An interpolation theorem related to the a.e. convergence of integral operators.

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Abstract

We show that for integral operators of general form the norm bounds in Lorentz spaces imply certain norm bounds for the maximal function. As a consequence, the a.e. convergence for the integral operators on the Lorentz spaces follows from the appropriate norm estimates.

1. Introduction

In this paper, we study the question of a.e. convergence of integral operators satisfying certain norm estimates in most general setting. The main tool in this studies is an interpolation theorem yielding the norm bounds on the maximal function of the int egral operator given norm estimates on the operator itself. The result we show looks so natural and is so general that we hope it may be useful in many situations. In particular, in the Appendix we sketch the application to the spectral analysis of Schrödinger operators. It was hard to believe that the result we prove here is not known, but discussions with many specialists in the field convinced me that at least the reference is not simple to find.

The main result we prove here is as follows.

Let us consider two measure spaces (X, μ) and (Y, ν) with μ, ν being positive measures. Let $A(k, x)$ be a measurable function on $M \times N$, integrable over the sets of finite measure in M for a.e. $k \in N$. Let an operator T be given by

$$(Tf)(k) = \int_X A(k,x) f(x)\, d\mu(x). \tag{1}$$

The operator T may be defined originally for all simple functions or bounded functions of finite support. Consider an extending family of measurable regions $\Omega_t \subset X$, depending on the real parameter t, so that $\Omega_{t_1} \subset \Omega_{t_2}$ if $t_2 > t_1$. Define a maximal function M_T, corresponding to the family $\{\Omega_t\}_{t \in R}$ and the operator T by

$$(M_T f)(k) = \sup_t \left| \int_{\Omega_t} A(k,x) f(x)\, dx \right|. \tag{2}$$

We use the notation $\| \, \|_{pq}$ for the norm in the Lorentz space L_{pq} or for a conventional quasinorm when the norm does not exist.

Then the following theorem holds:

Theorem. *Suppose that an operator T given by (1) satisfies the bounds*

$$\|Tf\|_{s_i \infty} \le C_i \|f\|_{p_i 1}, \quad i = 1, 2,$$

where all s_i, p_i are more than or equal to 1, $s_1 > s_2$, and $p_1 \neq p_2$. Then the maximal function M_T, given by (2), satisfies

$$\|(M_T f)\|_{s_r q} \le C_s(q) \|f\|_{p_r q} \tag{3}$$

where $s_r^{-1} = (1-r)s_1^{-1} + r s_2^{-1}$, $p_r^{-1} = (1-r)p_1^{-1} + r p_2^{-1}$, $0 < r < 1$, and q is any number satisfying $1 \le q \le \infty$.

Remark. The theorem above would turn into just generalized Marcinkiewicz interpolation theorem if we would change M_T to T in (3).

The motivation for studying this problem comes, in particular, from the applications of a.e. convergence results for certain integral operators to the study of the spectrum of

Schrödinger operators (continuous as well as discrete) [5], [?]. We briefly sketch this relationship in the Appendix, referring to the above works for details.

We note that the above theorem generalizes some old and well-known results on the a.e. convergence of the orhtogonal series and Fourier integrals, in particular, the results of Paley [7] and Zygmund [9]. Of course the theorem does not touch on the question of a.e. convergence at the corner points ($r = 0, 1$). These questions are very subtle even in the orthogonal series setting. In the case of trigonometric series the question about a.e. convergence of the series with coefficients from $l_2(Z)$ is equivalent to the celebrated Luzin's conjecture. Although in this case the a.e. convergence holds by the result of Carleson [2], it does not hold in general for orthonormal systems of functions (see, e.g. [4]). In particular, in the case of the multiple Fourier series, C.Fefferman showed that one can choose the extending sets Ω_t so that there is no a.e. convergence on $l_2(Z^n)$. [3].

2. Proof of the main result.

Proof. Fix an arbitrary real number $\tilde{r}$, $0 < \tilde{r} < 1$, and take s_3, s_4 so that $s_1 > s_3 > s_{\tilde{r}} > s_4 > s_2$. Let p_3, p_4 correspond to s_3, s_4 in a usual way. Also let us fix some number q' such that $q' > \max\{p_3, p_4\}$.

First, we prove the theorem under an additional assumption that $\mu(\Omega_t)$ depends continuously on t. The first step is to decompose the support of f into dyadic pieces and estimate certain auxilliary maximal functions. Such approach was used already by Paley [7] in his work on a.e. convergence of the series of orthonormal functions. Let f be a measurable bounded function of finite support and choose n so that $2^{n-1} \leq \mu(\text{supp}(f)) \leq 2^n$. Let the measurable set E be the support of the function $f : E = \{x| \; |f(x)| > 0\}$. For every integer $m < n$, we consider a partition of the set E into the dyadic pieces of size 2^m in a following way:

$$E_{m,l} = (\Omega_{t_{m,l+1}} \setminus \Omega_{t_{m,l}}) \cap E,$$

where $t_{m,l}$ is defined by a condition

$$t_{m,l} = \inf\{t|\mu(\Omega_t \cap E) = 2^m l.\}$$

The value $t_{m,l}$ is well-defined at least for $l \leq 2^{n-m-1}$ because $\mu(\Omega_t)$ depends on t continuously. The number of the sets $E_{m,l}$ is between 2^{n-m} and 2^{n-m-1}. For notational convenience, we will assume that this number is always 2^{n-m} and will define the missing $E_{m,l}$ as empty sets.

Let us define the functions $M_{m,l}f$ and $M_m f$ by

$$M_{m,l}f(k) = \int\limits_{E_{m,l}} A(k,x)f(x)\,dx.$$

and

$$M_m f(k) = \sup_l |M_{m,l}f(k)|$$

Considering a dyadic development of every real number N, it is easy to see that

$$(M_T f)(k) \leq \sum_{m=-\infty}^{n} M_m f(k). \tag{4}$$

Indeed, suppose that for a given value of k, the supremum in (2) is reached when the value of the parameter t is equal to N (clearly if f has finite support, the supremum is reached for some value of t). Define the real number a by $a = \mu(E \cap \Omega_N)$ and consider the dyadic development of a :

$$a = \sum_{j=-\infty}^{n} a_j 2^j,$$

where a_j is equal to 0 or 1 for every j. Then by construction, we can find disjoint sets $E_{m,l}$, at most one for each value of m, so that

$$\mu\left((E \cap \Omega_N) \setminus (\cup_m E_{m,l})\right) = 0.$$

In fact, for each m the set $E_{m,l}$ belongs to the union iff $a_m = 1$. The corresponding value of l then may be found by the formula $l = \sum_{j=m+1}^{n} a_j 2^{j-m}$. By the generalized Marcinkiewicz interpolation theorem (see, e.g., [1] or [8]) we have

$$\|Tf\|_{s_r q'} \leq C_r(q')\|f\|_{p_r q'}, \; 0 < r < 1.$$

Recall that the quasinorm $\|\cdot\|^*$ in Lorentz spaces may be defined by

$$\|f\|^*_{pq} = \left(\frac{q}{p}\int\limits_0^\infty |f^*(t)|^q t^{\frac{q}{p}-1}\,dt\right)^{\frac{1}{q}}.$$

Let us denote by $\chi(E)$ the characteristic function of the measurable set E. Using the equivalence of the quasinorm $\|\cdot\|^*_{pq}$ and norm $\|\cdot\|_{pq}$ for $p>1$ (note that p_i are automatically more than 1), we see that in particular

$$\|M_{m,l}f(k)\|^{s_iq'}_{q'} = \|T(f\chi(E_{m,l}))(k)\|^{q'}_{p_iq'} \leq C_i(q')^{q'}\left(\frac{q'}{p_i}\int\limits_0^\infty |(f\chi(E_{m,l}))^*|^{q'} t^{\frac{q'}{p_i}-1}\,dt\right) \leq$$

$$\leq C_i(q')^{q'} 2^{m(\frac{q'}{p_i}-1)}\frac{q'}{p_i}\|f\chi(E_{m,l})(t)\|^{q'}_{q'},\ \ i=1,2.$$

Note that

$$\|M_m f\|^{q'}_{s_iq'} \leq \sum_{l=1}^{2^{n-m}} \|M_{ml}f\|^{q'}_{s_iq'}.$$

Hence summing over l we obtain

$$\|M_m f\|^{q'}_{s_iq'} \leq C_i(q')^{q'}\frac{q'}{p_i}2^{m(\frac{q'}{p_i}-1)}\|f\|^{q'}_{q'}.$$

By (4), we have that

$$\|M_T f\|_{s_iq'} \leq C_i(q')\left(\frac{q'}{p_i}\right)^{\frac{1}{q'}}\|f\|_{q'}\sum_{m=-\infty}^{n} 2^{m(\frac{1}{p_i}-\frac{1}{q'})} = B_i(q')2^{n(\frac{1}{p_i}-\frac{1}{q'})}\|f\|_{q'}. \tag{5}$$

In the last inequality we denoted by $B_i(q')$ a new constant, which only depends on i and q'. Now we note that in a particular case when f is a characteristic function of a set, $f=\chi(E)$, (5) means

$$\|M_T\chi(E)\|_{s_iq'} \leq B_i(q')2^{n(\frac{1}{p_i}-\frac{1}{q'})}2^{\frac{n}{q'}} = B_i(q')2^{\frac{n}{p_i}} \leq 2^{\frac{1}{p_i}}B_i(q')\|\chi(E)\|_{p_i1}. \tag{6}$$

It is easy to verify that the operator $M_T f$, defined originally on the measurable bounded functions of compact support, is a sublinear operator. We remind that it means

$$|(M_T(f_1+f_2))(k)| \leq |(M_Tf_1)(k)| + |(M_Tf_2)(k)|,\ \ |(M_T(bf))(k)| = |b||(M_Tf)(k)|$$

for every scalar b. It is a well-known and simple to check fact that from the inequality (6) for sublinear operator it follows that $\|Mg\|_{s_i q'} \leq C\|g\|_{p_i 1}$ holds for all simple functions (see [8]) and hence by simple limiting argument for all bounded functions of finite support. Interpolating, we obtain that $\|Mf\|_{s_{\tilde{r}} q} \leq C_{\tilde{r}}(q)\|f\|_{p_{\tilde{r}} q}$ for every function f bounded and of compact support. It is straightforward to see that this relation is then extended to all functions $f \in L_{p_{\tilde{r}} q}$. This completes the proof under the assumption that $\mu(\Omega_t)$ is continuous.

Consider now the general case. The technical problem we face here is that now it is not so easy to apply the dyadic decomposition of the $\mathrm{supp}(f) = E$ into coherent pieces: the jumps of the monotone function $\mu(E \cap \Omega_t)$ may in general pose obstacles to that. We will handle this problem by constructing an auxilliary measure space and an auxilliary operator. We prove the estimates for the maximal function of this auxilliary operator, and then show that from these estimates follows the result for the original problem.

At every value of the parameter t the function $\mu(\Omega_t)$, as a monotone function, has limits from the left and from the right. Denote by $t_{\mp}$ the value of the jump on the left and on the right respectively: $t_+ = \mu(\Omega_{t+0} \setminus \Omega_t)$ and $t_- = \mu(\Omega_t \setminus \Omega_{t-0})$. The set of values of t where any jump may occur is clearly at most countable. Let the sequence $\{t_n\}_{n=1}^{\infty}$ denote these points. To each t_n corresponds the size of the jump of $\mu(\Omega_t)$ at this point, y_n. If for some values of t we have both t_- and t_+ nonzero, this value of t is encountered in the above sequence twice: say, $t_n = t_{n+1} = t$ and $y_n = t_-$, $y_{n+1} = t_+$. Hence, for each member of the sequence t_n only one of the two possible jumps takes place. We denote by Γ_{t_n} the portion of the measure space corresponding to $t_n : \Gamma_{t_n} = \Omega_{t_n+0} \setminus \Omega_{t_n}$ if we have a jump on the right at t_n and $\Gamma_{t_n} = \Omega_{t_n} \setminus \Omega_{t_n-0}$ otherwise.

Let us consider the following auxilliary measure space $\tilde{X}$ which we biuld out of X. At each value of t_n we replace $\Gamma_{t_n} \subset X$ by

$$B_{t_n} = \Gamma_{t_n} \times [0, 1]$$

with the structure of the product measure space. Hence

$$\tilde{X} = (X \setminus (\cup_n \Gamma_{t_n})) \cup (\cup_m B_{t_m}) .$$

The measure $\tilde{\mu}$ on $\tilde{X}$ coincides with measure μ on the measurable set

$$X_0 = X \setminus (\cup_n \Gamma_{t_n}) ,$$

while on B_{t_n} the measure $\tilde{\mu}$ equals the product measure $\mu \times dx$ (dx being a Lebesgue measure on $[0,1]$). Furthermore, we let the kernel $\tilde{A}(k,x)$, defined on $Y \times \tilde{X}$, be equal to $A(k,x)$ for all k when $x \in M_0$ and $\tilde{A}(k,x,y) = A(k,x)$ for all $k \in Y$ and all $y \in [0,1]$ if x belongs to Γ_{t_n} for some n. Define an integral operator $\tilde{T}$ by

$$(\tilde{T}f)(k) = \int_{\tilde{X}} \tilde{A}(k,x) f(x)\, dx.$$

Next, define a family $\tilde{\Omega}_u$ of the extending measurable sets in $\tilde{X}$. We construct $\tilde{\Omega}_u$ so that $\tilde{\mu}(\tilde{\Omega}_u) = u$. Let $t_0(u) = \sup_t \{t | \mu(\Omega_t) \le u\}$. If $t_0(u) \ne t_n$ for any n, we let

$$\tilde{\Omega}_u = \left(\Omega_{t_0(u)} \cap X_0\right) \bigcap \left(\cup_{t_m < t_0(u)} B_{t_m}\right) .$$

In this case $\tilde{\mu}(\tilde{\Omega}_u) = u$, since $\mu(\Omega_t)$ is continuous at every $t \notin \{t_n\}_{n=1}^{\infty}$. Suppose now that $t_0(u) = t_n$. If at t_n we have a jump on the left ($\mu(\Omega_{t_n} \setminus \Omega_{t_n - 0}) > 0$) we let

$$\tilde{\Omega}_u = (\Omega_{t_n - 0} \cap X_0) \cup (\cup_{t_m < t_n} \Gamma_{t_m} \times [0,1]) \cup \left(\Gamma_{t_n} \times [0, \frac{u - \mu(\Omega_{t_n - 0})}{\mu(\Gamma_{t_n})}]\right) .$$

Otherwise, if the jump is on the right ($\mu(\Omega_{t_n + 0} \setminus \Omega_{t_n}) > 0$), we define

$$\tilde{\Omega}_u = (\Omega_{t_n} \cap X_0) \cup (\cup_{t_m < t_n} \Gamma_{t_m} \times [0,1]) \cup \left(\Gamma_{t_n} \times [0, \frac{u - \mu(\Omega_{t_n})}{\mu(\Gamma_{t_n})}]\right) ,$$

essentially just changing $t_n - 0$ to t_n. Defined this way, $\tilde{\Omega}_u$ constitutes a measurable, extending family of sets such that $\tilde{\mu}(\tilde{\Omega}_u) = u$ and, in particular, is continuous. Define the maximal function $\tilde{X}$, corresponding to the operator $\tilde{T}$ and the family $\tilde{\Omega}_u$:

$$(\tilde{M}_{\tilde{T}} f)(k) = \sup_u \left| \int_{\tilde{\Omega}_u} \tilde{A}(k,x) f(x)\, d\mu(x) \right| .$$

We have

Lemma. *For every r, $1 > r > 0$, the operator $\tilde{T}$ satisfies the norm bounds*

$$\|\tilde{T}f\|_{s_r p_r} \le C(r)\|f\|_{p_r}.$$

Proof. Indeed,

$$\tilde{T}f(k) = \int\limits_{X_0} A(k,x)f(x)\,d\mu(x) + \sum_n \int\limits_{\Gamma_{t_n}} d\mu(x) \int\limits_0^1 \tilde{A}(k,x,y)f(x,y)\,dy =$$

$$= \int\limits_{X_0} A(k,x)f(x)d\mu(x) + \sum_n \int\limits_{\Gamma_{t_n}} d\mu(x)A(k,x) \int\limits_0^1 f(x,y)\,dy$$

by the definition of the kernel $\tilde{A}(k,x,y)$. Because of the norm bounds on the operator T and interpolation between them, we obtain

$$\|\tilde{T}f\|_{L_{s_r p_r}(Y)} \le C(r) \left(\int\limits_{X_0} |f(x)|^{p_r} d\mu(x) + \sum_n \int\limits_{\Gamma_{t_n}} d\mu(x) \left| \int\limits_0^1 f(x,y)\,dy \right|^{p_r} \right)^{\frac{1}{p_r}} \le$$

$$\le C(r) \left(\int\limits_{X_0} |f(x)|^{p_r} d\mu(x) + \sum_n \int\limits_{\Gamma_{t_n}} d\mu(x) \int\limits_0^1 |f(x,y)|^{p_r}\,dy \right)^{\frac{1}{p_r}} \le C(r)\|f\|_{L_{p_r}(\tilde{X})}.$$

We used Jensen inequality in the second step. $\square$

Now let us choose r_1, r_2 so that $s_{r_1} = s_3$ and $s_{r_2} = s_4$. By the lemma, we have the bounds

$$\|\tilde{T}f\|_{s_i \infty} \le C(i)\|f\|_{p_i},\ i = 3, 4.$$

Since the function $\tilde{\mu}(\tilde{\Omega}_u)$ is continuous in u, we can infer from the first part of the proof that the bound

$$\|\tilde{M}_{\tilde{T}}\|_{s_{\tilde{r}} q} \le C_{\tilde{r}}(q)\|f\|_{p_{\tilde{r}} q}$$

hold for every $q \in [1, \infty]$. But for every t, we have

$$\int\limits_{\Omega_t} A(k,x)f(x)d\mu(x) = \int\limits_{\tilde{\Omega}_{u(t)}} \tilde{A}(k,y)\tilde{f}(y)\,d\tilde{\mu}(y).$$

Here $u(t)$ is such that $\mu(\Omega_t) = u(t)$ and $\tilde{f}$ is a function which we define in a clear way. Namely, if $y = x \in X_0$ we let $\tilde{f}(y) = f(x)$ and if $y = (x_1, y_1) \in \Gamma_{t_n} \times [0,1]$, we let $\tilde{f}(x_1, y_1) = f(x_1)$. It is easy to see that $\|\tilde{f}\|_{L_{pq}(\tilde{X})} = \|f\|_{L_{pq}(X)}$ for all p, q because the distribution functions of $\tilde{f}$ and f coincide. Therefore,

$$(M_T f)(k) = \sup_t \left| \int\limits_{\Omega_t} A(k,x) f(x) d\mu(x) \right| \leq \sup_u \left| \int\limits_{\tilde{\Omega}_u} \tilde{A}(k,y) \tilde{f}(y) d\tilde{\mu}(y) \right| = \tilde{M}_{\tilde{T}} \tilde{f}(k)$$

and hence

$$\|M_T f\|_{L_{s_{\tilde{r}}q}(X)} \leq \|\tilde{M}_{\tilde{T}} \tilde{f}\|_{L_{s_{\tilde{r}}q}(\tilde{X})} \leq C_{\tilde{r}}(q) \|\tilde{f}\|_{L_{p_{\tilde{r}}q}(\tilde{X})} = C_{\tilde{r}}(q) \|f\|_{L_{p_{\tilde{r}}q}(X)}.$$

Hence we have shown the bound (3) in the general case and since the number $\tilde{r}$ is arbitrary between 0 and 1, the proof of the theorem is now complete. □

We may obtain the Zygmund's and Paley's theorems by specifying the measure spaces and the kernels. Let X and Y to be real lines equipped with Lebesgue measure and $A(k,x) = \exp(ikx)$. The $L_1 - L_\infty$ and $L_2 - L_2$ bounds are obvious and w e get the Zygmunds theorem, which says that the Fourier integral converges a.e. for functions f from L_p, $1 \leq p < 2$ and claims the corresponding estimate (3) for the maximal function.

Next let $X = Z^1$, $Y = (a,b)$ and $A(k,n) = \phi_n(k)$, where $\{\phi_n(k)\}_{n=1}^\infty$ is an orthonormal uniformly bounded system in $L_2(a,b)$. Again the same bounds as above hold and we get the Paley's theorem which says that the series of o rhtonormal functions $\sum_n c_n \phi_n(x)$ converges a.e. if $c_n \in l_p$, $1 \leq p < 2$.

The last example we would like to present here is that of a pseudodifferential operator in R^n with bounded symbol $a(k,x)$:

$$(Tf)(k) = \int\limits_{R^n} \exp(ikx) a(k,x) f(x)\, dx. \tag{7}$$

If the symbol $a(k,x)$ is from the class for which an $L_2 - L_2$ estimate is valid, we have that the integral defining (7) convergres a.e. k no matter which system of extending regions Ω_t

we take. This is the situation which appears (with $n = 1$) in some of the applications to Schrödinger operators [5].

Appendix.

In [5], a new approach to the investigation of the stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators under slowly decreasing pertubations is developed. The absolutely continuous spectrum corresponds to the infinite motion of the quantum particle. The question of the preservation of the absolutely continuous spectrum under decaying perturbations means, roughly speaking, finding out which local, i.e. decaying, pertubations are not strong e nough to destroy this infinite motion. It turns out that in many cases the problem of spectral analysis may be reduced to a problem of finding asymptotics of the solutions of a certain ODE system, depending on the parameter λ, for a.e. values of t his parameter (λ has a meaning of energy for Schrödinger equation). The latter problem may be tackled with asymptotic integration methods. The first step is an introduction of a transformation of the system, containing, in a simplest case, a function

$$q(x, \lambda) = \int_x^\infty \theta^2(y, \lambda) V(y)\, dy. \tag{8}$$

Here $\theta(x, \lambda)$ is a solution of the unperturbed Schrödinger equation, so that

$$\left(-\frac{d^2}{dx^2} + U\right) \theta = \lambda\theta$$

and $V(x)$ is a perturbation. It turns out that if the function $q(x, \lambda)$ may be defined and decays sufficiently fast for the values of λ from the support of the absolutely continuous spectrum of the unperturbed operator $(-\frac{d^2}{dx^2} + U)$, then this absolutely continuous spectrum is preserved under perurbation by V. For example, the simplest condition under which the a.c. spectrum is preserved is that $q(x, \lambda)V(x) \in L^1$ for a.e. λ from the support of the a.c. spectrum of the unperturbed operator. The interesting case is when V is not absolutely integrable; otherwise the stability is known for a long time and may be proven in a very simple way. However, although there has been a considerable attention to the subject, until recently no other general classes preserving the a.c. spectrum, given in terms of the rate of decay, were known even in the simplest situations (such as $U = 0$).

Thus one is led to the question of studying a.e. convergence and rate of decay estimates for the integral operators like (8). The paper [5] contains a simpler and less general version of the theorem we proved here. That result suffices for the app lications of that paper, in particular producing a new general class of potentials preserving the a.c. spectrum of the free ($U = 0$) and periodic (U periodic) Schrödinger operators: all potentials V satisfying $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$. The more general version we prese nt here, however, may also be applied to studying discrete Schrödinger operators [6].

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