Metric structures of inviscid flows

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Abstract

An intrinsic metric tensor, a flat connexion and the corresponding distance-like function are constructed in the configuration space formed by the velocity field **and** the thermodynamic variables of an inviscid fluid. The kinetic-energy norm is obtained as a limiting case; all physical quantities are Galilean invariant. Explicit expressions are given for the case of an ideal gas. The flat connexion is **not** metric-compatible. These results are achieved by applying the formalism of statistical manifolds [1, 2] to the statistical mechanics of a moving fluid.

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1 Introduction

Typically, the configuration of a simple fluid at a certain time t is described by its density field $\rho(\vec{x},t)$, its temperature field $T(\vec{x},t)$ and its velocity field $\vec{u}(\vec{x},t)$ where $\vec{x}$ is the position vector in a d-dimensional space. This d-dimensional space is often the everyday three-dimensional Euclidean space or a two-dimensional surface embedded in three-dimensional space, e.g., a plane or a sphere. In that d-dimensional space, it is possible to calculate the square of the distance between two nearby positions, say $\vec{x}$ and $\vec{x}+d\vec{x}$, in the form of an expression

$$M_{\lambda\nu}(\vec{x})dx^{\lambda}dx^{\nu} \tag{1}$$

where $M_{\lambda\nu}(\vec{x})$ is the metric tensor, dx^{λ} is the λ-component of $d\vec{x}$ in some coordinate system, $1 \leq \lambda$ and $\nu \leq d$, and the convention of summation over repeated lower and upper indices holds. For example, on a two-dimensional ($d=2$) sphere of radius r, in the coordinate system defined by the angles ϑ and ϕ (latitude and longitude), the non-vanishing components of the metric tensor are

$$M_{\vartheta\vartheta} = r^2, \tag{2}$$

$$M_{\phi\phi} = r^2 \sin^2 \vartheta, \tag{3}$$

and the corresponding volume element is $\sqrt{\det{(M_{\lambda\nu})}} = r^2 \sin\vartheta$. Such a metric tensor is also what is required in order to compute the scalar quantity we call the norm of a (tangent) vector, e.g., the norm of the velocity vector $d\vec{x}/dt$ is given by

$$M_{\lambda\nu}(\vec{x}) \frac{dx^\lambda}{dt} \frac{dx^\nu}{dt}. \tag{4}$$

Similarly, one can compute the angle between two (tangent) vectors, e.g., two velocities, at each position in the $\vec{x}$-space.

The situation in the configuration space of a fluid with coordinates $(\rho, T, \vec{u})$ is totally different from that in the $\vec{x}$-spaces described above as long as it is not known how to introduce, in a natural and intrinsic way, a geometric structure, e.g., something similar to the metric tensor $M_{\lambda\nu}(\vec{x})$ in (1) and (4). Such a lack of geometric structures imposes many restrictions on the kind of computations that one can perform: it is impossible to talk of "the distance" between two states of the fluid, i.e., between two positions in the space $(\rho, T, \vec{u})$; neither is it possible to talk of "the norm" of the vector formed by the rate of change of the dynamical variables $(d\rho/dt, dT/dt, d\vec{u}/dt)$; it is not possible to consider the angle between two such vectors; there is no volume element defined in configuration space, therefore, it does not make sense to talk about "the density" of a distribution of points in that space; etc. These limitations are too restrictive since, to name but a few examples: 1) A norm is needed when studying the (in)stability of flows, especially when the linear growth of the perturbations is only transient [3]; 2) The angle between two directions is required in order to determine how strongly a given perturbation projects along an optimal perturbation [4]; 3) A volume element is needed in order to determine the density distribution of ensemble simulations of flows [5]. Therefore, it is often opted to circumvent these limitations by introducing an acceptable, if somewhat arbitrary, metric tensor, e.g., the kinetic-energy metric in the case of incompressible, isentropic flows [3]. Such a situation is not unique to fluid mechanics: often our knowledge of a dynamical system is limited to the variables that define its phase space and to the equations that determine the time evolution of these variables while any information on the geometry of the phase space is lacking.

In this article it is shown how to generate, in a natural way, not only the analogue of the spatial metric tensor $M_{\lambda\nu}(\vec{x})$ but also a rich geometric structure in the configuration space of an inviscid, moving fluid starting from the fluid's probability distribution. The derivation of these results makes use of ideas and techniques from statistical mechanics, see e.g., [6, 7] and from statistical manifolds theory, see e.g., [1, 2], a particular branch of differential geometry, see e.g., [8, 9]. The presentation

is as self-contained as possible. A complete description of the required background material, however, was not attempted; detailed bibliographic references are given for the benefit of those readers willing to go deeper into some aspects.
The paper is structured as follows: In Section 2, the probability density of a moving fluid in local thermal equilibrium is presented. In Section 3, some fundamental concepts of the theory of statistical manifolds are introduced; additional information is given in Appendix A. In Section 4 these ideas are applied to the probability density ofa moving fluid, the natural metric tensor in the fluid's configuration space is derived and an application is described. In Section 5, it is shown that the probability density generates a flat, non-Riemannian, geometry in the configuration space. Basic concepts on flatness of affine connexions) are reviewed in Appendix B. Based on this flatness, a distance-like function is defined in Section 6. The computed metric and distance-like function are Galilean invariant, as they should. When the thermodynamic variables can be ignored, both the metric and the distance-like function reduce to the above-mentioned kinetic-energy norm times a conformal factor. For the purpose of illustration, the expressions corresponding to an ideal gas are explicitly given in Sections 4 and 6. In Section 7, we comment on related work done by others; in Appendix C, the differences between the approach developed in the present paper and a particular Hamiltonian formulation of fluid dynamics are pointed out. Finally, in Section 8, we review the results and discuss their generalizations, e.g., to systems far from thermal equilibrium.

2 Statistical mechanics of a moving fluid

The first step in our construction of an intrinsic geometric structure in the $(\rho, T, \vec{u})$ space consists in associating a probability density to these variables. In this Section we derive the fluid's probability distribution by applying statistical mechanics to a moving fluid in local thermal equilibrium.

Consider a simple fluid characterized, on a macroscopic level, by its density field $\rho(\vec{x}, t)$, temperature field $T(\vec{x}, t)$ and velocity field $\vec{u}(\vec{x}, t)$; from here onwards, we do not express the $(\vec{x}, t)$-dependence explicitly. We adopt the standard assumptions that make possible the derivation of the Euler equations and, less rigorously, of the Navier-Stokes equations from local thermal equilibrium statistical mechanics, see, e.g., [10, Chapters 1 to 3 of Part 1] and [11, Chapters 1 to 3], to wit: One considers systems in which the above-mentioned fields vary on length scales that are much larger than the intermolecular distances and on time scales that are much larger than the microscopic time scales needed for local thermal equilibration. Then the space can be divided into small volume elements ΔV with a typical size much smaller than the length scales of the macroscopic fields but much larger than the intermolecular distances so that the statistical mechanical description applies to them, i.e., on the length scales of ΔV, the fluid is in local thermal equilibrium. The dynamics of the fluid is assumed to be such that the following extensive quantities are conserved: a) the number of (indistinguishable) particles N, b) the total momentum of these particles

$$\vec{M} := m \sum_{k}^{N} \vec{v}_k, \tag{5}$$

where m is the mass of the particles, $\vec{v}_k$ is the velocity of the k-th particle, and c) their total energy

$$E := \sum_{k}^{N} \left(\frac{1}{2} m |\vec{v}_k|^2 + \frac{1}{2} \sum_{l \neq k}^{N} V_{kl} \right), \tag{6}$$

V_{kl} being the potential interaction between particles k and l. (The symbol to the left of $:=$ is defined by the expression to the right.)

Let us denote the conserved quantities by $\{H_i(\xi_N) \,|i = 1, \ldots, s\}$ where $\xi_N := \{\vec{x}_1, \ldots, \vec{x}_N, m\vec{v}_1, \ldots, m\vec{v}_N \,|N = 1, \ldots, \infty\}$ and the values of $\{H_i\}$ in a thermal equilibrium state by $\{\eta_i\}$. These two elements, i.e., the expressions defining the conserved quantities $\{H_i\}$ as functions of the microscopic variables $\{\xi_N\}$ and the values $\{\eta_i\}$ characterizing the macroscopic state, are the essential ingredients needed for the statistical mechanical description of the system. Statistical mechanics establishes that the probability density in the phase space ξ_N, of a system in ΔV, in thermal equilibrium with and free to exchange particles, momentum and energy with a surrounding thermal bath is

$$p(\xi_N; \theta) := \exp\left[\theta^i H_i(\xi_N)\right] / c^N N! \mathcal{Z}(\theta), \tag{7}$$

see, e.g., [6, Chapter 4], [7, Chapters 9, Section B.3]. We use the standard shorthand conventions: repeated lower and upper indices are summed up; θ stands for $\{\theta^i | i = 1, \ldots, s\}$. The $\mathcal{Z}(\theta)$ in the denominator, called the grand partition function, is the normalizing factor needed to make sure that $\sum_N^\infty \int d\xi_N \, p(\xi_N; \theta) = 1$, i.e.,

$$\mathcal{Z}(\theta) := \sum_{N}^{\infty} \int d\xi_N \, \exp\left[\theta^i H_i(\xi_N)\right] / c^N N!, \tag{8}$$

the constant c makes each of the terms contributing to $\mathcal{Z}$ dimensionless. From (7) and (8) we see that the average values of the conserved quantities $\{H_i(\xi_N)\}$, which we call $\{\eta_i\}$, can be written as

$$\eta_i(\theta) = \frac{\partial \ln \mathcal{Z}(\theta)}{\partial \theta^i}, \quad i = 1, \ldots, s. \tag{9}$$

Since the thermal equilibrium state of a system with s conserved quantities is completely characterized by their macroscopic values $\{\eta_i\}$, the last expression makes evident that the whole thermodynamics can be obtained from the grand partition function $\mathcal{Z}(\theta)$. In this way, one identifies $\ln \mathcal{Z}(\theta)$ as the thermodynamic potential which is minimized at fixed values of the θ-variables and the θ-variables as the intensive thermodynamic parameters (like temperature, pressure, chemical potential, etc). Since the system in ΔV is in thermal equilibrium with its surroundings, the values of these intensive variables must be equal to those of its thermal bath, see, e.g., [6, Chapter 4], [7, Chapters 9, Section B.3].

In the case of a moving fluid, the conserved quantities are $(N, E, \vec{M})$, confer (5) and (6), so that $s = d + 2$ and the probability density (7) reads

$$p(N, E, \vec{M}; \gamma, -\beta, \vec{\kappa}) = \exp[\gamma N - \beta E + \vec{\kappa} \cdot \vec{M}]/c^N N! \mathcal{Z}(\gamma, -\beta, \vec{\kappa}), \tag{10}$$

i.e., $\theta = (\gamma, -\beta, \vec{\kappa})$. It follows then that the average, macroscopic values of $(N, E, \vec{M})$ are given by

$$\langle N \rangle = \frac{\partial \ln \mathcal{Z}(\gamma, -\beta, \vec{\kappa})}{\partial \gamma} \tag{11}$$

$$\langle E \rangle = -\frac{\partial \ln \mathcal{Z}(\gamma, -\beta, \vec{\kappa})}{\partial \beta} \tag{12}$$

$$\langle M_\lambda \rangle = \frac{\partial \ln \mathcal{Z}(\gamma, -\beta, \vec{\kappa})}{\partial \kappa^\lambda}, \quad \lambda = 1, \ldots, d, \tag{13}$$

where the pointed brackets indicate average over the probability distribution (10). It is convenient to introduce the macroscopic velocity of the fluid $\vec{u}$ by

$$m \langle N \rangle \vec{u} := \langle \vec{M} \rangle \tag{14}$$

and the specific internal energy $\epsilon(\gamma, \beta, \vec{\kappa})$ as

$$\langle N \rangle (\epsilon + \frac{m}{2} u^2) := \langle E \rangle. \tag{15}$$

Relating these expressions to the thermodynamics of the system, one identifies $\ln \mathcal{Z}(\gamma, -\beta, \vec{\kappa})$ as the thermodynamic grand potential, i.e.,

$$\ln \mathcal{Z}(\gamma, -\beta, \vec{\kappa}) = \beta \Delta V P, \tag{16}$$

where $P = P(\gamma, \beta, \vec{\kappa})$ is the pressure, β as the inverse local temperature, $\beta \equiv (kT)^{-1}$, k is Boltzmann's constant and γ as $\gamma = \beta \mu$ with μ the local chemical potential; see, e.g., [6, Chapter 4], [7, Chapters 9, Section B.3]. Due to the simple dependence of $\vec{M}$

and of E upon the particles' velocities $\vec{v}_k$, confer (5) and (6), the integrals over the velocities in (8) can be performed; from (13) and (14) one obtains then that

$$\vec{\kappa} = \beta \vec{u} \tag{17}$$

and that

$$\mathcal{Z}(\gamma, -\beta, \vec{\kappa}) = \mathcal{Z}(\gamma - \frac{m\kappa^2}{2\beta}, -\beta, \vec{0}). \tag{18}$$

Galilean invariance implies then that

$$\gamma = \bar{\gamma} + \frac{m\kappa^2}{2\beta}, \tag{19}$$

where $\bar{\gamma}$ is the value of γ for the same system at rest. Making use of (17), one sees then that $\mu = \bar{\mu} + mu^2/2$, i.e., the chemical potential of the fluid moving with velocity $\vec{u}$ is shifted by an amount $mu^2/2$ with respect to $\bar{\mu}$, the chemical potential of the fluid at rest.

In the theory of statistical manifolds [1, 2] it is shown that a probability density like $p(\xi_N; \theta)$ in (7) induces a geometric structure in the parameter-space θ, i.e., in the space of the intensive thermodynamic variables. This is worked out in the following Sections.

3 Statistical manifolds

Given two probability densities, say $p(\xi_N, \theta_1)$ and $p(\xi_N, \theta_2)$, how should one quantify their (dis)similarity? Or given three distributions corresponding to θ_1, θ_2 and θ_3 respectively, is it possible to determine which pair of distributions is "closer" than the two other pairs? These and related questions have been extensively studied in statistics, see, e.g., [12, pages 290–296] and [13]. In this Section we give a fleeting overview of some concepts and results from the theory of statistical manifolds that we apply later to a moving fluid.

If $D(\theta_1, \theta_2)$ is a scalar quantity measuring the difference between two probability densities $p(\xi_N, \theta_1)$ and $p(\xi_N, \theta_2)$, then it should satisfy:
1) $D(\theta_1, \theta_2) \geq 0$, the equality holding if, and only if, $p(\xi_N, \theta_1) \equiv p(\xi_N, \theta_2)$,
2) $D(\theta_1, \theta_2)$ is sufficiently continuous in $p(\xi_N, \theta_1)$ and $p(\xi_N, \theta_2)$.
In the statistical literature such measures are often called divergences or contrast functionals; in order to avoid any possible confusion with the usual mathematical meaning of divergence, we shall keep to the second name.

We shall impose two additional constraints:
3) The value of $D(\theta_1, \theta_2)$ should be independent of our choice of coordinates in ξ_N-space, i.e., it should be invariant under general coordinate transformations in the space of the random variables.
4) We shall consider only contrast functionals that are local in the random variables $\{\xi_N\}$[1].
In order to satisfy these constraints, it is sufficient that the $D(\theta_1, \theta_2)$s be of the following form [1], [13, pp. 349–350],

$$D(\theta_1, \theta_2) = \int d\xi_N \, p(\xi_N, \theta_2) f\left(\frac{p(\xi_N, \theta_1)}{p(\xi_N, \theta_2)}\right), \tag{20}$$

where the function f must be sufficiently smooth and should satisfy $f(1) = 0$ and $f(t) - f'(1)(t-1) > 0$ if $t \neq 1$. Without any loss of generality, one normalizes f such that $f''(1) = 1$.

Consider next two distributions that are infinitesimally close, i.e., θ and $\theta + d\theta$. It turns out that, up to third order in $d\theta$, the corresponding contrast is given by [1, theorem 3.10]

$$D(\theta, \theta + d\theta) = \frac{1}{2} g_{ij}(\theta)\, d\theta^i\, d\theta^j + \frac{1}{2}\left\{[i, j; k] + \frac{\alpha}{3!} T_{ijk}(\theta)\right\} d\theta^i\, d\theta^j\, d\theta^k + O(d\theta^4), \tag{21}$$

where the components of $g_{ij}(\theta)$, known as the Fisher tensor [14, 15], are given by

$$g_{ij}(\theta) := \left\langle \frac{\partial \ln p}{\partial \theta^i} \frac{\partial \ln p}{\partial \theta^j} \right\rangle, \tag{22}$$

the pointed brackets indicate an average taken over the $p(\xi_N, \theta)$ distribution. From this expression it is evident that g_{ij} is symmetric in the indices i and j and that, since it is an addition of products of covariant vectors, under a general coordinates transformation it transforms like the product of two covariant vectors, i.e., it is a covariant tensor[2]. Moreover, any sensible choice of the parameters θ ensures that it is non-singular.
The first term in the coefficient of the third-order contribution to the expansion (21), $[i, j; k]$, is given by

$$[i, j; k] := \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial \theta^j} + \frac{\partial g_{jk}}{\partial \theta^i} - \frac{\partial g_{ij}}{\partial \theta^k}\right), \tag{23}$$

[1]C.R. Rao has considered also non-local contrast functionals, [2, pp. 226–227].
[2]For a more detailed meaning of these geometric concepts, see, e.g., [8, 9].

its meaning is discussed in Appendix B and in Section 5. The second term in the third-order coefficient is a totally symmetric covariant tensor, often called the skewness,

$$T_{ijk}(\theta) := \left\langle \frac{\partial \ln p}{\partial \theta^i} \frac{\partial \ln p}{\partial \theta^j} \frac{\partial \ln p}{\partial \theta^k} \right\rangle. \tag{24}$$

Both symmetric tensors, g_{ij} and T_{ijk}, will play an important role in the sequel.
The constant α in (21) is given by $\alpha := 2f'''(1) + 3$.

A number of points are worth noticing:
- The symmetric contravariant tensor $g_{ij}(\theta)$ has all the credentials for being the natural metric tensor in the θ-space. Besides the clear meaning ensuing from (21), it has an important statistical significance [15, 16] that is described in Appendix A.
- Up to second order, the expansion (21) is independent of the particular choice of the function f that defines the contrast D, confer (20). (Remember that f has been normalized so that $f''(1) = 1$.)
- Up to third order, the whole dependence upon the function f has been brought back to a single number, namely, to α.

4 Fisher's metric

In this and the following Sections, we apply the theory of statistical manifolds to the probability density $p(\xi_N, \theta)$ of a moving fluid.
As shown in the previous Section, the Fisher tensor appears naturally as the metric in θ-space. From (10) it follows that in the coordinate system $(\theta^1, \ldots, \theta^{d+2}) = (\gamma, -\beta, \vec{\kappa})$, the metric tensor (22) reads,

$$g_{ij}(\theta) = \frac{\partial^2 \ln \mathcal{Z}(\theta)}{\partial \theta^i \partial \theta^j} = \Delta V \frac{\partial^2 \beta P}{\partial \theta^i \partial \theta^j}, \tag{25}$$

see also (49) in Appendix A. We see then that the metric elements are expressed in terms of the derivatives of the thermodynamic potential. Moreover, from (11–13) or from (9) one has that, in the $(\gamma, -\beta, \vec{\kappa})$ coordinate system,

$$d\eta_i = g_{ij}(\theta) d\theta^j, \tag{26}$$

where η_i are the average values of the extensive quantities given in (11–13). I.e., in this coordinate system, the lowering of the indices corresponds to passing from the intensive variables θ to the extensive ones η. The inverse transformation is achieved by raising the indices by means of the inverse metric $g^{ij}(\theta)$.

It is instructive to work out the metric tensor for the case of an ideal gas, i.e., for a vanishing intermolecular potential V_{kl} in (6). In this case all the integrals over the molecular velocities $\vec{v}_k$ and over their positions $\vec{x}_k$ in (8) can be performed and one finds that

$$\ln \mathcal{Z}(\gamma, -\beta, \vec{\kappa}) = \Delta V\, \beta^{-d/2} \exp(\frac{m\kappa^2}{2\beta} - \gamma). \tag{27}$$

It is convenient to express the results in the coordinate system $(\rho, \beta, \vec{u})$, where ρ is the mass density, i.e., $\rho := m\langle N\rangle/\Delta V$. The non-vanishing elements of the metric are,

$$\begin{aligned} g_{\rho\rho} &= \Delta V (m\rho)^{-1}, \\ g_{\beta\beta} &= \Delta V \frac{d}{2}\frac{\rho}{m\beta^2}, \\ g_{u_i u_j} &= \Delta V \rho\beta\delta_{ij}. \end{aligned} \tag{28}$$

Therefore, each volume element ΔV contributes a squared distance $(dL)^2$ between two states $(\rho, \beta, \vec{u})$ and $(\rho + d\rho, \beta + d\beta, \vec{u} + d\vec{u})$ given by

$$\frac{\Delta V \rho}{m}\left[\left(\frac{d\rho}{\rho}\right)^2 + \frac{d}{2}\left(\frac{d\beta}{\beta}\right)^2 + m\beta\, d\vec{u}\cdot d\vec{u}\right]. \tag{29}$$

This expression agrees with our expectations: 1) The metric coefficients are independent of $\vec{u}$, as demanded by Galilean invariance[3]; 2) Since no prefered or external scales are available and taking into account the Galilean invariance, the differentials must always appear in the form of $d\rho/\rho$ and $d\beta/\beta$; the gas temperature determines the scale for measuring the kinetic energy associated with $d\vec{u}$; 3) Cross-terms like $d\rho\, d\vec{u}$ and $d\beta\, d\vec{u}$ cannot appear due to the rotational symmetry of the gas and 4) The factor multiplying the square brackets is just the number of particles in ΔV.
What could not have been guessed beforehand, is the absence of the cross-term $d\rho\, d\beta$ and the precise form of the coefficients.

One of the simplest applications of a metric tensor is the computation of the norm of a tangent vector, as in (4). Now we can do this in the configuration space of the moving fluid, i.e., we can compute the norm of the local rate of evolution of our sytem, that we denote by F, as

$$F^2 := g_{ij}\dot{\theta}^i\dot{\theta}^j, \tag{30}$$

where the dots indicate the time derivatives of the corresponding quantities as given by the Euler equations. In order to distinguish between this generalized velocity F and the standard velocities $\vec{v}_k$ and $\vec{u}$, we shall call this quantity *the rapidity.* The

[3]The Galilean invariance of the metric in the general case (non-ideal gas) is proven in Section 6, confer (46).

rapidity squared is a scalar with dimension $[\text{time}]^{-2}$. From (26) we see that it can also be written as

$$F^2 = \dot{\eta}_i \dot{\theta}^i, \tag{31}$$

i.e., it is the contraction between the velocities of the intensive variables with the velocities of their corresponding conjugate extensive variables.

Let us compute the rapidity associated with the material derivatives in the Euler equations with no external forcing, i.e.,

$$\begin{aligned} \frac{d\rho}{dt} &= -\rho \operatorname{div} \vec{u}, \\ \rho c_v \frac{dT}{dt} &= -P \operatorname{div} \vec{u}, \\ \rho \frac{d\vec{u}}{dt} &= -\vec{\nabla} P, \end{aligned} \tag{32}$$

where c_v is the specific heat at constant volume. In the case of an ideal gas, i.e., making use of (28) and inserting (32) into (30) leads to

$$F^2 = \frac{\Delta V \rho}{m} \left[\left(1 + \frac{d}{2} \left(\frac{R'}{c_v} \right)^2 \right) (\operatorname{div} \vec{u})^2 + R'T \left| \vec{\nabla} \ln(\rho T) \right|^2 \right], \tag{33}$$

where $R' := k/m$ is the specific constant of the gas.

5 Flatness of the manifold

Another standard application of a metric is the determination not only of an infinitesimal distance, as in (29), but also of finite distances, say $L(\mathbf{1}, \mathbf{2})$ between positions $\mathbf{1} := (\rho_1, -\beta_1, \vec{u}_1)$ and $\mathbf{2} := (\rho_2, -\beta_2, \vec{u}_2)$. This distance is defined as the length of the shortest path connecting $\mathbf{1}$ with $\mathbf{2}$,

$$L(\mathbf{1}, \mathbf{2}) = \min \int_1^2 d\tau \sqrt{g_{ij}(\theta) \frac{d\theta^i}{d\tau} \frac{d\theta^j}{d\tau}}, \tag{34}$$

where the minimum is taken over all paths from $\mathbf{1}$ to $\mathbf{2}$ and τ is any parametrization of these paths.

This calculation is not a simple one when there is no coordinate system in which the metric tensor $g_{ij}(\theta)$ takes a simple form; this happens in the case of our moving fluid since, as we will see in this Section, (35) and comments thereafter, (28) is *not* flat. Luckily, it turns out that the geometry of statistical manifolds offers more interesting and useful possibilities than (34); this is particularly so in the case of a probability density of the form (10) as is our case. In order to realize this, we need some concepts from differential geometry; for the sake of completeness, these concepts are listed in Appendix B.

Two simple calculations indicate that the connexions described in Appendix B may play an important role in the case of our moving fluid:
Let us compute first the $[ij;k]$ connexion compatible with the Fisher metric (25); from (50) one has that, in the coordinate system $(\theta^1, \ldots, \theta^{d+2}) = (\gamma, -\beta, \vec{\kappa})$,

$$[ij;k] = \frac{1}{2}\frac{\partial^3 \ln \mathcal{Z}(\theta)}{\partial\theta^i \partial\theta^j \partial\theta^k}. \tag{35}$$

One can compute the corresponding curvature tensor and check that, even for the ideal gas metric (28)-(29), this connexion is *not* flat.
Next, let us compute the skewness tensor T_{ijk} generated by (10), confer (24); one finds that

$$T_{ijk}(\theta) = \frac{\partial^3 \ln \mathcal{Z}(\theta)}{\partial\theta^i \partial\theta^j \partial\theta^k}. \tag{36}$$

From these two last results, we see that by taking $[ij;k] - (1/2)T_{ijk}$ one gets a connexion that vanishes identically in this coordinate system, i.e., *this connexion is flat.* It should be noted that the existence of a flat connexion implies that there is a prefered family of coordinate systems, in our case, the system of the intensive variables θ and their linear combinations. The physical relevance of these variables becomes even more evident when one considers the phenomenon of phase-coexistence, e.g., liquid-vapour coexistence: coexisting phases correspond to flat portions of the thermodynamic potential surface only in terms of the intensive variables.

Similarly, one can check that also $[ij;k]+(1/2)T_{ijk}$, notice the change in sign, vanishes identically in the *extensive* variables coordinate system η, i.e., also this connexion is flat. Connexions of the form $[ij;k] - (\alpha/2)T_{ijk}$ play an important role in the theory of statistical manifolds; they were introduced by Chentsov [17], Efron [18] and Amari [1, Chapter 3]. In the third reference it is shown that if such a connexion is flat for a certain α then it is also flat for $-\alpha$. In fact, the coordinate systems $\theta = (\gamma, -\beta, \vec{\kappa})$ and $\eta = (\langle N\rangle, \langle H\rangle, \langle\vec{M}\rangle)$ play totally symmetric roles: it can be shown [1, Theorems 3.4 and 3.5] that g^{ij}, the inverse of the metric tensor, is also given by the second-order derivatives of a function $\Phi(\eta)$, this time with respect to the extensive variables η, i.e.,

$$g^{ij}(\eta) = \frac{\partial^2 \Phi(\eta)}{\partial\eta_i \partial\eta_j}, \tag{37}$$

that the function $\Phi(\eta)$ is nothing else but the Legendre transform of $\ln \mathcal{Z}(\theta)$ and that the coordinate systems are related to each other as in a Legendre transformation, i.e.,

$$\theta^i(\eta) = \frac{\partial \Phi}{\partial \eta_i}, \tag{38}$$

$$\Psi(\theta) + \Phi(\eta) - \theta^i \cdot \eta_i = 0, \tag{39}$$

where we have introduced $\Psi := \ln \mathcal{Z}$, confer also (9). Recall that all functions are assumed to be sufficiently smooth so that (39) is good enough for the definition of the Legendre transform; for more complicated situations, see [19].

We have shown then that the probability density of the moving fluid generates not only a metric tensor (25) but also two flat connexions, namely $[ij;k] \pm (1/2)T_{ijk}$ and that all this is closely related to a Legendre transformation of thermodynamic potentials and variables. Some interesting consequences of these facts are presented in the next Section.

6 Distance-like function

While the computation of the distance L in (34) implies difficult calculations, it was pointed out by Amari [1, Section 3.5] that, when the connexions $[ij;k] \pm (1/2)T_{ijk}$ are flat, it is possible to define a distance-like function $D(\mathbf{1},\mathbf{2})$ between positions $\mathbf{1} := (\gamma_1, -\beta_1, \vec{\kappa}_1)$ and $\mathbf{2} := (\gamma_2, -\beta_2, \vec{\kappa}_2)$ as follows

$$D(\mathbf{1},\mathbf{2}) := \Psi(\mathbf{1}) + \Phi(\mathbf{2}) - \theta^i(\mathbf{1}) \cdot \eta_i(\mathbf{2}), \tag{40}$$

with $\Psi := \ln \mathcal{Z}$ as in (39), confer also (37), (38) and (9).

Amari has shown that this function shares some essential properties with the usual distance functions, to wit[4]:

$$\begin{gathered} 1) \qquad D(\mathbf{1},\mathbf{2}) \geq 0, \\ \text{the equality holds when, and only when,} \quad \mathbf{1} = \mathbf{2}, \end{gathered} \tag{41}$$

$$\begin{gathered} 2) \qquad D(\theta, \theta + d\theta) = \\ \frac{1}{2}\, g_{ij}(\theta)\, d\theta^i\, d\theta^j + \frac{1}{2}\left\{[i,j;k] + \frac{1}{3!}T_{ijk}(\theta)\right\} d\theta^i\, d\theta^j\, d\theta^k + O(d\theta^4), \end{gathered} \tag{42}$$

$$\begin{gathered} 3) \qquad D(\mathbf{1},\mathbf{2}) = D(\mathbf{1},Q) + D(Q,\mathbf{2}), \\ \text{where } Q \text{ is connected to } \mathbf{1} \text{ by a } \theta\text{−geodesic and to } \mathbf{2} \text{ by an} \\ \eta\text{−geodesic and these two geodesics intersect orthogonally at } Q \end{gathered} \tag{43}$$

$$\begin{gathered} \text{and} \qquad 4) \qquad \text{The } \min_{Q\in\Omega} D(\mathbf{1},Q) \\ \text{is obtained at a point on the boundary of the smooth closed region } \Omega \end{gathered}$$

[4] The first property is a special instance of Fenchel's theorem [19].

that is the projection of **1** along a $\theta-$geodesic orthogonal to the boundary of Ω. (44)

The θ-geodesics above is a linear interpolation from $\theta(\mathbf{1})$ to $\theta(Q)$ while the η-geodesic is a linear interpolation from $\eta(Q)$ to $\eta(\mathbf{2})$. The third property is a generalization of Pythagoras' theorem to a space with two biorthogonal coordinate bases, related to θ and η in our case. Analogously, the fourth property generalizes the notion of projection to such a space. The reader should refer to [1, Section 3.5] for the proofs of the properties listed above.

There is one important difference between the usual distance functions and the D defined above: In general, the D function is *not* symmetric, i.e., $D(\mathbf{1},\mathbf{2}) \neq D(\mathbf{2},\mathbf{1})$. This asymmetry is associated with the fact that the path going from **1** *first* along an η-geodesic to Q' and *then* along an orthogonal θ-geodesic to **2** is, in general, different from the path described in (43). If one is set on defining a symmetric distance, then taking the minimum of the values $D(\mathbf{1},\mathbf{2})$ and $D(\mathbf{2},\mathbf{1})$ seems to be the most satisfactory solution.

From properties (41) and (42), we recognize that this D may belong to the family of contrast functionals discussed in Section 3. In fact, using (9), (25), (38) and (39), one finds that $D(\mathbf{1},\mathbf{2})$ in (40) can be written as

$$D(\mathbf{1},\mathbf{2}) = (\beta_2-\beta_1)\langle H\rangle_1 - (\vec{\kappa}_2-\vec{\kappa}_1)\cdot\langle\vec{M}\rangle_1 - (\gamma_2-\gamma_1)\langle N\rangle_1 + \Delta V(\beta_2 P_2 - \beta_1 P_1), \quad (45)$$

where $\langle\cdots\rangle_1$ indicates that an average is taken over $p(N,E,\vec{M};\theta_1)$. One can check then that, in our case, the last expression for $D(\mathbf{1},\mathbf{2})$ is identical to (20) with $f(z) = z\ln z$. The circle is now complete.

The Galilean invariance of $D(\mathbf{1},\mathbf{2})$ follows from introducing (15), (17) and (19) into the last expression above; one finds then that the contrast functional (20) or (40) can be written as

$$D(\mathbf{1},\mathbf{2}) = (\beta_2-\beta_1)\langle N\rangle_1\epsilon_1 + \frac{m}{2}\beta_2|\vec{u}_2-\vec{u}_1|^2\langle N\rangle_1 - (\bar{\gamma}_2-\bar{\gamma}_1)\langle N\rangle_1 + \Delta V(\beta_2 P_2 - \beta_1 P_1). \quad (46)$$

Combining this with (21) and with (42), one sees that both $g_{ij}(\theta)$ and $T_{ijk}(\theta)$ are always independent of $\vec{u}$, i.e., Galilean invariant.

Notice also that when the variations in the thermodynamic variables can be ignored, as is the case in an (effectively) incompressible, isentropic flow, the contrast reduces to $D(\mathbf{1},\mathbf{2}) = (\Delta V\rho\beta/2)|\vec{u}_1-\vec{u}_2|^2$ with $\rho=\rho_1=\rho_2$ and $\beta=\beta_1=\beta_2$, i.e., the kinetic-energy norm times a conformal factor.

In the case of an ideal gas, the contrast function reads:

$$D(\mathbf{1},\mathbf{2}) = \frac{\rho_1}{m}\Delta V\left[\frac{m}{2}\beta_2|\vec{u}_2-\vec{u}_1|^2 + \left(\ln\frac{\rho_1}{\rho_2}+\frac{\rho_2}{\rho_1}-1\right) + \frac{d}{2}\left(\ln\frac{\beta_1}{\beta_2}+\frac{\beta_2}{\beta_1}-1\right)\right]. \quad (47)$$

7 Related work

Before closing, we review and comment some papers that deal with related questions: Weinhold [20] proposed to use the matrix of second-order derivatives of the internal energy as a metric in the space tangent to the equation-of-state surface at an equilibrium point. Some researchers tried then to attach a physical meaning to the curvature tensor derived from the Weinhold's metric-compatible connexion, confer Section 5. This led to a lengthy discussion which has been summarized in [21]; all these researchers ignored the flat connexions $[ij;k] \pm \frac{1}{2}T_{ijk}$ and the associated distance-like function D as discussed in Section 6. In the context of the present article, Weinhold's metric comes close to the Fisher metric (25) while Gilmore's approach, see [21], is closer to the distance-like function D (40).
The Fisher metric was introduced into thermodynamics by Ingarden [22, 23]. Janyszek and Mrugała [24] studied the curvature tensors of the corresponding metric-compatible connexions and tried to associate this curvature with some physical properties; they did not consider the flat connexions $[ij;k] \pm \frac{1}{2}T_{ijk}$ neither the associated distance-like function D.

On the 18th of December 1995, I presented this paper at the Technical University of Berlin. Prof. U. Simon informed me then about affine and projective differential geometry which deals, among other things, with conjugate connexions like $[ij;k] \pm \frac{1}{2}T_{ijk}$ and their flatness. Excellent, up-to-date reviews of this branch of differential geometry are [25] and [26].

There exists a rich literature on the Hamiltonian (symplectic) structure of hydrodynamics that can also be seen as one branch of differential geometry; see, e.g., the review articles [27, 28], the references therein and [29, 30] for another interesting application. This approach is based on the Poisson bracket and it does *not* require a metric tensor. By contraposition, the present article does not use the Poisson bracket and leads to a metric tensor, flat connexions, etc; i.e., the two approaches are not necessarily related and can be seen as complementary. In Arnold's analysis of incompressible flows [31], a metric plays an important role, however, this metric is just the metric of the ambient d-dimensional space that appears in the kinetic energy term, i.e., is not the type of metric developed in the present article. There is one instance of a Hamiltonian structure for fluid dynamics that apparently contains a candidate for a metric tensor in the configuration space of a moving fluid; this instance is discussed in Appendix C.

8 Summary and discussion

In this article, it has been shown that the grand canonical partition function (10) which describes the thermal fluctuations of a moving fluid in local thermal equilibrium generates a natural metric tensor $g_{ij}(\theta)$, two flat connexions $[ij;k] \pm \frac{1}{2}T_{ijk}$ and the corresponding distance-like contrast function D, confer (25), (35), (36) and (40). In the case of an ideal gas, the explicit expressions for these quantities have been given in (29) and in (47). It was also shown that these quantities are Galilean invariant, as they should, and that they reduce to the kinetic-energy norm times a conformal factor when one can ignore variations in variables other than the velocity field, confer (46). Noteworthy aspects of the geometric structure are that the flat connexions are *not* metric-compatible and that the distance-like function D is *not* symmetric. As pointed out by one of the referees, the analysis presented in this article can be applied to relativistic fluids, to plasmas, etc. In fact, some of the results listed above, e.g., the particular form (25) of the metric tensor, as well as (37)–(39) and the flatness of the connexions $[ij;k] \pm \frac{1}{2}T_{ijk}$, are valid for all systems characterized by Gibbs' probability distributions like (7).

The following considerations are relevant with regard to further applications of our approach. We have seen that all the components of the geometric structure are generated from $\ln \mathcal{Z} = \Delta V \beta P$ and since, in the absence of external forces, the equation of state $P = P(\bar{\gamma}, \beta)$ is all that is needed in order to write down the Euler equations of fluid motion, it is then clear that the metric, connexions and contrast function are closely related to the dynamical equations. When the system is far from thermal equilibrium, however, a different approach may be more appropriate. For example, when considering a fluid in a turbulent stationary state, the corresponding probability density should be used. Similarly, when studying the predictability properties of systems in the presence of noise, a time-dependent probability density should be employed [32]. In this sense, the geometric structures we have described are not universal; the specific physical conditions of the system under consideration determine whether a thermal-equilibrium, a stationary-state or a time-dependent probability density is the most appropriate starting point.

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Appendix A: On Fisher's metric

Besides the clear meaning due to (21), the Fisher metric tensor (22) plays an important role in statistics due to the following theorem [15, 16]:
Suppose that we perform n independent measurements of the random variables $(N, E, \vec{M})$ and use them in order to estimate the value of the parameters $(\gamma, -\beta, \vec{\kappa})$ in (10). Denote by $\{\hat{\theta}^i\}$ an unbiased estimate of these parameters. Then, for large enough n, the covariance of these estimated values with respect to the exact values $\{\theta^i\} = (\gamma, -\beta, \vec{\kappa})$ has the following lower bound

$$\mathrm{cov}\ \left[(\hat{\theta}^i - \theta^i)(\hat{\theta}^j - \theta^j)\right] \geq \frac{1}{n} g^{ij}, \tag{48}$$

where (g^{ij}) is the inverse of the matrix (g_{ij}). In other words, given a fixed tolerance error, the larger the distance between two probability distributions as measured by g_{ij}, the smaller the number of measurements needed in order to distinguish between them.

Finally, one should notice the following identity,

$$g_{ij}(\theta) = \left\langle \frac{\partial \ln p}{\partial \theta^i} \frac{\partial \ln p}{\partial \theta^j} \right\rangle = - \left\langle \frac{\partial^2 \ln p}{\partial \theta^i \partial \theta^j} \right\rangle. \tag{49}$$

This identity is obtained as follows: $\int dx\, p(x, \theta) = 1$ implies that $\langle \partial \ln p(x, \theta)/\partial \theta^i \rangle = \int dx\, \partial p(x, \theta)/\partial \theta^i = 0$; taking then the derivative of $\langle \partial \ln p(x, \theta)/\partial \theta^i \rangle$ with respect to θ^j leads to the identity in (49). (It is assumed that the order of derivation with respect to the $\{\theta^i\}$ and integration over x can be interchanged.)

Appendix B: On affine connexions

In some cases, the geometry of a manifold is not completely defined only by its metric tensor but also by another geometric object, called the connexion of the manifold[5], see, e.g., [8, Chapter 4, Sections 28 and 29] and [9, Chapters I–IX]. Briefly stated, the connexion expresses mathematically what it means "to move a vector along a curve in such a way that the vector remains constant", i.e., it defines the parallel transport of a vector.
We list some important properties of the connexions that we need in Section 5:
- Under a general transformation of the coordinates, a connexion does *not* transform like a tensor, i.e., a connexion is *not* a tensor. In particular: a symmetric connexion

[5]When this happens, we speak of a non-Riemannian manifold.

may vanish identically in a particular coordinate system *without* vanishing in a different one while a tensor that vanishes identically in a particular system of coordinates does so in all coordinate systems.

- The result of adding a third-rank tensor to a connexion is a connexion.
- We say that a manifold is *flat* with respect to a connexion when every vector that is parallel transported along every closed path returns to its original condition; otherwise, one says that the manifold is curved.
- A manifold is flat with respect to a connexion if, and only if, it is possible to find a coordinate system in which this connexion vanishes identically.
- Whether a given connexion is flat or not can be determined in a coordinate-system independent way: a connexion is flat if, and only if, it is symmetric in its two first indices, i.e., torsionless, and its curvature tensor vanishes identically.
- The partial derivatives of a tensor are *not* tensors, however, using the connexion it is possible to define the so-called covariant derivatives which are tensors. If the connexion vanishes identically in a given coordinate system then, in that coordinate system, the covariant derivatives coincide with the partial derivatives.
- There is only one connexion symmetric in its two first indices such that the covariant derivatives of the metric tensor vanish identically. If g_{ij} denotes the metric tensor, then this connexion[6] is given by

$$\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial \theta^j} + \frac{\partial g_{jk}}{\partial \theta^i} - \frac{\partial g_{ij}}{\partial \theta^k}\right). \tag{50}$$

This is precisely the $[ij;k]$ appearing in the third-order term in (21), confer (23). This connexion is called the metric-compatible or Levi-Civita connexion. For the proofs of these statements, refer to, e.g., [8, Chapter 4, Sections 28 and 29] and [9, Chapters I–IX].

Appendix C: On the Dubrovin-Novikov approach to flows

In an illuminating article [33], Dubrovin and Novikov introduced a novel Hamiltonian formalism for one-dimensional systems of hydrodynamic type described by s dynamical fields. (An extensive review of related ideas and developments can be found in [34].) One notable aspect of this formalism is the central role played by a symmetric covariant tensor of type (2,0), call it Λ^{ij}, $1 \leq (i,j) \leq s$: when $\det(\Lambda^{ij}) \neq 0$, the Jacobi identity that the Poisson bracket has to satisfy implies that the Levi-Civita connexion generated by $\Lambda_{ij} := (\Lambda^{-1})_{ij}$ must be flat. Moreover, when $\det(\Lambda^{ij}) \neq 0$ and other, relatively mild conditions are satisfied, it is possible to reconstruct (up

[6] Actually, this connexion is obtained by raising the k-index of the $[ij;k]$ in (50), i.e., by $g^{lk}[ij;k]$.

to a constant factor) the tensor Λ^{ij} directly from the hydrodynamical equations of motion [36]. Therefore, in such cases, it is possible to associate with the equations of motion, one metric in the phase space of the fluid: the inverse of (Λ^{ij}). In practice, these symmetric tensors cannot be used as metrics but in few, unrealistic cases due to the following reasons: 1) When the fluid exists in a d-dimensional space, d different symmetric tensors Λ^{ij} must be introduced [35] leading, in the best case, to d different metrics in the fluid phase space; 2) Even in the case of a one-dimensional physical space, one often has that $\det(\Lambda^{ij}) \equiv 0$, this is the case, e.g., of a one-dimensional nonbarotropic fluid [33, 35],[34, p. 59]. Moreover, it should be noticed that the Λ^{ij} tensor is part of the Poisson bracket definition, therefore, different dynamical systems generated by different Hamiltonians, i.e., fluids with different equations of state, but by the same Poisson bracket share the same Λ^{ij}; this property is obviously not desirable when one is trying to identify the geometric structures that characterize and distinguish between different hydrodynamical systems. Similarly, some hydrodynamical equations have a multi-Hamiltonian formulation, i.e., the same set of equations can be generated from different Hamiltonians by means of different Poisson brackets [37, 38]; in such cases, each of these Poisson brackets would lead to a different metric tensor for the same dynamical equations.

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