for $\epsilon = 0$ in the Sobolev space $X^1 \equiv H^1 \cap H^1_a$. Again, linear stability holds if and only if $1 - c\gamma + \gamma^2 \leq 0$, so the biggest perturbation space is obtained by choosing γ as in (1.8). Then, the equation (1.6) for $\epsilon = 0$ becomes

$$w_t \;=\; w_{xx} + \Delta_y w + \sqrt{c^2-4}\, w_x - 2gw - aw^2 \;. \tag{5.1}$$

It is known in this case that the origin is stable in X^1, with polynomial decay of the perturbations to zero as $t \to +\infty$.

Remark. In the case $c > 2$, it is also known that the origin is exponentially stable in X^1 if $1 - c\gamma + \gamma^2 < 0$. The best decay rate is obtained for the value $\gamma = c/2$ [Sa], which is precisely (1.10) for $\epsilon = 0$.

In Section 2, we have introduced various energy functionals for $\epsilon > 0$, which were used to estimate the different norms of the solution (w, w_t) of (1.9). These functionals are all well defined for $\epsilon = 0$ and allow us to control the norm of the solution w of (5.1). Since all the estimates are uniform in ϵ as ϵ goes to 0, we can follow exactly the lines of the proof of Theorem 1.1, and we arrive at the (already known) local stability result:

Theorem 5.1. *Assume that $n \leq 4$, and $c \geq 2$. Then there exist constants $\delta_0 > 0$ and $K_0 > 0$ such that the following holds: for all $\varphi_0 \in X^1$ satisfying $\|\varphi_0\|_{X^1} \leq \delta_0$, there exists a unique solution $w \in C^0(\mathbf{R}_+, X^1)$ of (5.1) with initial condition $w(0) = \varphi_0$. Moreover, one has $\|w(t)\|_{X^1} \leq K_0 \|\varphi_0\|_{X^1}$ for all $t \geq 0$, and*

$$\lim_{t \to +\infty} \left(\|\nabla w(t)\|_{X^0} + + \|w(t)\|_{L^2_a} \right) \;=\; 0\ .$$

In addition, if $c = 2$, one has

$$\lim_{t \to +\infty} \sqrt{t} \left(\|\nabla w(t)\|_{X^0} + \|w(t)\|_{L^2_a} \right) \;=\; 0\ .$$

Remark. Contrary to the hyperbolic case, a decay rate in time of the solution $w(t)$ of (5.1) is easily obtained for all $c \geq 2$. Indeed, following the ideas of Nash, it is a classical task to estimate the L^p-norm of solutions to parabolic equations for $p \geq 2$. In our case, if we know an upper bound on $\|w(t)\|_{L^p}$, then we can show that $\|w(t)\|_{L^{2p}}$ decays to zero (like an inverse power of t) as $t \to \infty$, see [FS]. Thus, using the L^2-bound of Theorem 5.1 and proceeding by recursion, we can show that

$$\|w(t)\|_{L^q} + \|w(t)\|_{L^q_a} \;=\; \mathcal{O}\left(t^{-\eta}\right)\ , \quad t \to +\infty\ ,$$

where $\eta = n(q-2)/(4q)$ and $q > 2$ is as in (1.14).

5.2. Global Stability

Like in the hyperbolic case, we obtain a global stability result when $n = 1$. But here we apply the Maximum Principle for parabolic equations on unbounded domains as given in [PW], Section 3.6. Remark that it is required that $w(x,t)$ does not grow faster than $\exp(Cx^2)$ as x goes to $\pm\infty$ (uniformly in t), a condition which is clearly satisfied in our case. Like in Paragraph 4.2, we denote by $\Sigma_d(t) \in \mathcal{L}(X^1, X^1)$ the linear semigroup associated with the equation

$$w_t \;=\; w_{xx} + \sqrt{c^2 - 4}\, w_x - (1 + d) g w\ , \quad d \in [-1, 1]\ .$$

For $\varphi_0 \in X^1$, we set $\Sigma_d(t)\varphi_0 = w_d(t)$, $\Sigma_d(t)\varphi_0^{\pm} = w_d^{\pm}(t)$, where $\varphi_0^{\pm}$ have been given in (1.19), (1.20). Then, following the lines of the proofs contained in Section 4, we obtain the global stability result below, which has already been known, though maybe not exactly in this form.

Theorem 5.2. *Assume that $n = 1$, and let $c \geq 2$, $d \in (0,1]$. Then, for any $\varphi_0 \in X^1$ satisfying (1.16), namely $\varphi_0(x) \geq -(1-d)a(x)^{-1}g(x)$ for all $x \in \mathbf{R}$, there exists a unique solution $w \in C^0(\mathbf{R}_+, X^1)$ of (5.1) with initial condition $w(0) = \varphi_0$. Moreover, one has $w(x,t) \geq -(1-d)a^{-1}(x)g(x)$, and*

$$w_{-1}^{-}(x,t) \leq w(x,t) \leq w_1(x,t) \leq w_{-1}^{+}(x,t) ,$$

for all $x \in \mathbf{R}$, $t \in \mathbf{R}_+$. In particular, if $d > 0$,

$$\lim_{t\to+\infty} t^{1/4} \left(\|w(t)\|_{L^\infty} + \|w(t)\|_{H_a^1} \right) = 0 .$$

Appendix A. Maximum Principle for a Hyperbolic Operator

We consider the following hyperbolic operator $\mathcal{L}$ with constant real coefficients

$$\mathcal{L}(u) = Au_{xx} + 2Bu_{xt} + Cu_{tt} + Du_x + Eu_t ,$$

where

$$C < 0 , \quad B^2 - AC > 0 . \tag{A.1}$$

We introduce a function $h \in C^0(\mathbf{R} \times [0,T])$ satisfying

$$h(x,t) \geq \underline{h}, \quad \text{for all } (x,t) \in \mathbf{R} \times [0,T] , \tag{A.2}$$

where T is a positive number and $\underline{h}$ is a real number. We suppose in addition, that the condition

$$(E^2 - 4C\underline{h})(B^2 - AC) \geq (BE - CD)^2 , \tag{A.3}$$

holds. Finally, we set $L = \mathcal{L} + h(x,t)$. The following Maximum Principle is a simple consequence of the one given by Protter and Weinberger (see [PW], Chapter 4, Theorem 1).

Theorem A.1. *Assume that the conditions (A.1), (A.2) and (A.3) hold. If the function* $(u(x,t), u_t(x,t))$ *belongs to* $C^0([0,T], H^1_{loc}(\mathbf{R}) \times L^2_{loc}(\mathbf{R}))$, *with* $Au_{xx} + 2Bu_{xt} + Cu_{tt}$ *in* $L^2_{loc}(\mathbf{R} \times (0,T))$, *and satisfies the following properties,*

$$L(u)(x,t) \geq 0 \ , \quad \text{a.e.}(x,t) \in \mathbf{R} \times [0,T] \ , \tag{A.4}$$

$$u(x,0) \leq 0 \ , \quad \forall\, x \in \mathbf{R} \ , \tag{A.5}$$

$$-Cu_t(x,0) - Bu_x(x,0) - \frac{1}{2}Eu(x,0) \leq 0 \ , \quad \text{a.e.} \ , \tag{A.6}$$

then $u(x,t) \leq 0$ *for all* $(x,t) \in \mathbf{R} \times [0,T]$.

Proof. Protter and Weinberger proved their Maximum Principle under the stronger assumption $u(x,t) \in C^2(\mathbf{R} \times (0,T)) \cap C^1(\mathbf{R} \times [0,T))$, but their proof generalizes easily to functions u satisfying the weaker regularity hypothesis $(u(x,t), u_t(x,t)) \in C^0([0,T], H^1_{loc}(\mathbf{R}) \times L^2_{loc}(\mathbf{R}))$, with $Au_{xx} + 2Bu_{xt} + Cu_{tt}$ in $L^2_{loc}(\mathbf{R} \times (0,T))$. Indeed their key identity (see [PW], Equation (3), page 202) still holds under these weaker regularity assumptions and is proved by a density argument.

If $E = D = 0$, the result of Theorem A.1 is a direct consequence of the above remark and of Theorem 1 of [PW]. Indeed, thanks to our assumptions (A.1), (A.2), (A.3), the condition of [PW] on the operator L, that is $h(x,t) \geq 0$, is clearly satisfied. Since $E = 0$, the conditions required on $u(x,t)$ are exactly the hypotheses (A.4) to (A.6).

If $E \neq 0$ or $D \neq 0$, we reduce our problem to the previous case by introducing the function

$$v(x,t) \ = \ e^{-\alpha t - \beta x} u(x,t) \ ,$$

where

$$\alpha \ = \ \frac{EA - BD}{2(B^2 - AC)} \ , \quad \beta \ = \ \frac{CD - EB}{2(B^2 - AC)} \ .$$

A short computation shows that

$$L(u)(x,t) \ = \ e^{\alpha t + \beta x} \tilde{L}(v)(x,t) \ ,$$

where $\tilde{L}(v) \ = \ Av_{xx} + 2Bv_{xt} + Cv_{tt} + \tilde{h}v$ and

$$\tilde{h}(x,t) \ = \ -\frac{1}{4C}\{(E^2 - 4Ch(x,t)) - \frac{(EB - CD)^2}{B^2 - AC}\} \ .$$

Now, we can apply the previous Maximum Principle, where $D = E = 0$, to the operator $\tilde{L}$ and to the function v. Indeed, due to the hypotheses (A.1), (A.2), (A.3), $\tilde{h}(x,t) \geq 0$ for all $(x,t) \in \mathbf{R} \times [0,T]$. Moreover, $\tilde{L}(v)(x,t) \geq 0$ a.e.$(x,t) \in \mathbf{R} \times [0,T]$ and v satisfies

$$v(x,0) \leq 0 \ , \quad -Cv_t(x,0) - Bv_x(x,0) \leq 0 \ ,$$

which are exactly the required conditions. Thus, we have proved that $v(x,t) \leq 0$, hence $u(x,t) \leq 0$ for all $(x,t) \in \mathbf{R} \times [0,T]$. □

Remark. Theorem A.1 suggests the following definition of a partial order in $H^1_{loc}(\mathbf{R}) \times L^2_{loc}(\mathbf{R})$. We say that $(\varphi_0, \varphi_1) \leq (\psi_0, \psi_1)$ if

$$\varphi_0(x) \leq \psi_0(x) \ , \quad \forall\, x \in \mathbf{R} \ ,$$
$$-C\varphi_1(x) - B\varphi_0'(x) - \frac{1}{2}E\varphi_0(x) \leq -C\psi_1(x) - B\psi_0'(x) - \frac{1}{2}E\psi_0(x) \quad \text{a.e.} \ ,$$

see (A.5), (A.6). Then, if $(\varphi_0, \varphi_1) \leq (\psi_0, \psi_1)$, the solution of the linear hyperbolic equation $L(u)(x,t) = 0$ satisfying $u(x,0) = \varphi_0(x)$, $u_t(x,0) = \varphi_1(x)$ stays for all $t \in \mathbf{R}_+$ below the corresponding solution with initial data (ψ_0, ψ_1). An important property of this order is that we can write any $(\varphi_0, \varphi_1) \in H^1_{loc} \times L^2_{loc}$ as the sum of a "positive" part $(\varphi_0^+, \varphi_1^+) \geq 0$ and a "negative" part $(\varphi_0^-, \varphi_1^-) \leq 0$. This decomposition is unique if we impose that $(\varphi_0^+, \varphi_1^+) = 0$ whenever $(\varphi_0, \varphi_1) \leq 0$ and $(\varphi_0^-, \varphi_1^-) = 0$ whenever $(\varphi_0, \varphi_1) \geq 0$. In the case of the operator L_d defined in (4.6), for which $C = -\epsilon$, $B = \epsilon c$, $E = -(1 + 2\epsilon c\gamma)$, the corresponding formulae for $(\varphi_0^\pm, \varphi_1^\pm)$ are given in (1.19), (1.20).

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