Emergence of Space-Time on the Planck Scale described as an Unfolding Phase Transition within the Scheme of Dynamical Cellular Networks and Random Graphs

Manfred Requardt

Institut für Theoretische Physik
Universität Göttingen
Bunsenstrasse 9
37073 Göttingen   Germany

Abstract

As in an earlier paper we start from the hypothesis that physics on the Planck scale should be described by means of concepts taken from *discrete mathematics*. This goal is realized by developing a scheme being based on the dynamical evolution of a particular class of *cellular networks* being capable of performing an *unfolding phase transition* from a (presumed) chaotic initial phase towards a new phase which acts as an *attractor* in total phase space and which carries a fine or super structure which is identified as the discrete substratum underlying ordinary continuous space-time (or rather, the physical vacuum). Among other things we analyze the internal structure of certain particular subclusters of nodes/bonds (maximal connected subsimplices, *mss*) which are the fundamental building blocks of this new phase and which are conjectured to correspond to the *physical points* of ordinary space-time. Their mutual entanglement generates a certain near- and far-order, viz. a causal structure within the network which is again set into relation with the topological/metrical and causal/geometrical structure of continuous space-time. The mathematical techniques to be employed consist mainly of a blend of a fair amount of *stochastic mathematics* with several relatively advanced topics of discrete mathematics like the *theory of random graphs* or *combinatorial graph theory*. Our working philosophy is it to create a scenario in which it becomes possible to identify both gravity and quantum theory as the two dominant but

derived(!) aspects of an underlying discrete and more primordial theory (dynamical cellular network) on a much coarser level of resolution, viz. continuous space-time..

1 Introduction

In a previous paper ([1]), starting from the hypothesis that both physics and mathematics are discrete on the Planck scale, we developed a certain framework in form of a class of '*cellular network models*' consisting of cells (nodes) interacting with each other via bonds according to a certain '*local law*' which governs their evolution.

Our personal philosophy in this endeavor is the following: If one wants to go beyond the metric continuum (a possibility which was already contemplated by B.Riemann in [2]), an entirely different and in some respects perhaps even new arsenal of physical/mathematical concepts and calculational tools is called for respectively has to be invented. (Ideas in a spirit similar to ours can e.g. be found in the work of R.Sorkin et al; [3] or C.Isham; [6]. A slightly different line of thought is persued in e.g. [5]; see further remarks at the end of section 2). Among other things one has to be extremely careful not to reimport notions and ideas through the backdoor which carry openly or implicitly a meaning or connotation having its origin in continuum concepts and, more generally, to obey Occam's principle, i.e. stick to as few as possible clear and natural hypotheses.

A case in point is the concept of *dimension* which appears to be, at least traditionally, a typical continuum concept. As ordinary physics can successfully be described by employing a framework which has as one of its cornerstones the postulate of space-time as a *metric continuum*, a serious candidate for an underlying discrete and more primordial substratum shall have the capability to generate a concept like dimension as an '*emergent*' collective quality without(!) having it already among its elementary buildingblocks in one or the other disguise.

We showed in [1] that our class of discrete model theories do have indeed a rich enough structure to accomplish, among other things, this goal. More specifically, scrutinizing what are in fact the crucial ingredients of something like dimension from the point of view of physics, in particular dynamics and interaction in physical systems, we managed to develop a notion of '*generalized dimension*' also in discrete and quite irregular systems and which mirrors both a characteristic and '*intrinsic*' property of the discrete network models under discussion by measuring its '*connectivity*' and, on the other side, does not resort to some sort of underlying dimension of a possible '*embedding space*'. In this sense it is in exactly the same way an '*intrinsic*' concept as, say, the concept of '*intrinsic curvature*' for manifolds. Furthermore it is indeed a true generalisation as it coincides in the regular and ordinary situations with the usual definition of dimension.

Remark: One can introduce of course other concepts over discrete structures which carry a certain dimensional flavor but which have their basis not so much in physics and dynamics but rather in the algebraic topology of, say, '*simplicial complexes*' (cf. e.g. [4]). To some extent such ideas were also persued in [1] but we have the (subjective) impression that the '*connectivity-dimension*' of the network is the really crucial property from the physical point of view (in particular when something like '*unfolding phase transitions*' and other large scale phenomena become relevant). On the other hand these different concepts may neatly complement each other in encoding appropriately various facets of '*complex systems*'.

In [1], besides introducing our network concept, possible dynamics on it and defining the concept of connectivity dimension, our main concern was the development of a kind of '*discrete analysis*' and the clarification of its relation to various other modern approaches which follow different lines of reasoning but seem, nevertheless, to be inspired by the same philosophy, i.e. the complex of ideas and methods called '*non-commutative geometry*'. On the other side, the possible dynamical processes in our network models, which are expected to lead, among other things, to the emergence of something like '*space-time*' and continuum physics in general were only briefly sketched.

The main purpose of this paper is a more detailed investigation of this '*unfolding*' of '*phase transition type*' and the development of the necessary mathematical, mostly topological/geometric, tools and concepts which will allow us to cast vague and qualitative reasoning into a more rigorous and precise framework.

Remark: As to references to the field in general cf. e.g. [1] and further references given in the papers mentioned there. A deserving bibliography of papers dealing with more or less non-orthodox ideas about space-time and related concepts is [7]. Various further mathematical sources we have found useful for our approach will be cited below.

A further clarifying remark (a kind of "petitio principii") concerning the (mathematical) rigor of the type of reasoning we are going to present in the following seems to be in order. In some sense one should regard it rather as a blending of some "educated speculation" with various pieces of more rigorous mathematics. The reasons for this are, as we think, almost inescapable. For one, we have to set up and fix both the complete abstract framework and conceptual tools and then to infer practical, observable consequences for ordinary space-time physics more or less at the same time in order to motivate our primordial assumptions. This is already difficult enough as we have to close a gap between Planck-scale physics and, say, ordinary quantum physics which extends over many orders of magnitude. On the other side we consider this to be of tantamount importance since it is, given the complete absence of reliable experimental data, the only selection principle allowing to judge the scientific value of the various assumptions in our model building process.

For another, many of the mathematical disciplines to be employed in the following are either almost in their infancies (like e.g. the correct statistical treatment of extremely complex and densely entangled dynamical systems, consisting of an extraordinarily large number of degrees of freedom, typical cases in point being our cellular networks), or do not belong to the usual armory of mathematical physics (as e.g. various branches of discrete mathematics) and have to be developed to a certain extent almost from scratch, at least as applications to our field are concerned.

Therefore our general strategy will be as follows: we will prefer to develop in this paper what we think is the generic scenario in broad outline up to the point that we can make transparent how several (partly long standing and puzzling) problems of ordinary space-time physics might be successfully approached within our much wider framework with, among other things, its deeper conception of the nature and role of space-time and the corrresponding fine structure of the physical vacuum.

There remain quite a number of technical details to be filled in, sometimes even veritable subtheories, which may have some interest of their own. This shall in part be postponed to subsequent work in case it would blow up the paper to much and after the general scheme has proved its value. For the time being we will content ourselves with inserting some preliminary remarks at appropriate places in the following sections which are to serve as hints as to where more detailed investigations seem to be worthwhile. In our view the central section of this paper (both with respect to applied mathematical techniques and physical content) is section four.

Concluding the introduction we would like to add the remark that after having nearly completed this paper we stumbled by chance over an exceptionally beautiful (but, as is probably the case with the whole discipline, perhaps not so widely known) book by Bollobas ([23]; unfortunately it is even out of print at the moment) about '*random graphs*', a field originally founded by Erdoes and Renyi in the late fifties. By brousing through the book we realized that the results, being presented there, may confirm, if appropriately translated, to a large extent the soundness of several of our speculations about the assumed generic behavior of our unfolding network. In section 4 of our paper we briefly mention some results from this book; we plan however to embark on a closer inspection of the relations between these two fields in subsequent work (note that the theory of random graphs was founded to tackle some deep problems in graph theory proper, not so much for its probabilistic aspects or even possible applications in physics).

2 The Class of Dynamical Cellular Networks

The idea to base '*discrete dynamics*' on the Planck scale on something like a cellular network was partly inspired by the role which its close relatives, the cellular automata, are playing in the science of complexity (see the references given in e.g. [1]) and the general philosophy to achieve complex, emergent behavior and pattern generation by means of surprisingly simple looking microscopic '*local laws*'.

On the other side, a cellular automaton proper appeared to be a far too regular and rigid array in our view, in particular in the context of possible formation of something like space-time as an ordered superstructure being embedded in or floating above a less organized and probably almost chaotic substratum. More specifically, the organisation of the network as such should be pronouncedly dynamical, most notably its kind of wiring. Furthermore, we maintain that really fundamental dynamical laws must have a quite specific structure, the basic ingredient being the mutual dynamical coupling of more or less two distinct classes of entities, in this context the one being of a local character (nodes or sites), the other of a pronouncedly "geometrical" quality (bonds or links); the state of the one class being governed by the previous state of the other class. Typical cases in point are e.g. general relativity and gauge theory. In contrast to our network models the bonds in a cellular automaton are typically both rigid and regular and not(!) dynamical so that a '*backreaction*' of the node states ("matterfields") on the bond states ("geometry/curvature") is supressed.

To achieve this we assumed our system to be made up of '*cells*' and elementary interactions mediating among these cells, both of which are taken to be dynamical variables. This qualitative picture is then condensed into the concept of a '*network*' consisting of '*nodes*' and '*bonds*' (synonyma being sites, vertices or links, edges respectively) with each bond connecting two *different* nodes.

At each node n_i (i running through a certain index set) the corresponding cell can be in a specific internal state $s_i \in \mathcal{S}$ ($\mathcal{S}$ typically being some countable or finite set not further specified at the moment). Correspondingly the bonds $\{b_{ik}\}$, b_{ik} the bond between node n_i and node n_k, carry a dynamical '*valuation*' $\{J_{ik} \in \mathcal{J}\}$. A dynamical '*local (global) law*' ll, (LL) is then introduced via the general evolution equation given below.

Remark: For the time being we employ a synchroneous '*clock time*' t, proceeding in discrete elementary steps τ, i.e. $t = n \cdot \tau$. That means, the whole system is updated in discrete steps. This clock time should *not*(!) be confused with socalled '*physical time*' which is expected to arise as an emergent collective concept on a possibly much coarser scale. In any case we conjecture that 'local' physical time will turn out to be some kind of '*order parameter field*' coming into existence via the dynamical interaction of many nodes/bonds within the local grains under discussion (see below). The choice of an external overall clock time is made mainly for technical convenience in order to keep already quite complicated matters reasonably simple and can be modified if necessary.

To sum up, we make the following definitions:

2.1 Definition (Cellular Network): In the following we will deal with the class of systems defined below:
i) "Geometrically" they are *graphs*, i.e. they consist of nodes $\{n_i\}$ and bonds $\{b_{ik}\}$ where pictorially the bond b_{ik} connects the nodes n_i and n_k with $n_i \neq n_k$ implied (there are graphs where this is not so), furthermore, to each pair of nodes there exists at most one bond connecting them. In other words the graph is 'simple' (schlicht). There is an intimate relationship between the theory of graphs and the algebra of

relations on sets. In this latter context one would call a simple graph a set carrying a homogeneous non-reflexive, (a)symmetric relation.

The graph is assumed to be *connected*, i.e. two arbitrary nodes can be connected by a sequence of consecutive bonds, and *regular*, that is it looks locally the same everywhere. Mathematically this means that the number of bonds being incident with a given node is the same over the graph (*'degree' of a node*). We call the nodes which can be reached from a given node by making one step the *1-order-neighborhood* $\mathcal{U}_1$ and by not more than n steps $\mathcal{U}_n$.
ii) On the graph we implant a class of dynamics in the following way:

2.2 Definition(Dynamics): As for a cellular automaton each node n_i can be in a number of internal states $s_i \in \mathcal{S}$. Each bond b_{ik} carries a corresponding bond state $J_{ik} \in \mathcal{J}$. Then we assume:

$$s_i(t+\tau) = ll_s(\{s'_k(t)\}, \{J'_{kl}(t)\}) \tag{1}$$

$$J_{ik}(t+\tau) = ll_J(\{s'_l(t)\}, \{J'_{lm}(t)\}) \tag{2}$$

$$(\underline{S}, \underline{J})(t+\tau) = LL((\underline{S}, \underline{J}(t)) \tag{3}$$

where ll_s, ll_J are two mappings (being the same all over the graph) from the state space of a local neighborhood of a given fixed node or bond to $\mathcal{S}, \mathcal{J}$, yielding the updated values of s_i and b_{ik}. s'_l and J'_{lm} denote the internal states of the nodes and bonds of this neighborhood under discussion, $\underline{S}, \underline{J}, LL$ the corresponding global extended states and law.

Remark: The theory of graphs is developed in e.g. [8, 9]. As to the connections to the algebra of relations see also [10]. There exist a lot of further concepts in '*graph theory*' and '*discrete mathematics*' in general which are useful in our context, some of which (together with some further relevant references)will be introduced below, especially in section 4.

2.3 Lemma: The kind of discrete analysis we developed in [1] made it necessary to give the bonds an '*orientation*' (which is extremely natural anyhow). Consistency then requires an analogous relation to hold for the interactions, i.e:

$$b_{ik} = -b_{ki} \quad \text{and hence} \quad J_{ik} = -J_{ki} \tag{4}$$

It is now our aim to attempt to motivate a kind of unfolding and pattern creation in such a network, which, starting from a certain initial state (or better: phase), is supposed to lead in the coarse of a dynamical process – among other things – to the emergence of something we are experiencing on a coarser scale of resolution as '*space-time*'. In order to achieve this we have in a first step to model a kind of concrete '*critical*' dynamics which, on the one hand, fits in the general scheme of Definition

2.2. On the other hand it turns out to be a subtle task to appropriately implement the crucial ingredients which are to catalyze the unfolding process.

We presented several types of local laws in [1]. We in fact studied quite a few other possible dynamical laws, the corresponding scenarios and mappings between the mathematical entities occurring in these scenarios and the concepts carrying a certain physical meaning on a possibly much coarser scale of resolution like e.g. space-time as such, matter, (quantum)fields, gravitation etc. The main difficulty was to implement the unfolding process of our '*network universe*' in a natural way and understand as what kind of '*order parameter manifold*' space-time was going to emerge in this evolving and quite chaotic background. In this selection or identification process an important role was played by '*Occam's principle*', i.e. to infer seemingly complicated phenomena from as simple as possible and, as we think, uncontorted microscopic laws. A crucial ingredient is the possible switching-on or -off of bonds b_{ik}, more specifically, of the corresponding elementary interactions J_{ik} in a '*hystheresis like*' manner (in catastrophy theory called a '*fold*').

These criteria can probably be fulfilled by a whole class of dynamical network laws, or expressed more carefully: there may exist a whole class of possible candidates, the behavior of which could be checked by simulation on a computer, from which we are presenting in the following the probably most simple ones. At the moment implementations of various possible network models on a computer are under the way. These investigations are quite time consuming for various reasons (complex behavior is difficult to study anyhow; cf. e.g. reference [17]), viz. it may turn out that the simple laws we are going to describe below have to be replaced in the end by another law from the same class in case they e.g. do not behave chaotically enough. These numerical investigations will be presented elsewhere.

The naive picture encoded in the following law is that a, for the time being, not further specified substance consisting of elementary quanta q is transported through the network according to the following law:

2.4 Definition of the 'Unfolding' Local Law: The local state space at each site n_i is assumed to be either

$$\mathcal{S} := \{n \cdot q; 0 \leq n \leq N, n \in \mathbb{Z}\} \quad \text{or} \quad q \cdot \mathbb{Z} \tag{5}$$

The bonds b_{ik} can carry the '*valuation*'

$$J_{ik} \in \{-1, 0, +1\} \tag{6}$$

Remark: As to the possible allowed range of the variables s_i some additional remarks are in order which will be given below.

Then the first half of the dynamical local law reads in the case $\mathcal{S} = q \cdot \mathbb{Z}$:

A)

$$s_i(t+\tau) - s_i(t) = -q \cdot \sum_k J_{ik}(t) \tag{7}$$

the sum extending over all the bonds being incident with the node n_i.
The second half, describing the backreaction of the node states on the bond states, is slightly more complicated:
B)

$$1) J_{ik}(t+\tau) = -sgn(s_k(t) - s_i(t)) \text{ if } J_{ik}(t) \neq 0 \text{ and} \tag{8}$$

$$\delta s_{ik} := |s_k(t) - s_i(t)| \geq \lambda_1 \text{ or } J_{ik} = 0 \text{ and } \delta s_{ik}(t) > \lambda_2 \tag{9}$$

with

$$0 < \lambda_1 < \lambda_2 \quad \text{and} \quad I_\lambda := [\lambda_1, \lambda_2] \tag{10}$$

denoting the '*hystheresis interval*'.

$$2) J_{ik}(t+\tau) = 0 \quad \text{if} \quad \delta s_{ik} < \lambda_1 \tag{11}$$

To say it in words: Below λ_1 bonds are temporarily annihilated, above λ_2 they are turned on again, and in between the bonds may only switch between $+1$ and -1, which is essentially a switch in orientation or local direction of transport which can be seen from the first half of the law.

C) The roles of λ_1, λ_2 are interchanged, i.e. bonds b_{ik} are switched off for $\delta s_{ik} > \lambda_2$ and switched on again for $\delta s_{ik} < \lambda_1$.

Remark: Both possible laws are presently carefully studied on the computer. As the simulations will take some time, the results will be published elsewhere. What one can however already say is that one of the many fascinating observations is the ability of systems like these to self-organize themselves and to find a variety of different attractors, i.e. to typically occupy a relatively small region of the potentially accessible huge phase space. Furthermore type **C)** leads to a totally different behavior as compared with type **B)**.

$\mathcal{S} := q \cdot \mathbb{Z}$ is probably appropriate if one does not want to bother about boundary conditions and should be considered as the idealisation of the scenario where typical local fluctuations of s, i.e.

$$\Delta s := < (s - < s >)^2 >^{1/2} \tag{12}$$

the average taken over some suitable time sequence and/or suitably large spacial array of nodes, are sufficiently far away from the possible internal boundaries of the concrete system with a finite local state space, e.g.

$$s \in \{n \cdot q, \quad 0 \leq n \leq N \gg 1\} \tag{13}$$

One could hence regard the former system as a certain limit for $N \to \infty$ and shifting at the same time the reference point from, say, $N/2$ to zero.

On the other side, for finite N one can introduce boundary conditions like the following ones, arguing that they do not change the qualitative behavior provided N is much larger than the generic local fluctuations about some average value $< s >$ with

$$0 \ll < s > \ll N \quad \text{,say, and} \quad < s > = \mathcal{O}(N/2) \tag{14}$$

2.4 A'):

$$s_i(t+\tau) = s_i(t) \quad \text{if} \quad -q \cdot \sum_k J_{ik}(t) > q \cdot N - s_i(t) \quad \text{or} \quad q \cdot \sum_k J_{ik}(t) > s_i \tag{15}$$

i.e. there is no transport if it would transcend the maximal/minimal '*charge*' of the node, $Q = q \cdot N$ or 0.

A"): Perhaps more natural are '*periodic boundary conditions*', i.e. the above equation has to be understood modulo N.

We want to conclude this section with a couple of additional remarks as to this particular type of local law:

Remarks: i) The above dynamical law shows that under certain circumstances '*elementary interactions*' J_{ik} may become zero for some lapse of time. It may even be possible that a substantial fraction becomes '*locked in*' at the value zero for a rather long time. This will then have some practical consequences for the (partial) representation of our network as a graph. On the one side one may consider the graph, in particular its wiring, to be given as a fixed static underlying substratum, i.e. its bonds b_{ik} "being there" as elements of the graph even if a substantial fraction of the corresponding elementary interactions J_{ik} is zero temporarily or even locked in at zero.

On the other side, if one wants to deal with e.g. phase transition like '*topological/geometrical*' changes of the whole wiring of the network it may be advantageous to change the point of view a little bit and regard not only the bond valuations J_{ik} as dynamical variables but rather the underlying graph as such, more specifically,its bonds b_{ik}, i.e. allow them to be annihilated or created. As a consequence the underlying graph will change its shape in the course of clock time which may perhaps be a more fruitful mode of representation. This latter point of view will be adopted in the following where it seems to be appropriate.

ii) There is a certain (faint) resemblance to non-linear electrical networks. One can in fact, if one likes so, consider the S-field as a charge distribution (or potential-field) with $s_i - s_k$ as kind of voltage difference between neighboring nodes. Part A) of the dynamical law reflects then the usual charge conservation. The interpretation of B) is more complicated. One could e.g. regard it as a non-linear dependence of the resistances of links on the applied voltages. In any case, we are confident that it be possible to realize or emulate our system as some kind of non-linear network,

thus producing possibly effects which resemble the unfolding of our universe in the laboratory!
iii) We would like to emphasize that our dynamical law is a genuine cellular network law and not(!) some Lagrangian field theory in disguise (which is frequently the case in other approaches). In our view it is not at all selfevident that it be advisable to model fundamental laws on the Planck scale according to one or the other kind of an (at best) '*effective quantum field theory*' which lives many scales above the Planck regime.

On the other side, we are quite confident that these known types of field theories will emerge as effective theories after some appropriate kind of '*renormalisation transformation*' on a much coarser level, describing the interaction of patterns (e.g. fields) which are themselves rather '*collective excitations*'. In this sense our approach is very much in the spirit of the philosophy of t'Hooft (cf. [12]).
iv) Our personal philosophy is "iconoclastic" in the sense described in the beautiful reviews about quantum gravity by Isham (see e.g. [11]) in sofar as we want to show that and how both quantum (field) theory and gravitation emerge as two different but related aspects of one and the same underlying and more primordial theory of the kind introduced above. The first steps of this endeavor will be undertaken in this paper, which is mainly concerned with the analysis of the kind of embedding of a certain '*order parameter manifold*' ST (space-time) in the background space QX (the cellular network). Most importantly we are concerned with the internal structure of certain subclusters of nodes/bonds which are on the "continuous" (coarse grained) level "experienced" as physical points and their mutual causal relations.

In this context a lot of topological analysis has to be carried out which is performed in a similar spirit as the work of Sorkin et al. ([3]) or Isham ([6]) at least as far as the general philosophy is concerned. The technical tools and concepts being employed may however be different. In our case it is mainly a blend of various fields of '*discrete mathematics*' as e.g. graph theory, finite geometry, calculus of relations and the like with arguments boroughed from statistical physics/mathematics and the science of complexity. Some of the concepts developed below may even be new but this is difficult to decide from the widely scattered literature, a substantial fraction of which may have escaped our notice up to now.

3 The Primordial (Chaotic) Network Phase QX_0

¿From the structure of the dynamical law we have elaborated in section 2 we can infer that our dynamical network, which we henceforth denote by QX ('*quantum space*'; at the moment this should however only be considered as a metaphor), is at each fixed clock time t a certain (dynamic) graph $G(t)$ if we do not take into account the details of the status of the fields of node and bond states but concentrate solely on its purely geometric content. The dynamics of $G(t)$ consists of the possible switching off or on of some bonds in the time step from t to $t + \tau$.

3.1 Definition: The full dynamical network (with its distribution of bond and node states being included) we call QX or $QX(t)$. If we want to concentrate on its purely geometric content we view it as a graph G or $G(t)$.

Our dynamical law is up to now '*deterministic*'. Therefore we should specify the kind of '*initial state*' (or rather a generic group of initial states) from which our network is supposed to evolve. As was the case with the class of admissible dynamical laws, we experimented with quite a few types of possible initial states, looking to what kind of scenarios they would probably lead. It is our impression that both the most natural (having Occam's principle in mind) and most promising assumption is to start from a maximally connected graph, a socalled '*complete graph*' or '*simplex*', i.e. each two nodes n_i, n_k are connected by a bond $b_{ik} = -b_{ki}$ or (see below) from a class of graphs which are '*almost complete*'.

3.2 Assumption: We assume that our '*initial phase* QX_0 consists generically (for reasons explained below) of '*almost complete graphs*', i.e. the '*average number*' of active bonds ($J_{ik} \neq 0$) is almost maximal (see the following remarks).

Some remarks are here in order: One knows already from the analysis of cellular automata that these systems behave typically in an extremely complicated manner and that for exactly the same reasons as in e.g. statistical mechanics it makes frequently only sense to study '*generic*' properties and behavior, mostly with the help of statistical means (cf. e.g. the references on cellular automata in [1], in particular the paper by Wolfram; [13]).

In other words, it seems more appropriate to speak of different '*phases*' instead of individual states. Our dynamical law shows that, typically, a certain percentage of bonds are switched on or off at each clock time step or during a certain interval. That means, it is more natural to assume that QX was "initially" or better: remained for possibly quite some time in a phase with two arbitrary nodes being connected on average with a probability '*near one*'. If we assume that the number of nodes in our network is a very large but finite number Λ, a simplex has $\Lambda(\Lambda-1)/2$ bonds. This means:

3.3 Assumption (Statistical Version): We assume that our network evolves from an initial phase QX_0 with the average number of bonds (taken e.g. over a suitable clock time interval) being "near" $\Lambda(\Lambda-1)/2$.

Henceforth our arguments will frequently (for obvious reasons) carry a markedly statistical flavor. Therefore we will introduce a couple of corresponding concepts and notations.

3.4 Definition: i) We denote the set of vertices (nodes) and edges (bonds) by V, E

respectively (we conform here to the usual notations of graph theory; note that in our case - simple graphs - E can be viewed as a certain subset of $V \times V$ and represents a homogeneous, non-reflexive, symmetric relation), their cardinalities by $|V|, |E|$. The degree of a node n, i.e. the number of bonds being *incident* with it, is $deg(n)$. The graph is called *regular* if $deg(n)$ is constant over the whole graph.
ii) As in our case $deg(n)$ or $|E|$ are time dependent (if we adopt the point of view developed in Remark i) at the end of the previous section, i.e. consider the underlying graph as such as a dynamical object), it makes sense to build the corresponding statistical concepts:

$$< deg >_s, < deg(\circ) >_t, < deg >_{st}, < E >_t \tag{16}$$

are the respective averages of deg, E with $< \circ >_s$ the "spacial" average over the graph at fixed clock time t, $< \circ >_t$ some temporal average (e.g. at a fixed node), $< \circ >_{st}$ a space-time average. The time average is assumed to be taken over an appropriate time interval, the length of which depends on the specific context under discussion (e.g. the particular '*phase*', various correlation lenghts etc.).
Our philosophy concerning the possible time dependence is, for the time being, that a bond b_{ik} is contributing in, say, deg or $|E|$ at clock time t if $J_{ik} \neq 0$.

Evidently there exist certain relations between these notions, e.g:

3.5 Observation: With

$$< deg >_s := \sum_V deg(n)/|V| \text{ and } \sum_V deg(n)/2 = |E| \tag{17}$$

we have

$$< deg >_s \cdot |V|/2 = |E| \tag{18}$$

and

$$< deg >_{st} \cdot |V|/2 = < E >_t \tag{19}$$

Remark: In [1] we gave each bond an orientation s.t. $b_{ik} = -b_{ki}$. We are however counting in Observation 3.5 these two differently oriented bonds as one and the same bond (as usual), hence the factor 1/2. In any case, this does not make a big difference as one can always associate an '*undirected*' graph with a '*bidirected*' graph.

3.6 Conclusion: With the help of what we have said above, Assumption 3.3 can now be framed this way:
$< deg >_s$ and/or $< deg >_{st}$ remain in a small neighborhood of $(|V| - 1)$ in the '*initial phase*' QX_0 for a possibly long time, where "small" has to be understood pragmatically, depending on the details of the model and the context.

Our network (graph) QX (G) carries a natural metric which makes it into a '*metric space*'. This was already employed in [1], to which the reader is referred for further details. Assuming that G is connected (i.e. every two nodes can be connected by a bond sequence) there exists a '*path*' of minimal '*length*' (i.e. number of bonds). Then we have:

$$d(n_i, n_k) := min(\text{length of bond sequences, connecting } n_i, n_k) \tag{20}$$

defines a metric on G.

With this metric G becomes a discrete topological space (even a '*Hausdorff space*') with a natural neighborhood structure of a given node, i.e:

$$U_m(n_0) = \{nodes;\ d(n_0, n) \leq m\} \tag{21}$$

The topology as such is however relatively uninteresting since in a discrete, finite Hausdorff space each one-element set is necessarily both open and closed. The simple proof runs as follows:

$$n_0 \neq n_i \text{ implies } \exists \text{ open set } O_i, n_0 \in O_i, n_i \notin O_i \tag{22}$$

G is finite hence $\cap O_i$ is a finite intersection of open sets and hence again open. By construction $\cap O_i$ contains no points other than n_0, that is n_0 is open. On the other side, with $\cup n_i$ open, $n_0 = G - \cup n_i$ is also closed.

But despite of this, '*finitary*' topological spaces are not(!) automatically trivial as can be seen from the beautiful analysis performed by Sorkin [3]. However they are typically only socalled T_0-spaces (as to these topological notions cf. any good textbook on topology as e.g. [14]).

At this place we want to make a proviso. The existence of a (physically) natural metric on G and a corresponding natural neighborhood structure neatly implementing the physically important concepts of "near by" and "far away" is perhaps much more important than the existence of a (mathematically) non-trivial topological structure. It may well be that on such genuinely discrete sets like networks or graphs natural(!) topological/geometric concepts should rather be adapted to the real discreteness of the substratum and perhaps not so much to an abstract axiomatic system defining a topology. This point of view will be more fully developed in the following section, to which we postpone the details of the (partly intricate) topological/geometric analysis.

In the rest of this section, however, we will try to sketch the expected qualitative behavior of our network QX by employing both statistical concepts and the concrete local law introduced above. Of particular interest in this respect are the initial phase QX_0 and the scenario when it starts to leave this (possibly '*metastable*') phase due to some sort of phase transition. The analysis is, for the time being, necessarily of a qualitative/statistical nature as only generic aspects should matter, especially as we are at the moment mainly interested in deriving observable consequences on much coarser scales which should not depend to much on microscopic details. For another - as is the case in statistical mechanics - it would be technically impossible (at least at the moment) to justify too detailed assumptions and draw precise conclusions from them.

Let us briefly recall the assumptions made about the phase QX_0:

$$\text{i) } |V| = \Lambda \,\text{extremely large, } |E| \approx \Lambda(\Lambda-1)/2 \;\rightarrow\; < deg > \approx \Lambda - 1 \tag{23}$$

i.e. QX_0 is almost a simplex or complete graph.
ii) The local state space at node n_i consists of multiples of an elementary quantum q, i.e:

$$s \in \{q \cdot n\},\; n \in \mathbb{N} \text{ or } \mathbb{Z} \tag{24}$$

ranging from 0 to an appropriately large number N or from $-N$ to $+N$ (cf. section 2). If we do not want to bother about boundary conditions we assume the accessible local state space to be the full $q \cdot \mathbb{Z}$.
iii) The bonds b_{ik} carry the valuation $\{-1, 0, +1\}$ modelling the elementary interactions J_{ik}.
iv) At each clock time step τ either an elementary quantum q is transported via b_{ik} depending on the sign of J_{ik} or the bond b_{ik} is "dead" if $J_{ik} = 0$. The bond states J_{ik} at clock time $t + \tau$ are dynamically coupled with the "charge difference" of the incident site states $s_i - s_k$ at clock time t.
v) Of crucial importance for pattern formation is the built-in "hysteresis law" and the hysteresis interval

$$[\lambda_1, \lambda_2]\;,\;\lambda_1\;,\;\lambda_2 \quad \text{being the lower and upper critical parameter} \tag{25}$$

For

$$\delta s_{ik} < \lambda_1 \text{ or } > \lambda_2 \tag{26}$$

the corresponding bond becomes extinct in the next step or becomes alive again if it had been extinct before, respectively the other way around in case of law **2.4 C)**. In the following we discuss law **2.4 B)**. The qualitative reasoning in case of **C)** would follow similar lines.

If the local fluctuations δs_{ik} are sufficiently large on average, i.e:

$$\Delta s_{ik} := < (s_i - s_k)^2 >^{1/2} > \lambda_2 \tag{27}$$

the corresponding phase is more or less stationary (a slightly more detailed analysis can be found below). Inspecting the situation assumed to prevail in the phase QX_0 with $< deg > \approx \Lambda - 1$ extremely large, we may infer that also the average fluctuation of the local charge s_i at an arbitrary node n_i is typically very large at each clock time step.

3.7 Conjecture/Assumption: We conjecture that in the (chaotic) regime QX_0 correlations have typically an extremely short range due to the enormous number of

links per node and the character of the local law.
Assuming then that the local "orientations" of the incident bond variables are almost statistically independent, both the fluctuations of the local charge at a typical node n_i and of the charge difference $(s_i - s_k)$ with respect to neighboring nodes can be inferred from the '*central limit theorem*'(see e.g. [15]), yielding among other things:

$$\Delta s := < (s - < s >)^2 >^{1/2} = \mathcal{O}(\Lambda^{1/2}) \tag{28}$$

$$\Delta s_{ik} = \mathcal{O}(\Lambda^{1/2}) \tag{29}$$

if we neglect possible phase boundaries ($\mathcal{S} = q \cdot \mathbb{Z}$). In other words, local fluctuations are expected to be enormous both with respect to (clock) time at an arbitrary but fixed node and among nearest neighbors as Λ is assumed to be extremely large.

Remark: Note that the following holds (assuming for simplicity that $< s_i > = < s_k >$):

$$< (s_i - s_k)^2 > = < [(s_i - < s_i >) - (s_k - < s_k >)]^2 > \tag{30}$$

which can be written as

$$\int [(s_i - < s_i >) - (s_k - < s_k >)]^2 \cdot p(s_i) \cdot p(s_k) ds_i ds_k \tag{31}$$

with $P(\circ)$ denoting the '*probability distribution*' under discussion, $p(\circ)$ the corresponding '*probability density*'. For convenience and for the sake of greater generality we write everything continuously while both our '*sample space*' and the occurring '*random variables*' are, by construction, discrete. (Being sloppy, we identify the random variable, say, s_i with the respective values it can acquire).

As, by assumption, the random variables s_i, s_k are independent we have:

$$p(s_i, s_k) = p(s_i) \cdot p(s_k) \tag{32}$$

and hence get with:

$$\int (s_i - < s_i >) p(s_i) ds_i = 0 \tag{33}$$

$$< (s_i - s_k)^2 > = < s_i^2 > + < s_k^2 > \tag{34}$$

i.e., up to a trivial factor the same kind of distribution.

The soundness of the above conjecture has to be confirmed by performing simulations with a variety of local configurations around a given typical node. Qualitatively one could test this conjecture as follows: Let us e.g. assume that s_0 at node n_0 happens to deviate from the surrounding $s_i's$ in a systematic way at time t; to be specific, we assume $s_0 > s_i$ for most of the neighboring nodes.

This local configuration makes it highly probable that, due tu part B) of the particular local law introduced above, most of the bonds b_{0i} are "pointing" from n_0

to n_i (more precisely: the corresponding J_{0i} will be positive) after one time step τ, which, after another step results via part A) in a huge discharge of node n_0. As a consequence the charge s_0 happens to be far below the typical charge of the neighboring sites. This then forces most of the incident bonds b_{0i} respectively J_{0i} to reorient themselves which results in a huge surplus charge of node n_0 after another two time steps and so on. Given that all the nodes under discussion are densely entangled with each other the local dynamics in the phase QX_0 may indeed be sufficiently chaotic to justify our conjecture. What is however not entirely clear is the statistical weight of such a systematic deviation from a more or less random distribution of the local charges (see the following remarks). Furthermore complex systems are capable of performing a lot of very surprising things (e.g. approaching attractors very quickly irrespective of their seemingly chaotic behavior; an observation made by e.g. Kauffmann in his study of '*switching nets*, see [17])

3.8 Some Annotations to the Probalisitic Framework: As to the soundness of the above conjecture some more remarks are appropriate:
i) This is one of the points, mentioned in the introduction, which comprises in effect a whole bunch of important questions which have a relevance of their own. The (partly intricate) technical details coming up in this context shall be postponed for the main part to forthcoming work (apart from some preliminary remarks following below) as we consider it to be our main task at the moment to develop in the rest of the paper a scenario in which space-time is to emerge as a kind of (coarse grained) order parameter manifold floating in the discrete network QX and to establish geometric/causal notions like "nearby" or "far away".
ii) Our philosophy is supported by observations made by e.g. S.Kauffmann and reported in ref. [17], p.109 or 112, viz. that too densely connected networks seem to support chaotic behavior while more sparsely connected ones are capable of generating complex behavior. The networks discussed there are however more rigid than ours in several respects and of a markedly different nature as to the details of their dynamical laws ('*switching nets*').
iii) It is of course not really crucial that something like the '*central limit theorem*' does strictly hold. What we actually do need is a guarantee that fluctuations tend to be very large and incoherent and depend to some extent on the density of the wiring. Note that e.g. for not necessarily independent random varariables X_i (with, for simplicity, vanishing mean and common variance σ^2) the following holds:

$$< (\sum_1^n X_i)^2 >= \sum_i < X_i^2 > + < \sum_{i\neq j} X_i X_j > \tag{35}$$

The first term on the rhs goes as $n \cdot \sigma^2$. If the X_i are '*uniformly weakly correlated*' in the following sense:

$$| < X_i \cdot \sum_{j\neq i} X_j > | \leq \varepsilon \tag{36}$$

uniformly in i and n with $\varepsilon < \sigma^2$, we have with $S_n := \sum_1^n X_i$:

$$n \cdot (\sigma^2 - \varepsilon) \leq < S_n^2 > \leq n(\sigma^2 + \varepsilon) \tag{37}$$

that is

$$\Delta S_n = \mathcal{O}(n^{1/2}) \tag{38}$$

Note that the above assumption is not particularly far fetched as products like $X_i X_j, i \neq j$, typically oscillate around zero. Furthermore, similar relations can be established for random variables which display a certain '*cluster behavior*' with respect to space or/and time (as e.g. in ordinary statistical mechanics).

iv) A careful treatment of many facetts of the central limit theorem can be found e.g. in [15]. Note that in our case the number of involved random variables is large but not really infinite. In this situation the '*Berry-Essen-Theorems*' can be applied (cf. e.g. [19]). Furthermore, in [20] some additional interesting remarks concerning the possible extension of the central limit theorem to weakly dependent random variables can be found. Note in particular that one consequence of the central limit theorem, viz. the normal distribution of the local fluctuations at, say, a typical node n_0, is not necessarily needed in our scenario, i.e. the fact that the small fluctuations are the most probable ones and dominate the behavior.

v) Quite the contrary, the particular kind of local fluctuations we discussed above may rather be an indication for a much more interesting behavior (not so frequently found in the usual examples of statistical mechanics), i.e. the tendency of amplification of small deviations, viz. a kind of '*positive feedback behavior*', which may play perhaps an important role in stabilizing certain phases or, alternatively, drive the system away from perhaps only '*metastable*' phase towards certain '*attractors*'. It is however not clear at the moment how frequently these special fluctuations actually do occur; put differently: how large their statistical weight is compared to the more chaotic fluctuations which would rather support a kind of gaussian behavior. This leads to the last point we want to mention.

vi) The above discussion shows that a more elaborated kind of statistical or stochastic theory is called for if one is dealing with such peculiar systems (some steps have already been taken in e.g. refs. [13] and [18] for a however simpler class, the cellular automata). Note that, in contrast to e.g. Gibbsian equilibrium statistical mechanics, many pieces of an a priori framework are missing as a natural probability measure on a suitable sample space, criteria concerning the relevance and statistical weight of the various initial configurations, their long-time effects (which are notoriously difficult to forecast in complex systems) and the like.

For that reason we made, for the time being, the above heuristic assumptions and suggest to calculate the various averages and probability distributions in a more practical way by assuming that the system behaves reasonable and that the states we are dealing with are sufficiently generic so that averages and probabilities taken with respect to e.g. "space" and/or "time" over, say, one concretely given actual state

of the network (or a time sequence of actual states) give sensible results (due to the assumption that the huge numbers of involved nodes and bonds, $|V|, |E|$ may serve as a substitute for ensemble averages). To give an exemple:
vii) The local law shows that fluctuations of the charge s_0 at a given node n_0 during one clock time step τ are given by $\sum J_{i0}(t)$, the sum extending over the neighboring nodes. If one wants to make statistical statements about fluctuations at a typical node, without having an allcomprising statistical framework at ones disposal, one can proceed as follows:

One chooses e.g. to concentrate on the situation at a fixed clock time t, i.e. make the statistics over the distribution of node and bond states at a fixed time.

Definition: a) The points ω of the local '*sample space*' Ω at an abstract typical node n_0 are all the possible '*bond configurations*'

$$\{J_{k0}\}_k,\ J_{k0} \in \{+1, 0, -1\} \text{ i.e. } |\Omega| = 3^{deg} \tag{39}$$

with *deg* the degree of the node, i.e. the number of incident bonds b_{k0}, k running from 1 to *deg*.
b) Probabilities of '*elementary events*' (i.e. a given configuration) are extracted simply from a frequency analysis over the array of nodes (frequency of occurrence of the various bond configurations under discussion) at time t with a suitable but more or less arbitrary numbering of bonds being implied. More general events can then be constructed by the usual additivity properties of measures.
c) The $J_{k0}(\omega)$ themselves are '*elementary random variables*' with

$$J_{k0}(\omega) := J_{k0} \tag{40}$$

With the help of the elementary probabilities calculated in b) each of the random variables $J_{k0}(\omega)$ has a (discrete) distribution and we can form corresponding sums, i.e:

$$J(\omega) := \sum_1^{deg} J_{ko}(\omega) \tag{41}$$

As charge fluctuations at a node have been linked with the above $J(\omega)$ which is, on the other side, a sum over elementary random variables (*deg* assumed to be very large) we can make the following observation:

Observation: It is $J(\omega)$ and the $J_{k0}(\omega)$ to which our above assumptions like e.g. the central limit theorem do apply.

3.9 Corollary to 3.7: In the perhaps more realistic case $\mathcal{S} = \{0, q, \ldots, q \cdot N\}$ and N large but nevertheless $\Lambda \gg N$, which we think is natural, given that we assume our whole universe to emerge from such a QX_0, the above estimate shows that in the regime QX_0 the entire local state spaces $\mathcal{S}_i$ are covered by the expected local

fluctuations of s_i with essentially equal probability $1/N$.
In other words, one may say that the '*local entropy*' at each node

$$-\sum_{i=0}^{N} w_i(s_i) \ln w_i(s_i) = -\sum_{0}^{N} 1/N \ln 1/N = \ln N \tag{42}$$

is maximal in the phase QX_0, thus reflecting the absence of any stable pattern.

Remarks:i) Such (information) entropy concepts may turn out to be quite useful in analyzing systems like our one (see e.g. [13] or [18] where related phenomena were analyzed in cellular automata).
ii) We would like to note that (vague) resemblances to the scenarios discussed in, say, '*synergetics*' and related fields are not accidental in our view (cf. e.g. [16] or the beautiful book about the working philosophy of the Santa Fe Institute; [17] and the references therein). To various paradigmatic catch words like '*order parameter*', '*slaving principle*' or '*selforganized criticality*' we hope to come back in forthcoming work.

4 The Transition from the Phase QX_0 to QX/ST and the Emergence of Space-Time as an Order Parameter Manifold

We provided arguments in the previous section that fluctuations tend to be extremely large in the primordial phase QX_0 and correlations so short lived that any kind of pattern formation will be obstructed as long as the fluctuations are on average substantially larger than the lower critical parameter λ_1 or, even better, larger than λ_2 (version **2.4 B)** of the local law):

$$\Delta s_{ik} = < (s_i - s_k)^2 >^{1/2} > \lambda_2 \tag{43}$$

(the average taken with respect to space, time or both).

Let us now assume that, by chance, a sufficiently extended and pronounced spontaneous fluctuation happens to be created in a '*subgraph*' $G'(t_0) \subset G$ by the network dynamics around clock time t_0.

4.1 Definition: With G a graph, V_G,E_G its sets of vertices (nodes) and edges (bonds) G' is called a *subgraph* if

$$V_{G'} \subset V_G \text{ and } E_{g'} \subset E_G \tag{44}$$

It is called a *section graph* if for every pair of nodes

$$(n_i, n_k) \in V_{G'} \;\Rightarrow\; b_{ik} \in E_{G'} \text{ provided that } b_{ik} \in E_G \tag{45}$$

We assume this fluctuation to consist of an array of anomalously small charge differences

$$\delta s_{ik} \lesssim \lambda_1 \text{ among the nodes of } G' \tag{46}$$

($\lesssim$ meaning that the charge differences are typically smaller than or approximately equal to λ_1).

Remark: Note that the subgraph G' need not be connected! Quite the contrary, it may well be that a subgraph consisting of an array of effectively distributed disconnected subclusters will turn out to serve its purpose much better.

If the surrounding network environment is favorable this fluctuation may then trigger an '*avalanche*' of rapidly increasing size in the following way:

$$\Delta_s s_{ik} \lesssim \lambda_1 \text{ in } G'(t_0) \tag{47}$$

(the index s in Δ_s denoting a spacial average) will have the effect that a possibly substantial fraction of bonds $b_{ik} \in E_{G'}$ become temporarily inactive after one clock time step τ (and, possibly, for a longer lapse of time).

By assumption we are still in the chaotic phase QX_0. So we expect that such local temporal deviations from the overall chaotic "equilibrium state" will typically quickly dissolve in the rapidly fluctuating background (at least for the specific network dynamics we proposed above and for most of the possible scenarios). If, however, the drop of the average node degree in $G'(t_0 + \tau)$ happens to be pronounced enough so that

$$\Delta_{norm} s_{ik}|_{G'} = \mathcal{O}(< deg >_{G'}^{1/2}) \lesssim \lambda_2 \tag{48}$$

where the lhs means the generic(!) amount of fluctuations which can be expected from the assumptions about the environment QX_0 (cf. Conjecture/Assumption 3.7), i.e. almost statistical independence of the orientations of the bonds being incident with the nodes belonging to G', we may have an entirely new situation!

Remark: $\Delta_s s_{ik}|_{G'(t_0)}$, i.e. the accidental particular fluctuation which happened to emerge spontaneously at time t_0, should not be confused with the above '*typical*' degree of fluctuations which one has to expect from probability theory. As to the order of their respective magnitudes we suppose that:

$$\Delta_s s_{ik}|_{G'(t_0)} < \Delta_{norm} s_{ik}|_{G'} \tag{49}$$

even after a certain amount of bonds in G' have already died off in the course of the phase transition.

4.2 Supposition: We expect that in the course of the phase transition, which started around clock time t_0 the amount of fluctuations in G' will typically lie below the upper critical parameter λ_2 but may on average lie above the lower critical parameter λ_1.

If the above described scenario actually happens to take place, then it may occur that the fluctuation pattern prevailing in G' will not be reabsorbed in the background after a few clock time steps (as has been probably the case many times before) but may represent the seed for a new and different kind of evolution. Given that the situation is as being described in Supposition 4.2 there is a realistic chance that on average more bonds are switched off in and around G' than are switched on again, i.e:

$$N_-(t) - N_+(t) > 0 \quad \text{on average for} \quad t > t_0 \tag{50}$$

It is obvious that the details of this process will depend on the details of the probability distribution (probability density) for the random variable s_{ik} restricted to G', this distribution being assumed to be constructed according to the principles described above. To be more specific, this kind of evolution will hold under the following proviso:

4.3 Observation: With $p(s_{ik})|_{G'}$ being the spacial probability density for the random variable s_{ik} for clock time $t > t_0$, the unfolding process (i.e. continuing annihilation of bonds) will go on if

$$\int_0^{\lambda_1} p(s_{ik})|_{G'} ds_{ik} > \int_{\lambda_2}^{\infty} p(s_{ik}) ds_{ik} \quad \text{on average} \tag{51}$$

Remarks: i) As we already mentioned at the end of section 3 such notions are not entirely easy to define rigorously. Such a probability distribution should rather be understood in the following way. On the one side we have a completely deterministic process, a result of which was (by assumption) this peculiar fluctuation phenomenon at clock time t_0. As it is practically impossible to follow the process in every detail step by step for $t > t_0$ one has to resort to probabilistic arguments.

On the basis of a given average node degree $< deg >_{G'}$ and the assumed statistical independence of bond orientations (in QX_0) one can e.g. calculate (as was shown in section 3) the probability distribution $p(s_{ik})$ of s_{ik} between two typical neighboring nodes in G'. If this $p(s_{ik})$ fulfills the above inequality for $t > t_0$ one has reason to expect that the unfolding process will go on (at least for a while).
ii) On the other hand, less stringent assumptions about the stability of the spontaneous fluctuation at t_0 will perhaps already suffice. What is actually important is that more bonds are annihilated on average than are created again. This will presumably both depend on the geometric structure of the subgraph G' and the overall network state of the graph G around that critical time t_0. We will however postpone a more detailed analysis in this direction and enter in a description of the new phase QX/ST which we expect to emerge from this phase transition.

In a first step we will simplify our task and concentrate solely on the geometry of the wiring diagram of the underlying graph $G(t)$, i.e. neglect the details of the internal states at the nodes and bonds apart from a bond being dead or alive (as was already done before to some extent), viz. a bond occurs in the wiring diagram of $G(t)$ if it is alive at time t. This will enable us to extract more clearly the underlying geometric content being encoded in the network. Furthermore, in order to limit the amount of technical notation, we will not always explicitly mention the statistical or fluctuating character of the various occurring quantities or notions. The correct meaning is assumed to be tacitly understood in the respective cases without extra mentioning.

What we are actually after is exhibiting the process in the course of which what we will later call '*physical points*' gradually emerge from certain protoforms. We assumed that in the primordial phase QX_0 almost all Λ nodes have been connected with each other, viz. the number of bonds in G_0 is roughly:

$$|E_{G_0}| \approx |V_{G_0}|(|V_{G_0}| - 1)/2 = \Lambda(\Lambda - 1)/2 \tag{52}$$

with Λ some huge number. Let us now assume that at the onset of the phase transition a certain (possibly small) fraction of bonds have become annihilated, say

$$1 \ll \alpha \ll \Lambda(\Lambda - 1)/2 \tag{53}$$

We make, in addition, the idealisation that G_0 is (for convenience, i.e. not only approximately) a complete graph (simplex). We have then that α arbitrarily selected bonds can at most connect $k \leq 2\alpha$ different nodes, hence there still exist at least $(\Lambda - k)$ nodes which are maximally connected, viz. they are spanning a still huge subsimplex $S' \subset G$. On the other hand there are at most $k \leq 2\alpha$ nodes with one or more incident bonds missing.

This picture will become increasingly intricate if more and more bonds are switched off. To keep track of a possible emerging '*superstructure*' we will proceed as follows. We want to cover the graph G (rather its node set) with a class of particular subgraphs $\{S_\nu\}$, constructed according to the following rule:

Starting from an arbitrary node, say n_0, we choose a node n_1 being connected with n_0 by a bond, and in the i-th step a node being connected with all(!) the preceeding ones; pictorially:

$$(n_0 \to n_1 \to \cdots \to n_k) \tag{54}$$

The extension process stops if there is no further node n_{k+1} being connectable with all the preceeding ones.

4.4 Definitions: i) A subgraph S is called a '*maximal subsimplex*' (mss) if there is no other simplex $\hat{S} \neq S$ contained in G with $\hat{S} \supset S$.
ii) A subgraph H is called to cover G if $V_H = V_G$ and at every node $\in V_G$ there exists at least one bond $\in E_H$.

Remark: Such mss are called in combinatorics '*cliques*' as we learned recently.

This class of mss, $S_\nu \subset G$, and their mutual entanglement will be a key concept to characterize the geometrical (large scale) structure of the unfolding graph G.

4.5 Definition: Let G_ν be a class of subgraphs of G.
i) $\cap G_\nu$ is the graph with $n \in V_{\cap G_\nu}$ if $n \in$ every V_{G_ν},
$b_{ik} \in E_{\cap G_\nu}$ if $b_{ik} \in$ every E_{G_ν}
ii) $\cup G_\nu$ is the graph with $n \in V_{\cup G_\nu}$ if $n \in V_{G_\nu}$ for at least one ν
$b_{ik} \in E_{\cup G_\nu}$ if $b_{ik} \in E_{G_\nu}$ for at least one ν.

4.6 Observations: i) Starting from an arbitrary node n_0 and performing the above described steps we get a certain mss being contained in G; described pictorially as

$$S(n_0 \to \cdots \to n_k) \tag{55}$$

With $S(n_0 \to \cdots \to n_k)$ given, each permutation will yield the same mss, i.e:

$$S(n_0 \to \cdots \to n_k) = S(n_{\pi(0)} \to \cdots \to n_{\pi(k)}) \tag{56}$$

Furthermore each mss can so be constructed, starting from one of its nodes. Evidently this could be done for each node and for all possible alternatives as to the choice of the next node in the above sequence.

It is however important to note that, starting e.g. from a given node, there may be several alternatives at each intermediate step, leading in the end to different(!) mss having some (or even many) nodes and/or bonds in common! We note in passing that
ii) $\cup S_\nu$ covers G with $\{S_\nu\}$ the class of mss. Note however that in general $\cup S_\nu$ is only a true subgraph of G, i.e.

$$E_{\cup S_\nu} \subset E_G \tag{57}$$

4.7 Example:

$$V_G = \{n_0, n_1, n_2, n_3\},\ E_G = \{b_{01}, b_{02}, b_{12}, b_{03}, b_{13}\} \tag{58}$$

There exist two mss

$$S(n_0 \to n_1 \to n_2) \text{ and } S(n_0 \to n_1 \to n_3) \tag{59}$$

where at n_1 two alternative choices can be made.

It turns out to be an ambitious task to calculate the cardinality of mss in a given graph as a function of the, say, α missing bonds and their distribution in G. the reasons for this are manifold. For one, this number depends sensitively on the way the missing bonds are distributed in G. For another, as can most easily be seen by studying examples, the prescription how the mss can be (re)constructed (cf. Observation 4.6) is strongly "path dependent" in the sense that they may be (in particular if they are large) intricately entangled. which makes an easy counting quite delicate. Nevertheless this is an important task.

4.8 Task: Estimate the number of mss spanning G by using only a few characteristics like the number of missing bonds and the like. It may well be that also in this situation a statistical approach would be the most appropriate.

Remark: This problem is presently under study by ourselves. We suppose that there is some veritable and interesting piece of mathematics buried in it which goes far beyond the particular context of our investigation and which would be of relevance in its own right. (Such problems are typically adressed in [23] which, we think, will be of help in this respect). We conjecture that there are, among other things, structural similarities to something like (co)homology theory to be built over such discrete structures.

While a general complete solution is at the moment not at our disposal, we are going to present some preliminary steps and estimates in that direction. As we already remarked, one might have the idea that it be possible to provide both a general and sensible upper bound on the number of mss in a graph G with, say, Λ nodes and $\Lambda(\Lambda-1)/2-\alpha$ bonds. This turns out to be a very difficult endeavor. It seems to be easier to shift, in order to get a better feeling for the crucial points of the problem, the point of view a little bit. That means we want to construct (typical) examples where the number of mss can be given, thus showing what order of magnitude one has to expect.

We construct a special graph as follows: Take $2k$ nodes, choose a subset G_1 consisting of exactly k nodes $(n_1.\ldots,n_k)$, make G_1 a simplex. With the remaining k nodes $(n'_1,\ldots,n'_k)$ we proceed in the same way, i.e. we now have two subsimplices G_1, G'_1.

We now choose a one-one-map from $(n_1,\ldots,n_k)$ to $(n'_1,\ldots,n'_k)$, say:

$$n_i \;\to\; n'_i \tag{60}$$

We now connect all the n_i with the n'_j except for the k pairs (n_i,n'_i). The graph G so constructed has

$$|E_G| = 2k(2k-1)/2 - k = 2k(2k-2)/2 \tag{61}$$

We see from this that, as in our network scenario, the number of missing bonds is a relatively small fraction, hence, the example may be not so untypical.

We can now make the following sequence of observations:

4.9 Observations i) $G_1 = S_1$ is already a mss as each $n'_i \in G'_1$ has one bond missing

with respect to G_1.
ii) One gets new mss by exchanging exactly one n_i with its partner n_i', pictorially:

$$[S_1 - n_i + n_i'] \tag{62}$$

yielding k further mss.
iii) One can proceed by constructing another class of mss, now deleting (n_i, n_j) and adding their respective partners, i.e:

$$[S_1 - n_i - n_j + n_i' + n_j'] \tag{63}$$

iv) This can be done until we end up with the mss

$$[S_1 - n_1 - \cdots - n_k + n_1' + \cdots + n_k'] = S_1' \tag{64}$$

The combinatorics goes as follows:

$$|\{mss\}| = \sum_{\nu=0}^{k} \binom{k}{\nu} = (1+1)^k = 2^k \tag{65}$$

i.e., our $2k-$node-graph (with k bonds missing) contains exactly 2^k mss.

4.10 Some Comments on Combinatorics: That the combinatorical problems we have adressed above are really ambitious can be seen with the help of the following argument. As a "warm-up exercise" one can tackle a seemingly much simpler problem, namely trying to find sensible bounds for the cardinality of certain coverings $C \in \mathcal{C}$ of a given finite (but, in general, large) set X with C being a covering of X, $C \in P(X)$, the '*power set*' of X. The constraint is that in a given covering C no subsets of already occurring sets are to be allowed, i.e.,

$$X_i \in C \text{ implies } C \ni X_j \not\subset X_i \; X_i \not\subset X_j \text{ for } j \neq i \tag{66}$$

The reason for this particular constraint stems from our mss (see Corollary 4.12 below).

We are e.g. interested in bounds of the following nature:

Give an effective bound for $sup(|C|;\, C \in \mathcal{C})$, i.e. a bound which is better than $|P(X)|$, by employing the above constraint.

As for our graph problem discussed above, this appears to be still quite difficult if one starts from first principles. Shifting the point of view one can instead try to construct coverings C with a large $|C|$ in order to get at least a sensible lower bound on $sup(|C|)$. It turns out to be effective to choose coverings with sets X_j with $|X_i| = |X_j|$ for all j and then take $|X_i| = k/2$ if $|X| = k$ with k being even. (This idea we owe to our coworker Th. Nowotny; note that in that case the above constraint is automatically implied). Then we have (k large):

$$|C| = k!/([k/2]!)^2 \approx (2/\pi k)^{1/2} \cdot 2^k \tag{67}$$

with the help of Stirling's formula. Note that a covering with non-overlapping sets (i.e. a '*partition*') has always a cardinality $|C| \leq k$.

A little bit surprisingly, this construction happens already to be optimal, as we found out recently when we stumbled by chance over an old but noteworthy paper by E.Sperner, [21] (see also the beautiful book of H.Lueneburg, [22] chapt.XVI and the remark on p.492 or [26]). In this paper the following result has been proven (which, by the way gave rise to a full-fledged subtheory in combinatorics); (in our notation):

Sperner's Theorem:

$$sup\,|C| = \binom{n}{k} \text{ with } k = [n/2] \tag{68}$$

and sup is attained for $P_k(X)$, a covering with $k-$sets, if n is even, $P_k(X)$ and $P_{k+1}(X)$ if n is odd.

Remark: It may well be that the methods developed there may be also helpful in our more general graph-problem.

Returning to our concrete network scenario we can now proceed as follows:
V_G can be split in the following way:

$$V_G = V_{S'} \cup V_N \tag{69}$$

with V_N the set of nodes with some of the α bonds among them missing, $V_{S'}$ the set of nodes being maximally connected (see the remarks after formula (53))

$$|V_N| = k \leq 2\alpha \tag{70}$$

Almost by definition, $V_{S'}$ generate a simplex $S' \subset G$.

4.11 Observation: i) The simplex S' is contained in each of the maximal subsimplices S_ν, i.e:

$$S' \subset \cap S_\nu \text{ with, in general, } S' \neq \cap S_\nu \tag{71}$$

ii) S' itself is not(!) maximal as $[S' \cup n_i]$ is always a larger simplex with $n_i \in N$ and $[S' \cup n_i]$ being the section graph spanned by $V_{S'}$ and n_i.
iii) To each maximal simplex $S_\nu \subset G$ belongs a unique maximal subsimplex $N_\nu \subset N$ with

$$S_\nu = [S' \cup N_\nu] \tag{72}$$

4.12 Corollary: From the maximality of the N_ν follows a general structure relation for the $\{S_\nu\}$ and $\{N_\nu\}$:

$$\nu \neq \mu \to S_\nu \neq S_\mu \to N_\nu \neq N_\mu \tag{73}$$

and neither

$$N_\nu \subset N_\mu \text{ nor } N_\mu \subset N_\nu \tag{74}$$

viz. there always exists at least one $n_\nu \in V_{N_\nu}$ s.t. $n_\nu \notin V_{N_\mu}$ and vice versa.

Proof of Observation 4.11: i) Starting from an arbitrary node $n \in G$, it is by definition connected with all the other nodes in S', since if say n, n' are not connected they both belong to N (by definition). I.e., irrespectively how we will proceed in the construction of some S_ν, S' can always be added at any intermediate step, hence $S' \subset \cap S_\nu$. On the other side one can easily construct scenarios where $S' \neq \cap S_\nu$.
ii) As $n \in N$ is connected with each $n' \in S'$ (by definition of N and S'), the section graph $[S' \cup n]$ is again a (larger) simplex.
iii) We have $S' \subset S_\nu$ for all ν, hence

$$S_\nu \neq S_\mu \text{ implies } N_\nu \neq N_\mu \tag{75}$$

with $N_{\nu,\mu}$ the corresponding section graphs in N.
With S_ν being a simplex, N_ν is again a subsimplex which is maximal in N. Otherwise S_ν would not be maximal in G.
On the other side each $S_\nu = [S' \cup N_\nu]$ is uniquely given by a maximal N_ν in N as each node in N is connected with all the nodes in S'.

We see from the above that as long as α, the number of dead (missing) bonds, is much smaller than the number of bonds in the initial simplex G_0, there does exist a considerable overlap S', among the class of mss S_ν. This overlap will become smaller with α increasing with clock time t; by the same token the number of mss will increase. To describe this unfolding process we make the following pictorial abbreviations and observations:

4.13 Abbreviation: We abbreviate n_i, n_k (not) connected by a bond by

$$n_i \sim n_k \quad (n_i \not\sim n_k) \tag{76}$$

We then have:

4.14 Observation: i) $n_i \not\sim n_k$ implies that they are lying in different S_ν's.
ii) S_ν, S_μ are disjoint, i.e. $S_\nu \cap S_\mu = \emptyset$ iff

$$\forall n_\nu \in S_\nu \; \exists \, n_\mu \in S_\mu \text{ with } n_\nu \not\sim n_\mu \tag{77}$$

or vice versa.

4.15 Consequence: This shows that it may well be that $S_\nu \cap S_\mu = \emptyset$ while the two *mss* have still a lot of '*interbonds*', i.e. bonds connecting the one with the other. The guiding idea is however that V_{S_ν} and V_{S_μ}, taken as a whole, will be generically considerably less strongly entangled with each other than the nodes within S_ν or S_μ among themselves after the unfolding process is fully developed.

4.16 The Physical Picture: For α small, viz. $S' \subset G$ still large, we regard the emerging S_ν's as '*protoforms*' of '*physical points*'. We suppose that this picture becomes more pronounced with increasing α, i.e. increasing clock time, when the entanglement between different proto points becomes weaker. For α sufficiently small all these proto points are hanging together via the non-empty(!) S' while for α sufficiently large (i.e. clock time large) and the dead bonds appropriately distributed over G it may happen that $S' = \emptyset$ and that, furthermore, a pronounced far- and near-order among the grains S_ν is established via their (varying) '*degree of connectedness*'.

This is the scenario which, we hope, will support a certain '*superstructure*' we dub ST (space-time), the whole complex we like to call QX/ST, i.e. a still wildly fluctuating "quantum underworld" with a both coarser and more smoothly behaving '*order parameter manifold*' being superimposed.

One can supply a rough estimate as to the threshold which (under certain conditions) may divide these two scenarios. With the primordial graph G_0 being nearly a simplex of node number, say, Λ, edge number $\Lambda(\Lambda-1)/2$ and α bonds already annihilated, we had the result (see above) that S' consists at least of

$$|V_{S'}| \geq \Lambda - 2\alpha \tag{78}$$

nodes as long as the rhs is positive. Hence:

4.17 Observation: There is a chance that $S' = \emptyset$, i.e an effective distribution of the annihilated bonds being assumed, when

$$\alpha \geq \Lambda/2 \tag{79}$$

We conjecture that this is a characteristic number which indicates where the transition zone will probably lie which divides the two regimes described above. The threshold value yields:

$$\Lambda(\Lambda-1)/2 - \Lambda/2 = \Lambda(\Lambda-2)/2 \tag{80}$$

bonds being still alive, which implies an average node degree:

$$< deg >_s = \Lambda - 2 \tag{81}$$

in other words, on the average there is only one bond missing per node!

Remark: The content of Observation 4.17 makes it clear again that relatively advanced stochastic graph concepts are called for, of a kind as they are developed in the beautiful book of Bollobas about '*random graphs*' ([23]), i.e. it has to be clarified whether e.g. $\alpha = \Lambda/2$ is actually representing what is called a '*threshold function*' for random graphs. These extremely important concepts will be discussed in more detail elsewhere, as we want to enter in the remaining part of this paper into a more thorough analysis of the nature of '*physical points*' we have introduced above and their mutual entanglement, in particular for large α i.e. far away from the phase transition regime.

Our strategy to relate the *mss* S_ν with what one may call on a much more coarse grained level physical points suggests the introduction of the following mathematical concepts, being of wide use in combinatorial mathematics and related fields (cf. e.g. [24] or [25]).

4.18 Definition: Over the '*ground set*' V we construct an '*incidence structure*' $(V, \mathcal{S}, I)$ or '*block space*' in the following way:
i) The '*blocks*' are the *mss* $\{S_\nu\}$
ii) The points of the incidence structure are the nodes
iii) The '*incidence relation*' I is $\in$.
In other words we say a node is incident with a block S_ν or belongs to S_ν. In our scenario the blocks are elements of the power set $P(V)$ (which needs not always be the case).
iv) With (n) we denote the blocks S_ν which are incident with n, i.e. $n \in S_\nu$. In the general theory (S_ν) would denote the nodes being incident with S_ν, in our case where $I \doteq \in$ we identify (S_ν) and S_ν; $|(n)|$, $|S_\nu|$ are their respective cardinalities.
v) The above is generalized in an obvious way to $(n_{i_1}, \ldots, n_{i_k})$ or $(S_{\nu_1}, \ldots, S_{\nu_k})$.

4.19 Definition: i) An '*automorphism*' A of a block space is a bijective map $V \to V$ s.t.

$$A: \; S_\nu \to S_{nu'} \, , \; \{\nu'\} \, \text{a permutation of} \, \{\nu\} \tag{82}$$

ii) The automorphism A is called '*inner*' if it acts within the respective blocks, i.e:

$$A: \; S_\nu \to S_\nu \quad \text{for all} \, \nu \tag{83}$$

4.20 Definition: With respect to the above block space we can speak of an
i) '*interior bond*' of a given S_ν, i.e:

$$b_{ik} \, \text{with} \, n_i, \, n_k \, \in S_\nu \tag{84}$$

ii) '*exterior bond*' with respect to a given S_ν, i.e:

$$b_{ik} \, \text{with} \, n_i, \, n_k \, \notin S_\nu \tag{85}$$

iii) an '*interbond*', i.e:

$$b_{ik} \text{ with } n_i \in S_\nu,\ n_k \in S_\mu,\ \nu \neq \mu \tag{86}$$

iv) a '*common bond*' of S_ν,S_μ if b_{ik} is an interior bond both of S_ν and S_μ.
v) a '*true interbond*' b_{ik} if for $\nu \neq \mu$:

$$n_i \in S_\nu,\ n_k \in S_\mu,\ n_k \notin S_\nu \tag{87}$$

vi) We then have the relation for given S_ν, S_μ:

$$\{interbonds\} - \{common\ bonds\} = \{true\ interbonds\} \tag{88}$$

Remarks: i) The relation between these classes describes the (time dependent) degree of entanglement among the blocks S_ν, viz. among the physical proto points and, as a consequence, the physical near- and far-order on the level of ST, the macroscopic causality structure of space-time and the non-local entanglement we observe in quantum mechanics.
ii) In Definition 2.1 we defined the notion of a '*simple graph*' and related it with the concept of a '*homogeneous*', '*non-reflexive*' relation. Note that, on the other side, a graph is a particular example of a block space, with the class of bonds being the blocks and the obvious incidence relation between nodes and bonds.

This identification generalizes easily to more complex scenarios in the following way: Following Bollobas in [26] we define a '*set system*' $\mathcal{F}$ over, say, V to be a subset of the powerset $P(V)$, i.e:

$$\mathcal{F} \subset P(V) \tag{89}$$

Such a set system is by the same token a block space, both being, on the other side, examples of a '*heterogeneous relation*' between V and $P(V)$ (a homogeneous relation corresponding to blocks $\subset P^{(2)}(V)$).

4.21 Definition: i) $P^{(k)}(V)$ is the class of $k-sets$ in $P(V)$
ii) A homogeneous relation is a subset of $V \times V$ or, by the same token, a subset of $P^{(2)}(V)$
iii) A heterogeneous relation is a relation between between two sets of, typically, different nature, i.e. a block space. If the set of blocks is contained in $P(V)$ one may also call it a '*hyper graph*' and its blocks, i.e. the occurring tuples of nodes its '*hyper edges*'. If the hyper edges are uniformly taken from $P^{(k)}(V)$ it is called a '*k-uniform*' hyper graph (see e.g. [26])
iv) As is the case for simple graphs and block spaces, one can now talk of the '*degree of a node*' in a hypergraph, i.e. the number of incident hyper edges, i.e. $|(n)|$ or the rank of a hyper edge, which is in our example simply the number of nodes belonging to it.

Remark: We would like to emphasize that, while this various definitions and introduced concepts seem to be, at first glance, more or less straightforward and related, they nevertheless turn out to be very efficient and economical tools to deal with a number of surprisingly deep questions in discrete mathematics from different perspectives (e.g. graphs, relational mathematics, incidence structures or block spaces; see the above mentioned literature).

4.22 Observation: As a simplex is uniquely given by its set of nodes, i.e. a set $\subset P(V)$, we can associate our complex QX/ST, i.e. the underlying graph G and the array of *mss*, with a hypergraph by identifying the S_ν's with its hyperbonds. This hypergraph is nothing but $\cup S_\nu \subset G$ (introduced in Observation 4.6), viz. it exhibits in general not the full wiring of G or QX but only the bonds occurring in the S_ν's, the true interbonds, however, are missing in $\cup S_\nu$.

In a next step one can go to a coarser level of resolution by defining the '*intersection graph*' of the set $\{S_\nu\}$; henceforth (as the notion hypergraph is already fixed in graph theory in the sense defined above) we prefer to call it the associated '*skeleton graph*' or '*super graph*'.

4.23 Observation: By shrinking the blocks S_ν to '*super nodes*' one gets another graph by saying that two super nodes S_ν, S_μ ($\nu \neq \mu$), are linked by a '*super bond*' if their intersection $S_\nu \cap S_\mu \neq \emptyset$. This '*super graph*' may then be associated with the manifold of '*unresolved*' physical points, ST, if we neglect their internal complexity (cf. this with what we said in 4.15, 4.16 above).

The above train of thought shows that one may impose on the underlying network QX a certain hierarchy of levels of varying resolution of the physical landscape which becomes more and more pronounced with increasing number of turned-off bonds, viz. with increasing clock time. Given an arbitrary but fixed block S_0, one can define its infinitesimal neighborhood $U_1(S_0)$ as consisting of the *mss* S_ν^0 with

$$S_\nu^0 \cap S_0 \neq \emptyset \tag{90}$$

in other words, the nodes adjacent to S_0 in the super graph ST of 4.23.

There may then exist *mss* S_μ with

$$S_\mu \cap S_\nu^0 \neq \emptyset \text{ for some } \nu \text{ but } S_\mu \cap S_0 = \emptyset \tag{91}$$

i.e. super nodes directly connected with some S_ν^0 but no longer with S_0 itself, viz. they belong to $U_1(S_\nu)$ but only to the second order neighborhood $U_2(S_0)$ of S_0 (as for this topological concept cf. Definition 2.1 above). Proceeding in this way it may be possible to impose some metrical near- and far-order on the network by taking in a certain approximation only the information into account which is encoded in the mutual overlap of the *mss* $S : \nu$ i.e. in $ST = \cup S_\nu$. What is projected out on this level are the (possibly still numerous) interbonds which may exist between the various disjunct mss. The details of the neighborhood structure encoded in the full QX/ST can then be rebuilt starting from the skeleton graph $ST = \cup S_\nu$.

In the physics of many degrees of freedom what typically matters is the strength of interaction between the various constituents. Furthermore this characteristic quantity is usually closely related with their spatial distance in the system under discussion and its dimension. Following these lines we will also organize our network.

Given two node sets A, B or the respective subgraphs we can count the number of bonds between them and regard this as a measure of their mutual dynamical coupling.

4.24 Definition (Connectivity): With A, B being two sets of the above kind we denote by$|A \sim B|$ the number of bonds connecting the nodes of A with the nodes of B and by $|A \sim B|_m$ their maximal possible number. Then we call

$$0 \leq c_{AB} := |A \sim B|/|a \sim B|_m \leq 1 \tag{92}$$

the '*connectivity*' of the pair A, B.
It represents the probability that a randomly picked up pair of nodes $n_A \in A$, $n_B \in B$ is connected by a bond.
$|A \sim B|_m$ depends however on the mutual relation between A and B!

4.25 Observation: i)

$$A \cap B = \emptyset \rightarrow |A \sim B|_m = |A| \cdot |B| \tag{93}$$

($|A|$, $|B|$ the respective number of nodes), hence

$$c_{AB} = |A \sim B|/|A| \cdot |B| \tag{94}$$

ii)

$$A = B \rightarrow |A \sim B|_m = \binom{|A|}{2} \tag{95}$$

iii)

$$\begin{aligned} A \cap B \neq \emptyset \rightarrow |A \sim B|_m &= |(A - B) \sim (B - A)|_m \\ &+ |(A\Delta B) \sim (A \cap B)|_m + |(A \cap B) \sim (A \cap B)|_m \end{aligned} \tag{96}$$

i.e:

$$|A \sim B|_m = |A - B| \cdot |B - A| + |A \cap B \cdot (|A - B| + |B - A|) + \binom{|A \cap B|}{2} \tag{97}$$

with $A\Delta B$ being the symmetric difference of A and B.

Remark: There exist of course other possibilities to quantify the degree of mutual influence among the various regions of the graph. One could e.g. admit not only direct bonds as connecting paths but paths up to a certain length etc. Furthermore there are a variety of (not primarily physically motivated) notions of connectedness in use in graph theory (e.g. the theorems of Menger, cf. [9]). The concept we developed above is adapted to our particular scenario with the mss as building blocks but with possibly a lot of surviving interbonds between disjunct grains $S_\nu, ?, S_\mu$ which may perhaps be quite a distance apart on the skeleton graph ST. Our notion is (among other things) able to measure these residual bonds which are in our view responsible for the observed '*non-locality*' of quantum theory.

4.26 The Metric/Topological Picture of QX/ST (a first Draft):
The picture we expect to emerge after the unfolding process is fully developed (i.e. α relatively large) is now the following:
i) It is one of the many fascinating observations made in [23] that, perhaps against the usual intuition, random graphs tend to be surprisingly regular, i.e. the generic(!) graph with, say, n nodes and m bonds tends to be almost '*translation invariant*', viz. the node degree is roughly the same over the graph (an effect of the peaked probability in phase space, similar to e.g. related phenomena in statistical mechanics). In other words, it is perhaps not too far-fetched to expect the same, at least on average, for the typical shape of the mss and their entanglement.
ii) Taking then a typical grain S_0 of ST, we expect its infinitesimal neighborhood $\{S_\nu^0\}$ to be densely connected with S_0 in the sense of Definition 4.24. In other words:

$$c_{S_0 S_0} = 1 \quad , \quad c_{S_0 S_\nu^0} \lesssim 1 \tag{98}$$

iii) Proceeding further with picking up grains S_μ in decreasing order of $c_{S_0 S_\mu}$ we can construct shell after shell around S_0 with weaker and weaker connectivity as regards to the central element S_0, i.e:

$$1 \geq c_{S_0 S_\nu^0} \geq c_{S_0 S_{\nu'}^1} \cdots \tag{99}$$

iv) We expect this process to be consistent with the neighborhood structure on the super graph ST which is defined by intersection, i.e. that node distance on ST corresponds more or less with the decrease in connectivity (properties like these have of course to be checked more systematically, analytically and/or via simulations on a computer which is under way).

5 Comments and Concluding Remarks

In this last section we would like to briefly recapitulate what has been accomplished in the preceeding sections and what remains to be done in forthcoming work. Furthermore we will focus on some longstanding goals which, we hope, can be acchieved by continuing the train of ideas developed and sketched above.

We have introduced and studied a class of cellular network models (rather, some simple candidates of a presumably much larger set) the dynamical laws of which are capable of performing some peculiar sort of *unfolding* in addition to the ordinary dynamics. This is implemented by introducing a socalled *hystheresis interval* given by the upper and lower critical parameters $\lambda_2,\ \lambda_1$ which regulate the switching-on and -off of elementary interactions between the nodes of the network.

We argue that the network evolution will drive the system for generic initial conditions towards a special type of attractor which, we conjecture, is the underlying hidden discrete substratum of our continuous space-time manifold (or rather, the physical vacuum) as we know it on a much coarser scale of resolution.

As particular constituents we describe the formation of physical (proto)points which consist in our scheme of densly entangled subclusters of, as we expect, roughly Planck scale size. The complex internal structure of these subclusters (mss) is supposed to be the carrier of the (quantum)fields which occur in continuum physics as elementary building blocks. This transition from the discrete network substratum to the continuum description shall be worked out in greater detail in forthcoming work together with a *correspondence principle* which relates both gravity and quantum theory with certain aspects of our scenario.

As to the technical side of the investigation, simulations and implementations of the various laws together with an appropriate sample of choices of the respective parameters and initial conditions are presently performed on a computer. They are designed to test and/or verify the conjectured scenarios described above. In this connection many details can and have to be checked like *length of transients, type of attractors, number, size and internal structure of the mss and their mutual overlap, cycle length, average node degree, number of extinct bonds, strength of fluctuations* and the like. The results shall be published in the near future.

Last but not least, we would like to relate our approach in more detail with the ideas of e.g. Sorkin et al., the conceptual scheme provided by Isham and the train of ideas developed by 't Hooft (see the references), not to mention the whole field of non-commutative geometry as such (see however our first paper [1]). We regret that this could not be done already in this paper due to lack of space. Furthermore, the deep results of the fascinating field of *random graphs* are only very briefly touched.

Acknowledgement: We would like to thank H.L.de Vries for many helpful remarks concerning the existing literature about various fields of discrete mathematics.

References

[1] M.Requardt: "Discrete Mathematics and Physics on the Planck-Scale exempli-

fied by means of a Class of 'Cellular Network Models' and their Dynamics", hep-th/9605103

[2] B.Riemann: "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen", section III,3 in 'B.Riemann Collected Papers, ed. R.Narasimhan, Springer/Teubner Berlin 1990

[3] L.Bombelli, J.Lee, D.Meyer, R.Sorkin: Phys.Rev.Lett. 59(1987)521
R.Sorkin: Int.Journ.Theor.Phys. 30(1991)923
A.P.Balachandran,G.Bimonte,E.Ercolesi,G.Landi,F.Lizzi,G.Sparano,P.Teotonio-Sobrinho:Journ.Geom.Phys. 18(1996)163

[4] F.Antonsen: Int.Journ.Theor.Phys. 33(1994)1189

[5] D.R.Finkelstein: "Quantum Relativity", Springer, Berlin 1995

[6] C.J.Isham: "An Introduction to General Topology and Quantum Topology" in "Physics,Geometry and Topology, ed. H.C.Lee, Plenum Pr., N.Y. 1990 or Class.Quant.Grav. 6(1989)1509, Class.Quant.Grav. 7(1990)1053 respectively

[7] P.Gibbs: "The Small Scale Structure of Space-Time: A Bibliographical Review, hep-th/9506171

[8] O.Ore:"Theory of Graphs", American Math. Soc., N.Y. 1962

[9] K.Wagner: "Graphentheorie", Bibliographisches Inst., Mannheim 1970

[10] G.Schmidt, Th.Stroehlein: "Relationen und Graphen", Springer, N.Y. 1989

[11] C.J.Isham: "Structural Issues in Quantum Gravity", Lectures given at the G14 Conference , Florence 1995, gr-qc/9510063 or
Lecures presented at the 1991 Schladming Winter School, in "Recent Aspects of Quantum Fields", eds. H.Mitter, H.Gausterer, Springer, Berlin 1992

[12] G.'t Hooft: J.Stat.Phys. 53(1988)323,
Nucl.Phys. B342(1990)471

[13] S.Wolfram: Physica 10D(1984)1

[14] J.L.Kelly: "General Topology", van Nostrand, N.Y. 1955 or
K.P.Grotemeyer: "Topologie", Bibl.Inst., Mannheim 1969

[15] L.Breiman: "Probability", Addison-Wesley, Reading Mass. 1968 or
H.Bauer: "Wahrscheinlichkeitstheorie", de Gruyter, N.Y. 1991

[16] H.Haken: "Synergetics", Springer, N.Y. 1978

[17] M.Waldrop: "Complexity", Viking Pr., London 1993

[18] D.Farmer: Zeitschr.Naturforsch. 37a(1982)1304

[19] W.Feller: "An Introduction to Probabiliy Theory and its Applications", Wiley, N.Y. 1971

[20] A.Renyi: "Wahrscheinlichkeitsrechnung", VEB Verlag, Berlin 1966

[21] E.Sperner: Math.Zeitschr. 27(1928)544

[22] H.Lueneburg: "Tools and Fundamental Constructions of Combinatorial Mathematics, BIWissenschaftsverlag, Mannheim 1989

[23] B.Bollobas: "Random Graphs", Academic Pr., London 1985

[24] A.Beutelspacher: "Einfuehrung in die endliche Geometrie", B.I.-Wissenschaftsverlag, Mannheim 1982

[25] H.-R.Halder, W.Heise: "Einfuehrung in die Kombinatorik", Hanser Verlag, Muenchen 1976

[26] B.Bollobas: "Combinatorics",Cambridge Univ. Pr., London 1986