Wavelet analysis of potential fields

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Abstract. It is shown how a continuous wavelet technique may be used to locate and characterize homogeneous point sources from the field they generate measured in a distant hyperplane. For this a class of wavelets is introduced on which the Poisson semi-group essentially acts as a dilation.

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1. Introduction

Consider a point source located at the origin, and assume now that you can measure the field it generates in a distant hyperplane. The problem is to recover the properties (type, strength, location, orientation,...) of the source from this measurement. A naive approach might be to use a deconvolution technique. However, because of the finite accuracy of measurements, this method cannot be really used in practice. The approach we propose is based on the continuous wavelet transform. Since this is a family of convolutions with well-localized functions, we shall not encounter the instabilities of deconvolutions. This kind of technique might have applications in remote sensing of sources. Obvious examples are subsurface imaging in geophysics from potential field measurements on the Earth's surface. Also, in medicine where stationary temperature fields obey the Poisson equation, applications to infrared thermography are in sight.

In the present study, we limit our discussion to homogeneous point sources. These constitute a natural family to represent local heterogeneities. However, we shall limit ourselves to the single source problem and leave the discussion of the multisource situation to a subsequent paper.

The main results of this paper can be summarized as follows. As is well-known in the case of signal analysis, the continuous wavelet transform allows detection and characterization of homogeneous singularities [7]. Indeed, the lines of constant phase or maxima or zero-crossings,... in the parameter half-space of the wavelet coefficients converge towards the point on the borderline where the singularity is located (figure 1). Now suppose this singularity is the source of a potential field measured in a hyperplane at a distance z. In this paper we show that there are wavelets such that the wavelet coefficients of this remote field also exhibit lines of constant phase or maxima or zero-crossings,... that converge, but now to a point located at a distance z outside the parameter half-space (figure 1).

2. Wavelet analysis: the basic formulas

2.1. *Wavelet transform and wavelet synthesis*

In this section the basic formulas of wavelet analysis are summarized for the convenience of the reader (see [8], [2] for general theory on the continuous wavelet transform). For the sake of generality we work in n dimensions and state the results on a formal level.

Let s and g be complex valued functions over $\mathbb{R}^n$. The wavelet transform of s with respect to the analyzing wavelet g is defined through [6], [7]

$$\mathcal{W}\left[g,s\right](b,a) \equiv \int_{\mathbb{R}^n} \mathrm{d}x \frac{1}{a^n} \overline{g}\left(\frac{x-b}{a}\right) s(x) \tag{1}$$

$$= \int_{\mathbb{R}^n} \mathrm{d}x \frac{1}{a^n} \widetilde{g}\left(\frac{b-x}{a}\right) s(x), \tag{2}$$

where $\widetilde{g}(x) = \overline{g}(-x)$ and dx is the n-dimensional Lebesgue measure. Here $b \in \mathbb{R}^n$ is a position parameter and $a \in \mathbb{R}_+$ is a scale parameter. The wavelet transform of a function over $\mathbb{R}^n$ is thus a function over the position-scale half-space $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$. The first formula expresses the wavelet transform in terms of a correlation function whereas the second is a convolution. If the wavelet is symmetric and real valued, $\widetilde{g} = g$ and both notions coincide.

By introducing the dilation (D_a) and translation operators (T_b) whose actions are respectively defined by

$$D_a s(x) \equiv a^{-n} s(x/a) \tag{3}$$

$$T_b s(x) \equiv s(x-b), \tag{4}$$

the wavelet transform may be written either as a family of scalar products or as a family of convolutions indexed by the scale parameter a:

$$\mathcal{W}[g,s](b,a) = \langle T_b D_a g \mid s \rangle, \tag{5}$$

$$= (D_a \widetilde{g} * s)(b). \tag{6}$$

Here the convolution product is defined as usual,

$$(\widetilde{g} * s)(x) \equiv \int_{\mathbb{R}^n} \mathrm{d}y \widetilde{g}(x-y) s(y) = (s * \widetilde{g})(x). \tag{7}$$

In the Fourier space the wavelet transform reads

$$\mathcal{W}[g,s](b,a) = \int_{\mathbb{R}^n} \mathrm{d}u\, \overline{\widehat{g}(au)}\, \mathrm{e}^{2i\pi ub}\, \widehat{s}(u), \tag{8}$$

where the independent variable u is dual of either x or b and the direct and inverse Fourier transforms are respectively defined by

$$\mathcal{F}[s(x)](u) \equiv \widehat{s}(u) \equiv \int_{\mathbb{R}^n} \mathrm{d}x \mathrm{e}^{-2i\pi ux} s(x), \tag{9}$$

$$\mathcal{F}^{-1}[\widehat{s}(u)](x) \equiv s(x) \equiv \int_{\mathbb{R}^n} \mathrm{d}u \mathrm{e}^{2i\pi ux} \widehat{s}(u). \tag{10}$$

The wavelet synthesis $\mathcal{M}$ maps functions $r(b,a)$ over $\mathbb{H}^{n+1}$ to functions over $\mathbb{R}^n$ and the synthesis of r with respect to the synthesizing wavelet h reads

$$\mathcal{M}[h,r](x) = \int_{\mathbb{H}^{n+1}} \frac{\mathrm{d}b\mathrm{d}a}{a} \frac{1}{a^n} h\left(\frac{x-b}{a}\right) r(b,a) \tag{11}$$

$$= \int_{\mathbb{H}^{n+1}} \frac{\mathrm{d}u\mathrm{d}a}{a} \widehat{h}(au)\, \mathrm{e}^{2i\pi ux}\, \widehat{r}(u,a). \tag{12}$$

2.2. Relation between $\mathcal{W}$ and $\mathcal{M}$

The wavelet synthesis $\mathcal{M}$ is the adjoint of the wavelet transform $\mathcal{W}$,

$$\int_{\mathbb{H}^{n+1}} \frac{\mathrm{d}b\mathrm{d}a}{a} \overline{\mathcal{W}\left[g,s\right]}\left(b,a\right) r(b,a) = \int_{\mathbb{R}^n} \mathrm{d}x\, \overline{s}\left(x\right)\; \mathcal{M}\left[g,r\right]\left(x\right), \tag{13}$$

and, in the Fourier space, the combination of these operators reads

$$\mathcal{M}\left[h, \mathcal{W}\left[g,s\right]\right] : \; \widehat{s}(u) \mapsto \widehat{m}_{g,h}(u)\widehat{s}(u), \tag{14}$$

with

$$\widehat{m}_{g,h}(u) = \int_0^\infty \frac{\mathrm{d}a}{a} \overline{\widehat{g}}\left(au\right)\, \widehat{h}(au). \tag{15}$$

Note that the Fourier multiplier $\widehat{m}_{g,h}$ only depends on the direction of u, $\widehat{m}_{g,h} = \widehat{m}_{g,h}(u/\left|u\right|)$. This is because the measure $\mathrm{d}a/a$ is scaling invariant.

In case that g and h are such that $\widehat{m}_{g,h}(u) = c_{g,h}$ with $0 < \left|c_{g,h}\right| < \infty$, we say that g, h are an analysis reconstruction pair, or that h is a reconstruction wavelet for g. We say that g is admissible if g is its own reconstruction wavelet, or (what is the same) if

$$\forall u \in \mathbb{R}^n \backslash \{0\} : \quad \int_0^\infty \frac{\mathrm{d}a}{a} \left|\widehat{g}(au)\right|^2 = c_{g,g} < \infty \tag{16}$$

Note that if $\widehat{g}$ is continuous at the origin, then we have necessarily $\widehat{g}(0) = 0$ if g is admissible. Note also that s may be recovered from its wavelet transform with respect to a non-admissible wavelet if a suitable reconstruction wavelet is chosen. Therefore in what concerns the analysis using wavelets, the admissibility of the wavelets is not mandatory.

If g and h are an analysis reconstruction pair, then the following formula holds

$$\int_{\mathbb{H}^{n+1}} \frac{\mathrm{d}b\mathrm{d}a}{a} \overline{\mathcal{W}\left[g,s\right]}\left(b,a\right) \mathcal{W}\left[h,r\right]\left(b,a\right) = c_{g,h} \int_{\mathbb{R}^n} \mathrm{d}x\, \overline{s}(x)\, r(x). \tag{17}$$

In particular if g is admissible, then we have conservation of energy

$$\int_{\mathbb{H}^{n+1}} \frac{\mathrm{d}b\mathrm{d}a}{a} \left|\mathcal{W}\left[g,s\right]\left(b,a\right)\right|^2 = c_{g,g} \int_{\mathbb{R}^n} \mathrm{d}x\, \left|s(x)\right|^2. \tag{18}$$

2.3. Covariance and homogeneous functions

The wavelet transform is covariant with respect to both translations and dilations,

$$\mathcal{W}[g, T_\beta s](b,a) \; = \mathcal{W}[g,s](b-\beta, a), \tag{19}$$

$$\mathcal{W}[g, D_\lambda s](b,a) = \frac{1}{\lambda^n}\mathcal{W}[g,s]\left(\frac{b}{\lambda}, \frac{a}{\lambda}\right). \tag{20}$$

The last invariance implies a certain behavior for wavelet transforms of homogeneous functions $s(x)$ of degree $\alpha \in \mathbb{R}$, i.e. such that

$$s(\lambda x) = \lambda^\alpha s(x)\; \forall \lambda > 0. \tag{21}$$

Indeed, the wavelet transform of homogeneous functions satisfies [5]

$$\mathcal{W}[g,s](\lambda b,\lambda a)=\lambda^{\alpha}\mathcal{W}[g,s](b,a), \tag{22}$$

and is fully determined by dilating and scaling one voice†:

$$\mathcal{W}[g,s](b,a)=a^{\alpha}\mathcal{W}\left[g,s\right](b/a,1). \tag{23}$$

As a consequence the points where $\mathcal{W}[g,s](b,a)=0$ are unions of straight lines converging towards the center of homogeneity. In the same way, the local maxima for each voice, that is the set of points where $\partial_b\mathcal{W}[g,s](b,a)=0$, form cone-like structures pointing towards the origin. More precisely, if the wavelet transform is locally maximum at position b and scale a, so it will be at position λb and scale λa. The set of points obtained as $\lambda>0$ varies is a line along which the wavelet transform scales with a power-law revealing the degree α of homogeneity of s.

A natural generalization of homogeneous functions are quasi-homogeneous functions where (21) is replaced by

$$s(\lambda x)=\lambda^{\alpha}s(x)+(\lambda x)^{\alpha}\log\lambda. \tag{24}$$

Obvious examples are the functions $x^{\alpha}\log x$. Note that for $\alpha\in\mathbb{N}$, the wavelet transform of a quasi-homogeneous function satisfies again [2]

$$\mathcal{W}[g,s](\lambda b,\lambda a)=\lambda^{\alpha}\mathcal{W}[g,s](b,a), \tag{25}$$

provided the wavelet has at least α vanishing moments,

$$\int \mathrm{d}x\, x^{\beta}g\left(x\right)=0,\ \ |\beta|\leq\alpha. \tag{26}$$

Indeed, in this case the second term in formula (24) is not visible in the wavelet transform.

3. Wavelet analysis based on the Poisson semi-group

3.1. Harmonic extension as a wavelet transform

Consider the following boundary value problem for a function of $q=(x,z)\in\mathbb{H}^{n+1}$:

$$\begin{array}{ll} i) & \Delta\phi\left(q\right)=0\ \ \forall q\in\mathbb{H}^{n+1}\\ ii) & \phi\left(x,z=0\right)=s\left(x\right)\\ iii) & \int_{\mathbb{R}^n}\mathrm{d}x\left|\phi\left(x,z\geq 0\right)\right|^2<K<\infty \end{array} \tag{27}$$

where s is a function in $\mathbb{R}^n$ that we suppose bounded and in $L^2\left(\mathbb{R}^n\right)$ and K is a constant. Condition $iii)$ implies that ϕ is of limited growth when $z\rightarrow+\infty$ and then that the field ϕ is uniquely determined by s and its boundary behavior at

† According to the primevial litterature concerning the wavelet transform [4], we shall hereafter call a line $\mathcal{W}\left[g,s\right](b,a=\text{constant})$ a "voice" of the wavelet transform.

infinity. Such a field is called the harmonic extension of s into the upper half-space $\mathbb{H}^{n+1}$ and can be obtained explicitly from s by means of the Poisson semi-group:

$$\begin{aligned} \phi(x,z) &= (D_z p * s)(x) && (28) \\ &= \mathcal{W}[p,s](x,z) && (29) \end{aligned}$$

where the Poisson kernels are defined by [1]

$$p(x) = c_{n+1}(1+|x|^2)^{-(n+1)/2}, \ \widehat{p}(u) = \mathrm{e}^{-2\pi|u|}, \tag{30}$$

and verify the semi-group property

$$D_z p * D_{z'} p = D_{z+z'} p. \tag{31}$$

As the remaining analysis will show this semi-group structure is the basic algebraic requirement and our analysis applies to the heat semi-group as well. Owing to both equation (6) and the symmetry properties of the Poisson kernel [see formula (30)], the harmonic extension of s may be written under the form of a wavelet transform with respect to p [equation (29)]. Let us remark that in the present case the analyzing wavelet is not admissible since $\widehat{p}(u)$ is continuous and $\widehat{p}(0) = 1$. Also, the scale parameter a plays the role of the physical dimension z along which the function s is upward-continued, while the translation parameter is the equivalent of x [compare equations (6) and (29)].

For later reference we note that relation (28) holds even in the case of tempered distributions as boundary values. More precisely, if ϕ satisfies $\Delta\phi = 0$ in $\mathbb{H}^{n+1}$ and if ϕ is of at most polynomial growth

$$|\phi(b,a)| \leq c\,(a+1/a)^K\,(1+|b|)^K \tag{32}$$

for some c and $K > 0$, then the limit

$$\phi(\cdot,z) \to \phi\left(\cdot,0^+\right), \quad (z \searrow 0) \tag{33}$$

holds in the sense of distributions [1], [3]. In addition, the field $\phi(\cdot,z)$ may be recovered, up to some polynomial, from the boundary distribution by means of the harmonic continuation formula,

$$\phi(\cdot,z) = D_z p * \phi\left(\cdot,0^+\right) + P. \tag{34}$$

Here P is a polynomial in $n+1$ variables. However, to give a precise meaning to the convolution, we have to regularize p in such a way that its Fourier transform is regular around the origin. That is if we consider the functions p_l defined through

$$\widehat{p_l}(u) = \mathrm{e}^{-2\pi|u| - 2\pi/|lu|}, \tag{35}$$

then $\widehat{p_l}(u) \to \widehat{p}(u)$ as $l \to \infty$. Now we have more precisely:

$$\phi(\cdot,z) = \lim_{l\to\infty} D_z p_l * \phi\left(\cdot,0^+\right) + P. \tag{36}$$

However, to simplify the notation, we shall not use this cumbersome formula and convolutions of distributions with the Poisson kernel always suppose this limit.

3.2. Homogeneous sources

In this section we recall the main properties of homogeneous distributions in $\mathbb{R}^m$. A distribution σ is called homogeneous of degree α if for all test functions ψ we have

$$\sigma(\psi_\lambda) = \lambda^\alpha \sigma(\psi), \ \psi_\lambda = \lambda^m \psi(\lambda \cdot), \ \lambda > 0. \tag{37}$$

If ψ is allowed to vary over all test functions with support in $\mathbb{R}^m$ we say that σ is a distribution in $\mathbb{R}^m$. If instead (37) holds only for those ψ whose support does not contain the origin, we say that σ is a distribution in $\mathbb{R}^m \backslash \{0\}$. For instance, every homogeneous function of degree $\alpha > -m$ defines a distribution of degree α. Other examples are given by the δ distribution and all its partial derivatives $\partial^\beta \delta$ with the multi-index $\beta \in \mathbb{N}^m$. Its degree of homogeneity is $-m-\beta_1-\beta_2-\cdots-\beta_m$. These are the only distributions having their support in one single point, and they can be identified with the classical multipoles. More general examples are given by replacing the ordinary derivative with the fractional derivative defined by

$$\partial^\beta : \ \widehat{s}(u) \mapsto (2i\pi u)^\beta \, \widehat{s}(u), \ \beta \in \mathbb{R}^m, \tag{38}$$

where the branch cut of the logarithm is taken on the negative part of the real axis. Note however that the distributions $\partial^\beta \delta$ with $\beta \in \mathbb{R}^m$ are not sharply localized anymore but exhibit a power law decay at large distances.

We now list some well known properties without proof [3]. The space of homogeneous distributions in respectively $\mathbb{R}^m$ and $\mathbb{R}^m \backslash \{0\}$ for a fixed degree α is a vector space. It can be shown that every homogeneous distribution in $\mathbb{R}^m$ is automatically tempered. Therefore its Fourier transform is defined in the sense of distributions. It follows, from the continuity of the Fourier transform and its covariance under dilation, that the Fourier transform of homogeneous distributions in $\mathbb{R}^m$ of degree α is again a homogeneous distribution but of degree $-m-\alpha$.

We now come to the problem of extensions. Every homogeneous distribution in $\mathbb{R}^m$ defines a homogeneous distribution in $\mathbb{R}^m \backslash \{0\}$ by restricting the set of test functions. However, the converse is not true. More precisely, not every homogeneous distribution σ_0 in $\mathbb{R}^m \backslash \{0\}$ has a homogeneous extension to a distribution σ in $\mathbb{R}^m$, and we have to distinguish two different situations. First, if $\alpha \neq -m, -m-1, -m-2 \cdots$ a homogeneous distribution σ_0 in $\mathbb{R}^m \backslash \{0\}$ can be extended to a homogeneous distribution σ in $\mathbb{R}^m$ of the same degree. Moreover, this extension is unique in the class of homogeneous distributions. In the remaining case, $m - \alpha \in \mathbb{N}$, the things are a little more complicated since not always does a homogeneous extension exist. Clearly an extension exists by the Hahn-Banach theorem, however the extension need not be homogeneous any more. However, there always exists an extension that is quasi-homogeneous in the sense that there is some distribution η such that

$$\sigma(\psi_\lambda) = \lambda^\alpha \sigma(\psi) + \eta(\psi) \log \lambda, \ \psi_\lambda = \lambda^m \psi(\lambda \cdot), \ \lambda > 0 \tag{39}$$

for all test functions in $\mathbb{R}^m$. Here η is a linear combination of derivatives $\partial^\beta \delta$ with the multi-index $\beta \in \mathbb{N}^m$ and $\beta_1 + \beta_2 + \cdots + \beta_m = -m - \alpha$. However, this quasi-homogeneous extension σ is not unique. Note that the class of those homogeneous distributions over $\mathbb{R}^m \backslash \{0\}$ that have a homogeneous extension to $\mathbb{R}^m$ can be completely characterized. But since we will not use this we will skip the details and we refer to the literature. For later reference note that the Fourier transform of a quasi-homogeneous distribution satisfies

$$\widehat{\sigma}\left(\psi_\lambda\right) = \lambda^{-m-\alpha}\widehat{\sigma}\left(\psi\right) + \theta\left(\psi\right)\log\lambda,\ \psi_\lambda = \lambda^m\psi\left(\lambda\cdot\right),\ \lambda > 0 \tag{40}$$

where $\theta = \widehat{\eta}$ is a homogeneous polynomial of degree $-m - \alpha$. It follows that the distribution $\upsilon = \widehat{\sigma} + \theta\log|\cdot|$ is homogeneous of degree $-m-\alpha$. Therefore we have that $\widehat{\sigma}$ can be written as a sum of a homogeneous distribution and a logarithmic correction:

$$\widehat{\sigma} = \upsilon - \theta\log|\cdot| \tag{41}$$

3.3. The field of homogeneous sources

Consider now the Poisson equation in $\mathbb{R}^{n+1}$

$$\Delta\phi(q) = -\sigma(q),\ q \in \mathbb{R}^{n+1}, \tag{42}$$

where the (generalized) function σ is a source term and this equation is to be understood in the sense of distributions. We work from now on in n dimensions because the $n+1$'th direction will play a privileged role and we shall write $q = (x, z)$, with $x \in \mathbb{R}^n$, $z \in \mathbb{R}$. Clearly, in application in geophysics we have $n = 2$, the two horizontal dimensions. The third dimension is the vertical direction. As it stands the solution ϕ in terms of σ is not unique. However, the tempered solutions to the homogeneous equation are the harmonic polynomials, and thus ϕ is essentially determined by σ, up to some polynomial. In addition, for physical reasons, we add the following requirements on the growth behavior at infinity. We suppose that as $q \to \infty$

$$|\phi(q)| \leq \begin{cases} c\log|q| & \text{for} \quad n = 1 \\ c & \text{for} \quad n > 1 \end{cases}. \tag{43}$$

Then the solution is actually unique up to a global constant.

We are particularly interested in homogeneous sources of the type discussed in the previous section. Suppose now that σ is a homogeneous distribution of degree α in $\mathbb{R}^{n+1}$. Consider first the case where $\alpha \notin \mathbb{N}$. We claim that there is a unique distribution ϕ which is homogeneous of degree $\alpha + 2$, and satisfies (42). Indeed, the Fourier transform $\widehat{\sigma}$ is a homogeneous distribution in $\mathbb{R}^{n+1}$of degree $-n-1-\alpha$. It follows that $\widehat{\phi} = \widehat{\sigma}/|u|^2$ defines a distribution in $\mathbb{R}^{n+1}\backslash\{0\}$. Now this distribution is homogeneous of degree $\rho = -n-3-\alpha$. Since now $\rho \notin \mathbb{N}$, we may extend $\widehat{\phi}$ to a homogeneous distribution in all of $\mathbb{R}^{n+1}$. Its inverse Fourier transform ϕ is then clearly a solution of the Poisson equation we have considered and the degree of homogeneity is $\alpha + 2$ as claimed.

Suppose now that $-n-1-\alpha \in \mathbb{N}$. Again we may set $\widehat{\phi} = \widehat{\sigma}/|u|^2$ to define a distribution in $\mathbb{R}^{n+1}\backslash\{0\}$. But now it is not clear wether or not it has a homogeneous extension. However a quasi-homogeneous extension exists. Its inverse Fourier transform satisfies therefore the decomposition of quasi-homogeneous distributions

$$\widehat{\phi}(\psi_\lambda) = \lambda^{\alpha+2}\phi(\psi) + \theta(\psi)\ln\lambda, \tag{44}$$

where θ is a polynomial of degree $\alpha+2$. Therefore in particular for $\alpha < -2$ the field is again homogeneous. Note that this general discussion is nicely exhibited by the Green's functions where $\sigma = \delta$ [1]:

$$G_0(q) = \begin{cases} c_2 \ln|q| & (n=1) \\ c_{n+1}|q|^{1-n} & (n>1) \end{cases}. \tag{45}$$

Until now we have discussed general homogeneous sources. However, for obvious physical reasons, we have to require that σ is supported by a subset of the lower half-space $z \leq 0$. As an additional property we may introduce the boundary distribution of the field ϕ in the hyperplane $z = 0$. More precisely the following limit exists in the sense of distributions

$$\phi(\cdot, z) \to \phi(\cdot, 0^+),\ (z \searrow 0). \tag{46}$$

This boundary distribution satisfies the same homogeneity and quasi-homogeneity properties as ϕ. In addition, the field $\phi(\cdot, z)$ may be recovered from the boundary distribution by means of the harmonic continuation formula,

$$\phi(\cdot, z) = D_z p * \phi(\cdot, 0^+). \tag{47}$$

4. Wavelet analysis of homogeneous fields.

4.1. Wavelets based on the Poisson semi-group

We now introduce a class of wavelets that behave nicely under the Poisson semi-group. This will be necessary to analyze homogeneous potential fields and will be used in the next section. More precisely, we say that a wavelet g satisfies the dilation-continuation condition if the following holds true

$$D_a g * D_{a'} p = c D_{a''} g. \tag{48}$$

Here $c = c(a, a')$ and $a'' = a''(a, a')$ are functions of the scales a and a'. This means that the continuation operator maps the dilated wavelet $D_a g$ into a wavelet at the same position but at scale a'' and amplitude c. In this section we will discuss some properties of wavelets that satisfy the dilation-continuation property. In particular we want to construct a large family of solutions of (48).

First note that an immediate solution is given by the Poisson kernel itself. Indeed the semi-group property

$$D_a p * D_{a'} p = D_{a+a'} p \tag{49}$$

shows that the Poisson kernel is a (non admissible) wavelet that satisfies the dilation-continuation property with $c = 1$ and $a'' = a + a'$. To obtain more general solutions, consider a linear operator $\mathcal{L}$ which satisfies the following properties with respect to the dilation and translation operators,

$$D_a\mathcal{L} = a^{\gamma}\mathcal{L}D_a,\ \gamma \in \mathbb{R} \tag{50}$$

$$T_b\mathcal{L} = \mathcal{L}T_b. \tag{51}$$

Property (51) means that $\mathcal{L}$ is a Fourier multiplier which by (50) is homogeneous of degree γ. Thus

$$\mathcal{L} : \widehat{s}(u) \mapsto m(u)\widehat{s}(u),\ m(\lambda u) = \lambda^{\gamma} m(u). \tag{52}$$

Now if g has the dilation-continuation property, we claim that $\mathcal{L}g$ too has the dilation-continuation property with the same function a'' and c replaced by $(a/a'')^{\gamma}\, c$. Indeed we may write

$$D_a\mathcal{L}g * D_{a'}p = a^{\gamma}\mathcal{L}\left(D_a g * D_{a'}p\right) \tag{53}$$

$$= \left(a/a''\right)^{\gamma} cD_{a''}\mathcal{L}g. \tag{54}$$

Therefore in particular all functions given by

$$\widehat{g}(u) = m(u)\mathrm{e}^{-|u|},\ m(\lambda u) = \lambda^{\gamma} m\left(u\right), \tag{55}$$

are solutions of equation (48). In the special case where we have in addition $c = c\left(a'\right)$ and $a'' = a + a'$, a second family of solutions can be obtained as follows: suppose g satisfies

$$D_a g * D_{a'}p = c\left(a'\right) D_{a+a'}g, \tag{56}$$

a differentiation with respect to a shows that $\left(x\partial_x\right) g$ is again solution of (56) with the same function c. Therefore, a general family of solutions is given by

$$P\left(x\partial_x\right)\mathcal{L}p, \tag{57}$$

where P is a polynomial in one variable and $\mathcal{L}$ is the operator (52).

4.2. Wavelet analysis of homogeneous fields

Both the covariance of the wavelet transform with respect to dilations (20) and the homogeneity of degree $\alpha + 2$ of the field ϕ ensure that

$$\mathcal{W}\left[g, \phi\left(\cdot, z\right)\right]\left(b, a\right) = \mathcal{W}\left[g, D_z p * \phi\left(\cdot, 0^{+}\right)\right]\left(b, a\right) \tag{58}$$

$$= \left[D_a\widetilde{g} * D_z p * \phi\left(\cdot, 0^{+}\right)\right]\left(b\right) \tag{59}$$

$$= c\left(a, z\right)\left[D_{a''(a,z)}\widetilde{g} * \phi\left(\cdot, 0^{+}\right)\right]\left(b\right) \tag{60}$$

$$= c\left(a, z\right)\mathcal{W}\left[g, \phi\left(\cdot, 0^{+}\right)\right]\left(b, a''\left(a, z\right)\right). \tag{61}$$

Now, using the covariance of the wavelet transform and the homogeneity of the boundary distribution $\phi(\cdot,0^+)$, we obtain

$$\mathcal{W}[g,\phi(\cdot,z)](b,a)=c(a,z)\,a''(a,z)^{-\alpha-2}\,\mathcal{W}\left[g,\phi\left(\cdot,0^+\right)\right]\left(\frac{b}{a''(a,z)},1\right) \tag{62}$$

To simplify the discussion, let us assume that the wavelet g belongs to the class defined by (57). Then the last equation becomes

$$\mathcal{W}[g,\phi(\cdot,z)](b,a)=\left(\frac{a}{a+z}\right)^{\gamma}(a+z)^{-\alpha-2}\,\mathcal{W}\left[g,\phi\left(\cdot,0^+\right)\right]\left(\frac{b}{a+z},1\right) \tag{63}$$

In order to get some insight into the geometry of this equation note that there are two functions f and F such that the wavelet transform can be written as

$$\mathcal{W}[g,\phi(\cdot,z)](b,a)=f(a)\,F\left(\frac{b}{z+a}\right), \tag{64}$$

where f and F read

$$f(a)\;=\;(a+z)^{-\alpha-2}\left(\frac{a}{a+z}\right)^{\gamma} \tag{65}$$

$$F(b)\;=\mathcal{W}\left[g,\phi\left(\cdot,0^+\right)\right](b,1)\,. \tag{66}$$

Note that the set of points (b,a) which satisfy $b/(z+a)=$ cst are located on the straight line in the half-space. For various constants cst, we obtain a family of lines that intersect at the point $(0,-z)$ outside the half-space $\mathbb{H}^{n+1}$. Therefore, the wavelet transform exhibits a cone-like structure where the top of the cone is shifted to the location of the source outside the half-space. This provides a natural geometric way to locate the source. The homogeneity α of the source, can be obtained either from the full expression of $f(a)$ using the formerly estimated z or from the asymptotic behavior $f(a)\simeq a^{-\alpha-2}$ in the limit $a\to\infty$.

It might be instructive to give a second derivation of these results using only the homogeneity of the field ϕ and not using the boundary field $\phi(\cdot,0^+)$. Again, the covariance of the wavelet transform with respect to dilations (20) and the homogeneity of the field ϕ imply that

$$\mathcal{W}[g,\phi(\cdot,z)](b,a)\;=\;\left(\frac{a'}{a}\right)^{n}\mathcal{W}\left[g,D_{a'/a}\phi(\cdot,z)\right]\left(\frac{ba'}{a},a'\right) \tag{67}$$

$$=\;\left(\frac{a'}{a}\right)^{n-\alpha-2}\mathcal{W}\left[g,\phi\left(\cdot,\frac{za'}{a}\right)\right]\left(\frac{ba'}{a},a'\right). \tag{68}$$

Here the dilation is acting on the first n variables only. Now, the harmonic extension relation (28) enables us to obtain $\phi(\cdot,za'/a)$ from $\phi(\cdot,z)$:

$$\mathcal{W}[g,\phi(\cdot,z)](b,a)\;=\;\left(\frac{a'}{a}\right)^{n-\alpha-2}\mathcal{W}\left[g,D_{z(a'/a-1)}p*\phi(\cdot,z)\right]\left(\frac{ba'}{a},a'\right) \tag{69}$$

$$=\;\left(\frac{a'}{a}\right)^{n-\alpha-2}\left(D_{a'}\widetilde{g}*D_{z(a'/a-1)}p*\phi(\cdot,z)\right)\left(\frac{ba'}{a}\right). \tag{70}$$

As before, assume that the wavelet g has the dilation-continuation property (48). Then we obtain

$$\mathcal{W}[g,\phi(\cdot,z)](b,a) = c\cdot\left(\frac{a'}{a}\right)^{n-\alpha-2}\left(D_{a''}\widetilde{g}*\phi(\cdot,z)\right)\left(\frac{ba'}{a}\right) \quad (71)$$

$$= c\cdot\left(\frac{a'}{a}\right)^{n-\alpha-2}\mathcal{W}[g,\phi(\cdot,z)]\left(\frac{ba'}{a},a''\right). \quad (72)$$

where c and a'' are functions of a', a and z (62). If, in addition, g belongs to the family (57) and verifies (52), this last expression simplifies

$$\mathcal{W}[g,\phi(\cdot,z)](b,a) = \left(\frac{a}{a''}\right)^{\gamma}\left(\frac{a''+z}{a+z}\right)^{\gamma+n-\alpha-2}\mathcal{W}[g,\phi(\cdot,z)]\left(b\frac{a''+z}{a+z},a''\right). \quad (73)$$

This equation is valid for all $a''>0$ which now plays the role of a parameter. In order to recover from this expression the geometry of the shifted cone-like structure in the wavelet transform, consider two points $\Omega=(b,a)$ and $\Omega''=(b(a''+z)/(a+z),a'')$. The straight line they define passes through the location of the source $(0,-z)$. From (73), we see that the ratio of the wavelet coefficients at these points can be written as

$$\frac{\mathcal{W}[g,\phi(\cdot,z)](\Omega)}{\mathcal{W}[g,\phi(\cdot,z)](\Omega'')} = \frac{f(a)}{f(a'')}. \quad (74)$$

5. Examples of source characterization

We shall now examine how the wavelet transform enables both a localization and a characterization of homogeneous sources responsible for an observed field. For an easy display of the results, we work in a 2-dimensional physical space (i.e. with $n=1$). Equation (73) is our basic working equation from which the horizontal and the vertical coordinates of the source are to be determined together with its homogeneity α. The wavelet used in the present example is displayed at the upper right corner of figures 2, 3, and 4 and is defined by

$$\widetilde{g}_1(x) = \frac{\mathrm{d}}{\mathrm{d}x}p(x), \quad (75)$$

and such that $\gamma=1$. For this wavelet, equation (73) reduces to

$$\mathcal{W}[g_1,\phi(\cdot,z)](b,a) = \frac{a}{a''}\left(\frac{a''+z}{a+z}\right)^{-\alpha}\mathcal{W}[g_1,\phi(\cdot,z)]\left(b\frac{a''+z}{a+z},a''\right). \quad (76)$$

We consider three examples of sources localized at the origin and corresponding to $\alpha=-3$, -3.5, and -4. The potential fields $\phi(x,z=20)$ created by these sources are respectively shown at the upper-left corner of figures 2, 3, and 4, and the wavelet transforms of the fields are shown at the middle-left of the same figures. Note that the field created by the source with a non-integer homogeneity $\alpha=-3.5$ has a non-symmetric wavelet transform (see figure 3). Two voices of these wavelet transforms corresponding to the dilations $a=6$ (solid lines) and $a=20$ (dashed lines) are displayed at the middle-right part of the figures. The voices of the wavelet transforms of the fields for the sources with $\alpha = -3$, and $\alpha = -3.5$ possess two extremas and the voices for $\alpha = -4$ have three. A two-step algorithm can be used to estimate both the homogeneity index α and the depth z to the source from the measurement plane (here, $z = 20$). First, the depth is geometrically determined by using equation (74) and is given by the location outside $\mathbb{H}^2$ where the lines of extrema of the wavelet transform cross (see the lower-left corner of figures 2, 3, and 4). Remark that the convergence of the lines of extrema of the wavelet transforms also gives a determination of the horizontal position of the source which creates the analyzed field. Once z is known, the exponent α is computed by examining the variation of the amplitude of the wavelet transform along a given line of extrema (see the lower-right corner of figures 2, 3, and 4). This procedure accurately (i.e. up to the numerical precision of the computer) restitutes the theoretical values of α and z.

6. Conclusion

We have shown that the wavelet transform of the potential field generated by a homogeneous source and measured in a hyperplane possesses truncated cone-like structures pointing towards the location of the source. In addition, the variation of the wavelet coefficients over the scales reflect the degree of homogeneity of the source. This simple geometric interpretation allows an easy localization and characterization of point sources. At the basis of this result is the construction of wavelets that behave nicely under the harmonic continuation. As we have shown, a large family of such wavelets exists.

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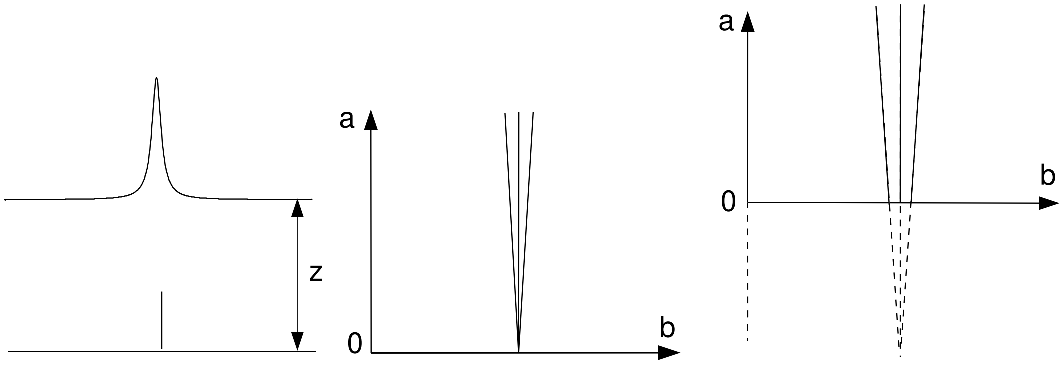

Figure 1. Wavelet transforms of singularities versus wavelet transforms of potential fields. The wavelet transforms (middle) of singularities (lower left) possess lines of maxima, of zero crossings,... which converge on the borderline of the (a, b) half-space at the location of the singularity. The wavelet transforms (right) of potential fields (upper left) due to singular sources (e.g. point mass, lower left) and measured at a distance z possess lines of maxima, of zero crossings,... which converge outside the (a, b) half-space at a distance z from the borderline. This property is satisfied only for a particular class of analyzing wavelets introduced in the remaining of this paper.

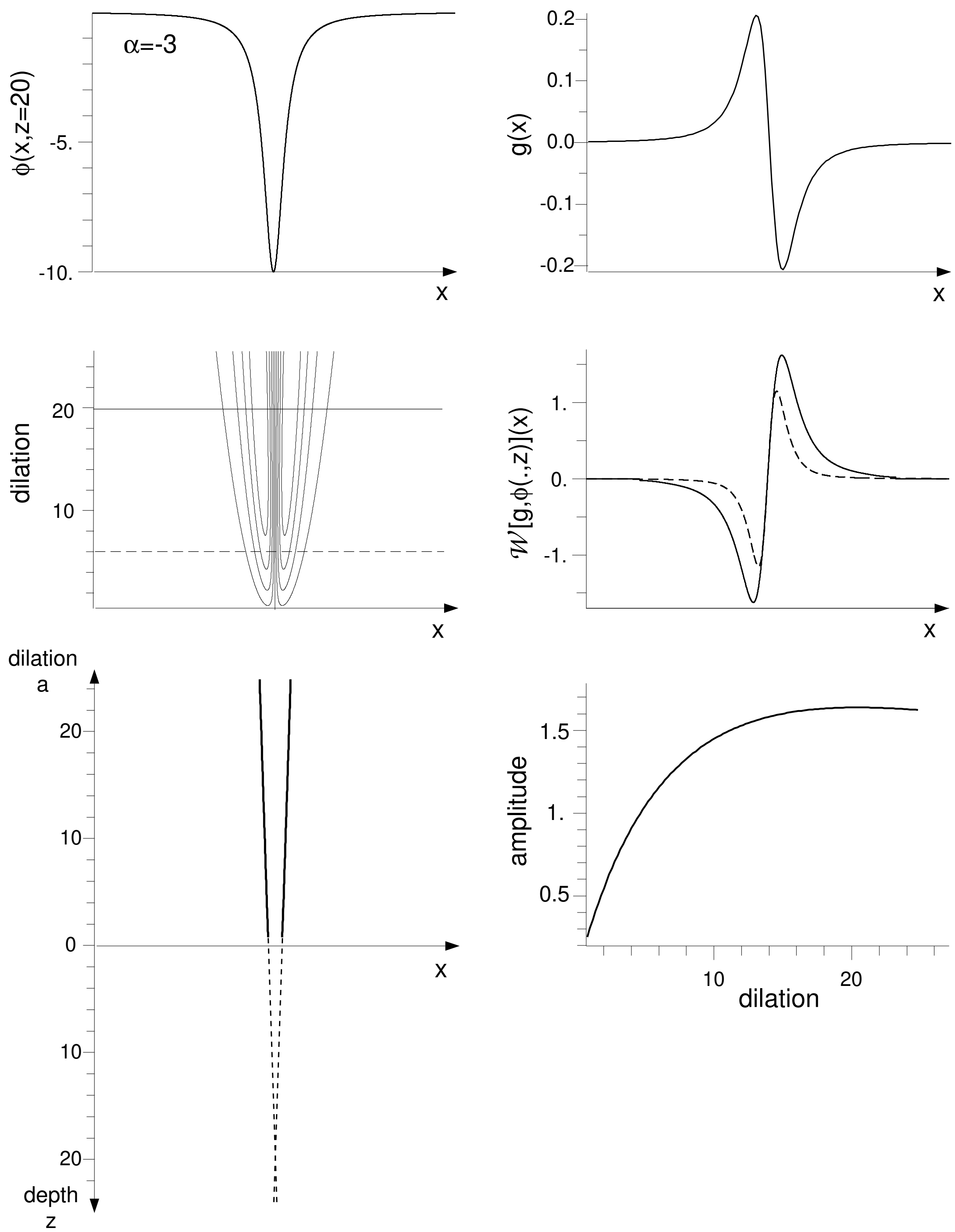

Figure 2. Wavelet analysis of the potential field measured at $z = 20$ and created by a homogeneous source with $\alpha = -3$ located at the origin (upper-left corner). Middle-left: wavelet transform of the field obtained with the analyzing wavelet shown at the upper-right corner and given by equation (75). Middle-right: two voices of the wavelet transform corresponding to $a = 6$ (dashed line) and $a = 20$ (solid line). Lower-left: the intersection of the straight lines formed by the extrema of the voices of the wavelet transform is at the source location. Lower-right: the variation of the amplitude of the wavelet transform along any line of extrema is controlled by both the depth z and the homogeneity α of the source (see equations 73 and 76).

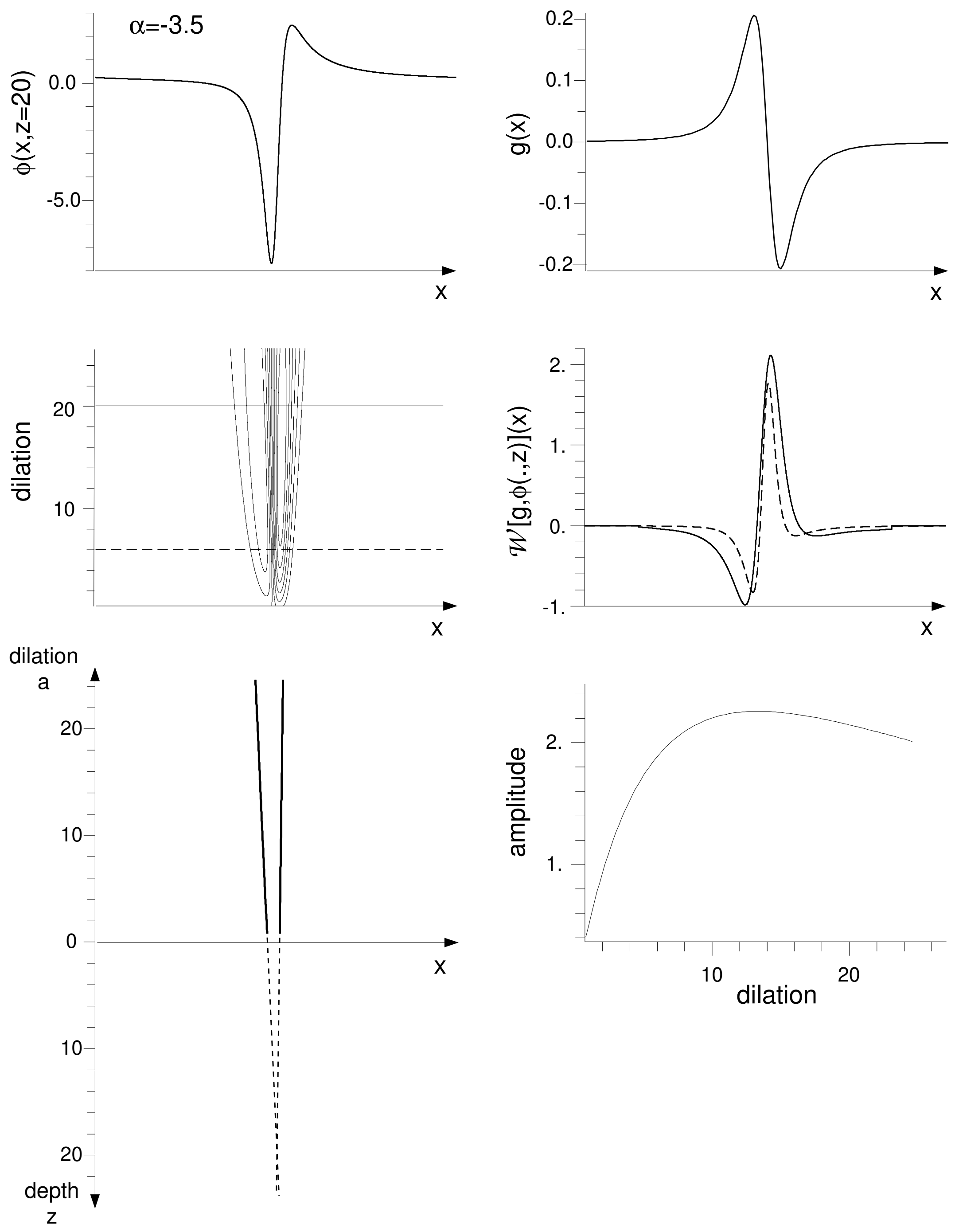

Figure 3. Same as figure 2 for $\alpha = -3.5$.

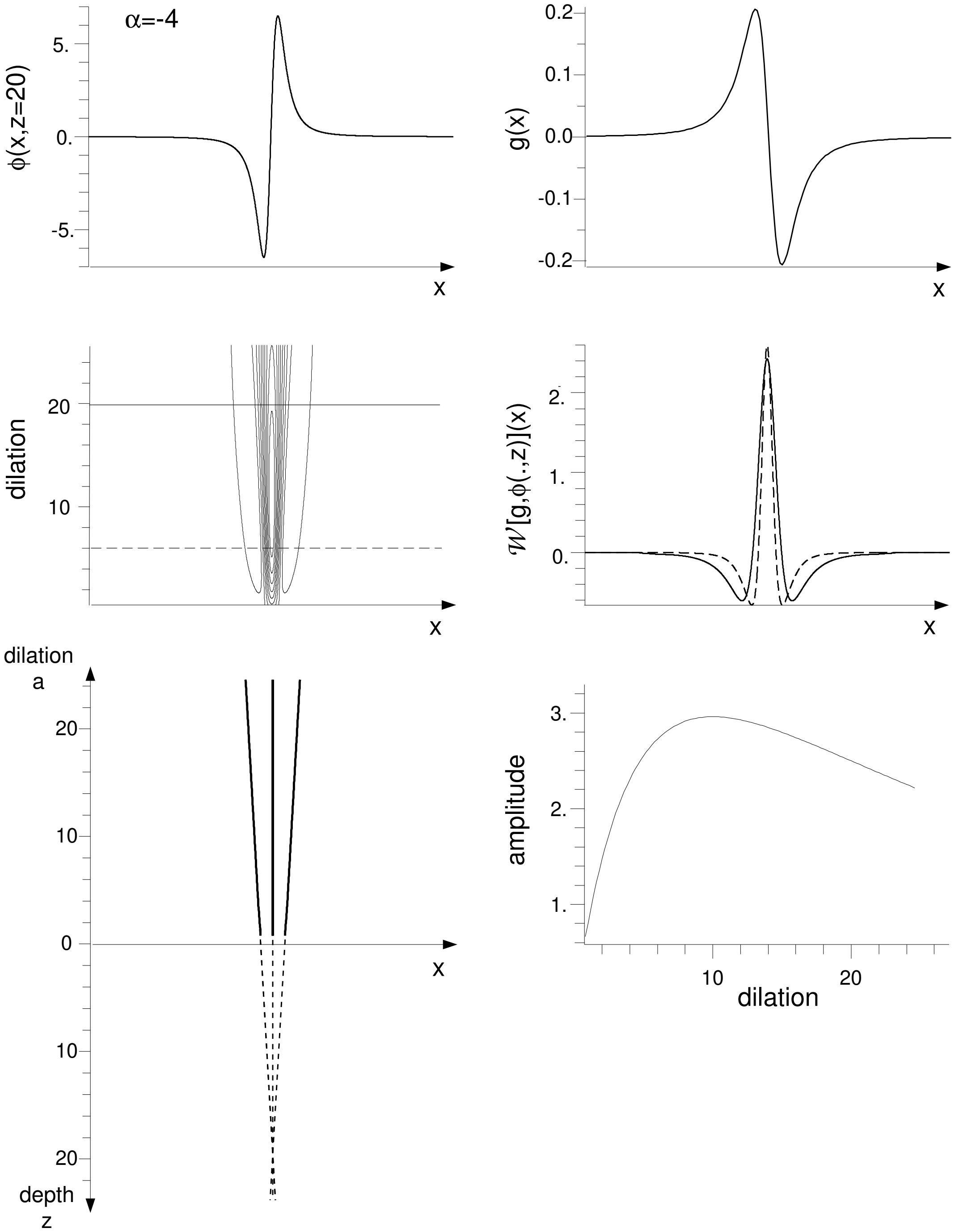

Figure 4. Same as figure 2 for $\alpha = -4$.