An infinite-dimensional extension of a Poincaré's result concerning the continuation of periodic orbits

Paolo Perfetti

Dipartimento di Matematica II Università di Roma
via della Ricerca Scientifica 00133 Roma, Italy

Abstract We study the existence of periodic solutions for the infinite-dimensional second order system $\ddot{x} = V_x,\ x \in \mathbb{T}^{\mathbb{Z}_+}$. Using the Implicit-Function-Theorem we prove the existence of time-periodic solutions at "high frequencies"; no "smallness condition" on $V(x)$ is required.

Contents

§0. Introduction

In this paper we study the following (infinite) second order ordinary differential equation

$$\ddot{x} = V_x(x), \quad x \in \mathbb{T}^{\mathbb{Z}_+} \equiv \bigotimes_{i=1}^{\infty} \mathbb{T}_i \tag{0.1}$$

Over $\mathbb{T}^{\mathbb{Z}_+}$ we introduce the topology induced by the quadratic metric, see (1.1), that makes it locally homeomorphic to l_2 (the space of the sequences $\{a_i\}$ such that $\sum a_i^2 < \infty$) so we indicate $\mathbb{T}^{\mathbb{Z}_+}$ as $\mathcal{T}_2$. $V(x)$ is a real function (sometimes called *interaction* or *potential*) of class C^2 over $\mathcal{T}_2$ in the sense of Fréchet and we look for *time-periodic* solutions at "high frequencies". The idea of studying solutions with "high frequencies" has been already used in [CP] but while there we find solutions with infinite energy (besides *maximal almost-periodic*), here the energy of each solution is finite. This difference sets a severe restriction on the potentials $V(x)$ that can be considered here. Now in fact $V(x)$ must be a true function of an infinite (countable) set of variables (1) and this causes a strong decay of the interaction among the variables x_j and $x_{j'}$ with $|j - j'|$ large (the bigger is $|j - j'|$ the smaller is the size of the interaction between x_j and $x_{j'}$). Here in fact we will consider the following potentials well defined over $\mathcal{T}_2$ ($g_j : \mathbb{T}^{2L_o+1} \to \mathbb{R}$, $x^{(L_o)} \equiv (x_{j-L_o}, \ldots, x_{j+L_o})$, L_o is a positive integer)

$$V(x) = \sum_{j=1}^{\infty} g_j(x^{(L_o)}), \qquad V(x) = \sum_{i,j=1}^{\infty} e^{-|i-j|}(1 - \cos(x_i - x_j)). \tag{0.2}$$

The first $V(x)$ is called *short-range interaction* while the second one is a kind of *long-range interaction* and in both of them the decay announced before is transparent. Technically speaking we apply the Implicit-Function-Theorem and look for solutions that are *continuation* of *unperturbed solutions*. The unperturbed solutions arise when all the variables x_i are decoupled and in this way the result contained in this paper can be considered an infinite-dimensional extension of a result of Poincaré, see [Po],[SM],[M].

¿From a physical point of view the system (0.1) represents a collection of identical rotators, whose mass can be different from zero and interacting via the potential $V(x)$. Roughly speaking we find solutions in which the first rotator x_1 rotates so fast that its motion "almost decouples" from the others one that instead give rise to librations at "high frequency" and of definitively small amplitude (in the asymptotic regime the frequency of rotation of x_1 goes to infinity and the variables are decoupled). Hence we can

(1) in [CP] the system of equations $\ddot{x}_i = f_i(x)$ $(i \geq 1)$ is studied and what must be well defined is $f_i : \mathbb{T}^{\mathbb{Z}_+} \to \mathbb{R}$; the function of an infinite countable set of variables $V : \mathbb{T}^{\mathbb{Z}_+} \to \mathbb{R}$ such that $\frac{\partial V}{\partial x_i}(x) = f_i(x)$ may not even exist.

consider the previous dynamic generated by (0.1) as representative of a one-dimensional dynamical system on which acts an infinite-dimensional perturbation. This is quite clear writing the solutions as

$$
\begin{aligned}
x_1(t) &= [\omega t + u_1(\omega t)], \qquad && \omega + \omega \frac{du_1}{d(\omega t)} \neq 0 \quad \forall t \\
x_i(t) &= [u_i(\omega t)] && (i \geq 2)
\end{aligned}
$$

ω (*the frequency*) is a real ("large") number, $u_i : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic, smooth enough, function while $[\cdot]$ is the standard projection of $\mathbb{R}$ over $\mathbb{T}$. Practically we find periodic, parameterized by ω, solutions such that when $\omega \to \infty$ they reduces to $x_1(t) = [ct + x_1^o]$, $\quad x_j(t) \equiv [x_j^o] \quad (j \geq 2)$ and this will be the unperturbed solution. The introduction of the quadratic metric over $\mathbb{T}^{\mathbb{Z}_+}$ makes quite natural the search of "l_2" solutions $x(t)$; briefly this means that $\sup_{t\in[0,2\pi]} |u_j(t)| \to 0$ as $j \to \infty$ in such a way that $\sum_j \sup_{t\in[0,2\pi]} |u_j(t)|^2 < \infty$.

Equations like (0.1) often arise in the study of energy propagation (sound, heat) in crystal lattices (see [FSW]). In an other context the equation (0.1) can be considered as the discretization of a partial differential equation and some features of the solutions of the partial equation can be recovered performing some suitable limits (*Lagrange limit* for example; see [G] and [Ga]) starting from the solutions of the discrete model.

The existence of periodic solutions in systems of differential equations (ordinary as well as partial but both infinite), is an argument extensively studied. From the "local methods" point of view (as it is the *continuation method* used in the present paper) we mention [AF], [AFS], [Ku], [Wa], [CW], [Pö]. We just point out that most of the models studied in the given references involve the presence of small divisors while in our case such a kind of problem is absent.

Acknowledgements It is a pleasure to thank Prof. Luigi Chierchia for having proposed to me the problem, for useful suggestions and advises.

§1. Setup and definitions

Let $\mathbb{T}^{\mathbb{Z}_+}$ denotes the Cartesian product of infinitely many copies of the one-dimensional flat torus

$$
\mathbb{T}^{\mathbb{Z}_+} = \bigotimes_{i=1}^{\infty} \mathbb{T}_i, \qquad \mathbb{T}_i = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.
$$

On $\mathbb{T}^{\mathbb{Z}_+}$ we introduce the distance between two points $[x],[x'] \in \mathbb{T}^{\mathbb{Z}_+}$ (2)

$$\rho([x],[x']) = \big(\sum_{i=1}^{\infty} \rho^2([x_i],[x'_i])\big)^{\frac{1}{2}}, \qquad \rho([x_i],[x'_i]) = \inf_{k\in\mathbb{Z}} \{|x_i - x'_i + 2\pi k|\}$$

and consider the following metric and vector spaces respectively (3)

$$\mathcal{T}_2 \doteq \{x \in \mathbb{T}^{\mathbb{Z}_+} \mid \sum_{i=1}^{\infty} \rho^2(x_i) < \infty\}, \qquad l_2 \doteq \{y \in \mathbb{R}^{\mathbb{Z}_+} \mid \sum_{i=1}^{\infty} y_i^2 < \infty\}$$

The space $\mathcal{T}_2 \times l_2$ is a complete metric with distance

$$\rho\big((x,y),(x',y')\big) = \big(\sum_{i=1}^{\infty} \rho^2(x_i,x'_i)\big)^{\frac{1}{2}} + \big(\sum_{i=1}^{\infty} (y_i - y'_i)^2\big)^{\frac{1}{2}} \tag{1.1}$$

and denoting $x \equiv (x_1, \hat{x})$ we define $S(\hat{x}^o,\delta) = \{x \in \mathcal{T}_2 | (\sum_{i=2}^{\infty} \rho^2(x_i,x_i^o))^{\frac{1}{2}} \leq \delta,\ 0 \leq \delta < \infty\}$ (4).
Let $V\colon \mathcal{T}_2 \to \mathbb{R}$ be a continuous function ($V(x) \equiv V(x_1,x_2,x_3,\ldots) \equiv V(x_1,\hat{x})$) that satisfies
i) $V(x) \in C^2(\mathcal{T}_2,\mathbb{R})$, $\sup_{x\in\mathcal{T}_2} \|V_{xx}\| \doteq V_2 < \infty$
The second of i) implies that:
1) V_x a *uniform Lipschitz* gradient: $\|V_x(x) - V_x(x')\|^2 \leq L^2\rho^2(x,x')\ \forall x,x' \in \mathcal{T}_2$ (L is a constant).
2) $\{\sum_{i=1}^{\infty} V_{x_i}^2(x)\}^{\frac{1}{2}} \leq a\rho(x,0) + b \quad a$ and b constants. (5)

In this paper we will consider two kind of interactions $V\colon \mathcal{T}_2 \to \mathbb{R}$: *"short-range interaction"* and *"long-range interaction"*. In the first case we have

$$V(x) = \sum_{j=1}^{\infty} g_j(x^{(L_o)})$$

where $g_j\colon \mathbb{T}^{2L_o+1} \to \mathbb{R}$ depends only on the variables $x_{j-L_o},\ldots,x_{j+L_o}$ (L_o positive integer). In the second case the bound represented by L_o is relaxed and each variable

(2) $[x] \in \mathbb{T}^{\mathbb{Z}_+}$ is the equivalence class mod.2π of $x \in \mathbb{R}^{\mathbb{Z}_+}$
(3) For notational convenience we suppress the square parenthesis understanding $x \in \mathbb{T}^{\mathbb{Z}_+}$
(4) In the following we will sometimes write $\|\hat{p}\|$ which is $\rho((\hat{0},\hat{p}),0) = (\sum_{i=2}^{\infty} p_i^2)^{\frac{1}{2}}$
(5) Being $V(x) \in C^2(\mathcal{T}_2,\mathbb{R})$ there exists a continuous map $V_{xx}\colon \mathcal{T}_2 \to \mathcal{L}(l_2)$ ($\mathcal{L}$ is the space of linear bounded maps over l_2) defined by means of the expression $\|V_x(x+h) - V_x(x) - V_{xx}(x)h\| < \epsilon\|h\|$ for $\|h\| < \delta_\epsilon$. In the following we will write $\{\sum_{i=1}^{\infty} \sup_{S(\hat{x}^o,\delta)} V_{x_i}^2(x)\}^{\frac{1}{2}} \doteq V^1(x^o,\delta)$

x_i can interacts with every other variable without any constraint on the distance of their sites e.g.

$$V(x) = \sum_{i,j=1}^{\infty} e^{-|i-j|}(1 - \cos(x_i - x_j))$$

§2. The model

Consider the following real hamiltonian $H: \mathcal{T}_2 \times l_2 \to \mathbb{R}$

$$H(x,y) = \frac{1}{2}y^2 - V(x)$$

where V satisfies i) of §**1**. As we look for *time-periodic* solutions of the Newton's equations

$$\frac{d^2 x_i}{dt^2} \equiv \ddot{x}_i = \frac{\partial V}{\partial x_i} \equiv V_{x_i}, \quad i \in \mathbb{Z}_+ \tag{2.1}$$

we write them as

$$\begin{cases} x_1(t) = [\omega t + u_1(\omega t)], & \omega + \omega \dfrac{du_1}{d(\omega t)} \neq 0 \quad \forall t \\ x_i(t) = [u_i(\omega t)] & (i \geq 2) \end{cases} \tag{2.2}$$

where $\omega \in \mathbb{R}$ is the *frequency*, $u_i : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function and $[\cdot]$ is the standard projection of $\mathbb{R}$ over $\mathbb{T}$ (from now on this fact will be omitted). If we for a moment suppose (as it will be) that $u \in C^2(\mathbb{T}, \mathcal{T}_2)$ then (2.2) represents a one-dimensional torus $\mathbb{T}_\omega$ *embedded* in the ambient-space $\mathcal{T}_2 \times l_2$ i.e. $\mathbb{T}_\omega = \{(x,y) \in \mathcal{T}_2 \times l_2 | x_1 = \tau + u_1(\tau),\ x_j = u_j(\tau),\ y_1 = \omega(1 + \frac{du_1}{d\tau}),\ y_j = \omega \frac{du_j}{d\tau},\quad \tau \in \mathbb{T}\}$. Inserting (2.2) into (2.1) we get the system [6]

$$\frac{d^2 u_i}{d\tau^2} = \varepsilon V_{x_i}(\tau + u_1, \hat{u}) \quad i \in \mathbb{Z}_+, \qquad (\varepsilon \equiv \omega^{-2}, \quad u \equiv (u_1, \hat{u})) \tag{2.3}$$

and in order to apply the Implicit-Function-Theorem we look at (2.3) as a system of equations of the first order and then we rewrite it as

$$(u_i \doteq Q_i, \quad \frac{du_i}{d\tau} \doteq P_i, \quad \dot{Q}_i = \frac{dQ_i}{d\tau}, \quad \dot{P}_i = \frac{dP_i}{d\tau}) \Rightarrow \begin{cases} \dot{Q}_i = P_i \\ \dot{P}_i = \varepsilon V_{x_i}(\tau + Q_1, \hat{Q}). \end{cases} \tag{2.4}$$

(6) We define $\tau = \omega t$

Now (2.4) can be viewed as a *non-autonomous hamiltonian* system with hamiltonian function $H = \frac{1}{2}P^2 - \varepsilon V(\tau + Q_1, \hat{Q})$ for which, of course, $\dot{H} \not\equiv 0$. Instead the function

$$G(\tau, P, Q) = P_1 + \frac{1}{2}P^2 - \varepsilon V(\tau + Q_1, \hat{Q}) \tag{2.5}$$

is a first integral for (2.4).

The system (2.4) is naturally defined over the phase-space $\mathbb{T} \times \mathcal{T}_2 \times l_2 \doteq \mathcal{M}$ which is a manifold (see [LA] for a treatment of infinite-dimensional manifolds). Its tangent space $\mathcal{M}_x$ at $x \doteq (\tau, P, Q) \in \mathcal{M}$ is $\mathbb{R} \times l_2 \times l_2$. Over $\mathrm{T}\mathcal{M} = \bigcup_{x \in \mathcal{M}} (x, \mathcal{M}_x)$ the *non-singular* (see [A]) 2-form $\omega^2 \equiv \sum_{j=1}^{\infty} dP_j \wedge dQ_j - dH(\tau, P, Q) \wedge d\tau$ is defined [(7)].

Definition *A periodic-solution of (2.4) will be a $2\pi-$periodic map $(P, Q) \in C^1(\mathbb{R}, l_2 \times \mathcal{T}_2)$.*

Global existence and uniqueness for the Cauchy problem associated to (2.4) are standard applications of the contraction techniques (see Appendix 1):

Theorem 2.1 *Let $V: \mathcal{T}_2 \to \mathbb{R}$ be a continuous function that verifies the property i) in §**1**. Then there exists a unique solution $(P(t), Q(t)) \in C^1(\mathbb{R}, l_2 \times \mathcal{T}_2)$ of the Cauchy problem*

$$\begin{cases} \dot{Q} = P \\ \dot{P} = \varepsilon V_x(\tau + Q_1, \hat{Q}) \qquad (P(0), Q(0)) = (P^o, Q^o) \in l_2 \times \mathcal{T}_2. \end{cases} \tag{2.6}$$

Remark

In Appendix 1, besides the statement of Theorem 2.1 we will show that (P, Q) are C^1 functions of ε and of the initial data (P^o, Q^o). This fact will be used in §**4** where it will proved the statement of the next Theorem

§3. Main Results

Following Poincaré and Lyapunov (see [Po], [LY], [SM] and references therein) we look for solutions that are *continuation* ($\varepsilon \neq 0$) of *periodic solutions* with $\varepsilon = 0$. At $\varepsilon = 0$ the most general solution of the system (2.6) is

$$Q(\tau) = Q^o + P^o \tau, \quad P(\tau) = P^o$$

---

(7) $\omega^2 = d\omega^1$ where $\omega^1 \equiv \sum_{j=1}^{\infty} P_j dQ_j - H(\tau, P, Q) d\tau$; at every point of $\mathcal{M}$ the tangent vector $\xi(\tau, P, Q) = \alpha(1, \varepsilon V_x(\tau + Q_1, \hat{Q}), P)$ ($\alpha \in \mathbb{R}$) is defined; $\xi(x) \in \mathcal{M}_x$ ($x = (\tau, P, Q)$) and the following relation holds $\omega(\xi, \eta) = 0 \forall \eta \in \mathcal{M}_x$. The integral curves, lying in $\mathcal{M}$, of the field of directions associated to the set of $\xi-$vectors are the solutions of the system (2.4).
The manifold $\mathcal{M}$ is locally homemomorphic to $\mathbb{R} \times l_2 \times l_2$ through a homeomorphism $\varphi: \mathcal{M} \to \mathbb{R} \times l_2 \times l_2$. A function $F: \mathcal{M} \to \mathbb{R}$ is said to be Fréchet differentiable if $F \circ \varphi^{-1}$ is Fréchet differentiable as a function defined over the Banach space $\mathbb{R} \times l_2 \times l_2$.

(we denote it as $\gamma(\tau; P^o, Q^o, 0) \doteq \big(\phi(\tau; P^o, Q^o, 0), \psi(\tau; P^o, Q^o, 0)\big)$, the *flow* of the system, where 0 refers to $\varepsilon = 0$) and if we want it periodic it is necessary that

$$\begin{cases} \phi(2\pi; P^o, Q^o, 0) - \phi(0; P^o, Q^o, 0) = 2\pi P^o \\ \psi(2\pi; P^o, Q^o, 0) - \psi(0; P^o, Q^o, 0) = 0. \end{cases} \tag{3.1}$$

Now two facts are clear: i) the equations (3.1) imply that $P_n^o \in \mathbb{Z}$ so that $2\pi P^o = 0$ (mod.2π). We indicate with $\bar{P}_n^o$ a particular value of $P_n^o \in \mathbb{Z}$.
ii) $P^o \in l_2$, see (2.6) and i) implies that there must exist an integer N such that $\bar{P}_n^o = 0$ when $n > N$ i.e.$\{\bar{P}_n^o\} = \big(\bar{P}_1^o, \bar{P}_2^o, \bar{P}_3^o \ldots, \bar{P}_N^o, 0, 0, \ldots\big)$. This fact imposes a severe restriction on the periodic solutions that can be continuated at $\varepsilon \neq 0$.

When $\varepsilon \neq 0$ the system of equations (3.1) becomes

$$\begin{aligned} \phi(2\pi; P^o, Q^o, \varepsilon) - \phi(0; P^o, Q^o, \varepsilon) &= \int_0^{2\pi} d\tau\, \dot{\phi}(\tau; P^o, Q^o, \varepsilon) = \\ &= \int_0^{2\pi} d\tau \psi(\tau; P^o, Q^o, \varepsilon) = 0(mod.2\pi) \\ \frac{1}{\varepsilon}(\psi(2\pi; P^o, Q^o, \varepsilon) - \psi(0; P^o, Q^o, \varepsilon)) &= \frac{1}{\varepsilon}\int_0^{2\pi} d\tau\, \dot{\psi}(\tau; P^o, Q^o, \varepsilon) = \\ &= \int_0^{2\pi} d\tau V_Q(\tau + \phi_1, \hat{\phi}) = 0 \end{aligned} \tag{3.2}$$

where $(P^o, Q^o) \equiv (Q(0), P(0)) = \gamma(0; P^o, Q^o, \varepsilon)$ is the initial datum of the flow with $\varepsilon \neq 0$.
Of course if we had a solution $(\bar{P}^o, \bar{Q}^o, \bar{\varepsilon})$ of (3.2) $\gamma(\tau; \bar{P}^o, \bar{Q}^o, \bar{\varepsilon})$ would be the looked for 2π-periodic solution.
Hence we want to view (3.2) as a system of equations for the variables (P^o, Q^o, ε) in the sense that there exist a real number ε_o (sufficiently small) and two functions $(P^o(\varepsilon), Q^o(\varepsilon))$ such that $(P^o(\varepsilon), Q^o(\varepsilon), \varepsilon)$ identically solve (3.2) for any value $\varepsilon \in (-\varepsilon_o, \varepsilon_o)$ [(8)].
In order to formulate our main result we must introduce a linear map $\Gamma: l_2 \times l_2 \to l_2 \times l_2$ whose entries are (see (4.10)):

$$\begin{aligned} &\Gamma_{2i-1,2i} = 2\pi \ (i \geq 2), \qquad \Gamma_{2i,2j} = \int_0^{2\pi} d\theta\theta V_{\hat{Q}_i,\hat{Q}_j}(\pm\theta|1 + \bar{P}_1|, \hat{\bar{Q}}) \ (i, j \geq 2), \\ &\Gamma_{2i,2j-1} = \int_0^{2\pi} d\theta V_{\hat{Q}_i,\hat{Q}_j}(\theta, \hat{\bar{Q}}) \ (i, j \geq 2), \qquad 0 \ \text{(otherwise)} \end{aligned} \tag{3.3}$$

---

(8) It may be useful to say something about the notations. When we write Q^o we mean the variable $Q^o \in \mathcal{T}_2$ that can vary over some domain of initial data. Instead when we write $\bar{Q}$ we mean some fixed initial datum. Moreover we will write, sometimes, $Q^o = (Q_1^o, \hat{Q}^o)$. An analogous argument is valid for the variables P^o.

(about the meaning $\bar{P}_1$ and $\hat{\bar{Q}}$ see Theorem 3.1).
A critical point of the map $\hat{Q}^o \to \int_0^{2\pi} d\tau V_{\hat{Q}}(\tau,\hat{Q}^o)$ is said to be *non-degenerate* iff the map Γ has bounded inverse. The entries of the map Γ are given by the derivatives of a suitable set of equations obtained after that (3.2) has gone through some coordinate-transformations which will be explained in §**4**. We then have

Theorem 3.1 *Let $(\bar{Q},\bar{P}) \in \mathcal{T}_2 \times l_2$ such that: $\bar{P}_1 \in \mathbb{Z}\backslash\{-2,-1,0\}$, $\hat{\bar{P}} = 0$, $\hat{\bar{Q}}$ is a non-degenerate critical point of the map*

$$\hat{Q}^o \to \int_0^{2\pi} d\tau V_{\hat{Q}}(\tau,\hat{Q}^o) \tag{3.4}$$

*(where $V(x)$ satisfies i) of §**1**).*
Then there exist an $\varepsilon_o > 0$ and two functions $(P^o,Q^o)\colon(-\varepsilon_o,\varepsilon_o) \to l_2 \times \mathcal{T}_2$, continuous with the first derivative in $\varepsilon = 0$, solutions of (3.2), such that: $\gamma(\tau;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon) \in l_2 \times \mathcal{T}_2$ $\forall\tau \in \mathbb{R}$ is a $2\pi-$periodic solution of (2.4), $Q^o(0) = \bar{Q}, P^o(0) = \bar{P}$, and

$$\max_{\tau\in[0,2\pi],p=0,1}\Big\{\rho\Big(\frac{d^p}{d\tau^p}\gamma(\tau;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon),\frac{d^p}{d\tau^p}\gamma(\tau;\bar{P},\bar{Q},0)\Big)\Big\}\le C(\varepsilon,L) \tag{3.5}$$

[9] with $C(\varepsilon,L) \to 0$ as $\varepsilon \to 0$.
Remarks
i) The proof will be such that $P_1^o(\varepsilon) \equiv \bar{P}_1$ $Q_1^o(\varepsilon) \equiv \bar{Q}_1$.
ii) It could be taken a slightly more general $\big(Q^o(0),P^o(0)\big)$ i.e. $\bar{Q}$ as in the statement of the theorem but $\bar{P}_1 \in \mathbb{Z}\backslash\{-2,-1,0\}$, $\bar{P}_i \in \mathbb{Z}$ $2 \le i \le N$, $\bar{P}_i = 0$ $i > N$ for some integer N. However it can be easily shown that with a suitable canonical change of coordinates the former initial datum can be reduced to that one given in the statement of the theorem. The proof will be not given but it is an easy matter in the framework of canonical mappings [10]
iii) P_1^o is different from -1 because, otherwise, we would not have a one-dimensional torus *embedded* in the ambient space $\mathcal{T}_2 \times l_2$. Instead the values $-2,0$ must be avoided in order to have a well defined Poincaré-map (see (4.5) and in particular the equation for $q_1(\theta)$ of §**4**).
iv) The inequality (3.5) means that $\gamma(\tau;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon)$ is a *continuation* of the solution with $\varepsilon = 0$ and initial datum $(\bar{P},\bar{Q})$.

[9] Being $\phi \in \mathcal{T}_2$, for $p = 0$ the distance is given in (1.1) while for $p = 1$ it is the usual distance of $l_2 \times l_2$

[10] Let's define an operator $\mathcal{A}\colon l_2 \times \mathcal{T}_2 \to l_2 \times \mathcal{T}_2$, $\mathcal{A}(P,Q) = (P',Q')$ that satisfies the following requests: 1) it is linear and bounded (over $\mathcal{T}_2$ the linearity is restricted to the vector sum) 2) preserves the structure of $\mathcal{T}_2$ (i.e. $Q_i' \in \mathbb{T}$) 3) preserves the 2-form $\omega = \sum_{j=1}^{\infty} dP_j \wedge dQ_j$ 4)$\mathcal{A}(\bar{P}_1^o,0,\bar{P}_2^o,0,\bar{P}_3^o,\dots,\bar{P}_N^o,0,0,0,\dots)^T = (\bar{P}_1^o,0,0,0,\dots)^T$ (T denotes the transposition). As a consequence we can take initial data for which $\hat{P}^o = 0$.

v) The statement of Theorem 3.1) resembles that one relative to the finite-dimensional theory but it is different in two points. The first one is the necessity of avoiding the values $-2,0$ of $\bar{P}_1$. The second one is that with a finite number of degrees of freedom the *non-degenerateness* condition would become the non-singularity of the matrix $\int_0^{2\pi} d\theta V_{\hat{Q},\hat{Q}}(\theta,\hat{\bar{Q}})$. Of course our *non-degenerateness* condition implies that the submatrix $\Gamma_{2i,2j-1}$ is non-singular.
vi) We will show how actually the functions $(P^o(\varepsilon),Q^o(\varepsilon))$ are continuous up to the derivative of order one in $(-\varepsilon_o,\varepsilon_o)$ and this depends on the fact that $V\in C^2(\mathcal{T}_2,\mathbb{R})$.

§4. Proofs

The proof splits in two parts. In the first one we show how $\big(P^o(\varepsilon),Q^o(\varepsilon),\varepsilon\big)$ solve (3.2) for all $\varepsilon\in(-\varepsilon_o,\varepsilon_o)$ and this fact implies that $\gamma(\tau,P^o(\varepsilon),Q^o(\varepsilon),\varepsilon)$ is a $2\pi-$periodic solution. In the second part of the proof we get (3.5).
We begin with the second part. It is a straightforward computation of the following three quantities

$$\begin{aligned}&\rho\big(\phi(t;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon),\phi(t;\bar{P},\bar{Q},0)\big)\\ &\|\psi(t;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon)-\psi(t;\bar{P},\bar{Q},0)\|,\qquad \|\dot{\psi}(t;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon)-\dot{\psi}(t;\bar{P},\bar{Q},0)\|.\end{aligned}$$

Using i) of §**1** we obtain the following bounds (we will introduce some numerical constants B_i $i\geq 1$)

$$\begin{aligned}\max_{t\in[o,2\pi]}\rho\big(\phi(t;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon),\phi(t;\bar{P},\bar{Q},0)\big)&\leq B_1\big[\rho\big((\hat{P}^o(\varepsilon),\hat{Q}^o(\varepsilon)),(\hat{0},\hat{\bar{Q}})\big)+\\ &\qquad+\varepsilon V^1(Q^o(\varepsilon),\bar{\Delta})\big]<\infty\\ \max_{t\in[o,2\pi]}\|\psi(t;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon)-\psi(t;\bar{P},\bar{Q},0)\|&\leq\|\hat{P}^o(\varepsilon)\|+\varepsilon 2\pi V^1(Q^o(\varepsilon),\bar{\Delta})<\infty\\ \max_{t\in[o,2\pi]}\|\dot{\psi}(t;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon)-\dot{\psi}(t;\bar{P}^o,\bar{Q}^o,0)\|&\leq\varepsilon V^1(Q^o(\varepsilon),\bar{\Delta})<\infty\end{aligned}$$

$\bar{\Delta}=\Delta(\varepsilon,L,T,P^o(\varepsilon),Q^o(\varepsilon))|_{T=2\pi}$ and $\Delta(\varepsilon,L,T,P^o(\varepsilon),Q^o(\varepsilon))$ is an upper bound of $\sup_{t\in[-T,T]}\rho\big(Q(t;P^o(\varepsilon),Q^o(\varepsilon),\varepsilon),Q^o(\varepsilon)\big)$ (see (6.3)) while for the definition of $V^1(Q^o(\varepsilon),\bar{\Delta})$ and L see §**1**. Being $Q^o(\varepsilon)$, $P^o(\varepsilon)$ and V_Q continuous functions of their arguments we could compute $C(\varepsilon,L)$.

The discussion about the smoothness of $(P^o(\varepsilon),Q^o(\varepsilon))$ of Remark vi) will be postponed at the end of Appendix 1.

Now we show the proof of the first part of Theorem 3.1.

First of all we rewrite the system (2.6) as autonomous [11]

$$\dot Q = P \qquad \dot P = \varepsilon V_x(Q_o + Q_1, \hat Q) \qquad \dot Q_o = 1 \quad (Q_o = Q_o(\tau) \equiv \tau) \tag{4.1}$$

then we make the following change of coordinates [12] $(p,q) = U(P,Q)\colon \mathbb{T} \times \mathcal{T}_2 \times l_2 \to \mathbb{T} \times \mathcal{T}_2 \times l_2$

$$q = Q, \qquad p_1 = G(P,Q) \qquad \hat p = \hat P. \tag{4.2}$$

The map $U(P,Q)$ is Fréchet differentiable for every $(P,Q) \in l_2 \times \mathbb{T} \times \mathcal{T}_2$. It is non-singular if $1 + P_1 \neq 0$ so we require $P_1(\tau; Q^o, P^o, \varepsilon) \neq -1 \forall \tau \in [0, 2\pi]$ which is possible for ε sufficiently small. In the new (p,q) coordinates the system (4.1) becomes

$$\begin{array}{lll} \dot q_o = 1 & \dot q_1 = -1 \pm \left(1 - \hat p^2 + 2\varepsilon V(q_o + q_1, \hat q) + 2G^o\right)^{\frac{1}{2}} & \\ \dot p_1 = 0 & \dot{\hat q} = \hat p & \dot{\hat p} = \varepsilon V_{\hat q}(q_o + q_1, \hat q) \end{array} \tag{4.3}$$

$G^o = G(P^o, Q^o) \doteq G(P(0), Q(0))$. Setting $\varepsilon = 0$ in (4.3) we obtain

$$\dot q_o = 1 \quad \dot q_1 = -1 \pm \left(1 - \hat p^2 + 2G^o\right)^{\frac{1}{2}} \quad \dot p_1 = 0 \qquad \dot{\hat q} = \hat p \qquad \dot{\hat p} = 0$$

i.e.

$$\begin{cases} q_o(\theta) = \theta + q_o^o \qquad q_1(\theta) = \left(-1 \pm (1 - \hat p^2 + 2p_1^o)^{\frac{1}{2}}\right)\theta + q_1^o \\ p_1(\theta) = p_1^o \qquad \hat q(\theta) = \hat q^o + \hat p^o \theta \qquad \hat p(\theta) = \hat p^o \end{cases}$$

and taking into account the change (4.2) we get

$$\begin{cases} q_o(\theta) = \theta + Q_o^o, \qquad p_1(\theta) = P_1^o + \dfrac{1}{2}(P_1^o)^2, \qquad \hat q(\theta) = \hat Q^o + \hat P^o \theta \\ q_1(\theta) = \left(-1 \pm \left(1 + 2P_1^o + (P_1^o)^2 - \hat P^2\right)^{\frac{1}{2}}\right)\theta + Q_1^o, \qquad \hat p(\theta) = \hat P^o, \end{cases} \tag{4.4}$$

then (recall that $\hat P^o = 0$ and $P_1^o = \bar P_1 \in \mathbb{Z} \backslash \{-2, -1, 0\}$ and that $\bar{\hat Q}$ is the critical point of the map (3.4))

$$\begin{cases} q_o(\theta) = \theta + \bar Q_o \qquad \hat q(\theta) = \bar{\hat Q} \qquad p_1(\theta) = \bar P_1 + \dfrac{1}{2}\bar P_1^2 \\ q_1(\theta) = \left(-1 \pm |1 + \bar P_1|\right)\theta + Q_1^o, \qquad \hat p(\theta) = 0. \end{cases} \tag{4.5}$$

The equations (4.5) represent the periodic solution when $\omega \to \infty$ (the "unperturbed solution") and consequently the interaction among the variables disappears. We look for 2π-periodic solutions $\gamma(\tau, P^o, Q^o, \varepsilon)$ near the unperturbed solution in the sense that

---

(11) from now on $Q \equiv (Q_o, Q_1, \hat Q)$

(12) $G(P,Q) = G(P, Q_o, Q_1, \hat Q)$; the integral of motion $G(\tau, P, Q)$, see (2.5), is assumed as new coordinate in order to eliminate a Floquet multiplier that inevitably is equal to 1

when $\omega \to \infty$ $(\varepsilon \to 0)$ $\gamma(\tau, P^o, Q^o, \varepsilon)$ tends to the unperturbed one. Moreover we note that taking the plus sign in the equation for $q_1(\theta)$ we are forced to exclude $\bar{P}_1 = 0, -2$.

Now denoting the flux generated by the system of equations (4.3) as $\lambda(\theta; p^o, q^o, \varepsilon) \equiv \big(\xi(\theta; p^o, q^o, \varepsilon), \eta(\theta; p^o, q^o, \varepsilon)\big)$ the equations that have to be solved are

$$
\begin{aligned}
&\xi_o(2\pi; p^o, q^o, \varepsilon) - \xi_o(0; p^o, q^o, \varepsilon) = 2\pi (= 0\, mod.\, 2\pi);\\
&\xi(2\pi; p^o, q^o, \varepsilon) - \xi(0; p^o, q^o, \varepsilon) = \int_0^{2\pi} d\theta\, \dot{\xi}(\theta; p^o, q^o, \varepsilon) =\\
&\qquad\qquad = \int_0^{2\pi} d\theta \eta(\theta; p^o, q^o, \varepsilon) = 0 \ (mod. 2\pi)\\
&\frac{1}{\varepsilon}(\eta(2\pi; p^o, q^o, \varepsilon) - \eta(0; p^o, q^o, \varepsilon)) = \frac{1}{\varepsilon}\int_0^{2\pi} d\theta\, \dot{\eta}(\theta; p^o, q^o, \varepsilon) =\\
&\qquad\qquad = \int_0^{2\pi} d\theta V_Q(\xi_o + \xi_1, \hat{\xi}) = 0.
\end{aligned}
\tag{4.6}
$$

Now we do the *Poincaré-map* taking as *surface-map*

$\Sigma = \big\{(p, q) \in l_2 \times \mathbb{T} \times \mathcal{T}_2 | q_o = 0,\ q_1 = 0,\ p_1 = \bar{P}_1 + \frac{1}{2}\bar{P}_1^2\ , \bar{P}_1 \neq -2, -1, 0\big\}$.

and in this way the first three equations of (4.6) are excluded giving

$$
\begin{aligned}
&\hat{\xi}(2\pi; \hat{p}^o, \hat{q}^o, \varepsilon) - \hat{\xi}(0; \hat{p}^o, \hat{q}^o, \varepsilon) = \int_0^{2\pi} d\theta \hat{\eta}(\theta; \hat{p}^o, \hat{q}^o, \varepsilon) = 0 \ (mod. 2\pi)\\
&\frac{1}{\varepsilon}(\hat{\eta}(2\pi; \hat{p}^o, \hat{q}^o, \varepsilon) - \hat{\eta}(0; \hat{p}^o, \hat{q}^o, \varepsilon)) = \int_0^{2\pi} d\theta V_{\hat{Q}}(\xi_o + \xi_1, \hat{\xi}) = 0.
\end{aligned}
\tag{4.7}
$$

We indicate (4.7) as[13] $\Gamma(\hat{p}^o, \hat{q}^o, \varepsilon) \doteq (\Phi(\hat{p}^o, \hat{q}^o, \varepsilon), \Psi(\hat{p}^o, \hat{q}^o, \varepsilon)) = 0$ and to solve it means to show that there exist two functions $(\hat{p}^o(\varepsilon), \hat{q}^o(\varepsilon))$ so that $\Gamma(\hat{p}^o(\varepsilon), \hat{q}^o(\varepsilon), \varepsilon) \equiv 0$ for every ε sufficiently small. The resolution is a standard application of the Implicit-Function-Theorem whose statement is:

Let us consider a map $F: X \times Y \to Z$ (X, Y, Z are Banach spaces) such that 1) $F(x^o, y^o) = 0$, 2) F is Fréchet differentiable respect to y and x in (x^o, y^o); $F_y(x, y)$ and $F_x(x, y)$ are continuous in (x^o, y^o) 3) $F_y(x^o, y^o)$ has bounded inverse. Then there exists a function $f: U \to U'$, $x_o \in U \subset X$, $y_o \in U' \subset Y$, continuous and differentiable in (x_o, y_o), such that $f(x_o) = y_o$ and $F(x, f(x)) \equiv 0$ $\forall x \in U$.

Doing the suitable analogies[14] we have to find under what conditions the following

---

(13) $\Gamma_{2i-1} = \Phi_i$, $\Gamma_{2i} = \Psi_i$ $i \geq 2$, $\Phi_i = \int_0^{2\pi} d\theta \hat{\eta}_i$, $\Psi_i = \int_0^{2\pi} d\theta V_{\hat{Q}_i}$

(14) We should consider $X = l_2 = \{\hat{p}_i\}_{i=2}^{\infty}$, $Y = l_2 = \{\hat{\phi}_i \circ \hat{q}_i\}_{i=2}^{\infty}$ and $Z = X \times Y$. ϕ_i is the homeomorphism of $\mathbb{T}$ into $\mathbb{R}$ such that $\otimes_{i=1}^{\infty} \phi_i \doteq \phi$ makes $\mathcal{T}_2$ locally homeomorphic to l_2 (see footnote (7)). Actually we consider $X = \mathcal{T}_2$ because it is a linear manifold respect to vector sum. Moreover what in the statement of the Implicit- Function-Theorem is x in our case is ε while F is represented by the system (4.7)

relations are true: there exist $\hat{\bar{p}}, \hat{\bar{q}}, \bar{\varepsilon}$ such that
1)$\Gamma(\hat{\bar{p}}, \hat{\bar{q}}, \bar{\varepsilon}) = 0$
2)$\Gamma(\hat{p}^o, \hat{q}^o, \varepsilon)$ is Fréchet differentiable respect to $(\hat{p}^o, \hat{q}^o)$ and ε in $(\hat{\bar{p}}, \hat{\bar{q}}, \bar{\varepsilon})$; $\Gamma_{\hat{p}^o,\hat{q}^o}(\hat{\bar{p}}, \hat{\bar{q}}, \bar{\varepsilon})$ and $\Gamma_{\varepsilon}(\hat{\bar{p}}, \hat{\bar{q}}, \bar{\varepsilon})$ continuous in $(\hat{\bar{p}}, \hat{\bar{q}}, \bar{\varepsilon})$;
3)$\Gamma_{\hat{p}^o,\hat{q}^o}(\hat{\bar{p}}, \hat{\bar{q}}, \bar{\varepsilon})$ has bounded inverse

1) We solve $\Gamma(\hat{p}^o, \hat{q}^o, \varepsilon) = 0$ for $\varepsilon = \bar{\varepsilon} = 0,\ \hat{q}^o = \hat{\bar{q}} = \hat{\bar{Q}},\ \hat{p}^o = 0 = \hat{\bar{p}} = \hat{\bar{P}}$

$$\begin{cases} \Phi(0, \hat{\bar{Q}}, 0) = 0: \quad \hat{\eta}(2\pi; 0, \hat{\bar{Q}}, 0) - \hat{\eta}(0; 0, \hat{\bar{Q}}, 0) = 0 \\ \Psi(0, \hat{\bar{Q}}, 0) = 0: \quad \dfrac{1}{\varepsilon}\big(\hat{\eta}(2\pi; 0, \hat{\bar{Q}}, 0) - \hat{\eta}(0; 0, \hat{\bar{Q}}, 0)\big) = 0 \Rightarrow \\ \qquad\qquad \Rightarrow \displaystyle\int_0^{2\pi} d\theta V_{\hat{Q}}(\pm|1 + \bar{P}_1|\theta, \hat{\bar{Q}}) = 0. \end{cases} \tag{4.8}$$

The periodicity of $V(x)$ and $\bar{P}_1 \in \mathbb{Z}\backslash\{-2, -1, 0\}$ imply that (4.8) is satisfied if (see (3.4))

$$\int_0^{2\pi} d\theta V_{\hat{Q}}(\theta, \hat{\bar{Q}}) = 0 \tag{4.9}$$

3) $\Gamma_{\hat{p}^o,\hat{q}^o}(\hat{\bar{p}}, \hat{\bar{q}}, 0)$ with bounded inverse means that the linear map whose representation is given by the following matrix has to be non-singular and the inverse of it bounded

$$\frac{\partial \Gamma}{\partial(\hat{p}^o, \hat{q}^o)}\Big|_{\varepsilon=0,\hat{p}^o=0,\hat{q}^o=\hat{\bar{q}}} = \begin{pmatrix} 0 & \Gamma_{34} & 0 & 0 & \dots & & & & \\ \Gamma_{43} & \Gamma_{44} & \Gamma_{45} & \Gamma_{46} & \dots & & & & \\ 0 & 0 & 0 & \Gamma_{56} & 0 & 0 & 0 & \dots & \\ \Gamma_{63} & \Gamma_{64} & \Gamma_{65} & \Gamma_{66} & \Gamma_{67} & \Gamma_{68} & \dots & & \\ 0 & 0 & 0 & 0 & 0 & \Gamma_{78} & 0 & 0 & \dots \\ \vdots & \vdots & \dots & & & & & & \end{pmatrix} \tag{4.10}$$

$$\Gamma_{2i-1,2i} = 2\pi,\ \Gamma_{2i,2j} = \int_0^{2\pi} d\theta\theta V_{\hat{Q},\hat{Q}}(\pm\theta|1 + \bar{P}_1|, \hat{\bar{Q}}),\ \Gamma_{2i,2j-1} = \int_0^{2\pi} d\theta V_{\hat{Q}_i,\hat{Q}_j}(\theta, \hat{\bar{Q}}).$$

A necessary condition is the non-singularity of the matrix (4.10) and a short meditation shows as this is equivalent to the injectivity of the following matrix

$$M = \begin{pmatrix} \Gamma_{43} & \Gamma_{45} & \Gamma_{47} & \Gamma_{49} & \Gamma_{4,11} & \dots \\ \Gamma_{63} & \Gamma_{65} & \Gamma_{67} & \Gamma_{69} & \Gamma_{6,11} & \dots \\ \vdots & \vdots & \dots & & & \end{pmatrix} \tag{4.11}$$

Being the problem infinite-dimensional, we have to show that the inverse of the matrix (4.10) is bounded. This explains the *non-degenerateness* condition of the critical point $(0, \hat{\bar{q}}, 0)$ for the map in (3.4). In the next section two examples at which to apply this scheme.

2) Being $V \in C^2(\mathcal{T}_2, \mathbb{R})$, by (4.7) we realize that the continuity of $\Gamma(\hat{p}^o, \hat{q}^o, \varepsilon)$ and its derivatives involved in point *2)* are consequences of the same properties of the flow $\lambda(\theta; p^o, q^o, \varepsilon)$. Then by means of the following relation

$$\lambda(\theta; p^o, q^o, \varepsilon) = U(\gamma(\theta; P^o, Q^o, \varepsilon))|_{U^{-1}(p^o, q^o)}$$

(which is the transformation law of the flows λ and γ through the map (4.2)) we reduce ourselves to study the flow $\gamma(\theta; P^o, Q^o, \varepsilon)$ (note that if $|\varepsilon|$ and $\|\hat{p}^o\|$ are small enough then $|1 + 2p_1^o - \hat{p}^2 + 2\varepsilon V(q_o + q_1, \hat{q})| \geq |\frac{1}{2}(1 + 2p_1^o)|$ and this makes the change U well defined for all times, differentiable and invertible).

Now using standard techniques in the theory of ordinary differential equations we will show that γ, $\frac{\partial\gamma}{\partial\varepsilon}$ and $\frac{\partial\gamma}{\partial(P^o, Q^o)}$ are continuous as functions of (P^o, Q^o, ε) and a fortiori in $P^o = (\bar{P}_1, \hat{0})$ $Q^o = (\bar{Q}_1, \hat{\bar{Q}})$, $\varepsilon = 0$. In fact supposing that $\gamma(\cdot; P^o, Q^o, \varepsilon)$ is continuous (the proof will be given in Appendix 1) we have

Corollary 4.1 *Let be $\gamma(\theta; y^o, \varepsilon)$ the flow of the system (2.6). Then*

$$\frac{\partial\gamma}{\partial\varepsilon}(\theta; P^o, Q^o, \varepsilon) \quad \text{and} \quad \frac{\partial\gamma}{\partial y^o}(\theta; P^o, Q^o, \varepsilon)$$

are continuous as functions of P^o, Q^o, ε.

Proof

We introduce the so called *linearized equation* [15]

$$\frac{d}{d\theta}\frac{\partial\gamma}{\partial y^o}(\theta; y^o, \varepsilon) = \frac{\partial f}{\partial y}(\gamma(\theta; y^o, \varepsilon), \varepsilon)\frac{\partial\gamma}{\partial y^o}(\theta; y^o, \varepsilon) \tag{4.12}$$

whose solution is the series

$$\frac{\partial\gamma}{\partial y^o}(\theta; y^o, \varepsilon) = \sum_{k=0}^{\infty}\int_0^\theta d\theta_1 \int_0^{\theta_1} d\theta_2 \ldots \int_0^{\theta_{k-1}} d\theta_k f_y^{(1)} \ldots f_y^{(k)} \tag{4.13}$$

$f_y^{(i)} \doteq f_y(\gamma(\theta_i; y^o, \varepsilon), \varepsilon)$ and $f(y, \varepsilon)$ is symbolically the right hand side of (2.6).

First of all we prove that $\frac{\partial\gamma}{\partial y^o}$ is a bounded (and of course linear) operator over $\mathbb{R} \times l_2 \times l_2$. Suppose in fact we know that $f_y^{(i)}$ maps $\mathbb{R} \times l_2 \times l_2$ into itself, then the sequence

[15] we set $(\hat{P}^o, \hat{Q}^o) = y^o$ i.e. $y_{2i}^o = \hat{P}_i^o$, $y_{2i-1}^o = \hat{Q}_i^o$ $(i \geq 2)$

$\{\sum_{k=0}^{n}\int_0^\theta d\theta_1\int_0^{\theta_1} d\theta_2\dots\int_0^{\theta_{k-1}} d\theta_k f_y^{(1)}\dots f_y^{(k)}\}$ is a Cauchy sequence in $\mathcal{L}(\mathbb{R}\times l_2\times l_2)$ (the space of the linear bounded operators on $\mathbb{R}\times l_2\times l_2$) which is complete and then it converges to a well defined bounded linear operator that is $\frac{\partial\gamma}{\partial y^o}$. So all the work is reduced to show that $f_y^{(i)}$ maps $\mathbb{R}\times l_2\times l_2$ into itself and this can be achieved by means of the following bound

$$\sup_{\theta\in[0,2\pi]}\|f_y(\gamma(\theta;P^o,Q^o,\varepsilon),\varepsilon)\|\le 2\max\{1,|\varepsilon|V_2\}.$$

Indeed, via the continuity of V_{xx} in x and of γ in P^o,Q^o,ε, f_y is continuous too

$$\sup_{\theta\in[0,2\pi]}\|f_y(\gamma(\theta_i;y^o,\varepsilon),\varepsilon)-f_y(\gamma(\theta_i;\tilde y^o,\tilde\varepsilon),\tilde\varepsilon)\|\le A(|\varepsilon-\tilde\varepsilon|,\rho(y^o,\tilde y^o)) \tag{4.14}$$

and $A(|\varepsilon-\tilde\varepsilon|,\rho(y^o,\tilde y^o))\to 0$ as $|\varepsilon-\tilde\varepsilon|+\rho(y^o,\tilde y^o)\to 0$. The uniform convergence of the series (4.12) makes the operator $\frac{\partial\gamma}{\partial y^o}$ continuous.

About the continuity of $\frac{\partial\gamma}{\partial\varepsilon}(\theta;P^o,Q^o,\varepsilon)$ we have to solve the equation

$$\frac{d}{d\theta}\frac{\partial\gamma}{\partial\varepsilon}(\theta;y^o,\varepsilon)=\frac{\partial f}{\partial y}(\gamma(\theta;y^o,\varepsilon),\varepsilon)\frac{\partial\gamma}{\partial\varepsilon}(\theta;y^o,\varepsilon)+\frac{\partial f}{\partial\varepsilon}(\theta;y^o,\varepsilon)$$

whose solution is

$$\frac{\partial\gamma}{\partial\varepsilon}(\theta;y^o,\varepsilon)=\sum_{k=0}^{\infty}\int_0^\theta d\theta_1\int_0^{\theta_1} d\theta_2\dots\int_0^{\theta_{k-1}} d\theta_k f_y^{(1)}\dots f_y^{(k-1)}f_\varepsilon^{(k)}$$

and repeating the argument used for (4.12) we can prove that the series converges uniformly and being each term continuous in P^o,Q^o,ε it defines a continuous vector function i.e. $\frac{\partial\gamma}{\partial\varepsilon}$.

§5. Applications

Consider the following potential function (it represents an infinite countable set of point masses all rotating over a circle of radius one and positioned on a vertical plane; κ is a positive constant, m is the mass and g is the gravity constant)

$$V(Q)=-\sum_{i=1}^{\infty}mg(1-\cos Q_i)+\sum_{i=1}^{\infty}\kappa(1-\cos(Q_{i+1}-Q_i)) \tag{5.1}$$

$$\begin{aligned}V_{Q_i}&=-mg\sin Q_i+\kappa\big(\sin(Q_i-Q_{i-1})-\sin(Q_{i+1}-Q_i)\big)\quad i\ne 1\\ V_{Q_1}&=-mg\sin Q_1+\kappa\big(\sin(Q_1-Q_2)\big)\end{aligned}$$

$$V_{Q_jQ_i}=\begin{cases} i=j & -mg\cos Q_j+\kappa\big(\cos(Q_j-Q_{j-1})+\cos(Q_{j+1}-Q_j)\big)\\ i=j+1 & \quad -\kappa\cos(Q_j-Q_{j+1})\\ i=j-1 & \quad -\kappa\cos(Q_j-Q_{j-1})\end{cases}$$

It is easy to check the property i) of §**1** ($V_2\le 6(2\kappa+mg)$) and then Theorem 2.1 applies together with the smoothness property of the flow γ.
$\hat{Q}^o\equiv 0$ is a critical point of the map (3.4) while the matrix associated to (4.11) and calculated in 0 gives

$$\begin{matrix} (-mg+\kappa)2\pi & -2\kappa\pi & 0 & 0 & \dots & \dots & \dots\\ -2\kappa\pi & (-mg+2\kappa)2\pi & -2\kappa\pi & 0 & \dots & \dots & \dots\\ 0 & -2\kappa\pi & (-mg+2\kappa)2\pi & -2\kappa\pi & 0 & \dots & \dots\\ 0 & 0 & -2\kappa\pi & (-mg+2\kappa)2\pi & -2\kappa\pi & 0 & \dots\\ \vdots & \vdots & \dots & \dots & & & \end{matrix}$$

and if 2κ is small compared to $|mg|$ then the matrix is non-singular proving that the matrix (4.10) is non-singular to. The next step consists in showing that (4.10) has bounded inverse. In fact a short calculation gives $\|N\tilde{v}\|\ge C\|\tilde{v}\|$ with $C=(2\pi)^2(|2\kappa-mg|)$ hence $\|N^{-1}\|\le C^{-1}$ [16].

Now for the potential (5.1) we compute the first order in ε of the functions $\hat{P}^o(\varepsilon),\hat{Q}^o(\varepsilon)$. As it should be clear, these functions are the initial values that originates *periodic solutions* of the problem with potential function (5.1). $0\equiv\Gamma(y^o(\varepsilon),\varepsilon)=\Gamma(\bar{y},0)+\big(\frac{\partial\Gamma}{\partial y^o}(y^o(\varepsilon),\varepsilon)\frac{dy^o}{d\varepsilon}+\frac{\partial\Gamma}{\partial\varepsilon}\big)|_{\varepsilon=0}\varepsilon+\circ(\varepsilon)$ and then $\frac{dy^o}{d\varepsilon}|_{\varepsilon=0}=-\big(\frac{\partial\Gamma}{\partial y^o}(y^o(\varepsilon),\varepsilon)|_{\varepsilon=0}\big)^{-1}\frac{\partial\Gamma}{\partial\varepsilon}|_{\varepsilon=0}=-\big(\frac{\partial\Gamma}{\partial y^o}(\bar{y},0)\big)^{-1}\frac{\partial\Gamma}{\partial\varepsilon}(\bar{y},0)$.
¿From the last equality we derive two relations. The first one is

$$\frac{\partial}{\partial\varepsilon}\Psi(y^o(\varepsilon),\varepsilon)=\frac{\partial}{\partial\varepsilon}\int_0^{2\pi}d\theta V_{\hat{Q}}(\theta,\hat{y}^o)=0$$

(as ε is not explicitly present but only implicitly in $\hat{y}^o$) while the second one is

$$\frac{\partial}{\partial\varepsilon}\Phi(y^o(\varepsilon),\varepsilon)|_{\varepsilon=0}=\frac{\partial}{\partial\varepsilon}\int_0^{2\pi}d\theta\ \hat{\psi}(\theta,y^o(\varepsilon))|_{\varepsilon=0}=\frac{\partial}{\partial\varepsilon}\int_0^{2\pi}d\theta\int_0^{\theta}d\theta'\ \dot{\hat{\psi}}(\theta',y^o(\varepsilon))|_{\varepsilon=0}$$
$$=\frac{\partial}{\partial\varepsilon}\int_0^{2\pi}d\theta\int_0^{\theta}d\theta'\ \varepsilon V_{\hat{Q}}(\theta',\hat{q}^o(\varepsilon))|_{\varepsilon=0}=\int_0^{2\pi}d\theta\int_0^{\theta}d\theta' V_{\hat{Q}}(\theta',\hat{\bar{q}})\doteq\hat{b}.$$

[16] recall that for the non-singularity of the matrix (4.11) we have imposed $\frac{2\kappa}{|mg|}\ll 1$; with N we indicate the matrix (4.10)

Consequently we have $\frac{\partial\Gamma}{\partial\varepsilon}|_{\varepsilon=0} = -(b_3, 0, b_5, 0, b_7, 0, b_9, 0, \ldots)$ and considering the potential (5.1) we have $b_3 = -2\kappa\pi$ while $b_j = 0$ for $j \neq 3$; $\frac{\partial\Gamma}{\partial\varepsilon}|_{\varepsilon=0} = -(b_3, 0, \ldots)$ and then it results

$$\frac{dy^o}{d\varepsilon}|_{\varepsilon=0} = (0, 2\kappa\pi, 0, \ldots) \quad \text{i.e.} \ \frac{dQ^o}{d\varepsilon}|_{\varepsilon=0} = 0 \quad \frac{dP_2^o}{d\varepsilon}|_{\varepsilon=0} = 2\kappa\pi \quad \frac{dP_i^o}{d\varepsilon}|_{\varepsilon=0} = 0 \ i \geq 3 \tag{5.2}$$

Now we briefly investigate the meaning of the latter computation.

With the Theorem 3.1 we have shown that on the surface Σ there are initial data that originate periodic solutions. The local nature of the Implicit Function Theorem implies that these initial data are located close with ε, to the point $\bar{Q}, \bar{P}$ with $\bar{Q}_1 = 0(mod.2\pi)$, $\bar{P}_1 \in \mathbb{Z}\backslash\{-2,-1,0\}$, $\hat{\bar{P}} = 0$ $\hat{\bar{Q}}$ is a non degenerate critical point of the map (3.4).

Suppose now to expand a component $\hat{Q}_j^o(\varepsilon)$ or $\hat{P}_j^o(\varepsilon)$ in power series of ε; for example we have $Q_j^o(\varepsilon) = \bar{Q}_j + \varepsilon \frac{dQ_j^o}{d\varepsilon}|_{\varepsilon=0} + \circ(\varepsilon)$ and it follows that the unique contribution to first order in ε comes from $P_2^o(\varepsilon)$ i.e. $\frac{dP_2^o}{d\varepsilon}|_{\varepsilon=0} = -2\pi\kappa$ (the others one are all equal to 0). The velocities $\dot{x}_i$, where x_i is the generic angle of (2.2), assume the form

$$\begin{cases} \dot{x}_1 = \omega + \omega\dfrac{du_1}{d(\omega t)} = \omega + \omega P_1(\tau, P^o(\varepsilon), Q^o(\varepsilon), \varepsilon) = \omega + \omega\bar{P}_1(\tau) + \omega O(\varepsilon) \equiv \\ \quad = \omega + \omega\bar{P}_1 + O(\omega^{-1}) \\ \dot{x}_2 = \omega\dfrac{du_2}{d(\omega t)} = \omega P_2(\tau, P^o(\varepsilon), Q^o(\varepsilon), \varepsilon) = \omega(\bar{P}_2 + 2\pi\kappa\varepsilon + \circ(\varepsilon)) = \dfrac{2\pi\kappa}{\omega} + \circ(\omega^{-1}) \\ \dot{x}_j = \omega\dfrac{du_j}{d(\omega t)} = \omega P_j(\tau, P^o(\varepsilon), Q^o(\varepsilon), \varepsilon) = \omega(\bar{P}_j + \circ(\varepsilon)) = \circ(\omega^{-1}) \quad j > 2. \end{cases} \tag{5.3}$$

The various $\circ(\cdot)$ and $O(\cdot)$ are all 2π periodic functions of ωt. We then conclude that the rotator x_1 moves very fast while the velocities of x_j with $j \geq 2$ are small and tend to zero as $\omega \to \infty$.

Remarks

i) $\hat{Q}^o = 0$ is not the only critical point of the map (3.4). For example also the point $Q_2^o = \pm\pi$, $\{Q_i^o\}_{i=3}^{\infty} = 0$ is a good one. In general, for the potential (5.1) we could take as critical point any one in the set $\mathcal{Q} = \{x \in \mathcal{T}_2 | \exists N \in \mathbb{N} s.t. \ x_j = 0, \pm\pi \ 1 \leq j \leq N, \ x_j = 0 \ j > N\}$.

For the *long-range potential* $V(Q) = \sum_{i,j=1}^{\infty} e^{-|i-j|}(1 - \cos(Q_i - Q_j))$ we have

$$V_{Q_k} = 2\sum_{j=1}^{\infty} e^{-|k-j|}\sin(Q_k - Q_j), \qquad \begin{cases} V_{Q_k Q_k} = 2\displaystyle\sum_{j=1}^{\infty} e^{-|k-j|}\cos(Q_k - Q_j) \\ V_{Q_k Q_l} = -2e^{-|k-l|}\cos(Q_k - Q_l) \end{cases}$$

and property i) of §**1** is verified ($V_2 \leq 2\sqrt{2}\frac{e+1}{e-1}$. $\hat{Q}^o = 0$ is a critical point (3.4) and the matrix (4.11) can be written as $4\pi\frac{e+1}{e-1}I + B$ where I is the identity and $B_{kk} = -4\pi\frac{e^{2-k}}{e-1}$ while $B_{k,l} = -4\pi e^{-|k-l|}$ $k \neq l$. Being $\|B\| < 4\pi\frac{e+1}{e-1}$ the operator (4.11) is non-singular; the matrix (4.10) has a bounded inverse and Theorem 3.1 applies in this case of *long-range potential* too. A calculation analogous to (5.2) and consequently (5.3) is less available now because of the long-range feature of the model.

Appendix 1. Existence of the dynamics

In this Appendix we prove the Theorem 2.1 and some useful bounds relative to Corollary 4.1.

Let us define the space of continuous function $\mathcal{Q}_T = \{Q : [-T, T] \subset \mathbb{R} \to \mathcal{T}_2\}$; on $\mathcal{Q}_T$ the following metric is given

$$\rho(Q, Q') = \Big(\sum_{i=1}^{\infty} \sup_{t\in[-T,T]} \rho^2(Q_i(t), Q'_i(t))\Big)^{\frac{1}{2}}$$

and makes $\mathcal{Q}_T$ a Fréchet space. For what concerns the existence of the dynamic (i.e. the existence of the functions (P, Q) of Theorem 2.1) we study the system (2.6) as a second order system

$$\ddot{Q} = \varepsilon V_x(\tau + Q_1, \hat{Q}) \qquad\qquad \big(\dot{Q}(0), Q(0)\big) = \big(P^o, Q^o\big) \in l_2 \times \mathcal{T}_2$$

and we show that this system admits a unique solution $Q \in C^2([-T, T], \mathcal{T}_2)$ so the $P(t)$ of theorem 2.1 is simply $\dot{Q} \in C^1([-T, T], l_2)$. Then using point 2) of i) in §**1** we can affirm that the solution exists for all $t \in \mathbb{R}$.

Let us define the operator $F\colon \mathcal{Q}_T \to \mathcal{Q}_T$ by means of the sequence of functions $Q^{(n)}(\tau)$ as follows

$$\begin{cases} Q^{(n)}(\tau) \doteq F(Q^{(n-1)})(\tau) \equiv Q^{(o)}(\tau) + \varepsilon \displaystyle\int_0^{\tau} dt \int_0^{t} dt' V_x(t' + Q_1^{(n-1)}(t'), \hat{Q}^{(n-1)}(t')) \\ Q^{(o)}(\tau) \doteq Q^o + P^o\tau \qquad n \geq 1. \end{cases}$$

First we impose that
$\sup_{t\in[-T,T]} \rho(Q^{(n)}(\tau), Q^o) \leq \varepsilon\frac{T^2}{2}V^1\big(Q^o, \bar{\rho}\big) + T\|P^o\| \leq \bar{\rho}$ which gives an upper bound over T; then consider the subset $\mathcal{Q}_{T,\bar{\rho}} = \{Q : [-T, T] \subset \mathbb{R} \to \mathcal{T}_2 \ |\rho(Q, Q^o) \leq \bar{\rho}\} \subset \mathcal{Q}_T$. $\mathcal{Q}_{T,\bar{\rho}}$ is a closed subset in a complete metric space and then it is complete. We then

look for the solution $Q(t) \in \mathcal{Q}_{T,\bar\rho}$.

$$\begin{aligned}
&\rho(Q^{(n+1)}(\tau), Q^{(n+1)}(\tau')) \le \\
&\le |\tau-\tau'|\|P^o\| + \varepsilon \int_{\tau'}^{\tau} dt \int_0^t dt' \big[\sum_{i=1}^{\infty} V^2_{x_i}(t' + Q_1^{(n)}(t'), \hat Q^{(n)}(t'))\big]^{\frac12} \le \\
&\le |\tau-\tau'|\|P^o\| + \varepsilon\frac{1}{2}|\tau^2 - \tau'^2| V^1\big(Q^o,\bar\rho\big)
\end{aligned} \tag{6.1}$$

so $Q^{(n)} \in \mathcal{Q}_{T,\bar\rho} \Rightarrow Q^{(n+1)} \in \mathcal{Q}_{T,\bar\rho}$.

Local Existence

We will show that the sequence $\{Q^{(n)}(\tau)\}$ is uniformly convergent to a function $Q \in C^2([-T,T], \mathcal{T}_2)$ and being T arbitrary the existence is established for $\tau \in \mathbb{R}$.

$$\begin{aligned}
&\rho(Q^{(n+1)}(\tau), Q^{(n)}(\tau)) \le \\
&\le \varepsilon\Big(\sum_{i=1}^{\infty}\Big|\int_0^{\tau} dt_1 \int_0^{t_1} dt_2 V_{x_i}(t_2 + Q_1^{(n)}(t_2), \hat Q^{(n)}(t_2)) - \\
&\qquad\qquad - V_{x_i}(t_2 + Q_1^{(n-1)}(t_2), \hat Q^{(n-1)}(t_2))\Big|^2\Big)^{\frac12} \le \\
&\le \varepsilon \int_0^{\tau} dt_1 \int_0^{t_1} dt_2 \Big[\sum_{i=1}^{\infty}\Big|V_{x_i}(t_2 + Q_1^{(n)}(t_2), \hat Q^{(n)}(t_2)) - \\
&\qquad\qquad - V_{x_i}(t_2 + Q_1^{(n-1)}(t_2), \hat Q^{(n-1)}(t_2))\Big|\Big]^{\frac12} \\
&\le \varepsilon L \int_0^{\tau} dt_1 \int_0^{t_1} dt_2 \rho(Q^{(n)}(t_2), Q^{(n-1)}(t_2)) \\
&\vdots \\
&\le (\varepsilon L)^n \int_0^{\tau} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2n-1}} dt_{2n} \rho(Q^{(1)}(t_{2n}), Q^{(o)}(t_{2n})) \\
&\le \frac{(\varepsilon L T^2)^n}{(2n)!}\frac{\varepsilon T^2}{2} \sup_{S(Q^o, T\|\hat P^o\|)} \big(\sum_{i=1}^{\infty} V^2_{x_i}(t_{2n} + Q_1^o + P_1^o t_{2n}, \hat Q^o + \hat P_1^o t_{2n})\big)^{\frac12} \\
&\le \frac{(\varepsilon L T^2)^n}{(2n)!}\frac{\varepsilon T^2}{2} V^1(Q^o, T\|\hat P^o\|).
\end{aligned} \tag{6.2}$$

(6.2) makes uniform in $[-T,T]$ the existence of the following limit defining a function $Q \in \mathcal{Q}_{T,\bar\rho}$

$$\lim_{n\to\infty} Q^{(n)}(\tau) = \lim_{n\to\infty} \sum_{k=1}^{n}\big(Q^{(k)}(\tau) - Q^{(k-1)}(\tau)\big) + Q^{(o)}(\tau) \doteq Q(t);$$

Uniqueness

Let $Q(\tau)$ $\bar{Q}(\tau)$ be two different solutions of the problem (2.6). We can easily prove the bound

$$\rho\big(Q(\tau),\bar{Q}(\tau)\big)\leq (\varepsilon L)^{n-1}\frac{T^{2n}}{(2n)!}\rho(Q,\bar{Q})$$

which is a contradiction for n big enough unless $\rho(Q,\bar{Q})=0$.

About the global existence and uniqueness we can say that a solution of the system (2.6) cannot be extended over $\mathbb{R}$ if and only if it diverges in a finite time: there exists t_o such that $\lim_{t\to t_o^-}|\rho(Q(t),Q^o)|=+\infty$. But the upper bound on V_x in §**1** forbids the divergence at t_o.

Now we check that $Q\in C^2([-T,T],\mathcal{T}_2)$.

Let us consider (of course $\dot{Q}^{(k)}$ is continuous for $\tau\in[-T,T]$)

$$\lim_{n\to\infty}\dot{Q}^{(n)}(\tau)=\lim_{n\to\infty}\sum_{k=1}^{n}\big(\dot{Q}^{(k)}(\tau)-\dot{Q}^{(k-1)}(\tau)\big)+\dot{Q}^{(o)}(\tau)\doteq\dot{Q}(\tau)$$

and by means of the following bound

$$\rho(\dot{Q}^{(n)}(\tau),\dot{Q}^{(n-1)}(\tau))\leq\frac{(\varepsilon LT^2)^{n-1}}{(2n-2)!}\frac{\varepsilon^2T^3L}{2}V^1(Q^o,T\|\hat{P}^o\|)$$

the limit, by definition $\dot{Q}$, exists uniformly in $[-T,T]$ defining a continuous function. The last point of Theorem 2.1 is to show that $\dot{Q}\in C^1([-T,T],l_2)$. We then consider the series

$$\begin{aligned}\ddot{Q}(\tau)&=\lim_{n\to\infty}\ddot{Q}^{(n)}(\tau)=\lim_{n\to\infty}\sum_{k=1}^{n}\big(\ddot{Q}^{(k)}(\tau)-\ddot{Q}^{(k-1)}(\tau)\big)\\&=\lim_{n\to\infty}\sum_{k=1}^{n}\varepsilon\big(V_x(\tau+Q_1^{(k-1)},\hat{Q}^{(k-1)})-V_x(\tau+Q_1^{(k-2)},\hat{Q}^{(k-2)})\big)\end{aligned}\tag{6.3}$$

and because of (see footnote of §**1**)

$$\rho\big(V_x(\tau+Q_1^{(k-1)},\hat{Q}^{(k-1)}),V_x(\tau+Q_1^{(k-2)},\hat{Q}^{(k-2)})\big)\leq L\rho\big(Q^{(k-1)},Q^{(k-2)}\big)$$

the limit in (6.3) exists uniformly in $[-T,T]$ determining the continuity of $\ddot{Q}(\tau)$ too.

Here we prove the continuity of γ respect to ε. The continuity respect to P^o,Q^o can be achieved exactly in the same way.

$$\sup_{t\in[-T,T]}\rho(Q^{n+1}(t;P^o,Q^o,\varepsilon),Q^{n+1}(t;P^o,Q^o,\tilde{\varepsilon}))\leq|\varepsilon-\tilde{\varepsilon}|\frac{T^2}{2}V^1(Q^o,\Delta)\sum_{k=0}^{n}\frac{(\tilde{\varepsilon}LT^2)^{2k}}{(2k)!}$$

and then

$$\sup_{t\in[-T,T]} \rho(Q(t;P^o,Q^o,\varepsilon),Q(t;P^o,Q^o,\tilde{\varepsilon})) \leq |\varepsilon-\tilde{\varepsilon}|\frac{T^2}{2}V^1(Q^o,\Delta)\cosh\sqrt{\tilde{\varepsilon}LT^2}).$$

We can also perform the following estimates

$$\begin{aligned} \sup_{t\in[-T,T]} \rho(Q(t),Q^o) &\leq \sum_{n=1}^{\infty}\frac{(\varepsilon T^2L)^{n-1}}{(2n-2)!}\frac{\varepsilon}{2}T^2V^1(Q^o,T\|\hat{P}^o\|)+T\|P^o\| = \\ &= \frac{\varepsilon}{2}T^2V^1(Q^o,T\|\hat{P}^o\|)\cosh\sqrt{T^2L\varepsilon}+T\|P^o\| \doteq \Delta(\varepsilon,L,T,Q^o,P^o). \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} \rho(\dot{Q}(\tau),P^o) &\leq \sum_{n=1}^{\infty}\frac{(\varepsilon LT^2)^{n-1}}{(2n-2)!}\frac{\varepsilon^2T^3L}{2}V^1(Q^o,T\|\hat{P}^o\|) = \\ &= \frac{\varepsilon^2T^3L}{2}V^1(Q^o,T\|\hat{P}^o\|)\cosh\sqrt{\varepsilon LT^2} \equiv \Xi(Q^o,P^o,T,\varepsilon). \end{aligned} \tag{6.5}$$

References

[A] V.I.ARNOL'D, "Mathematical Methods of Classical Mechanics", Springer-Verlag, New York second ed. 1980 (chapter 9 p.233-235)

[AF] C.ALBANESE and J.FRÖLICH, *Periodic Solutions of Some Infinite- Dimensional Hamiltonian Systems Associated with Non-Linear Partial Difference Equations I*, Commun. Math. Phys., **116** (1988), 475-502

[AFS] C.ALBANESE, J.FRÖLICH and T.SPENCER *Periodic Solutions of Some Infinite- Dimensional Hamiltonian Systems Associated with Non-Linear Partial Difference Equations II*, Commun. Math. Phys., **119** (1988), 677-699

[CP] L.CHIERCHIA and P.PERFETTI, *Second Order Hamiltonian Equations on* $\mathbb{T}^\infty$ *and Almost-Periodic Solutions*, Journal of Differential Equations Vol.**116** No.1 1995, 172-201

[CW] W.CRAIG and C.E.WAYNE, *Newton's method and periodic solutions of nonlinear wave equations*, Commun. Pure Appl. Math. **46**(1993) 1409-1498

[FSW] J.FRÖLICH, T.SPENCER and C.E.WAYNE, *Localization in Disordered Nonlinear Dynamical Systems*, J. Stat. Phys. **42** (1986), 247-274

[G] G.GALLAVOTTI, "The Elements of Mechanics", Springer-Verlag, New York 1983

[Ga] F.GANTMACHER, "Lectures in Analytical Mechanics", Mir Publishers Moscow, second printing 1975 (chapter 6 p.227)

[Ku] S.B.KUKSIN, *Nearly Integrable Infinite-Dimensional Hamiltonian Systems*, Lecture Notes in Mathematics 1556, Springer-Verlag, New York 1993

[LA] S.LANG, "Differential Manifolds", Addison-Wesley Eds., 1972

[LY] A.LYAPUNOV, "Problème Général de La Stabilité du Movement", Princeton Univ.Press, 1947

[M] J.K.MOSER, "Dynamical Systems", Courant Institute of Mathematical Sciences, 1979-1980

[Po] H.POINCARÉ, "Les Méthodes Nouvelles de La Mécanique Céleste" Tome I No.36-42, Gauthier-Villars, 1892 (Reprinted by "Librairie Albert Blanchard", rue de Médicis, 75006 Paris (1987))

[Pö] J.PÖSCHEL, *Some Recent Results concerning Quasi-periodic Solution for a Nonlinear String Equation*, Preprint 95-1 Universität Stuttgart.

[SM] C.L.SIEGEL and J.K.MOSER, "Lectures on Celestial Mechanics" §21, Springer-Verlag, New York, 1971

[Wa] C.E.WAYNE, *Periodic and Quasi-Periodic Solutions of Nonlinear Wave Equations via KAM Theory*, Commun. Math. Phys. **127** (1990) 479-528

E-mail: perfetti@mat.utovrm.it