BOUNDARY VALUES OF REGULAR RESOLVENT FAMILIES

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Abstract. We study properties of the boundary values $(H-\lambda\pm i0)^{-1}$ of the resolvent of a self-adjoint operator H for λ in a real open set Ω on which H admits a locally strictly conjugate operator A (in the sense of E. Mourre, i.e. $\varphi(H)^*[H,iA]\varphi(H)\geq a|\varphi(H)|^2$ for some real $a>0$ if $\varphi\in C_0^\infty(\Omega)$). In particular, we determine the Hölder-Zygmund class of the $B(\mathcal{E};\mathcal{F})$-valued maps $\lambda\mapsto(H-\lambda\pm i0)^{-1}$ and $\lambda\mapsto\Pi_\pm(H-\lambda\pm i0))^{-1}$ in terms of the regularity properties of the map $\tau\mapsto e^{-iA\tau}He^{iA\tau}$. Here $\mathcal{E}$, $\mathcal{F}$ are spaces from the Besov scale associated to A and $\Pi_\pm$ are the spectral projections of A associated to the half-lines $\pm x>0$.

1. Introduction

1.1. Let H be a self-adjoint operator in a (complex) Hilbert space $\mathcal{H}$. We denote $\sigma(H)$ the spectrum of H and for $z\in\mathbb{C}\setminus\sigma(H)$ we set $R(z)=(H-z)^{-1}$. The boundary values "$\lim_{\mu\to+0}$"$R(\lambda\pm i\mu)\equiv R(\lambda\pm i0)$ and their regularity properties as functions of the spectral parameter $\lambda\in\sigma(H)$ play an important role in many areas of mathematical physics (for example in quantum scattering theory, where H is the hamiltonian of a physical system and λ its energy). Such kind of problems have been studied by many authors, in particular Povzner, Kato, Kuroda, Agmon, Hörmander, etc. (see chapter 14 in [H] and also [BD], [K], [RS], [Y] and the references there). The limit "$\lim_{\mu\to+0}$", if it exists, is taken in a suitable space (and for a suitable topology) which may depend on the problem. A common procedure is to choose Banach spaces $\mathcal{E}$, $\mathcal{F}$ with dense embeddings $\mathcal{E}\subset\mathcal{H}\subset\mathcal{F}$, thus identifying $B(\mathcal{H})$ with a subspace of $B(\mathcal{E};\mathcal{F})$, and to take the limit for some natural topology on $B(\mathcal{E};\mathcal{F})$. This procedure is quite efficient in many concrete situations but it has the drawback that the choice of the space $\mathcal{E}$, $\mathcal{F}$ and the techniques used in the proof of the existence of the limit are ad hoc. The abstract schemes which were proposed before the eighties had, in general, a perturbative character, and so they did work only for rather limited classes of hamiltonians. This partly explains the fact that there was a rather complete theory for short-range perturbations of constant coefficients partial differential operators, while the theory in the case of long-range perturbations or N-body hamiltonians was in a much less satisfactory state.

The situation improved significantly with the paper [M1] of E. Mourre, where the notion of *conjugate operator* was introduced. Roughly speaking,

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a self-adjoint operator A is conjugated to H on an interval J if there exist a strictly positive real number a and a compact operator K such that $E(J)[H, iA]E(J) \geq aE(J) + K$, where E is the spectral measure of H. We ignore for the moment the regularity conditions that the couple (A, H) has to satisfy; these are, in fact, conditions on the map $\tau \mapsto e^{-iA\tau}He^{iA\tau}$ and are quite important in the theory. The results of Mourre in this context may be considered as a far reaching generalization of those of C.R. Putnam and T. Kato concerning couples of operators with a positive commutator, while his techniques are strongly related to the theory of dilation analytic Schrödinger operators due to J. Aguilar, E. Balslev and J.-M. Combes. One may find an account of Mourre's theory in (essentially) its original form in [CFKS]. The main feature is that it gives a systematic (and simple) procedure to construct the spaces $\mathcal{E}$, $\mathcal{F}$ once a conjugate operator is known. For example, one may take $\mathcal{E} = D(A)$, the domain of A equipped with the graph norm, and $\mathcal{F} = D(A)^*$ (the adjoint, or anti-dual space). On the other hand it is quite easy to construct conjugate operators for large classes of hamiltonians (like pseudo-differential operators or systems of operators, N-body Schrödinger hamiltonians, etc.). One may find interesting examples in the references listed at the end of this article, but one should consult [CFKS], [ABG] and the references there in order to get a correct perspective about the possibilities of the theory.

1.2. The version of the "conjugate operator method" that will be considered in this paper has been introduced in [BG2] (see also [BGS]). It has the advantage that only the resolvent operator $R(z)$ (and not H) is involved in the regularity conditions that the couple (A, H) has to satisfy. This allows one to treat very singular hamiltonians: the operator H need not be densely defined and the closure $\overline{D(H)}$ of its domain $D(H)$ need not be stable under the unitary group generated by A (N-body hamiltonians with strong interactions belong to this category, see [BGS]).

Let $\{R(z)\}_{z\in\mathbb{C}\setminus\mathbb{R}}$ be a self-adjoint resolvent family in $\mathcal{H}$, i.e. a family of bounded operators $R(z) \in B(\mathcal{H})$ such that $R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2)$ and $R(z)^* = R(\overline{z})$. The range of $R(z)$ is a subspace of $\mathcal{H}$ independent of z and, if we denote by P the orthogonal projection of $\mathcal{H}$ onto the closure of $R(z)\mathcal{H}$, there is a unique self-adjoint operator H in the Hilbert space $P\mathcal{H}$ such that $R(z) = (H - z)^{-1}P$. In the rest of the paper an operator like H will be just called self-adjoint (in $\mathcal{H}$); if $P = 1$ we shall say that H is a densely defined self-adjoint operator. For any bounded Borel complex function φ on $\mathbb{R} \cup \{\infty\}$ we set $\varphi(H) \equiv \varphi(H)P + \varphi(\infty)(1 - P)$.

Now let A be a densely defined self-adjoint operator in $\mathcal{H}$ and let us set $W_\tau = e^{i\tau A}$. The family $\{R(z)\}$ (or H) is of class $C^1(A)$ if the map $\tau \mapsto W_\tau^* R(z) W_\tau \in B(\mathcal{H})$ is Lipschitz for some $z \in \mathbb{C} \setminus \mathbb{R}$. If this is the case $D(A) \cap D(H)$ is a dense subspace of $D(H)$ (for the graph topology) and the sesquilinear form $[H, A] = HA - AH$ defined on $D(A) \cap D(H)$ by the expression $\langle Hf, Ag\rangle - \langle Af, Hg\rangle$ is continuous for the topology induced by $D(H)$. We denote by $\mathcal{A}[H]$ the unique continuous sesquilinear form on D(H) which extends $[H, A]$ and observe that $\varphi(H)i\mathcal{A}[H]\varphi(H)$ is canonically identified with a continuous (everywhere defined) symmetric operator in $\mathcal{H}$ if φ is a real function in $C_0^\infty(\mathbb{R})$.

If H is of class $C^1(A)$ we denote by $\tilde{\mu}^A(H)$ the set of $\lambda \in \mathbb{R}$ such that there exist a real function $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda) \neq 0$, a strictly positive number a and a compact operator K in $\mathcal{H}$ such that $\varphi(H) i\mathcal{A}[H]\varphi(H) \geq a\varphi^2(H) + K$. And $\mu^A(H)$ is the subset of those λ for which the preceding property holds with $K = 0$. So $\tilde{\mu}^A(H)$ is the set of all real points that have neighbourhoods on which A is conjugated to H. Clearly $\mu^A(H)$ and $\tilde{\mu}^A(H)$ are open real sets with $\mu^A(H) \subset \tilde{\mu}^A(H)$ and it can be easily shown that they differ very little. Indeed, *$\tilde{\mu}^A(H) \setminus \mu^A(H)$ is a countable set consisting of eigenvalues of H of finite multiplicity, and it does not have accumulation points inside $\tilde{\mu}^A(H)$. The operator H has no eigenvalues in $\mu^A(H)$.* Note that in quite general situations it is rather easy to describe $\tilde{\mu}^A(H)$ explicitly, because this set is stable under a large class of perturbations; but this is not the case for $\mu^A(H)$.

We expect that H has nice spectral properties in the set $\mu^A(H)$, e.g. that it has no singularly continuous spectrum, but this has not yet been shown. What is certain is that the regularity class $C^1(A)$ is too weak to assure the validity of the so-called limiting absorption principle for $\{R(z)\}$ in spaces of the Sobolev scale associated to A (see Chapter 7 in [ABG] for a detailed discussion of this problem).

A minimal (on the Besov scale) regularity condition under which estimates of the form $|\langle f, R(\lambda + i\mu)f\rangle| \leq C(\lambda, f) < \infty$ hold for each $\lambda \in \mu^A(H)$ and each $f \in \cap_{n=1}^\infty D(A^n)$ has been isolated in [BG1]. Resolvent families which fulfil this condition will be called regular. More precisely, we say that $\{R(z)\}$ is an *A-regular resolvent family* if for some z the following condition is satisfied:

$$\int_0^1 ||W_{2\tau}^* R(z) W_{2\tau} - 2W_\tau^* R(z) W_\tau + R(z)|| \tau^{-2} d\tau < \infty.$$

Let $\mathcal{K}$ be the space of vectors $f \in \mathcal{H}$ with $\int_0^1 ||W_\tau f - f|| \tau^{-3/2} d\tau < \infty$ equipped with the natural topology, so that $\mathcal{K}$ is continuously and densely embedded in $\mathcal{H}$. We identify $\mathcal{K} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{K}^*$, in particular $B(\mathcal{H}) \subset B(\mathcal{K}; \mathcal{K}^*)$. Set $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}$. *If $\{R(z)\}$ is an A-regular resolvent family which has a spectral gap* (i.e. $\sigma(H) \neq \mathbb{R}$), *then the holomorphic maps $\mathbb{C}_\pm \ni z \mapsto R(z) \in B(\mathcal{K}; \mathcal{K}^*)$ have weak* continuous extensions to the sets $\mathbb{C}_\pm \cup \mu^A(H)$.* This result has been proved in [BG2] (see also [ABG] and references therein). Results of the same nature, but with stronger regularity hypotheses on H, have been proved in [M1], [PSS], [JP]; see [CFKS] for a survey.

Our purpose in this paper is to study, in the framework of the preceding theorem, the operators $R(\lambda \pm i0) = \text{w*-}\lim_{\mu \to +0} R(\lambda \pm i\mu) \in B(\mathcal{K}; \mathcal{K}^*)$, with $\lambda \in \mu^A(H)$, and the regularity properties of the maps $\lambda \mapsto R(\lambda \pm i0)$ when considered as $B(\mathcal{E}; \mathcal{F})$-valued, with $\mathcal{E}$, $\mathcal{F}$ spaces from the Besov scale associated to A. We shall state these regularity properties in terms of Hölder-Zygmund spaces, which gives at the same time natural and optimal results.

1.3. We begin with the definition of the Hölder-Zygmund classes Λ^α, $\alpha > 0$ real. Let $\mathbf{E}$ be a Banach space and $\phi : \mathbb{R} \to \mathbf{E}$ a bounded continuous function. If $0 < \alpha < 1$ then ϕ is of class Λ^α if there is a finite constant

c such that $||\phi(x+\varepsilon)-\phi(x)|| \leq c|\varepsilon|^\alpha$ for all $x,\varepsilon \in \mathbb{R}$. ϕ is of class Λ^1 if $||\phi(x+\varepsilon)+\phi(x-\varepsilon)-2\phi(x)|| \leq c|\varepsilon|$ for a constant c and all x,ε. Note that Λ^1 is Zygmund's class of "smooth" functions and is sensibly larger than the class of Lipschitz functions (of order 1). If $\alpha > 0$ is arbitrary, write $\alpha = k+\sigma$ where $k \in \mathbb{N}$ and $0 < \sigma \leq 1$. Then ϕ is of class Λ^α if its derivative of order k is of class Λ^σ. If Ω is a real open set and $\phi : \Omega \to \mathbf{E}$, we say that ϕ is locally of class Λ^α if $\theta\phi$ is of class Λ^α for each $\phi \in C_0^\infty(\Omega)$.

One may define classes $\Lambda^{\alpha,p}$, with $\alpha > 0$ real and $1 \leq p \leq \infty$, by a natural extension of the preceding procedure, see §2.1. We mention only two facts, namely $\Lambda^\alpha = \Lambda^{\alpha,\infty}$ and $\Lambda^{s,p} \subset \Lambda^{t,q}$ if and only if $s > t$ or $s = t$ but $p \leq q$. The classes $\Lambda^{\alpha,p}$ are convenient for a unified presentation of some Besov type spaces of vectors and operators associated to A. Let s be a strictly positive real number and let $1 \leq p \leq \infty$. Then $\mathcal{H}_{s,p}$ is the set of vectors $f \in \mathcal{H}$ such that the map $\tau \mapsto W_\tau f \in \mathcal{H}$ is of class $\Lambda^{s,p}$. And $\mathcal{C}^{s,p}(A)$ is the set of operators $S \in B(\mathcal{H})$ such that the map $\tau \mapsto W_\tau^* S W_\tau \in B(\mathcal{H})$ is of class $\Lambda^{s,p}$. We shall say that a resolvent family is of class $\mathcal{C}^{s,p}(A)$ if one of the operators $R(z)$ is of class $\mathcal{C}^{s,p}(A)$. Then the resolvent family is A-regular if and only if it is of class $\mathcal{C}^{1,1}(A)$.

It is possible to extend the scale $\mathcal{H}_{s,p}$ to $s \leq 0$, see §2.2 (we do not need the classes $\mathcal{C}^{s,p}(A)$ for $s \leq 0$). We mention the following facts. Each space $\mathcal{H}_{s,p}$ has a natural Banach space topology such that if $s > t$ or $s = t$ and $p \leq q$ then $\mathcal{H}_{s,p}$ is continuously embedded in $\mathcal{H}_{t,q}$. If $q < \infty$ then this embedding is dense. The space $\mathcal{H}$ is assumed to be identified with its adjoint space $\mathcal{H}^*$ with the help of the Riesz isomorphism. Let $\mathcal{H}_\infty = \cap_{n=1}^\infty D(A^n)$, then $\mathcal{H}_\infty \subset \mathcal{H}_{s,p}$ for all s,p and, if we denote $\mathcal{H}_{s,p}^\circ$ the closure of $\mathcal{H}_\infty$ in $\mathcal{H}_{s,p}$, then $\mathcal{H}_{s,p}^\circ = \mathcal{H}_{s,p}$ if $p < \infty$. One has canonically $[\mathcal{H}_{s,p}^\circ]^* = \mathcal{H}_{-s,p'}$, where $1/p + 1/p' = 1$. For real $s \geq 0$ we have $\mathcal{H}_{s,2} = D(|A|^s)$.

In the next two theorems we state the main results of this paper. Note that in these theorems the resolvent family $\{R(z)\}$ is at least of class $\mathcal{C}^{1,1}(A)$, so it is A-regular, and that we always assume that it has a spectral gap (i.e. $\sigma(H) \neq \mathbb{R}$), so that the operators $R(\lambda \pm i0)$ are well defined elements of $B(\mathcal{K};\mathcal{K}^*)$ if $\lambda \in \mu^A(H)$ (according to the results described in §1.2). Moreover, one has $\mathcal{K} = \mathcal{H}_{1/2,1}$, and so $\mathcal{K}^* = \mathcal{H}_{-1/2,\infty}$. If $s > 1/2$ then for all $p,q \in [1,\infty]$ we have $\mathcal{H}_{s,p} \subset \mathcal{K}$ continuously and densely and $\mathcal{K}^* \subset \mathcal{H}_{-s,q}$ continuously, so we have a canonical continuous embedding $B(\mathcal{K};\mathcal{K}^*) \subset B(\mathcal{H}_{s,p};\mathcal{H}_{-s,q})$ and we may consider $\lambda \mapsto R(\lambda \pm i0)$ as $B(\mathcal{H}_{s,p};\mathcal{H}_{-s,q})$-valued functions with domain $\mu^A(H)$.

Theorem A. *Let $\{R(z)\}$ be a resolvent family having a spectral gap.*

(a) *If $\{R(z)\}$ is of class $\mathcal{C}^{s+1/2}(A)$ for some real $s > 1/2$, then the maps $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_{s,\infty};\mathcal{H}_{-s,1})$ are locally of class $\Lambda^{s-1/2}$ on $\mu^A(H)$.*

(b) *Assume that $\{R(z)\}$ is of class $\mathcal{C}^{s+1/2,1}(A)$ for a real number s such that $s-1/2$ is an integer ≥ 1. Then the maps $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_{s,1};\mathcal{H}_{-s,\infty})$ are of class $C^{s-1/2}$ in the weak* topology and their derivatives of order $k = 0,1,2,\ldots,s-1/2$ are given by*

$$\frac{d^k}{d\lambda^k} R(\lambda \pm i0) = \lim_{\mu \to +0} k! R(\lambda \pm i\mu)^{k+1},$$

where the limits exist weakly in $B(\mathcal{H}_{s,1};\mathcal{H}_{-s,\infty})$, locally uniformly in $\lambda \in \mu^A(H)$.*

The weak* topology we refer to in part (b) above is, of course, determined by the identification $\mathcal{H}_{-s,\infty} = [\mathcal{H}_{s,1}]^*$. Clearly one can reformulate part (b) in an apparently stronger form: *the holomorphic functions $\mathbb{C}_\pm \ni z \mapsto R(z) \in B(\mathcal{H}_{s,1};\mathcal{H}_{-s,\infty})$ extend to functions of weak* class $C^{s-1/2}$ on $\mathbb{C}_\pm \cup \mu^A(H)$.* Similarly, part (a) says that the holomorphic functions $\mathbb{C}_\pm \ni z \mapsto R(z) \in B(\mathcal{H}_{s,\infty};\mathcal{H}_{-s,1})$ extend to functions locally of class $\Lambda^{s-1/2}$ on $\mathbb{C}_\pm \cup \mu^A(H)$ (the local Λ^α classes have an obvious definition for functions defined on manifolds with boundary, like $\mathbb{C}_\pm \cup \mu^A(H)$). When we speak about the Λ^α class of a map with values in $B(\mathcal{E};\mathcal{F})$, where $\mathcal{E}$, $\mathcal{F}$ are Banach spaces, we have in mind the norm topology of $B(\mathcal{E};\mathcal{F})$. However, due to the uniform boundedness principle, we get the same class if we consider on $B(\mathcal{E};\mathcal{F})$ the weak (or weak*, if $\mathcal{F} = \mathcal{G}^*$) topology.

The assertions of Theorem A are optimal on each of the scales $\mathcal{C}^{t,q}(A)$, $\mathcal{H}_{s,p}$ and Λ^α. A detailed discussion of this question may be found in [BGSh2] and [BG3]; see also §1.4 below. Under stronger regularity conditions on the operator H the continuity properties of the maps $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_{s,2};\mathcal{H}_{-s,2})$ have been studied before in [PSS], [W], [JMP] (see also [ABG]). In [W], for example, it is shown that this map is locally Λ^α with $\alpha = (s-1/2)(s+1/2)^{-1}$ and $1/2 < s \le 1$. In [JMP] similar results are obtained in the region $s > 1$. In [BG3] a slightly weaker version of part (a) of Theorem A has been proved.

If λ is a spectral value of H which belongs to $\mu^A(H)$ then $R(\lambda \pm i0)\mathcal{H}_{1/2,1} \subset \mathcal{H}_{-1/2,\infty}$ and this assertion is optimal on the Besov scale: indeed, $R(\lambda \pm i0)\mathcal{H}_\infty$ is not included in $\mathcal{H}^\circ_{-1/2,\infty}$ in general. Roughly speaking, the vectors of the form $R(\lambda \pm i0)f$ do not decay at infinity in the spectral representation of A. However, simple examples, like those treated in §1.4 below, suggest that there should be an asymmetry between the behaviour in the region where $A \to +\infty$ and that where $A \to -\infty$. More precisely, a vector like $R(\lambda + i0)f$ does not decay in the region $A \to +\infty$ but behaves quite well as $A \to -\infty$. The first general result of this nature has been obtained in [M2]; see [JMP] and [J] for refinements. Our next theorem is an optimal result along these lines.

We denote $\Pi_\pm$ the spectral projections of A associated to the intervals $\pm[0,\infty)$. Note that $\Pi_\pm$ induce bounded operators in each space $\mathcal{H}_{s,p}$, so the products $\Pi_\mp R(\lambda \pm i0)$ are well defined. On a space of the form $B(\mathcal{H}_{s,p};\mathcal{H}_{t,q})$ we shall consider, besides the norm topology, the w-topology, which is defined by the family of seminorms $T \mapsto |\langle g, Tf\rangle|$ with $f \in \mathcal{H}_{s,p}$ and $g \in \mathcal{H}_{-t,q'}$ $(1/q + 1/q' = 1)$. It can be shown that *$B(\mathcal{H}_{s,p};\mathcal{H}_{t,q})$ is sequentially complete in the w-topology* (see §2.6 in [BGSh2]).

Theorem B. *Let $\{R(z)\}$ be a resolvent family with a spectral gap and of class $\mathcal{C}^{s+1/2,p}(A)$ for some real $s > 1/2$ and some $p \in [1,\infty]$. Then one has*

$$\Pi_\mp R(\lambda \pm i0)\mathcal{H}_{s,p} \subset \mathcal{H}_{s-1,p}$$

for all $\lambda \in \mu^A(H)$ and the maps $\lambda \mapsto \Pi_\mp R(\lambda \pm i0) \in B(\mathcal{H}_{s,p};\mathcal{H}_{s-1,p})$ are w-continuous. If α is an integer such that $0 \le \alpha < s - 1/2$, then the maps

$$\lambda \mapsto \Pi_\mp R(\lambda \pm i0) \in B(\mathcal{H}_{s,p};\mathcal{H}_{s-1-\alpha,p})$$

are of class C^α in the w-topology. If α is an arbitrary real number such that $0 < \alpha < s - 1/2$, then the functions

$$\lambda \mapsto \Pi_\mp R(\lambda \pm i0) \in B(\mathcal{H}_{s,\infty}; \mathcal{H}_{s-1-\alpha,1})$$

are locally of class Λ^α on $\mu^A(H)$.

The assertions of Theorem B are optimal in a sense described in [BGSh1,2]. In [BGSh1] one may find several propagation theorems that are corollaries of the preceding result. For applications in quantum scattering theory, see [JMP] and [K]. In the Theorems 5.7 and 5.8 below (see Section 5) we discuss properties of the operators $\Pi_\mp R(\lambda \pm i0)\Pi_\pm$; these are extensions of results from [M2], [JMP], [J].

There is one unnatural condition in the preceding theorems, the spectral gap condition, and this is annoying for some applications (e.g. it excludes Stark effect hamiltonians). If H is densely defined and of class $\mathcal{C}^\alpha(A)$ with $1 < \alpha < 3/2$ this condition has been eliminated in [Sh1,2].

1.4. In order to clarify the significance and the implications of the Theorems A and B we shall now discuss two examples. The first one is elementary but fundamental: the general theory may be considered as a noncommutative version of this model. Let H be the operator of multiplication by the real measurable function h in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and let A be the usual self-adjoint realization of $i(d/dx)$. Although by our definition of regularity we could treat quite singular functions h (e.g. H is of class $C^\infty(A)$ if h is an arbitrary rational function), we shall assume, for simplicity, that h is bounded. Then H is of class $C^1(A)$ if and only if h is a Lipschitz function, and H is A-regular if and only if h belongs to the (usual) Besov space $B^{1,1}_\infty(\mathbb{R})$ (this implies $h \in C^1(\mathbb{R})$). Again for simplicity we assume that $h \in C^1(\mathbb{R})$ and that $h'(x) > 0$ $\forall x \in \mathbb{R}$. Then the range of h is a bounded open interval $I = (a, b)$, the spectrum of H is $\sigma(H) = [a, b]$, and $\mu^A(H) = \mathbb{R} \setminus \{a, b\}$. Let $f, g \in L^2(\mathbb{R})$ and let us set

$$u(x) = \overline{g(h^{-1}(x))} f(h^{-1}(x)) [h'(h^{-1}(x))]^{-1} \text{ for } x \in I.$$

Clearly for $z \notin \mathbb{R}$ we have

$$\langle g, R(z)f\rangle = \int_I u(x)(x-z)^{-1}dx$$

hence, if we denote $\tilde{u}$ and $\hat{u}$ the Hilbert and the Fourier transform of u, we formally get

$$\pi^{-1}\langle g, R(\lambda + i0)f\rangle = \tilde{u}(\lambda) + iu(\lambda) = 2i\pi^{-1} \int_0^\infty e^{i\lambda x}\hat{u}(x)dx.$$

By using these explicit expressions one may check the optimality of the Theorems A and B and one can understand the connection between part (a) of Theorem A and some classical properties of the Hilbert transformation (e.g. the fact that it leaves invariant the Λ^α classes). On the other hand, these theorems allow one to treat fairly easily the n-dimensional version of the preceding example. As a consequence one may get, in the context of Theorem 7.6.2 from [ABG], quite precise continuity properties of the distributions $[h(x) - \lambda \mp i0)]^{-k}$, where $k \geq 1$ is an integer.

It is much more interesting, however, to consider generalized Schrödinger hamiltonians of the form $H = h(P) + V$ in $\mathcal{H} = L^2(\mathbb{R}^n)$, with $P = (-i\partial_1, \ldots, -i\partial_n)$. We work in the setting of §7.6.3 from [ABG] and we recall several hypotheses and notations. The function $h : \mathbb{R}^n \to \mathbb{R}$ is assumed to be of class C^m for some integer $m \geq 2$ and such that:

(a) $h(x) \to +\infty$ as $|x| \to \infty$;

(b) the derivatives $h^{(\alpha)}$ of order $|\alpha| = m$ are bounded;

(c) $|h^{(\alpha)}(x)| \leq C(1 + |h(x)|)$ if $|\alpha| < m$.

We denote H_0 the usual self-adjoint realization of $h(P)$ in $\mathcal{H}$, $\mathcal{G} = D(|H_0|^{1/2})$ is the form domain of H_0 (equipped with the graph topology), and we identify $\mathcal{G} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{G}^*$. Then $\mathcal{G}_{s,p}$ and $\mathcal{G}^*_{s,p}$ ($s \in \mathbb{R}$, $1 \leq p \leq \infty$) are the Besov scales associated to the operator $\langle Q \rangle$ of multiplication by $\langle x \rangle = (1 + |x|^2)^{1/2}$ in $\mathcal{G}$ and $\mathcal{G}^*$ respectively (e.g. the norm in $\mathcal{G}_{s,2}$ is $||\langle Q \rangle^s f||_{\mathcal{G}}$; see page 343 in [ABG] for details). We consider only a rather restricted class of perturbations V (see [Sh1] for the case of non-local singular perturbations satisfying natural extensions of the conditions (7.6.19), (7.6.20) in [ABG]). Let σ be a real number such that $0 < \sigma \leq m - 1$ and let $V = \sum V_k$, where the sum runs over all the integers k satisfying $0 \leq k < \sigma + 1$. We assume that each V_k is a bounded real function on $\mathbb{R}^n$ which tends to zero at infinity and which satisfies $\sum_{|\alpha|=k} |V_k^{(\alpha)}(x)| \leq C\langle x \rangle^{-1-\sigma}$. Now let $s = \sigma + 1/2$ and denote by $\kappa(H)$ the union of the set $\kappa(h)$ of critical values of the function h and of the set of eigenvalues of H; $\kappa(H)$ is a closed real set and its accumulation points belongs to $\kappa(h)$. Then for each $\lambda \in \mathbb{R} \setminus \kappa(H)$ the limit $\lim_{\mu \to +0}(H - \lambda \mp i\mu)^{-1} \equiv R(\lambda \pm i0)$ exists in norm in $B(\mathcal{G}^*_{s,\infty}; \mathcal{G}_{-s,1})$, locally uniformly in λ, and the maps $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{G}^*_{s,\infty}; \mathcal{G}_{-s,1})$ are locally of class Λ^σ on $\mathbb{R} \setminus \kappa(H)$. These assertions follow easily from part (a) of Theorem A, as explained in §7.6.3 from [ABG]. Similar results hold for N-body hamiltonians. For the case of Dirac operators, Stark effect hamiltonians and simply characteristic operators, see [Sh1].

1.5. The paper is organized as follows. In Section 2 we introduce two Besov type scales associated to a self-adjoint operator A: one consisting of vectors from $\mathcal{H}$ (these are the spaces $\mathcal{H}_{s,p}$ with $s > 0$) and one consisting of bounded operators on $\mathcal{H}$. We briefly recall those properties that will be needed later on (a complete treatment in a general setting may be found in [ABG], but see also [BB] and [BGSh2]) and we prove two estimates (Theorems 2.1 and 2.2) which will be important for the proofs of the main theorems. In §2.4 we recall a regularization procedure introduced in [BG1] and a result from [BG3]: these are the main technical tools of our paper. In Section 3 one may find a description of the regularity classes of arbitrary self-adjoint operators and of the Mourre estimate in the context of general resolvent families: the main result of this section is Proposition 3.1, which improves a result from [BGS]. Section 4 is the most technical one. We consider there a symmetric bounded A-regular operator $H \in B(\mathcal{H})$, we regularize it by considering the operator $H(\varepsilon) = \theta(\varepsilon\mathcal{A})H$ (in the commutative situation of §1.4 this means that we approximate the function h by an entire function of exponential type; see the comment after Theorem 2.3), and then we "twist" $H(\varepsilon)$ in the spirit of the theory of dilation analytic Schrödinger operators, i.e. we introduce the operator $H_\varepsilon = e^{-\varepsilon A}H(\varepsilon)e^{\varepsilon A} = e^{\varepsilon\mathcal{A}}\theta(\varepsilon\mathcal{A})H \equiv \xi(\varepsilon\mathcal{A})H$. All Section 4 is devoted to estimating the resolvent of the non-self-adjoint operator H_ε. Finally, the main results of the paper are proved in Section 5. Note that the spectral gap hypothesis allows a rather straightforward reduction to the case when H is a bounded (everywhere defined) operator.

2. Regularity Classes Associated to a Selfadjoint Operator A

2.1. Let $\mathbf{E}$ be a Banach space and $\phi : \mathbb{R} \to \mathbf{E}$ a bounded continuous function. We recall that the modulus of continuity (or smoothness) of order m (integer ≥ 1) of ϕ is given by $\omega_m(\varepsilon) = \sup_{x\in\mathbb{R}} || \sum_{k=0}^{m}(-1)^k C_m^k \phi(x+k\varepsilon)||_{\mathbf{E}}$ for $\varepsilon > 0$. Let $s > 0$ be a real number and $p \in [1,\infty]$. Then ϕ *is of class* $\Lambda^{s,p}$ if there is an integer $m > s$ such that $[\int_0^1(\varepsilon^{-s}\omega_m(\varepsilon))^p\varepsilon^{-1}d\varepsilon]^{1/p} < \infty$ (if $p = \infty$ this means $\omega_m(\varepsilon) \leq c\varepsilon^s$ for a finite constant c). One has $\Lambda^{s,p} \subset \Lambda^{t,q}$ if and only if $s > t$ or $s = t$ but $p \leq q$. The classes $\Lambda^{s,\infty} \equiv \Lambda^s$ are called *Hölder-Zygmund classes*.

If $k \geq 1$ is an integer the classes BC^k and $\mathrm{Lip}^{(k)}$ are defined as follows: $\phi \in BC^k$ means that the derivatives of order $\leq k$ of ϕ exist and are bounded and norm continuous; $\phi \in \mathrm{Lip}^{(k)}$ means that $\omega_k(\varepsilon) \leq c\varepsilon^k$ for a constant c and all $\varepsilon > 0$ (this is a k-th order Lipschitz condition). Then we have $\Lambda^{k,1} \subset BC^k \subset \mathrm{Lip}^{(k)} \subset \Lambda^k$, all embeddings being strict and optimal on the scale $\Lambda^{s,p}$ (the spaces $\Lambda^{k,p}$ are not comparable with BC^k and $\mathrm{Lip}^{(k)}$ if $1 < p < \infty$). Now assume that $s = k + \sigma$ where $k \geq 1$ is an integer and $\sigma > 0$ is a real number. Then $\phi \in \Lambda^{s,p}$ if and only if $\phi \in BC^k$ and $\phi^{(k)} \in \Lambda^{\sigma,p}$. In particular, if $0 < \sigma < 1$ then $\phi \in \Lambda^s$ means $\phi \in BC^k$ and $||\phi^{(k)}(x+\varepsilon) - \phi^{(k)}(x)||_{\mathbf{E}} \leq C|\varepsilon|^\sigma$; if $\sigma = 1$ then $\phi \in \Lambda^s$ means that $\phi \in BC^k$ and $||\phi^{(k)}(x+\varepsilon) + \phi^{(k)}(x-\varepsilon) - 2\phi^{(k)}(x)||_{\mathbf{E}} \leq C|\varepsilon|$, i.e. $\phi^{(k)}$ has to verify a Zygmund condition.

There are natural local classes associated to the preceding ones. For example, if Ω is an open real set and $\phi : \Omega \to \mathbf{E}$ is a continuous function, then ϕ is locally of class $\Lambda^{s,p}$ if $\theta\phi \in \Lambda^{s,p}$ for each $\theta \in C_0^\infty(\Omega)$.

2.2. Let $\mathcal{H}$ be a complex Hilbert space and A a densely defined self-adjoint operator in $\mathcal{H}$ (these objects are fixed from now on). We denote by E_A the spectral measure of A and we set $W_\sigma = e^{iA\sigma}$ for $\sigma \in \mathbb{R}$. We identify $\mathcal{H}$ with its adjoint space $\mathcal{H}^*$ (space of anti-linear continuous functionals on $\mathcal{H}$) with the help of the Riesz isomorphism and we denote by $||\cdot||$ the norm in $\mathcal{H}$ and in $B(\mathcal{H})$. The scalar product $\langle\cdot,\cdot\rangle$ in $\mathcal{H}$ is linear in the second variable.

We denote $\mathcal{H}_\infty$ the vector space $\cap_{k\in\mathbb{N}}D(A^k)$ equipped with the natural Fréchet space topology and then we define $\mathcal{H}_{-\infty}$ as the adjoint space $[\mathcal{H}_\infty]^*$ equipped with the strong topology. Since the embedding $\mathcal{H}_\infty \subset \mathcal{H}$ is continuous and dense, we obtain by transposition a continuous dense embedding $\mathcal{H} = \mathcal{H}^* \subset \mathcal{H}_{-\infty}$. The Besov spaces $\mathcal{H}_{s,p}$ (with $s \in \mathbb{R}$ and $p \in [1,\infty]$) associated to A are Banach spaces continuously embedded in $\mathcal{H}_{-\infty}$. To define them, note first that for each compact real set K the spectral projection $E_A(K)$ extends to a continuous linear map $E_A(K) : \mathcal{H}_{-\infty} \to \mathcal{H}_\infty$. For real τ we set $E_A(\tau) = E_A([-2\tau,-\tau] \cup [\tau,2\tau])$ and for each $f \in \mathcal{H}_{-\infty}$, $s \in \mathbb{R}$ and

$1 \le p \le \infty$ we define the number $||f||_{s,p} \in [0,\infty]$ by:

$$||f||_{s,p} \equiv ||E_A([-2,2])f|| + \Big[\int_1^\infty ||\tau^s E_A(\tau)f||^p \tau^{-1} d\tau\Big]^{1/p}. \tag{2.1}$$

If $p=\infty$ then the second term on the r.h.s. should be read $\sup_{\tau\ge 1}||\tau^s E_A(\tau)f||$. Finally, we may define $\mathcal{H}_{s,p}$ as the space of vectors $f \in \mathcal{H}_{-\infty}$ such that $||f||_{s,p} < \infty$, equipped with the Banach space structure associated to (2.1). We have $\mathcal{H}_\infty \subset \mathcal{H}_{s,p}$ and we denote by $\mathcal{H}^\circ_{s,p}$ the closure of $\mathcal{H}_\infty$ in $\mathcal{H}_{s,p}$. In fact $\mathcal{H}^\circ_{s,p} = \mathcal{H}_{s,p}$ if $1 \le p < \infty$.

We recall that $\mathcal{H}_{s,p} \subset \mathcal{H}_{t,q}$ (continuous embedding) if $s > t$ or if $s = t$ but $p \le q$. If $1 \le p < \infty$ we have a continuous dense embedding $\mathcal{H}_\infty \subset \mathcal{H}_{s,p}$, hence one can realize $[\mathcal{H}_{s,p}]^*$ as a subspace of $\mathcal{H}_{-\infty}$; indeed, one has $[\mathcal{H}_{s,p}]^* = \mathcal{H}_{-s,p'}$ if $p^{-1}+p'^{-1}=1$. Similarly $[\mathcal{H}^\circ_{s,\infty}]^* = \mathcal{H}_{-s,1}$.

If $s>0$ then $\mathcal{H}_{s,p}$ is the set of vectors $f \in \mathcal{H}$ such that the map $\tau \mapsto W_\tau f \in \mathcal{H}$ is of class $\Lambda^{s,p}$. Another equivalent description is as follows: assume that $0 < s < m$ with m integer and let $f \in \mathcal{H}$; then $f \in \mathcal{H}_{s,p}$ if and only if $[\int_0^1 ||\varepsilon^{-s}(W_\varepsilon - 1)^m f||^p \varepsilon^{-1} d\varepsilon]^{1/p} < \infty$. If $s = k$ is an integer ≥ 1 and $p=2$ then $f \in \mathcal{H}_{k,2}$ if and only if $\tau \mapsto W_\tau f \in \mathcal{H}$ is of class BC^k (or, equivalently, of class $\mathrm{Lip}^{(k)}$). In particular $\mathcal{H}_{k,2} = D(A^k)$. It will be convenient to set $\mathcal{H}_s = \mathcal{H}_{s,2}$ for all $s \in \mathbb{R}$; in particular $\mathcal{H}_0 = \mathcal{H}$ (as topological vector spaces).

Theorem 2.1. *Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a locally bounded Borel function and α a real number, and assume that there are real numbers $\nu > 0$ and c such that $|\varphi(x)| \le c|x|^\alpha \min(|x|^\nu, |x|^{-\nu})$ for all $x \in \mathbb{R}$. Then there is a constant $C < \infty$ such that for all $s \in \mathbb{R}$, all $p \in [1,\infty]$ and all $f \in \mathcal{H}_{-\infty}$:*

$$\Big[\int_0^1 ||\varepsilon^{-\alpha}\varphi(\varepsilon A)f||^p_{s,1}\varepsilon^{-1}d\varepsilon\Big]^{1/p} \le C||f||_{s+\alpha,p}. \tag{2.2}$$

In particular $||\varphi(\varepsilon A)f||_{s,1} \le C\varepsilon^\alpha ||f||_{s+\alpha,\infty}$ for all $\varepsilon \in (0,1)$ and $f \in \mathcal{H}_{-\infty}$.

Proof. (i) Set $\varrho(u) = \min(u^\nu, u^{-\nu})$ for $u > 0$ and observe that for $0 < a < x < 2a$ we have $\varrho(x) \le 2^\nu \varrho(a)$. We show that for all $\varepsilon > 0$, $\tau > 0$ and $f \in \mathcal{H}_{-\infty}$ one has

$$||E_A(\tau)\varphi(\varepsilon A)f|| \le c2^\nu \max(1,2^\alpha)(\varepsilon\tau)^\alpha \varrho(\varepsilon\tau)||E_A(\tau)f||. \tag{2.3}$$

Indeed, we have by hypothesis $|\varphi(x)| \le c|x|^\alpha \varrho(|x|)$ and, if we denote by χ the characteristic function of the set $[-2,-1]\cup[1,2]$, then the l.h.s. of (2.3) can be estimated as follows

$$\begin{aligned} ||\chi(A/\tau)\varphi(\varepsilon A)f|| &\le c||\chi(A/\tau)|\varepsilon A|^\alpha \varrho(\varepsilon|A|)f|| \\ &\le c\max(1,2^\alpha)(\varepsilon\tau)^\alpha ||\chi(A/\tau)\varrho(\varepsilon|A|)f|| \\ &\le c\max(1,2^\alpha)(\varepsilon\tau)^\alpha \sup_{\tau<x<2\tau} \varrho(\varepsilon x)||\chi(A/\tau)f|| \end{aligned}$$

which clearly implies (2.3).

(ii) Let us set $C_1 = c2^\nu \max(1, 2^\alpha)$. Then from (2.1) and (2.3) we get

$$||\varepsilon^{-\alpha}\varphi(\varepsilon A)f||_{s,1} = ||E_A([-2,2])\varepsilon^{-\alpha}\varphi(\varepsilon A)f|| + \int_1^\infty ||\tau^s E_A(\tau)\varepsilon^{-\alpha}\varphi(\varepsilon A)f||\tau^{-1}d\tau$$
$$\leq ||E_A([-2,2])\varepsilon^{-\alpha}\varphi(\varepsilon A)f|| + C_1\int_1^\infty \varrho(\varepsilon\tau)||\tau^{s+\alpha}E_A(\tau)f||\tau^{-1}d\tau$$
$$= ||E_A([-2,2])\varepsilon^{-\alpha}\varphi(\varepsilon A)f|| + C_1\int_0^\infty \varrho(\tau)||\chi_1(\tau/\varepsilon)(\tau/\varepsilon)^{s+\alpha}E_A(\tau/\varepsilon)f||\tau^{-1}d\tau$$

where χ_1 is the characteristic function of the interval $(1,\infty)$. Now observe that if g is a positive Borel function on $(0,\infty)$ then for $\tau \geq 1$ we have

$$\Big[\int_0^1 g(\tau/\varepsilon)^p\varepsilon^{-1}d\varepsilon\Big]^{1/p} = \Big[\int_\tau^\infty g(\sigma)^p\sigma^{-1}d\sigma\Big]^{1/p} \leq \Big[\int_1^\infty g(\sigma)^p\sigma^{-1}d\sigma\Big]^{1/p}.$$

Since $\int_0^\infty \varrho(\tau)\tau^{-1}d\tau = 2/\nu$ we see that the l.h.s. of (2.2) is smaller than

$$[\int_0^1 ||E_A([-2,2])\varepsilon^{-\alpha}\varphi(\varepsilon A)f||^p\varepsilon^{-1}d\varepsilon]^{1/p} + 2C_1\nu^{-1}[\int_1^\infty ||\sigma^{s+\alpha}E_A(\sigma)f||^p\sigma^{-1}d\sigma]^{1/p}.$$

If $\alpha < 0$ then our hypotheses imply that φ is a bounded function and, by using the inequality $(1-\alpha p)^{-1/p} \leq 1$, we see that the first term above is bounded by $\sup|\varphi| \cdot ||E_A([-2,2])f||$. If $\alpha \geq 0$ we use the inequality $|\varphi(x)| \leq c|x|^{\alpha+\nu}$ and we bound this term by

$$c2^{\alpha+\nu}[\int_0^1 ||\varepsilon^\nu E_A([-2,2])f||^p\varepsilon^{-1}d\varepsilon]^{1/p} \leq c2^{\alpha+\nu}(1+\nu p)^{-1/p}||E_A([-2,2])f||. \quad \square$$

2.3. Let s be a strictly positive real number and $p \in [1,\infty]$. We say that an operator $S \in B(\mathcal{H})$ is of class $\mathcal{C}^{s,p}(A)$, and we write $S \in \mathcal{C}^{s,p}(A)$, if the map $\tau \mapsto S(\tau) \equiv W_\tau^* S W_\tau \in B(\mathcal{H})$ is of class $\Lambda^{s,p}$. We set $\mathcal{C}^s(A) = \mathcal{C}^{s,\infty}(A)$. It is clear that $\mathcal{C}^{s,p}(A)$ is a full involutive subalgebra of $B(\mathcal{H})$ (a subalgebra $\mathcal{C}$ of $B(\mathcal{H})$ is full if each invertible in $B(\mathcal{H})$ operator S from $\mathcal{C}$ has the property $S^{-1} \in \mathcal{C}$).Moreover, one has $\mathcal{C}^{s,p}(A) \subset \mathcal{C}^{t,q}(A)$ if $s > t$ or if $s = t$ but $p \leq q$.

Now let $k \geq 1$ be an integer. We say that $S \in B(\mathcal{H})$ is of class $C^k(A)$ if the map $S(\cdot) : \mathbb{R} \to B(\mathcal{H})$ (defined above) is of class $\mathrm{Lip}^{(k)}$. This is equivalent with asking that $S(\cdot)$ be strongly of class C^k; if this map is norm C^k we say that S is of class $C^k_{\mathrm{u}}(A)$ (this makes sense and is not trivial even if $k = 0$; on the other hand $C^0(A) = B(\mathcal{H})$). We have $\mathcal{C}^{k,1}(A) \subset C^k_{\mathrm{u}}(A) \subset C^k(A) \subset \mathcal{C}^k(A)$ and, if A is not bounded, these embeddings are strict and optimal on the scale $\mathcal{C}^{s,p}(A)$ (i.e. the spaces $\mathcal{C}^{k,p}$ with $1 < p < \infty$ are not comparable with $C^k_{\mathrm{u}}(A)$ and $C^k(A)$). Clearly $C^k(A)$ and $C^k_{\mathrm{u}}(A)$ are full involutive subalgebras of $B(\mathcal{H})$ and $C^0_{\mathrm{u}}(A)$ is a C^*-subalgebra of $B(\mathcal{H})$. We set $C^\infty(A) = \cap_{k\in\mathbb{N}} C^k(A)$.

For each $S \in B(\mathcal{H})$ we define a continuous sesquilinear form $\mathrm{ad}_A^k S$ on $\mathcal{H}_k$ (or, equivalently, a continuous operator $\mathcal{H}_k \to \mathcal{H}_{-k} = \mathcal{H}_k^*$) by induction over k: $ad_A^0 S = S$, $\mathrm{ad}_A S = [A,S] = AS - SA$ and $\mathrm{ad}_A^{k+1} S = \mathrm{ad}_A \mathrm{ad}_A^k S$. One has for all $f,g\in\mathcal{H}_k=D(A^k)$:

$$\langle f,(\operatorname{ad}_A^k S)g\rangle=\sum_{i+j=k}\frac{k!(-1)^j}{i!j!}\langle A^i f,SA^j g\rangle. \tag{2.4}$$

We have $S\in C^k(A)$ if and only if the sesquilinear form $\operatorname{ad}_A^k S$ is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{H}_k$. In this case, and if we denote by $\mathcal{A}^k[S]\equiv\mathcal{A}^k S$ the unique operator in $B(\mathcal{H})$ such that $\langle f,(\mathcal{A}^k S)g\rangle=(-1)^k\langle f,(\operatorname{ad}_A^k S)g\rangle$ for all $f,g\in\mathcal{H}_k$, then we have $\mathcal{A}^k S=(-id/d\tau)^k S(\tau)|_{\tau=0}$ (the rather pedantic notation $\mathcal{A}^k S$ is convenient for later purposes). Now assume that $s=k+\sigma$ with $k\geq 1$ integer and $\sigma>0$. Then one has $S\in\mathcal{C}^{s,p}(A)$ if and only if $S\in C^k(A)$ and $\mathcal{A}^k S\in\mathcal{C}^{\sigma,p}(A)$.

Let $S\in\mathcal{C}^{\alpha,p}(A)$ for some $\alpha>0$, $p\in[1,\infty]$. Then S leaves $\mathcal{H}_{\alpha,p}$ invariant and has a canonical extension to a continuous operator $S:\mathcal{H}_{-\alpha,p'}\to\mathcal{H}_{-\alpha,p'}$ (if $1<p\leq\infty$ then the extension is uniquely determined by its continuity; if p=1 we have to require it to be weak* continuous, $\mathcal{H}_{-\alpha,\infty}$ being considered as the adjoint of $\mathcal{H}_{\alpha,1}$). This extension has the property $S\mathcal{H}_{t,q}\subset\mathcal{H}_{t,q}$ if $-\alpha<t<\alpha$ and $1\leq q\leq\infty$, or if $t=\alpha$ and $p\leq q\leq\infty$ (note that under these conditions and if $t\neq 0$, we have $\mathcal{C}^{\alpha,p}\subset\mathcal{C}^{|t|,q}$). Now we prove that the part of S which is off-diagonal relatively to A has better properties.

Theorem 2.2. *Let us set $\Pi_+=E_A([0,\infty))$ and $\Pi_-=E_A((-\infty,0])$. If $S\in\mathcal{C}^{\alpha,p}(A)$ for some real $\alpha>0$ and some $p\in[1,\infty]$, then $\Pi_\mp S\Pi_\pm\mathcal{H}\subset\mathcal{H}_{\alpha,p}$. In particular, if $S\in\mathcal{C}^{\alpha,2}(A)$ then $\Pi_\mp S\Pi_\pm\in B(\mathcal{H}_s;\mathcal{H}_{s+\alpha})$ for all real s such that $-\alpha\leq s\leq 0$.*

Proof. (i) We first prove a weak-type estimate, namely we show that $S_0\equiv\Pi_-S\Pi_+$ sends $\mathcal{H}$ into $\mathcal{H}_{m,\infty}$ if $S\in C^m(A)$ for some integer $m\geq 1$. Let χ be the characteristic function of the real set defined by $1\leq|x|\leq 2$. Then it suffices to show that $||\chi(\varepsilon A)S_0||\leq C\varepsilon^m$ for some constant C and all $0<\varepsilon<1$. Set $S_\tau=\exp(\tau A)S_0\exp(-\tau A)$ for $\tau\geq 0$. Then $\tau\mapsto S_\tau$ is strongly of class C^m on $[0,\infty)$ and its k-th order derivative ($0\leq k\leq m$) is equal to $\operatorname{ad}_A^k S_\tau=\exp(\tau A)\Pi_-(\operatorname{ad}_A^k S)\Pi_+\exp(-\tau A)$. By making a Taylor expansion up to order m we get :

$$S_0=\sum_{k=0}^{m-1}\frac{(-1)^k}{k!}\operatorname{ad}_A^k S_1+\frac{(-1)^m}{(m-1)!}\int_0^1\operatorname{ad}_A^m S_\tau\cdot\tau^{m-1}d\tau.$$

The operators $\operatorname{ad}_A^k S_1$ clearly send $\mathcal{H}_{-\infty}$ into $\mathcal{H}_{+\infty}$, so it suffices to consider the contribution of the integral term. We have:

$$\int_0^1||\chi(\varepsilon A)\operatorname{ad}_A^m S_\tau||\tau^{m-1}d\tau\leq||\operatorname{ad}_A^m S_0||\int_0^1||\chi(\varepsilon A)\Pi_-e^{\tau A}||\tau^{m-1}d\tau\leq$$

$$\leq||\operatorname{ad}_A^m S_0||\int_0^1\sup_{x>0}\chi(\varepsilon x)e^{-\tau x}\tau^{m-1}d\tau=||\operatorname{ad}_A^m S_0||\int_0^1 e^{-\tau/\varepsilon}\tau^{m-1}d\tau\leq C\varepsilon^m,$$

which is the desired estimate.

(ii) Let $\mathcal{P}:B(\mathcal{H})\to B(\mathcal{H})$ be the linear continuous operator given by $\mathcal{P}S=\Pi_-S\Pi_+$. Then $||\mathcal{P}||=1$ and $\mathcal{P}C^m(A)\subset B(\mathcal{H};\mathcal{H}_{m,\infty})$ (by what we have shown above and the closed graph theorem). On the space $C^m(A)$ there is a natural Banach space structure such that the embedding $C^m(A)\subset B(\mathcal{H})$ be continuous. Then one can obtain the spaces $\mathcal{C}^{\alpha,p}(A)$ by real interpolation:

$\mathcal{C}^{\alpha,p} = (C^m(A), B(\mathcal{H}))_{\theta,p}$ with $\theta = 1 - \alpha/m$ if $0 < \alpha < m$ (see (5.2.22) in [ABG]). Similarly $(\mathcal{H}_{m,\infty}, \mathcal{H})_{\theta,p} = \mathcal{H}_{\alpha,p}$. Now fix a vector $f \in \mathcal{H}$ and consider the map $S \mapsto \mathcal{P}(S)f$. It sends $B(\mathcal{H})$ into $\mathcal{H}$ continuously and $C^m(A)$ into $\mathcal{H}_{m,\infty}$ continuously. By interpolation it will send $\mathcal{C}^{\alpha,p}$ into $\mathcal{H}_{\alpha,p}$. For the last assertion of the theorem note that S^* is of the same class as S. □

2.4. We shall denote by $\mathcal{A}$ the operator acting in the Banach space $B(\mathcal{H})$ according to the following rule: an element $S \in B(\mathcal{H})$ belongs to the domain of $\mathcal{A}$ if and only if the sesquilinear form $\langle f, SAg\rangle - \langle Af, Sg\rangle$ (with domain $D(A)$) is continuous for the topology induced by $\mathcal{H}$; and then $\mathcal{A}[S] \equiv \mathcal{A}S$ is the unique element of $B(\mathcal{H})$ such that $\langle f, \mathcal{A}[S]g\rangle = \langle f, SAg\rangle - \langle Af, Sg\rangle$ for all $f, g \in D(A)$. Clearly $C^k(A)$ coincides with the domain of the power $\mathcal{A}^k$ of $\mathcal{A}$, for each $k \in \mathbb{N}$. Moreover, the following identity holds in $B(\mathcal{H}_k; \mathcal{H}_{-k})$:

$$\mathcal{A}^k[S] \equiv \mathcal{A}^k S = (-1)^k \operatorname{ad}_A^k S = \sum_{i+j=k} \frac{k!}{i!j!}(-1)^i A^i S A^j \tag{2.5}$$

The operator $\mathcal{A}$ can be interpreted as the infinitesimal generator of a one-parameter group of automorphisms of $B(\mathcal{H})$. For each real τ we define $\mathcal{W}_\tau : B(\mathcal{H}) \to B(\mathcal{H})$ by $\mathcal{W}_\tau[S] \equiv \mathcal{W}_\tau S = W_\tau^* S W_\tau$. Then $\mathcal{W}_0 = 1$, $\mathcal{W}_{\tau+\sigma} = \mathcal{W}_\tau \mathcal{W}_\sigma$ for all $\tau, \sigma \in \mathbb{R}$, and the function $\tau \mapsto \mathcal{W}_\tau S \in B(\mathcal{H})$ is strongly continuous (but not norm continuous in general). For $S \in B(\mathcal{H})$ we have $S \in C^1(A)$ (= domain of $\mathcal{A}$) if and only if $\lim_{\varepsilon \to 0} (i\varepsilon)^{-1}(\mathcal{W}_\varepsilon - 1)S$ exists in $B(\mathcal{H})$ in the ultraweak (or weak, or strong) operator topology, and then the limit is just $\mathcal{A}S$ and one has $\mathcal{W}_\tau S = S + i\int_0^\tau \mathcal{W}_\sigma \mathcal{A} S d\sigma$ for all $\tau \in \mathbb{R}$. So $\mathcal{A}$ is the infinitesimal generator of the "weak" one-parameter group $\{\mathcal{W}_\tau\}_{\tau\in\mathbb{R}}$ in the Banach space $B(\mathcal{H})$ and this justifies the notation $\mathcal{W}_\tau = \exp(i\tau\mathcal{A})$ (the notion of weak semigroup is introduced in [BB]; note that $B(\mathcal{H})$ is identified with the adjoint of the space of trace class operators).

We shall define a functional calculus for the operator $\mathcal{A}$ with the help of the group $\{\mathcal{W}_\tau\}_{\tau\in\mathbb{R}}$. Let $\mathcal{M} = \mathcal{M}(\mathbb{R})$ be the unital subalgebra of the algebra $BC(\mathbb{R})$ (bounded continuous complex functions on $\mathbb{R}$) consisting of Fourier transforms of bounded Borel measures. The algebraic operations in $\mathcal{M}$ are those inherited from the embedding $\mathcal{M} \subset BC(\mathbb{R})$, but we take as norm in $\mathcal{M}$ of $\varphi(t) = \int_\mathbb{R} e^{it\tau}\mu(d\tau)$ the total variation of the measure μ. This makes $\mathcal{M}$ an abelian Banach algebra with unit. If φ is given by the preceding expression we define a linear continuous operator $\varphi(\mathcal{A}) : B(\mathcal{H}) \to B(\mathcal{H})$ by setting $\varphi(\mathcal{A})S \equiv \varphi(\mathcal{A})[S] \equiv \int_\mathbb{R} W_\tau^* S W_\tau \mu(d\tau)$; the integral exists in the strong operator topology. Observe that the notation $\mathcal{W}_\tau = e^{i\mathcal{A}\tau}$ is consistent with the functional calculus, i.e. $\varphi(\mathcal{A}) = \mathcal{W}_\tau$ if φ is the function $\varphi(t) = e^{it\tau}$. It is easily checked that the map $\varphi \mapsto \varphi(\mathcal{A})$ is a unital homomorphism such that $(\varphi(\mathcal{A})[S])^* = \varphi^+(\mathcal{A})[S^*]$ if $\varphi^+(t) = \overline{\varphi(-t)}$, and that the norm of the operator $\varphi(\mathcal{A})$ (acting in the Banach space $B(\mathcal{H})$) is $\leq ||\varphi||_\mathcal{M}$.

It is clear that $\varphi(\mathcal{A})$ commutes with $\mathcal{A}$, in fact if $k \in \mathbb{N}$ then $\varphi(\mathcal{A})C^k(A) \subset C^k(A)$ and $\mathcal{A}^k\varphi(\mathcal{A})S = \varphi(\mathcal{A})\mathcal{A}^k S$ for $S \in C^k(A)$. On the other hand, if φ decays at infinity then it improves regularity with respect to $\mathcal{A}$. For $m \geq 1$ integer we define $\varphi_{(m)}$ by $\varphi_{(m)}(x) = x^m\varphi(x)$. Then if $\varphi \in \mathcal{M}$ and $\varphi_{(m)} \in \mathcal{M}$, we have $\varphi(\mathcal{A})C^k(A) \subset C^{k+m}(A)$ for all $k \in \mathbb{N}$ and $\mathcal{A}^m\varphi(\mathcal{A}) = \varphi_{(m)}(\mathcal{A})$. In particular, if $\varphi \in C_0^\infty(\mathbb{R})$ then $\varphi(\mathcal{A})B(\mathcal{H}) \subset C^\infty(A)$.

The main purpose of the functional calculus introduced above is to allow us to construct operators of class $C^\infty(A)$ which approximate a given operator $S \in B(\mathcal{H})$ rapidly enough, in a sense that we shall make precise below.

Note first that if $\varphi \in \mathcal{M}$ and $\varepsilon \in \mathbb{R}$ then the function $x \mapsto \varphi(\varepsilon x)$, denoted φ^ε, belongs to $\mathcal{M}$ and $||\varphi^\varepsilon||_{\mathcal{M}} \leq ||\varphi||_{\mathcal{M}}$ (the equality holds if $\varepsilon \neq 0$). We set $\varphi(\varepsilon\mathcal{A}) = \varphi^\varepsilon(\mathcal{A})$, in other terms $\varphi(\varepsilon\mathcal{A})S = \int_{\mathbb{R}} W^*_{\varepsilon\tau} S W_{\varepsilon\tau}\mu(d\tau)$. So for each $S \in B(\mathcal{H})$ the map $\varepsilon \mapsto \varphi(\varepsilon\mathcal{A})S \in B(\mathcal{H})$ is strongly continuous and $\varphi(0\mathcal{A})S = \varphi(0)S$. In particular, if $\varphi(0) = 1$ and $\varphi_{(m)} \in \mathcal{M}$ for some integer $m \geq 1$, then by what we have seen before we have $\varphi(\varepsilon\mathcal{A})S \in C^m(A)$ for $\varepsilon \neq 0$ and $\varphi(\varepsilon\mathcal{A})S \to S$ in the strong operator topology as $\varepsilon \to 0$. Operators of the form $S_\varepsilon = \varphi(\varepsilon\mathcal{A})S$ with $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi(0) = 1$ will be called *regularizations of* S $(\varepsilon \neq 0)$.

The rapidity of the convergence of S_ε to S is determined by the degree of regularity of S with respect to A. We explain this fact in rather rough terms. Assume that $S \in C^k(A)$ for some integer $k \geq 1$ and let $\varphi \in \mathcal{M}$ be of the form $\varphi(x) = 1 + x^k\eta(x)$ for some $\eta \in \mathcal{M}$. Then $\varphi(\varepsilon\mathcal{A})S = S + \varepsilon^k\mathcal{A}^k\eta(\varepsilon\mathcal{A})S = S + \varepsilon^k\eta(\varepsilon\mathcal{A})\mathcal{A}^k S$ for $\varepsilon \neq 0$ hence $||\varphi(\varepsilon\mathcal{A})S - S|| \leq |\varepsilon|^k||\eta||_{\mathcal{M}}||\mathcal{A}^k S||$. So if $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi(x) - 1 = O(x^k)$ as $x \to 0$, then for each $S \in C^k(A)$ we have $||\varphi(\varepsilon\mathcal{A})S - S|| = O(\varepsilon^k)$ as $\varepsilon \to 0$. The case $p = \infty$ of the next theorem says that this behaviour characterizes the class of operators $\mathcal{C}^k(A)$, which is slightly larger than $C^k(A)$.

Theorem 2.3. *Let $S \in B(\mathcal{H})$, s a strictly positive real number, and $p \in [1,\infty]$. If there is a function $\theta \in \mathcal{M}(\mathbb{R})$, which is not identically zero on $(0,\infty)$ and on $(-\infty,0)$, such that*

$$\left[\int_0^1 ||\varepsilon^{-s}\theta(\varepsilon\mathcal{A})S||^p\varepsilon^{-1}d\varepsilon\right]^{1/p} < \infty \tag{2.6}$$

then $S \in \mathcal{C}^{s,p}(A)$. Reciprocally, if $S \in \mathcal{C}^{s,p}$ and if $m > s$ is an integer, then (2.6) holds for each θ such that $\theta^{(k)} \in \mathcal{M}(\mathbb{R})$ for $0 \leq k \leq m$ and $\theta^{(k)}(0) = 0$ for $0 \leq k \leq m-1$.

For the proof, see [BG 3].

This theorem characterizes the property $S \in \mathcal{C}^{s,p}(A)$ in terms of the rapidity of the convergence of the regularizations S_ε of S. Let us say that an operator $T \in B(\mathcal{H})$ has *A-exponential type less than r* if there is a holomorphic function $T(\cdot) : \mathbb{C} \to B(\mathcal{H})$ such that $T(\tau) = W^*_\tau T W_\tau$ for $\tau \in \mathbb{R}$ and, moreover, there is a constant C such that $||T(\zeta)|| \leq C\exp(r|\zeta|)$ for all $\zeta \in \mathbb{C}$. One may show that T has this property if and only if $\varphi(\mathcal{A})T = T$ for all $\varphi \in \mathcal{M}$ such that $\varphi(x) = 1$ on a neighbourhood of the interval $|x| \leq r$. Hence Theorem 2.3 is an extension of classical results of Jackson and Zygmund concerning the best approximation of Hölder-Zygmund functions by trigonometric polynomials. Note also that by taking $\theta(x) = (e^{ix} - 1)^m$ we get that $S \in \mathcal{C}^{s,p}(A)$ if and only if $[\int_0^1 ||\varepsilon^{-s}(\mathcal{W}_\varepsilon - 1)^m S||^p\varepsilon^{-1}d\varepsilon]^{1/p} < \infty$.

3. Resolvent Families and Mourre Estimates

3.1. A family $\{R(z) \mid z \in \mathbb{C} \setminus \mathbb{R}\}$ of bounded operators in the Hilbert space $\mathcal{H}$ will be called a *(self-adjoint) resolvent family* if the following two conditions are satisfied: $R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2)$ (first resolvent identity) and $R(z)^* = R(\overline{z})$ for all $z_1, z_2, z \in \mathbb{C} \setminus \mathbb{R}$. It follows easily from these relations that the map $R(\cdot) : \mathbb{C} \setminus \mathbb{R} \to B(\mathcal{H})$ is holomorphic and $(d/dz)^k R(z) = k! R(z)^{k+1}$. The *spectrum* of the resolvent family is the set of real numbers λ such that the function $R(\cdot)$ has no holomorphic extension to any neighbourhood of λ. Clearly, the first resolvent identity holds for all complex numbers z_1, z_2 not in the spectrum of the resolvent family. Note also that we have $||R(z)|| \leq |\operatorname{Im} z|^{-1}$ as a consequence of this identity; moreover, one has $\lim_{\varepsilon \to 0} ||R(\lambda + i\varepsilon)|| = \infty$ if and only if λ belongs to the spectrum of $\{R(z)\}$.

It is most convenient to think of resolvent families in terms of possibly non-densely defined self-adjoint operators in $\mathcal{H}$. To be precise, we shall work with a slight extension of the standard notion of self-adjoint operator: for us a *self-adjoint operator in* $\mathcal{H}$ is a linear operator H defined on a linear subspace $D(H)$ of $\mathcal{H}$ with values in $\mathcal{H}$, such that $HD(H) \subset \overline{D(H)}$ (= closure of $D(H)$ in $\mathcal{H}$) and such that, when considered as operator in the Hilbert space $\overline{D(H)}$, H is self-adjoint in the usual sense (so a *densely defined* self-adjoint operator is a "usual" self-adjoint operator). Note that $H - z : D(H) \to \overline{D(H)}$ is bijective if $\operatorname{Im} z \neq 0$. The resolvent family associated to such an operator is defined by $R(z)f = (H-z)^{-1}f$ if $f \in \overline{D(H)}$ and $R(z)f = 0$ if f is orthogonal to $D(H)$. Reciprocally, if $\{R(z)\}$ is a resolvent family then there is a unique self-adjoint operator H such that $R(z)$ be of the above form for $z \in \mathbb{C} \setminus \mathbb{R}$. Note that the spectrum of the densely defined self-adjoint operator H in the Hilbert space $\overline{D(H)}$ coincides with the spectrum of the resolvent family $\{R(z)\}$.

If φ is a complex continuous function on $\mathbb{R}$ which tends to zero at infinity, then $\varphi(H)$ is a well-defined bounded operator in $\overline{D(H)}$ (by the functional calculus associated to the densely defined selfadjoint operator H in $\overline{D(H)}$). We extend $\varphi(H)$ to a bounded operator on $\mathcal{H}$ by setting $\varphi(H)f = 0$ if f is orthogonal to $D(H)$. Then clearly we have

$$\varphi(H) = \operatorname*{w-lim}_{\varepsilon \to +0} \frac{1}{\pi} \int_{\mathbb{R}} \varphi(\lambda) \operatorname{Im} R(\lambda + i\varepsilon) d\lambda \tag{3.1}$$

where the integral exists in the weak topology. This formula expresses $\varphi(H)$ directly in terms of the resolvent family but is not convenient for our purposes here. A more useful representation for $\varphi(H)$ can, however, be easily deduced from (3.1). Let r be a strictly positive number. We shall use Taylor's formula for the function $\mu \mapsto R(\lambda + i\mu)$ on the interval $[\varepsilon, r]$ with $0 < \varepsilon < r$. Since, by holomorphy, we have $(d/d\mu)R(\lambda + i\mu) = i(d/d\lambda)R(\lambda + i\mu)$, we get for any integer $m \geq 1$ (with $\partial_\lambda = d/d\lambda$):

$$\begin{aligned} R(\lambda + i\varepsilon) = & \sum_{k=0}^{m-1} \frac{(r-\varepsilon)^k}{k!} (-i\partial_\lambda)^k R(\lambda + ir) \\ & + \frac{1}{(m-1)!} \int_\varepsilon^r (-i\partial_\lambda)^m R(\lambda + i\mu)(\mu - \varepsilon)^{m-1} d\mu. \end{aligned}$$

So if φ is of class $C_0^m(\mathbb{R})$ we get after an integration by parts:

$$\int_{\mathbb{R}} \varphi(\lambda) R(\lambda + i\varepsilon) d\lambda = \sum_{k=0}^{m-1} \frac{(r-\varepsilon)^k}{k!} \int_{\mathbb{R}} i^k \varphi^{(k)}(\lambda) R(\lambda + ir) d\lambda + \frac{i^m}{(m-1)!} \int_{\mathbb{R}} \int_{\varepsilon}^{r} \varphi^{(m)}(\lambda) R(\lambda + i\mu) \mu^{m-1} d\mu d\lambda.$$

Since $||R(\lambda + i\mu)|| \leq |\mu|^{-1}$ the double integral exists in norm for $m \geq 2$ even if $\varepsilon = 0$. Hence for each $m \geq 2$ we have (cf. [BG1] and [ABG])

$$\varphi(H) = \sum_{k=0}^{m-1} \frac{1}{\pi k!} \int_{\mathbb{R}} \varphi^{(k)}(\lambda) \operatorname{Im}[(ir)^k R(\lambda + ir)] d\lambda + \frac{1}{\pi m!} \int_{\mathbb{R}} \int_0^r \varphi^{(m)}(\lambda) \operatorname{Im}[i^m R(\lambda + i\mu)] d\mu^m d\lambda, \tag{3.2}$$

where the integrals exist in norm and $d\mu^m = m\mu^{m-1}d\mu$. This representation of $\varphi(H)$ is similar to the Helffer-Sjöstrand formula (see [D] and references therein).

Assume now that a densely defined self-adjoint operator A is given in $\mathcal{H}$ and let $\{R(z)\}$ be the resolvent family associated to a self-adjoint operator H. We shall say that {R(z)} (or H) is of class $C^k(A)$, $C_{\mathrm{u}}^k(A)$, or $\mathcal{C}^{s,p}(A)$, if there is a complex number z_0 outside the spectrum of H such that the bounded operator $R(z_0)$ is of class $C^k(A)$, $C_{\mathrm{u}}^k(A)$, or $\mathcal{C}^{s,p}(A)$ respectively. Note that if this property holds for some z_0 then it holds for all complex numbers z outside the spectrum of H. Indeed, the operator $1-(z-z_0)R(z_0)$ will then be invertible in $\mathcal{H}$ with inverse equal to $1+(z-z_0)R(z)$. Hence the assertion follows from the fact that $C^k(A), C_{\mathrm{u}}^k(A)$ and $\mathcal{C}^{s,p}(A)$ are full subalgebras of $B(\mathcal{H})$.

The following property is a straightforward consequence of (3.2): if $\varphi \in C_0^\infty(\mathbb{R})$ and H is of class $C^k(A)$, $C_{\mathrm{u}}^k(A)$, or $\mathcal{C}^{s,p}(A)$, then $\varphi(H)$ is of class $C^k(A)$, $C_{\mathrm{u}}^k(A)$, or $\mathcal{C}^{s,p}(A)$ respectively. We mention the following example. Let $\mathcal{H} = L^2(\mathbb{R})$, $A = id/dx$ and let H be the operator of multiplication by the function $h : \Omega \to \mathbb{R}$, where Ω is an open real set (think that $h(x) = \infty$ if $x \notin \Omega$). If h is a rational function and Ω is the complement of the set of poles of h, then H is of class $C^\infty(A)$. If $h : \Omega \to \mathbb{R}$ is of class C^∞ and proper (i.e. $|h(x)|$ diverges when x approaches the boundary of Ω) then $\varphi(H)$ is of class $C^\infty(A)$, for all $\varphi \in C_0^\infty(\mathbb{R})$. In connection with a question left open in §6.2.1 of [ABG] (see the discussion before Example 6.2.8) consider now the function $h(x) = x^{-1}$ (so $\Omega = \mathbb{R} \setminus \{0\}$). We clearly get a densely defined self-adjoint operator H of class $C^\infty(A)$ such that $\varphi(H) \notin C^1(A)$ if $\varphi : \mathbb{R} \to \mathbb{C}$ is a function of class C^1 which has finite but distinct limits at $+\infty$ and $-\infty$.

3.2. We now show that if H is of class $C^1(A)$ then one can give a meaning to the commutator $[H, A]$ as a continuous sesquilinear form on the domain $D(H)$ of H endowed with the graph topology, i.e. the topology associated to the norm $||f||_H = (||f|^2 + ||Hf||^2)^{1/2}$.

Proposition 3.1. *Let A, H be selfadjoint operators in $\mathcal{H}$ such that A is densely defined and H is of class $C^1(A)$. Then $D(A) \cap D(H)$ is a dense* *subspace of $D(H)$ (for the graph topology) and there is a constant $C < \infty$ such that*

$$|\langle Hf, Ag\rangle - \langle Af, Hg\rangle| \le C||f||_H||g||_H \quad \forall f, g \in D(A) \cap D(H). \tag{3.3}$$

Proof. (i) Let $\mathcal{D}$ be the set of $f \in D(A) \cap D(H)$ such that $Hf \in D(A)$. Each bounded operator of class $C^1(A)$ leaves $D(A)$ invariant, hence for any $z \in \mathbb{C} \setminus \sigma(H)$ one has $R(z)^2 D(A) \subset \mathcal{D} \subset R(z)D(A)$ (note that $\mathcal{D} = R(z)D(A)$ if H is densely defined; moreover, it follows from the first resolvent identity that the spaces $R(z)^2D(A)$ and $R(z)D(A)$ are independent of the choice of z). The operator $R(z)^2$ is a continuous surjective map of $\mathcal{H}$ onto $D(H^2)$ (equipped with its graph topology), so it sends a dense subspace (e.g. $D(A)$) of $\mathcal{H}$ onto a dense subspace of $D(H^2)$. Hence $R(z)^2D(A)$ is a dense subspace of $D(H^2)$, and so of $D(H)$. In particular $\mathcal{D}$ is dense in $D(H)$.

(ii) Let $f, g \in \mathcal{D}$. Then for $z \in \mathbb{C} \setminus \sigma(H)$ one has

$$\begin{aligned} \langle Hf, Ag\rangle - \langle Af, Hg\rangle &= \langle A(H-\overline{z})f, R(z)(H-z)g\rangle \\ &\quad - \langle (H-\overline{z})f, R(z)A(H-z)g\rangle \\ &= -\langle H-\overline{z})f, \mathcal{A}[R(z)](H-z)g\rangle. \end{aligned} \tag{3.4}$$

By taking $z = i$ we get

$$|\langle Hf, Ag\rangle - \langle Af, Hg\rangle| \le ||\mathcal{A}[R(i)]|| \cdot ||f||_H||g||_H. \tag{3.5}$$

Our purpose is to show that this remains true for $f, g \in D(A) \cap D(H)$.

(iii) We now point out two relations that will be needed below. If $z_1, z_2 \in \mathbb{C} \setminus \sigma(H)$ then by applying $\mathcal{A}$ to the first resolvent identity we get

$$\mathcal{A}[R(z_1)]\{1 + (z_2 - z_1)R(z_2)\} = \{1 + (z_1 - z_2)R(z_1)\}\mathcal{A}[R(z_2)]. \tag{3.6}$$

On the other hand, we clearly have $\{1+(z_2-z_1)R(z_2)\}\cdot\{1+(z_1-z_2)R(z_1)\} = 1$. Hence

$$\mathcal{A}[R(z_2)] = \{1 + (z_2 - z_1)R(z_2)\}\mathcal{A}[R(z_1)]\{1 + (z_2 - z_1)R(z_2)\}. \tag{3.7}$$

(iv) For real $\varepsilon \neq 0$ we set $R_\varepsilon = (i\varepsilon)^{-1}R(i/\varepsilon) = (1 + i\varepsilon H)^{-1}P$, where P is the orthogonal projection of $\mathcal{H}$ onto $\overline{D(H)}$. By using (3.6) and (3.7) we get

$$\begin{aligned} \mathcal{A}[R_\varepsilon]R_1 &= \{1 + (\varepsilon - 1)R_\varepsilon\}\mathcal{A}[R_1]R_\varepsilon, \\ \varepsilon\mathcal{A}[R_\varepsilon] &= \{1 + (\varepsilon - 1)R_\varepsilon\}\mathcal{A}[R_1]\{1 + (\varepsilon - 1)R_\varepsilon\}. \end{aligned}$$

When $\varepsilon \to 0$ the operator R_ε converges strongly to P. Hence

$$\operatorname*{s-lim}_{\varepsilon\to 0} \mathcal{A}[R_\varepsilon]R_1 = P^\perp\mathcal{A}[R_1]P, \tag{3.8}$$

$$\operatorname*{s-lim}_{\varepsilon\to 0} \mathcal{A}[R_\varepsilon] = P^\perp\mathcal{A}[R_1]P^\perp. \tag{3.9}$$

Now set $S_\varepsilon = (1 + i\varepsilon A)^{-1}$ for $\varepsilon \in \mathbb{R}$. Then for each $f \in \mathcal{H}$ one has

$$\langle f, [S_\varepsilon, R_1]f\rangle = \langle S_\varepsilon^* f, R_1 i\varepsilon A S_\varepsilon f\rangle + \langle i\varepsilon A S_\varepsilon^* f, R_1 S_\varepsilon f\rangle,$$

and this clearly implies

$$[S_\varepsilon, R_1] = i\varepsilon S_\varepsilon\mathcal{A}[R_1]S_\varepsilon. \tag{3.10}$$

(v) Finally, set $J_\varepsilon = R_\varepsilon^2 S_\varepsilon$ for $\varepsilon \neq 0$, let $f \in D(A) \cap D(H)$ and denote $f_1 = i(H-i)f \in \overline{D(H)}$, so that $f = R_1 f_1$. We have $\text{s-lim}_{\varepsilon\to 0} J_\varepsilon = P$ and (3.10) gives

$$\begin{aligned}(H-i)J_\varepsilon f &= (H-i)J_\varepsilon R_1 f_1 \\ &= (H-i)R_1 J_\varepsilon f_1 + i\varepsilon(H-i)R_\varepsilon \cdot R_\varepsilon S_\varepsilon \mathcal{A}[R_\varepsilon] S_\varepsilon f_1.\end{aligned}$$

When $\varepsilon \to 0$ the operator $i\varepsilon(H-i)R_\varepsilon = \{1 + (\varepsilon - 1)R_\varepsilon\}P$ converges to zero and $J_\varepsilon f_1 \to f_1$, hence $(H-i)J_\varepsilon f \to (H-i)R_1 f_1 = (H-i)f$. So $J_\varepsilon f \to f$ in $D(H)$. On the other hand the bounded operator J_ε is clearly of class $C^1(A)$ and $\mathcal{A}[J_\varepsilon] = \mathcal{A}[R_\varepsilon^2]S_\varepsilon$, hence $AJ_\varepsilon f = J_\varepsilon Af - \mathcal{A}[R_\varepsilon^2]S_\varepsilon f$. By using (3.10) again we obtain

$$\begin{aligned}\mathcal{A}[R_\varepsilon^2]S_\varepsilon R_1 &= \mathcal{A}[R_\varepsilon^2]R_1 S_\varepsilon + \mathcal{A}[R_\varepsilon^2] i\varepsilon S_\varepsilon \mathcal{A}[R_1]S_\varepsilon \\ &= \mathcal{A}[R_\varepsilon]R_1 R_\varepsilon S_\varepsilon + R_\varepsilon \mathcal{A}[R_\varepsilon]R_1 S_\varepsilon \\ &\quad + \varepsilon\mathcal{A}[R_\varepsilon]R_\varepsilon i S_\varepsilon \mathcal{A}[R_1]S_\varepsilon + R_\varepsilon \varepsilon \mathcal{A}[R_\varepsilon] i S_\varepsilon \mathcal{A}[R_1]S_\varepsilon.\end{aligned}$$

Now we use (3.8), (3.9) and the relations $R_\varepsilon \to P$, $S_\varepsilon \to 1$ strongly as $\varepsilon \to 0$; we get $\text{s-lim}_{\varepsilon\to 0} \mathcal{A}[R_\varepsilon^2]S_\varepsilon R_1 = P^\perp \mathcal{A}[R_1]P$. Hence we get

$$\operatorname*{s-lim}_{\varepsilon\to 0} AJ_\varepsilon f = PAf - P^\perp \mathcal{A}[R(i)](H-i)f,$$

in particular $||PAJ_\varepsilon f - PAf|| \to 0$.

(vi) We can now prove the validity of (3.5) for all $f, g \in D(A) \cap D(H)$. (3.5) holds if f, g are replaced by $f_\varepsilon = J_\varepsilon f$ and $g_\varepsilon = J_\varepsilon g$ (because f_ε, g_ε belong to $\mathcal{D}$). In the inequality obtained in this way we make $\varepsilon \to 0$ and take into account that by what we have proved at step (v) we have:

$$\langle Hf_\varepsilon, Ag_\varepsilon\rangle = \langle Hf_\varepsilon, PAg_\varepsilon\rangle \to \langle Hf, PAg\rangle = \langle Hf, Ag\rangle. \quad \square$$

If H is a selfadjoint operator of class $C^1(A)$ we shall denote by $\mathcal{A}[H]$ the unique continuous sesquilinear form on $D(H)$ such that $\langle f, \mathcal{A}[H]g\rangle = \langle Hf, Ag\rangle - \langle Af, Hg\rangle$ for all $f, g \in D(A) \cap D(H)$. If $S, T \in B(\mathcal{H})$ and their ranges are contained in $D(H)$ then $S^*\mathcal{A}[H]T$ will be identified with a bounded operator in $\mathcal{H}$ by using Riesz lemma. It follows easily from (3.4) and from the argument at step (vi) of the preceding proof that

$$\mathcal{A}[R(z)] = -R(z)\mathcal{A}[H]R(z) \text{ if } z \in \mathbb{C} \setminus \sigma(H). \tag{3.11}$$

Now let $\varphi, \psi \in C_0^\infty(\mathbb{R})$ real and such that $x\varphi(x) = \psi(x)\varphi(x)$ for all real x. Then $\varphi(H) \in C^1(A)$, hence for $f \in D(A)$ we have $\varphi(H)f \in D(A) \cap D(H)$ and

$$\begin{aligned}\langle \varphi(H)f, i\mathcal{A}[H]\varphi(H)f\rangle &= 2\,\mathrm{Re}\langle H\varphi(H)f, iA\varphi(H)\rangle \\ &= 2\,\mathrm{Re}\langle \psi(H)\varphi(H)f, iA\varphi(H)f\rangle \\ &= \langle \varphi(H)f, i\mathcal{A}[\psi(H)]\varphi(H)f\rangle.\end{aligned}$$

Note that $\psi(H) \in C^1(A)$. So we have $\varphi(H)\mathcal{A}[H]\varphi(H) = \varphi(H)\mathcal{A}[\psi(H)]\varphi(H)$.

We now define the *strict Mourre set* $\mu^A(H)$ *of* H *with respect to* A as the set of real numbers λ such that there are a real function $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ and a strictly positive real number a such that $\varphi(H)i\mathcal{A}[H]\varphi(H) \geq a\varphi(H)^2$. This is clearly an open subset of $\mathbb{R}$.

In non-trivial practical situations it is impossible to find explicitly the set $\mu^A(H)$. For this reason it is useful to introduce the *Mourre set* $\tilde{\mu}^A(H)$ *of* H

with respect to A, defined as the set of real numbers λ for which there are a real function $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda) \neq 0$, a strictly positive real number a, and a compact operator K in $\mathcal{H}$, such that $\varphi(H) i\mathcal{A}[H]\varphi(H) \geq a\varphi(H)^2 + K$. It turns out that in many interesting cases one can describe $\tilde{\mu}^A(H)$ rather explicitly (this is related to the following invariance property: if H, H_0 are self-adjoint operators of class $C_\mathrm{u}^1(A)$ and if $(H+i)^{-1} - (H_0+i)^{-1}$ is compact, then $\tilde{\mu}^A(H) = \tilde{\mu}^A(H_0)$; see Theorem 7.2.9 in [ABG]). For this reason the next result is important. Note that $\tilde{\mu}^A(H)$ is an open set and $\mu^A(H) \subset \tilde{\mu}^A(H)$.

Proposition 3.2. *The set $\tilde{\mu}^A(H) \setminus \mu^A(H)$ does not have accumulation points inside $\tilde{\mu}^A(H)$ and it consists of eigenvalues of H of finite multiplicity. The spectrum of H in $\mu^A(H)$ is purely continuous.*

Proof. The assertions of the proposition follow easily (see [M1]) once we have shown that the *virial theorem* is valid, namely that if $f \in D(H)$ is an eigenvector of H then $\langle f, \mathcal{A}[H]f\rangle = 0$. Let $\varphi, \psi \in C_0^\infty(\mathbb{R})$ be real functions such that $x\varphi(x) \equiv \psi(x)\varphi(x)$ and $\varphi(\lambda) = 1$, where $\lambda \in \mathbb{R}$ is such that $Hf = \lambda f$. Then

$$\begin{aligned}\langle f, \mathcal{A}[H]f\rangle &= \langle f, \varphi(H)\mathcal{A}[H]\varphi(H)f\rangle = \langle f, \varphi(H)\mathcal{A}[\psi(H)]\varphi(H)f\rangle \\ &= \langle f, \mathcal{A}[\psi(H)]f\rangle = \lim_{\varepsilon\to 0}\langle f, [\psi(H), (i\varepsilon)^{-1}(W_\varepsilon - 1)]f\rangle = 0. \quad \square\end{aligned}$$

We shall say that the self-adjoint operator H (or the resolvent family $\{R(z)\}$ associated to it) *has a spectral gap* if its spectrum is not equal to $\mathbb{R}$. Fix such an H, let λ_0 be a real number outside the spectrum of H, and set $R = -R(\lambda_0)$. Then R is a bounded self-adjoint operator $R : \mathcal{H} \to \mathcal{H}$ and for $\operatorname{Im} z \neq 0$:

$$R(z) = (\lambda_0 - z)^{-1} R[R - (\lambda_0 - z)^{-1}]^{-1}. \tag{3.12}$$

Proposition 3.3. *H is of class $C^1(A)$ if and only if R is of class $C^1(A)$. A real number $\lambda \neq \lambda_0$ belongs to $\mu^A(H)$ (resp. $\tilde{\mu}^A(H)$) if and only if $(\lambda_0 - \lambda)^{-1}$ belongs to $\mu^A(R)$ (resp. $\tilde{\mu}^A(R)$).*

The proof of this result is straightforward and will not be given; see Proposition 8.3.4 in [ABG] and note that in our context one can replace the class C_u^1 by the class C^1 (cf. Propositions 7.2.5 and 7.2.7 of [ABG] for the case of densely defined operators).

4. The Twisted Hamiltonian

The main technical estimates of this article will be derived in this section. We consider a bounded everywhere defined self-adjoint operator H in $\mathcal{H}$, we denote by E its spectral measure, and we assume that H is of class $C^1(A)$. Furthermore, *we fix a real open set J and a real number $a > 0$ such that the following condition is satisfied: there is an open set J_0 with* $\operatorname{dist}(J, \mathbb{R} \setminus J_0) \equiv \inf\{|x-y| \mid x \in J, y \notin J_0\} = \delta > 0$ *and there is a number $a_0 > a$ such that $E(J_0)i\mathcal{A}[H]E(J_0) \geq a_0 E(J_0)$.*

We shall need a version of the so-called quadratic estimate of Mourre. The proof of the next proposition can be found in §4.4 of [BG3]; see [M1], [ABG] for similar results.

Proposition 4.1. *Let $\{H_\varepsilon\}_{\varepsilon\geq 0}$ be a family of bounded operators in $\mathcal{H}$ such that $H_0 = H$, $||H_\varepsilon - H|| \to 0$ and $||\varepsilon^{-1}\operatorname{Im} H_\varepsilon + i\mathcal{A}[H]|| \to 0$ as $\varepsilon \to 0$. Then there are strictly positive numbers ε_0, b such that, for each $\varepsilon \in [0,\varepsilon_0]$ and each $z \in \mathbb{C}$ with $\operatorname{Re} z \in J$ and $\operatorname{Im} z > -a\varepsilon$, the operator $H_\varepsilon - z : \mathcal{H} \to \mathcal{H}$ is bijective and its inverse $G_\varepsilon = G_\varepsilon(z) = (H_\varepsilon - z)^{-1} \in B(\mathcal{H})$ satisfies the estimates*

$$(4.1)\quad ||G_\varepsilon^{(\pm)}f||^2 \leq \pm\frac{1}{a\varepsilon + \operatorname{Im} z}\operatorname{Im}\langle f, G_\varepsilon f\rangle + \frac{b\varepsilon}{(a\varepsilon + \operatorname{Im} z)[\delta^2 + (\operatorname{Im} z)^2]}||f||^2$$

for all $f \in \mathcal{H}$. We have set $G_\varepsilon^{(+)} = G_\varepsilon$, $G_\varepsilon^{(-)} = G_\varepsilon^*$. In particular, one has

$$(4.2)\qquad ||G_\varepsilon(z)|| \leq \frac{1}{a\varepsilon + \operatorname{Im} z} + \left[\frac{b\varepsilon}{(a\varepsilon + \operatorname{Im} z)[\delta^2 + (\operatorname{Im} z)^2]}\right]^{1/2}$$

The following consequences of the inequalities (4.1) and (4.2) will be especially useful later on: if $\operatorname{Im} z \geq 0$ then for $0 < \varepsilon \leq \varepsilon_0$ one has

$$(4.3)\qquad ||G_\varepsilon^{(\pm)}f||^2 \leq \pm\frac{1}{a\varepsilon}\operatorname{Im}\langle f, G_\varepsilon f\rangle + \frac{b}{a\delta^2}||f||^2,$$

$$(4.4)\qquad ||G_\varepsilon|| \leq \frac{1}{a\varepsilon} + \left(\frac{b}{a\delta^2}\right)^{1/2}.$$

Now let us assume that the family $\{H_\varepsilon\}$ from Proposition 4.1 has two more properties:

(1) H_ε is of class $C^1(A)$ if $0 < \varepsilon < \varepsilon_0$;

(2) the map $\varepsilon \mapsto H_\varepsilon \in B(\mathcal{H})$ is strongly C^1 on $(0,\varepsilon_0)$.

Let z be a complex number with $\operatorname{Re} z \in J$ and $\operatorname{Im} z \geq 0$ and let $0 < \varepsilon < \varepsilon_0$. Then $G_\varepsilon \in C^1(A)$ and $\mathcal{A}[G_\varepsilon] = -G_\varepsilon\mathcal{A}[H_\varepsilon]G_\varepsilon$. Indeed, if for $\tau \neq 0$ we set $A_\tau = (i\tau)^{-1}(e^{i\tau A} - 1)$ then we clearly have $[A_\tau, G_\varepsilon] = G_\varepsilon[H_\varepsilon, A_\tau]G_\varepsilon$ and the result follows by taking the limit as $\tau \to 0$ and by using, for example, the fact that $[H_\varepsilon, A_\tau] \to \mathcal{A}[H_\varepsilon]$ strongly as $\tau \to 0$. Furthermore, the map $\varepsilon \mapsto G_\varepsilon \in B(\mathcal{H})$ is strongly C^1 on $(0,\varepsilon_0)$ and its derivative is given by $G'_\varepsilon \equiv \frac{d}{d\varepsilon}G_\varepsilon = -G_\varepsilon H'_\varepsilon G_\varepsilon$ (this is an easy consequence of (4.4)). In particular we get

$$(4.5)\qquad G'_\varepsilon = \mathcal{A}[G_\varepsilon] + G_\varepsilon(\mathcal{A}[H_\varepsilon] - H'_\varepsilon)G_\varepsilon.$$

This equation plays a fundamental role in the theory.

In this paper we shall choose H_ε (for $\varepsilon \in \mathbb{R}$) of the form $H_\varepsilon = \xi(\varepsilon\mathcal{A})H$, where ξ is a function of the form $\xi(x) = e^x\theta(x)$ with $\theta \in C_0^\infty(\mathbb{R})$ real even and such that $\theta(x) = 1$ on a neighbourhood of zero (a rather detailed motivation of this choice can be found in [BGSh2]). Note that the operator H_ε is not self-adjoint in general and that we have $H_\varepsilon^* = H_{-\varepsilon}$. We shall also need the function η given by $\eta(x) = x(\xi(x) - \xi'(x)) = -e^x x\theta'(x)$, so that $\eta \in C_0^\infty(\mathbb{R}\setminus\{0\})$. Formally, (4.5) becomes:

$$(4.6)\qquad G'_\varepsilon = \mathcal{A}[G_\varepsilon] + \varepsilon^{-1}G_\varepsilon\eta(\varepsilon\mathcal{A})[H]G_\varepsilon.$$

It is not yet clear whether the so-defined family $\{H_\varepsilon\}$ satisfies or not the hypotheses of Proposition 4.1. In fact it does not if H is only of class $C^1(A)$, as we explain in the Proposition 4.2. We first state a lemma which can be proven without difficulty and which will be needed below.

Lemma. *Let $\varphi \in \mathcal{M}$ be a function of class C^1 and such that its derivative φ' and the function $\tilde{\varphi}(x) = x\varphi'(x)$ belong to $\mathcal{M}$. Then for each $S \in B(\mathcal{H})$ the map $\varepsilon \mapsto \varphi(\varepsilon\mathcal{A})S \in B(\mathcal{H})$ is strongly C^1 on $\mathbb{R} \setminus \{0\}$, for $\varepsilon \neq 0$ the operator $\varphi'(\varepsilon\mathcal{A})S$ is of class $C^1(A)$, and we have $(d/d\varepsilon)\varphi(\varepsilon\mathcal{A})S = \varepsilon^{-1}\tilde{\varphi}(\varepsilon\mathcal{A})S = \mathcal{A}\varphi'(\varepsilon\mathcal{A})S$. In particular, if $\varphi \in C_0^\infty(\mathbb{R})$ then $\varphi^{(k)}(\varepsilon\mathcal{A})S \in C^\infty(A)$ if $\varepsilon \neq 0$ and $k \in \mathbb{N}$, the map $\varepsilon \mapsto \varphi(\varepsilon\mathcal{A})S \in B(\mathcal{H})$ is of class C^∞ on $\mathbb{R} \setminus \{0\}$ and we have $(d/d\varepsilon)^k\varphi(\varepsilon\mathcal{A})S = \mathcal{A}^k\varphi^{(k)}(\varepsilon\mathcal{A})S$.*

Proposition 4.2. *The family $\{H_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ defined above satisfies the hypotheses of Proposition 4.1 if and only if the operator H is of class $C_{\mathrm{u}}^1(A)$. Assume that $H \in C_{\mathrm{u}}^1(A)$ and let $z \in \mathbb{C}$ with $\operatorname{Re} z \in J$ and $\operatorname{Im} z > 0$.*

(a) *For $0 \leq \varepsilon \leq \varepsilon_0$ one has $G_\varepsilon \in C_{\mathrm{u}}^1(A)$ and $\mathcal{A}[G_\varepsilon] = -G_\varepsilon\mathcal{A}[H_\varepsilon]G_\varepsilon$; if $0 < \varepsilon < \varepsilon_0$ then $G_\varepsilon \in C^\infty(A)$.*

(b) *The map $\varepsilon \mapsto H_\varepsilon$ is of class C^1 in norm on $\mathbb{R}$ and is of class C^∞ on $\mathbb{R} \setminus \{0\}$. The map $\varepsilon \mapsto G_\varepsilon$ is of class C^1 in norm on the closed interval $[0, \varepsilon_0]$, where its derivative is given by $G'_\varepsilon = -G_\varepsilon H'_\varepsilon G_\varepsilon$, and is of class C^∞ on $(0, \varepsilon_0]$.*

(c) *Set $K_\varepsilon = \varepsilon^{-1}\eta(\varepsilon\mathcal{A})H$ for $\varepsilon \neq 0$, where $\eta \in C_0^\infty(\mathbb{R} \setminus \{0\})$ is given by $\eta(x) = -e^x x\theta'(x)$. Then $K_\varepsilon \in C^\infty(A)$, $\varepsilon \mapsto K_\varepsilon$ is of class C^∞ on $\mathbb{R} \setminus \{0\}$, and for $0 < \varepsilon \leq \varepsilon_0$ one has*

$$G'_\varepsilon = \mathcal{A}[G_\varepsilon] + G_\varepsilon K_\varepsilon G_\varepsilon. \tag{4.7}$$

(d) *Set $K_\varepsilon^{(j)} = (d/d\varepsilon)^j K_\varepsilon$ and let $\alpha > -1$ real and $p \in [1, \infty]$. Then H is of class $\mathcal{C}^{1+\alpha,p}(A)$ if and only if the condition*

$$\Big[\int_0^1 ||\varepsilon^{-\alpha+j}K_\varepsilon^{(j)}||^p \varepsilon^{-1} d\varepsilon\Big]^{1/p} < \infty \tag{4.8}$$

holds for $j = 0$. If this is the case then (4.8) holds for each integer $j \geq 0$.

Proof. We define a real even function $\varrho \in C_0^\infty(\mathbb{R})$ by $\varrho(0) = 1$ and $\varrho(x) = x^{-1}\operatorname{sh} x \cdot \theta(x)$ if $x \neq 0$. Then for an arbitrary bounded self-adjoint operator H we have $\varepsilon^{-1}\operatorname{Im} H_\varepsilon^* = i\mathcal{A}\varrho(\varepsilon\mathcal{A})H \equiv S_\varepsilon$. Assume first that $\lim_{\varepsilon\to 0} S_\varepsilon$ exists in norm in $B(\mathcal{H})$ and denote by S the limit. Since $C^\infty(A)$ is a subspace of the norm-closed space $C_{\mathrm{u}}^0(A)$ and $S_\varepsilon \in C^\infty(A)$ if $\varepsilon \neq 0$, we get $S \in C_{\mathrm{u}}^0(A)$. For $f \in D(A)$ we have

$$\langle f, S_\varepsilon f\rangle = \langle f, [\varrho(\varepsilon\mathcal{A})H, iA]f\rangle = 2\operatorname{Re}\langle(\varrho(\varepsilon\mathcal{A})H)f, iAf\rangle$$

which converges to $2\operatorname{Re}\langle Hf, iAf\rangle$ as $\varepsilon \to 0$. So we have $2\operatorname{Re}\langle Hf, iAf\rangle = \langle f, Sf\rangle$ for $f \in D(A)$, i.e. $i\mathcal{A}H = S \in C_{\mathrm{u}}^0(A)$. This clearly means $H \in C_{\mathrm{u}}^1(A)$. Reciprocally, if $H \in C_{\mathrm{u}}^1(A)$ then H is of class $C_{\mathrm{u}}^0(A)$ hence $||H_\varepsilon - H|| \to 0$ as $\varepsilon \to 0$. Moreover, we shall also have $S_\varepsilon = i\varrho(\varepsilon\mathcal{A})\mathcal{A}H$ and $\mathcal{A}H \in C_{\mathrm{u}}^0(A)$, so $||S_\varepsilon - i\mathcal{A}H|| \to 0$ as $\varepsilon \to 0$. Hence the family $\{H_\varepsilon\}_{\varepsilon\geq 0}$ satisfies the hypotheses of Proposition 4.1.

The proof of the assertions (a), (b) and (c) is easy, see the Lemma stated before Proposition 4.2. For part (d) we use the Theorem 2.3. Observe that $\eta \in C_0^\infty(\mathbb{R} \setminus \{0\})$ is not identically zero on $(-\infty, 0)$ and on $(0, \infty)$, so if (4.8) holds with $j = 0$ then $H \in \mathcal{C}^{1+\alpha,p}(A)$. Reciprocally, if H has this property then we have (4.8) for all j because $\varepsilon^j K_\varepsilon^{(j)} = \eta_j(\varepsilon\mathcal{A})H$ for some $\eta_j \in C_0^\infty(\mathbb{R} \setminus \{0\})$. $\square$

We denote by $|||\cdot|||$ either the norm in the Banach space $\mathcal{K}=\mathcal{H}_{1/2,1}$ or the norm associated to it in $B(\mathcal{K};\mathcal{K}^*)$, and we recall that we have continuous embeddings $\mathcal{K}\subset\mathcal{H}\subset\mathcal{K}^*$ and $B(\mathcal{H})\subset B(\mathcal{K};\mathcal{K}^*)$. *From now on we assume that H is (at least) of class $\mathcal{C}^{1,1}(A)$.* We write $z=\lambda+i\mu$ and the numbers ε,λ,μ are supposed to verify $0<\varepsilon<\varepsilon_0$, $\lambda\in J$, $\mu>0$. One should think of μ rather as a parameter, but it is important that the various constants that appear below are independent of μ. If F is a function of $(\lambda,\varepsilon)\in J\times(0,\varepsilon_0)$ we denote by $F^{(k,m)}\equiv\partial_\lambda^k\partial_\varepsilon^m F$ its derivative of order k with respect to λ and of order m with respect to ε. We also set $F^{(m)}=F^{(0,m)}$. The operator $G_\varepsilon=G_\varepsilon(z)=G_\varepsilon(\lambda+i\mu)$ will be considered as a function of $(\lambda,\varepsilon)\in J\times(0,\varepsilon_0)$; we clearly have for $k\in\mathbb{N}$:

$$G_\varepsilon^{(k,0)}=\partial_\lambda^k G_\varepsilon(\lambda+i\mu)=k!G_\varepsilon^{k+1}. \tag{4.9}$$

Proposition 4.3. *If H is of class $\mathcal{C}^{1,1}(A)$ then for each $k,m\in\mathbb{N}$ there is a number $C<\infty$, independent of $\varepsilon\in(0,\varepsilon_0)$, $\lambda\in J$ and $\mu>0$, such that*

$$|||G_\varepsilon^{(k,m)}|||\le C\varepsilon^{-k-m}, \tag{4.10}$$

$$||G_\varepsilon^{(k,m)}||_{\mathcal{K}\to\mathcal{H}}+||G_\varepsilon^{(k,m)}||_{\mathcal{H}\to\mathcal{K}^*}\le C\varepsilon^{-k-m-1/2}. \tag{4.11}$$

Proof. (i) We first prove (4.10), (4.11) in the case $k=m=0$. Fix a number $\varepsilon_1\in[0,\varepsilon_0)$ and a family $\{f_\varepsilon\}_{\varepsilon_1<\varepsilon\le\varepsilon_0}$ of vectors in $D(A)$ such that the function $\varepsilon\mapsto f_\varepsilon\in\mathcal{H}$ is of class C^1. We set $F_\varepsilon=\langle f_\varepsilon,G_\varepsilon f_\varepsilon\rangle$ for $\varepsilon_1<\varepsilon\le\varepsilon_0$ and we get by using (4.7):

$$F'_\varepsilon=\langle f'_\varepsilon-Af_\varepsilon,G_\varepsilon f_\varepsilon\rangle+\langle G_\varepsilon^*f_\varepsilon,f'_\varepsilon+Af_\varepsilon\rangle+\langle G_\varepsilon^*f_\varepsilon,K_\varepsilon G_\varepsilon f_\varepsilon\rangle.$$

Denote $\ell_\varepsilon=||f'_\varepsilon||+||Af_\varepsilon||$. Then (4.3) implies

$$\begin{aligned}|F'_\varepsilon|&\le\ell_\varepsilon(||G_\varepsilon f_\varepsilon||+||G_\varepsilon^*f_\varepsilon||)+||K_\varepsilon||\cdot||G_\varepsilon f_\varepsilon||\cdot||G_\varepsilon^*f_\varepsilon||\\&\le 2\ell_\varepsilon a^{-1/2}(\varepsilon^{-1/2}|F_\varepsilon|^{1/2}+b^{1/2}\delta^{-1}||f_\varepsilon||)+||K_\varepsilon||a^{-1}(\varepsilon^{-1}|F_\varepsilon|+b\delta^{-2}||f_\varepsilon||^2).\end{aligned}$$

So there is a constant $c>0$, depending only on a,b and d, such that for $\varepsilon_1<\varepsilon\le\varepsilon_0$:

$$c^{-1}|F'_\varepsilon|\le\ell_\varepsilon||f_\varepsilon||+||K_\varepsilon||\cdot||f_\varepsilon||^2+\ell_\varepsilon\varepsilon^{-1/2}|F_\varepsilon|^{1/2}+||K_\varepsilon||\varepsilon^{-1}|F_\varepsilon|.$$

According to Lemma 7.A.1 from [ABG] the preceding estimate implies

$$\begin{aligned}|F_{\varepsilon_1}|\le 2\Big\{|F_{\varepsilon_0}|&+c\int_{\varepsilon_1}^{\varepsilon_0}[\ell_\tau||f_\tau||+||K_\tau||\cdot||f_\tau||^2]d\tau\\&+c^2\Big[\int_{\varepsilon_1}^{\varepsilon_0}\ell_\tau\tau^{-1/2}d\tau\Big]^2\Big\}\exp\int_{\varepsilon_1}^{\varepsilon_0}c||K_\tau||\tau^{-1}d\tau.\end{aligned} \tag{4.12}$$

By Proposition 4.2 (d) we have $\int_0^{\varepsilon_0}||K_\tau||\tau^{-1}d\tau$ if and only if $H\in\mathcal{C}^{1,1}(A)$. Now let $f\in\mathcal{H}_{1/2,1}$ and $f_\varepsilon=\theta((\varepsilon-\varepsilon_1)A)f$, with the same function θ as in the definition of H_ε. If we set $\tilde\theta(x)=x\theta'(x)$ and $\theta_{(1)}(x)=x\theta(x)$, then

$$\begin{aligned}\int_{\varepsilon_1}^{\varepsilon_0}\ell_\tau\tau^{-1/2}d\tau&=\int_0^{\varepsilon_0-\varepsilon_1}(||\tilde\theta(\sigma A)f||+||\theta_{(1)}(\sigma A)f||)\frac{d\sigma}{\sigma(\sigma+\varepsilon_1)^{1/2}}\\&\le c'||f||_{\mathcal{H}_{1/2,1}}=c'|||f|||\end{aligned}$$

where c' is a finite constant depending only on ε_0 and θ. Now by using (4.12) we easily see that there is a constant $c''<\infty$ such that $|\langle f,G_\varepsilon f\rangle|\le c''|||f|||^2$

for $0 < \varepsilon \le \varepsilon_0$, $\lambda \in J$, $\mu > 0$ and $f \in \mathcal{K}$. The polarization identity will then give $|||G_\varepsilon||| \le \text{const}$. Finally, the estimate (4.11) with $k = m = 0$ is an immediate consequence of the preceding one and of (4.3).

(ii) Now we treat the case where one of the numbers k, m is not zero. If $m = 0$ then the estimates follow easily from those with $k = m = 0$ by taking into account (4.4) and (4.9), so we can assume $m \ge 1$. Then by Proposition 4.2 (b) the operator $G_\varepsilon^{(m)}$ is a linear combination of terms of the form $G_\varepsilon H_\varepsilon^{(m_1)} G_\varepsilon H_\varepsilon^{(m_2)} \dots G_\varepsilon H_\varepsilon^{(m_n)}$ with $m_1, \dots, m_n \ge 1$ integers and $m_1 + \dots + m_n = m$. So from (4.9) it follows that $G_\varepsilon^{(k,m)}$ is a linear combination of terms of the form

$$G_\varepsilon^{k_0+1} H_\varepsilon^{(m_1)} G_\varepsilon^{k_1+1} H_\varepsilon^{(m_2)} G_\varepsilon^{k_2+1} \dots H_\varepsilon^{(m_n)} G_\varepsilon^{k_n+1}$$

with $m_1, \dots, m_n$ as above and $k_0, k_1, \dots, k_n \in \mathbb{N}$ such that $k_0 + k_1 + \dots + k_n = k$. The norm in $B(\mathcal{K}; \mathcal{K}^*)$ of such a term is bounded by

$$\begin{aligned} &||G_\varepsilon||_{\mathcal{H}\to\mathcal{K}^*} ||G_\varepsilon||^{k_0} ||H_\varepsilon^{(m_1)})|| \cdot ||G_\varepsilon||^{k_1+1} \dots ||H_\varepsilon^{(m_n)}|| \cdot ||G_\varepsilon||^{k_n} ||G_\varepsilon||_{\mathcal{K}\to\mathcal{H}} \\ &\le \text{const.}\, \varepsilon^{-1/2} \cdot \varepsilon^{-k_0} ||H_\varepsilon^{(m_1)}|| \cdot \varepsilon^{-k_1-1} \dots ||H_\varepsilon^{(m_n)}|| \varepsilon^{-k_n} \cdot \varepsilon^{-1/2} \end{aligned}$$

where we have used (4.11) with $k = m = 0$ and (4.4). Similarly, the norm in $B(\mathcal{K}; \mathcal{H})$ is bounded by

$$\begin{aligned} &||G_\varepsilon||^{k_0+1} ||H_\varepsilon^{(m_1)}|| \cdot ||G_\varepsilon||^{k_1+1} \dots ||H_\varepsilon^{(m_n)}|| \cdot ||G_\varepsilon||^{k_n} ||G_\varepsilon||_{\mathcal{K}\to\mathcal{H}} \le \\ &\le \text{const.} \varepsilon^{-k_0-1} ||H_\varepsilon^{(m_1)}|| \cdot \varepsilon^{-k_1-1} \dots ||H_\varepsilon^{(m_n)}|| \cdot \varepsilon^{-k_n} \cdot \varepsilon^{-1/2}. \end{aligned}$$

We see that the assertions of the proposition are a consequence of the estimate $||H_\varepsilon^{(m)}|| \le c_m \varepsilon^{1-m}$ for $m \ge 1$ integer and $\varepsilon > 0$. But we have

$$\begin{aligned} H_\varepsilon^{(m)} &= \partial_\varepsilon^m \xi(\varepsilon\mathcal{A}) H = \mathcal{A}^m \xi^{(m)}(\varepsilon\mathcal{A}) H \\ &= \varepsilon^{1-m} (\varepsilon\mathcal{A})^{m-1} \xi^{(m)}(\varepsilon\mathcal{A}) \mathcal{A} H = \varepsilon^{1-m} \varphi(\varepsilon\mathcal{A}) \mathcal{A} H. \end{aligned}$$

where $\varphi(x) = x^{m-1} \xi^{(m)}(x)$ is a function of class $C_0^\infty(\mathbb{R})$. Hence

$$||H_\varepsilon^{(m)}|| \le \varepsilon^{1-m} ||\varphi||_{\mathcal{M}} ||\mathcal{A}H||. \quad \square$$

Lemma 4.4. *Set $\tilde{G}_\varepsilon = G_\varepsilon K_\varepsilon G_\varepsilon$, where K_ε is as in Proposition 4.2 (c). Then for each $k, m \in \mathbb{N}$ there is a finite constant C, independent of $\varepsilon, \lambda, \mu$, such that*

$$|||\tilde{G}_\varepsilon^{(k,m)}||| \le C \varepsilon^{-k-m-1} \sum_{j=0}^{m} ||\varepsilon^j K_\varepsilon^{(j)}||. \tag{4.13}$$

In particular, if $H \in \mathcal{C}^{1+\alpha}(A)$ for some $\alpha > 0$, then we have $|||\tilde{G}_\varepsilon^{(k,m)}||| \le c\varepsilon^{\alpha-k-m-1}$.

Proof. By Leibnitz formula, and since K_ε does not depend on λ, $\tilde{G}_\varepsilon^{(k,m)}$ is a linear combination of terms of the form $G_\varepsilon^{(a,u)} K_\varepsilon^{(w)} G_\varepsilon^{(b,v)}$ with $a, b, u, v, w \in \mathbb{N}$ and $a + b = k$, $u + v + w = n$. Then Proposition 4.3 implies

$$\begin{aligned} |||G_\varepsilon^{(a,u)} K_\varepsilon^{(w)} G_\varepsilon^{(b,v)}||| &\le ||G_\varepsilon^{(a,u)}||_{\mathcal{H}\to\mathcal{K}^*} ||K_\varepsilon^{(w)}|| \cdot ||G_\varepsilon^{(b,v)}||_{\mathcal{K}\to\mathcal{H}} \\ &\le \text{const.} \varepsilon^{-a-u-1/2} ||K_\varepsilon^{(w)}|| \cdot \varepsilon^{-b-v-1/2} \\ &= \text{const.} \varepsilon^{-k-m-1} ||\varepsilon^w K_\varepsilon^{(w)}||. \quad \square \end{aligned}$$

For the proof of the next estimates we need a generalization of the identity (4.7). Assume that we are under the hypotheses of Proposition 4.2 and let $\tilde{G}_\varepsilon = G_\varepsilon K_\varepsilon G_\varepsilon$. Then for all $\ell, k \in \mathbb{N}$ with $k \geq 1$ and all $\varepsilon \in (0, \varepsilon_0)$, $z = \lambda + i\mu$, $\lambda \in J$, $\mu > 0$ we have

$$G_\varepsilon^{(\ell,k)} = \ell! \mathcal{A}^k [G_\varepsilon^{\ell+1}] + \sum_{r=0}^{k-1} \mathcal{A}^{k-r-1} [\tilde{G}_\varepsilon^{(\ell,r)}]. \tag{4.14}$$

If $\ell = 0$, $k = 1$ this is just (4.7). (4.14) follows from this special case by taking successively derivatives with respect to ε and λ and by using the following simple result: *Let $[a, b]$ be a real interval and $\{S_x\}_{a \leq x \leq b}$ a family of bounded operators on $\mathcal{H}$ having the following properties:*

(i) *$x \mapsto S_x \in B(\mathcal{H})$ is strongly of class C^1, with derivative $S'_x = \partial_x S_x$;*
(ii) *S_x and S'_x are of class $C^1(A)$ for all $x \in [a, b]$;*
(iii) *$x \mapsto \mathcal{A} S'_x \in B(\mathcal{H})$ is strongly continuous.*

Then the map $x \mapsto \mathcal{A} S_x \in B(\mathcal{H})$ is strongly C^1 and its derivative is given by $\partial_x \mathcal{A} S_x = \mathcal{A} S'_x$.

Now let us fix two functions $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and let us define the operator $L_\varepsilon \equiv L_\varepsilon(z) : \mathcal{H}_{-\infty} \to \mathcal{H}_{+\infty}$ by

$$L_\varepsilon(z) = \varphi(\varepsilon A) G_\varepsilon(z) \psi(\varepsilon A) \tag{4.15}$$

for $0 < \varepsilon < \varepsilon_0$ and $z = \lambda + i\mu$ with $\lambda \in J$ and $\mu > 0$. Let $\ell, m \in \mathbb{N}$. By using Leibnitz formula and by taking into account the relation $\partial_\varepsilon^i \varphi(\varepsilon A) = A^i \varphi^{(i)}(\varepsilon A) = \varepsilon^{-i} \varphi_i(\varepsilon A)$ with $\varphi_i(x) = x^i \varphi^{(i)}(x)$ we obtain

$$L_\varepsilon^{(\ell,m)} = \sum_{i+j+k=m} \frac{m!}{i!j!k!} \varepsilon^{k-m} \varphi_i(\varepsilon A) G_\varepsilon^{(\ell,k)} \psi_j(\varepsilon A),$$

where the indices i, j, k run over $\mathbb{N}$. If we use (4.14) the expression in the r.h.s. above becomes

$$\begin{aligned} L_\varepsilon^{(\ell,m)} = & \sum_{i+j+k=m} \frac{\ell! m!}{i!j!k!} \varepsilon^{k-m} \varphi_i(\varepsilon A) \mathcal{A}^k [G_\varepsilon^{\ell+1}] \psi_j(\varepsilon A) \\ & + \sum_{\substack{i+j+k=m \\ k \geq 1 \\ n+r=k-1}} \frac{m!}{i!j!k!} \varepsilon^{k-m} \varphi_i(\varepsilon A) \mathcal{A}^n [\tilde{G}_\varepsilon^{(\ell,r)}] \psi_j(\varepsilon A). \end{aligned}$$

Then by taking into account the identity (2.5) we get

(4.16)

$$\begin{aligned} \varepsilon^m L_\varepsilon^{(\ell,m)} = & \sum_{i+j+p+q=m} \frac{\ell! m!}{i!j!p!q!} (-\varepsilon A)^p (\varepsilon A)^i \varphi^{(i)}(\varepsilon A) G_\varepsilon^{\ell+1} (\varepsilon A)^{j+q} \psi^{(j)}(\varepsilon A) \\ & + \sum_{i+j+p+q+r=m-1} \frac{m!(p+q)!(-1)^p \varepsilon^{r+1}}{i!j!p!q!(m-i-j)!} (\varepsilon A)^{i+p} \varphi^{(i)}(\varepsilon A) \tilde{G}_\varepsilon^{(\ell,r)} (\varepsilon A)^{j+q} \psi^{(j)}(\varepsilon A). \end{aligned}$$

Proposition 4.5. *Let $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and let $L_\varepsilon = L_\varepsilon(z)$ be defined by $L_\varepsilon = \varphi(\varepsilon A) G_\varepsilon \psi(\varepsilon A)$. Then for each $\ell, m \in \mathbb{N}$ there is a constant C, independent* *of $\varepsilon, \lambda, \mu$, such that for all $f, g \in \mathcal{H}_{-\infty}$:*

$$|\langle g, \varepsilon^{\ell+m} L_\varepsilon^{(\ell,m)} f\rangle| \leq C \sum_{\substack{a+b=m\\0\leq i\leq a\\0\leq j\leq b}} |||\varphi_{i,a}(\varepsilon A)g||| \cdot |||\psi_{j,b}(\varepsilon A)f|||$$

$$+\, C \sum_{\substack{a+b+c\leq m-1\\0\leq i\leq a\\0\leq j\leq b}} |||\varphi_{i,a}(\varepsilon A)g||| \cdot |||\psi_{j,b}(\varepsilon A)f||| \cdot ||\varepsilon^c K_\varepsilon^{(c)})||. \tag{4.17}$$

Here the functions $\varphi_{i,a}$ and $\psi_{j,b}$ are defined by $\varphi_{i,a}(x) = x^a \varphi^{(i)}(x)$ and $\psi_{j,b}(x) = x^b \psi^{(j)}(x)$.

Proof. We use (4.16) and the estimates

$$|||\varepsilon^\ell G_\varepsilon^{\ell+1}||| \leq C(\ell) \text{ and } |||\varepsilon^{\ell+r+1} \tilde{G}_\varepsilon^{(\ell,r)}||| \leq C(\ell, r) \sum_{0\leq c\leq r} ||\varepsilon^c K_\varepsilon^{(c)}||$$

which have been obtained in Proposition 4.3 and Lemma 4.4. □

It is clear that the first sum from (4.16) becomes much simpler if φ is a function such that $\varphi^{(i)}(x) = \varphi(x)$ for all x. But the only function which has this property is $\varphi(x) = e^x$ and it does not belong to $\mathcal{S}(\mathbb{R})$. However, one can circumvent this difficulty if in place of L_ε one considers the operator $\Pi_- L_\varepsilon$, where $\Pi_- = E_A((-\infty, 0])$ is the spectral projection of A associated with the interval $(-\infty, 0]$. Then we take a function $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\varphi(x) = e^x$ if $x \leq 0$. Observe that for j, q fixed with $n = m - j - q > 0$ one has $\sum_{i+p=n} (i!p!)^{-1}(-x)^p x^i = 0$. Hence, after left multiplication by Π_- of (4.16), in the first sum on the r.h.s. will remain only terms with $j + q = m$, so $i = p = 0$. On the other hand:

$$\sum_{j+q=m} \frac{m!}{j!q!} x^{j+q} \psi^{(j)}(x) = x^m \Big(1 + \frac{d}{dx}\Big)^m \psi(x) \equiv \zeta(x). \tag{4.18}$$

Hence we obtain:

$$\varepsilon^m \Pi_- L_\varepsilon^{(\ell,m)} = \ell! \Pi_- e^{\varepsilon A} G_\varepsilon^{\ell+1} \zeta(\varepsilon A)$$
$$+ \sum_{i+j+p+q+r=m-1} \frac{m!(p+q)!(-1)^p \varepsilon^{r+1}}{i!j!p!q!(m-i-j)!} \Pi_-(\varepsilon A)^{i+p} e^{\varepsilon A} \tilde{G}_\varepsilon^{(\ell,r)} (\varepsilon A)^{j+q} \psi^{(j)}(\varepsilon A).$$

By the same argument as in the proof of Proposition 4.5 we get, with a slight change of notation:

Proposition 4.6. *Let $\psi \in \mathcal{S}(\mathbb{R})$, define ζ by (4.18), and set $L_\varepsilon = \Pi_- e^{\varepsilon A} G_\varepsilon \psi(\varepsilon A)$. Then for each $\ell, m \in \mathbb{N}$ there is a constant C, independent of $\varepsilon, \lambda, \mu$, such that for all $f, g \in \mathcal{H}_{-\infty}$:*

(4.19)

$$|\langle g, \varepsilon^{\ell+m} L_\varepsilon^{(\ell,m)}) f\rangle| \leq C |||\Pi_- e^{\varepsilon A} g||| \cdot |||\zeta(\varepsilon A) f|||$$
$$+\, C \sum_{\substack{a+b+c\leq m-1\\0\leq j\leq b}} |||\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot |||(\varepsilon A)^b \psi^{(j)}(\varepsilon A) f||| \cdot ||\varepsilon^c K_\varepsilon^{(c)}||.$$

This estimate can be further simplified by a special choice of ψ. Note that if $\psi(x) = e^{-x}$ then $\zeta = 0$. Of course this choice is not allowed by the condition $\psi \in \mathcal{S}(\mathbb{R})$. However, if we take ψ of class $\mathcal{S}(\mathbb{R})$ and such that $\psi(x) = e^{-x}$ if $x \geq 0$, then $\Pi_+\zeta(\varepsilon A)f = 0$ for each $f \in \mathcal{H}_{-\infty}$. Hence Proposition 4.6 immediately implies the next one. Here $\Pi_+ = E_A([0,\infty))$.

Proposition 4.7. *Let $L_\varepsilon = \Pi_- e^{\varepsilon A} G_\varepsilon e^{-\varepsilon A}\Pi_+$. Then for each $\ell, m \in \mathbb{N}$ with $m \geq 1$ there is $C < \infty$, independent of $\varepsilon, \lambda, \mu$, such that for all $f, g \in \mathcal{H}_{-\infty}$:*

$$(4.20) \quad |\langle g, \varepsilon^{\ell+m} L_\varepsilon^{(\ell,m)} f\rangle| \leq \\ \leq C \sum_{a+b+c\leq m-1} |||\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot |||\Pi_+(\varepsilon A)^b e^{-\varepsilon A} f||| \cdot ||\varepsilon^c K_\varepsilon^{(c)}||.$$

5. Boundary Values of Resolvent Families

5.1. Throughout this section $\{R(z)\}$ is a resolvent family on the Hilbert space $\mathcal{H}$, we denote by H the self-adjoint operator associated to it, and *we assume that H has a spectral gap (its spectrum $\sigma(H)$ is not the whole real line)*. We shall make several hypotheses concerning the regularity class of H with respect to A, but these hypotheses will always imply that H is A-regular (i.e. of class $\mathcal{C}^{1,1}(A)$). In particular the open real set $\mu^A(H)$ is well defined and contains $\mathbb{R} \setminus \sigma(H)$. If $f \in \mathcal{H}$ then $z \mapsto \langle f, R(z)f\rangle$ is a well defined holomorphic map on the open complex set $\mathbb{C} \setminus \sigma(H)$ and this set contains the upper ($\mathbb{C}_+$) and lower ($\mathbb{C}_-$) half-planes (we set $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}$). Our first purpose is to prove the existence of the limits $\lim_{\mu\to\pm 0}\langle f, R(\lambda + i\mu)f\rangle \equiv \langle f, R(\lambda \pm i0)f\rangle$ for $\lambda \in \mu^A(H)$ and to discuss the continuity and differentiability properties of the maps $\lambda \mapsto \langle f, R(\lambda \pm i0)f\rangle$ in terms of the regularity properties of H and f with respect to A. Due to the relation $\langle f, R(\lambda + i\mu)f\rangle^* = \langle f, R(\lambda - i\mu)f\rangle$ we may restrict ourselves to the case $\mu \to +0$. Note also that, due to the polarization identity, it is not necessary to consider the case of $\langle g, R(z)f\rangle$ with $g \neq f$.

Theorem 5.1. *Assume that H is of class $\mathcal{C}^{1+\ell,1}(A)$ for some integer $\ell \geq 0$ and set $s = \ell + 1/2$. Then for each $f \in \mathcal{H}_{s,1}$ the holomorphic map $\mathbb{C}_+ \ni z \mapsto \langle f, R(z)f\rangle$ extends to a function of class C^ℓ on $\mathbb{C}_+ \cup \mu^A(H)$, i.e. for each integer $0 \leq k \leq \ell$ the holomorphic function on $\mathbb{C}_+$ given by $(d/dz)^k\langle f, R(z)f\rangle = \langle f, k!R(z)^{k+1}f\rangle$ has a continuous extension to $\mathbb{C}_+ \cup \mu^A(H)$. The limit $\lim_{\mu\to+0}\langle f, R(\lambda+i\mu)f\rangle \equiv \langle f, R(\lambda+i0)f\rangle$ exists locally uniformly in $\lambda \in \mu^A(H)$, the boundary value function $\lambda \mapsto \langle f, R(\lambda + i0)f\rangle$ is of class C^ℓ on $\mu^A(H)$, and for $0 \leq k \leq \ell$ integer one has*

$$(5.1) \qquad \frac{d^k}{d\lambda^k}\langle f, R(\lambda + i0)f\rangle = \lim_{\mu\to+0}\langle f, k!R(\lambda + i\mu)^{k+1}f\rangle$$

locally uniformly in $\lambda \in \mu^A(H)$.

Proof. (i) We first show that it suffices to prove the theorem under the assumption that H is a bounded everywhere defined operator. For this we use the identity (3.12) which can be written $\langle f, R(z)f\rangle = \zeta\langle f, (R-\zeta)^{-1}Rf\rangle$, where $\zeta = (\lambda_0 - z)^{-1}$. The map $z \mapsto \zeta$ is a holomorphic diffeomorphism of $\mathbb{C}\setminus\{\lambda_0\}$ onto $\mathbb{C}\setminus\{0\}$ which leaves $\mathbb{C}_+$ (and $\mathbb{C}_-$) invariant and, by Proposition 3.3, restricts to a C^∞ diffeomorphism of $\mu^A(H)\setminus\{\lambda_0\}$ onto $\mu^A(R)\setminus\{0\}$. The operator R belongs to $\mathcal{C}^{1+\ell,1}(A)$, hence $Rf \in \mathcal{H}_{s,1}$ (see the discussion before Theorem 2.2). So, by taking into account the polarization identity, it suffices to prove the theorem with H replaced by R, which is bounded.

(ii) From now on we assume that H is a bounded (everywhere defined) operator. By considering a small enough neighbourhood J of a point from $\mu^A(H)$, we may assume that the hypotheses made at the beginning of Section 4 are satisfied. For the rest of the proof we use the notations and the results of Section 4. Let $L_\varepsilon = L_\varepsilon(z) = \varphi(\varepsilon A)G_\varepsilon(z)\varphi(\varepsilon A)$ where φ is a function in $\mathcal{S}(\mathbb{R})$ with $\varphi(0) = 1$ and $0 \le \varepsilon \le \varepsilon_0$, $z = \lambda + i\mu$ with $\lambda \in J$, $\mu > 0$. Clearly

$$(5.2) \quad L_\varepsilon^{(\ell,0)} = \partial_\lambda^\ell L_\varepsilon = (\frac{d}{dz})^\ell \varphi(\varepsilon A)G_\varepsilon(z)\varphi(\varepsilon A) = \varphi(\varepsilon A)\ell! G_\varepsilon(z)^{\ell+1}\varphi(\varepsilon A).$$

Note that by Proposition 4.2 (b) the map $\varepsilon \mapsto L_\varepsilon^{(\ell,0)} \in B(\mathcal{H})$ is strongly C^1 on the closed interval $[0,\varepsilon_0]$ and $L_0^{(\ell,0)} = \partial_z^\ell R(z) = \ell! R(z)^{\ell+1}$.

Now let us fix $f \in \mathcal{H}_{s,1}$ and define $h(\varepsilon) = \langle f, L_\varepsilon^{(\ell,0)} f\rangle$ for $0 \le \varepsilon \le \varepsilon_0$. Then for $\varepsilon > 0$ and $m \ge 0$ integer we have $h^{(m)}(\varepsilon) = \langle f, L_\varepsilon^{(\ell,m)} f\rangle$ which can be estimated as in (4.17). So there is $C < \infty$, independent of $\varepsilon, \lambda, \mu$ and f, such that

$$(5.3) \qquad |\varepsilon^m h^{(m)}(\varepsilon)| \le C \sum_{\substack{a+b=m \\ i \le a \\ j \le b}} \varepsilon^{-\ell} |||\varphi_{i,a}(\varepsilon A) f||| \cdot |||\varphi_{j,b}(\varepsilon A) f||| + C|||f|||^2 \sum_{0 \le j \le m-1} \varepsilon^{-\ell} ||\varepsilon^j K_\varepsilon^{(j)}||.$$

By Proposition 4.2 (d) the condition $H \in \mathcal{C}^{1+\ell,1}(A)$ is equivalent to the integrability with respect to the measure $\varepsilon^{-1}d\varepsilon$ on $(0,\varepsilon_0)$ of the second term in the r.h.s. of (5.3). We claim that if $m > 2\ell$ then each term of the first sum from (5.3) is also integrable (with respect to the same measure). Indeed, if $a + b = m$ then either $a > \ell$ or $b > \ell$. In the first case we have

$$\int_0^1 \varepsilon^{-\ell} |||\varphi_{i,a}(\varepsilon A) f||| \cdot |||\varphi_{j,b}(\varepsilon A) f||| \varepsilon^{-1} d\varepsilon \le C' |||f||| \int_0^1 ||\varepsilon^{-\ell}\varphi_{i,a}(\varepsilon A) f||_{1/2,1} \varepsilon^{-1} d\varepsilon \le C'' |||f||| \cdot ||f||_{s,1}$$

due to the Theorem 2.1 (observe that $\varphi_{i,a}$ has a zero of order $\ge a > \ell$ at the origin).

Let us fix an integer $m > 2\ell$. We have seen that there is a function $\chi : (0,\varepsilon_0) \to \mathbb{R}$, independent of λ and μ, such that $|\varepsilon^m h^{(m)}(\varepsilon)| \le \chi(\varepsilon)$ and $\int_0^{\varepsilon_0} \chi(\varepsilon)\varepsilon^{-1} d\varepsilon < \infty$. So we can apply Lemma 5.2 (see below) and thus obtain

$$(5.4) \quad \langle f, \partial_z^\ell R(z) f\rangle = \sum_{k=0}^{m-1} \frac{(-\varepsilon_0)^k}{k!} \langle f, L_{\varepsilon_0}^{(\ell,k} f\rangle + \frac{(-1)^m}{(m-1)!} \int_0^{\varepsilon_0} \langle f, L_\varepsilon^{(\ell,m)} f\rangle \varepsilon^{m-1} d\varepsilon.$$

According to Proposition 4.1, for each $\varepsilon \in [0,\varepsilon_0]$ the function $z \mapsto G_\varepsilon(z) = (H_\varepsilon - z)^{-1}$ is holomorphic in the region $\lambda \in J$, $\mu > -a\varepsilon$, where $a > 0$. So each term in the sum from (5.4) extends to a holomorphic function of z below the real axis if $\operatorname{Re} z \in J$ (see (4.16) for example). For the integral in (5.4) we can use the dominated convergence theorem in order to deduce that its limit as $\mu \to +0$ exists uniformly in $\lambda \in J$.

We have shown that $\lim_{\mu\to 0}\langle f, \partial_z^\ell R(z) f\rangle$ exists uniformly in $\lambda \in J$. Clearly the arguments still work if ℓ is replaced by a smaller integer. $\square$

In the preceding proof we used the following elementary fact:

Lemma 5.2. *Let $h : (0, \varepsilon_0] \to \mathbb{C}$ be a function of class C^m for some integer $m \geq 1$ and some real $\varepsilon_0 > 0$. Assume that $\int_0^{\varepsilon_0} |\varepsilon^{m-1} h^{(m)}(\varepsilon)| d\varepsilon < \infty$. Then $\lim_{\varepsilon\to 0} h(\varepsilon) \equiv h(0)$ exists and*

$$h(0) = \sum_{k=0}^{m-1} \frac{(-\varepsilon_0)^k}{k!} h^{(k)}(\varepsilon_0) + \frac{(-1)^m}{(m-1)!} \int_0^{\varepsilon_0} h^{(m)}(\varepsilon)\varepsilon^{m-1} d\varepsilon. \tag{5.5}$$

It is convenient to reformulate Theorem 5.1 in slightly different terms. For an arbitrary self-adjoint operator H the map $z \mapsto R(z) \in B(\mathcal{H})$ is holomorphic on $\mathbb{C}_+$. Recall that we have continuous embeddings

$$B(\mathcal{H}) \subset B(\mathcal{K}; \mathcal{K}^*) \subset B(\mathcal{H}_{s,1}; \mathcal{H}_{-s,\infty}) \tag{5.6}$$

if $s \geq 1/2$. So, for example, $z \mapsto R(z) \in B(\mathcal{K}; \mathcal{K}^*)$ is a holomorphic map on $\mathbb{C}_+$. Now assume that $H \in \mathcal{C}^{1,1}(A)$, i.e. the hypothesis of Theorem 5.1 holds with $\ell = 0$. Then the theorem says that the preceding function extends to a weak* continuous function on $\mathbb{C}_+ \cup \mu^A(H)$, in fact $\lim_{\mu\to+0} R(\lambda + i\mu) \equiv R(\lambda + i0) \in B(\mathcal{K}; \mathcal{K}^*)$ exists in the weak* topology of $B(\mathcal{K}; \mathcal{K}^*)$, locally uniformly in $\lambda \in \mu^A(H)$. So the boundary value function $\lambda \mapsto R(\lambda + i0) \in B(\mathcal{K}; \mathcal{K}^*)$ is well defined and weak* continuous on $\mu^A(H)$. According to (5.6), we may consider the map $\lambda \mapsto R(\lambda + i0) \in B(\mathcal{H}_{s,1}; \mathcal{H}_{-s,\infty})$ for each $s \geq 1/2$; clearly it is a weak* continuous function (recall that $\mathcal{H}_{-s,\infty} = \mathcal{H}_{s,1}^*$, which defines the weak* topology of the preceding space). Now assume that $H \in \mathcal{C}^{1+\ell,1}(A)$ for some integer $\ell \geq 1$. Then the Theorem 5.1 says that the map $\lambda \mapsto R(\lambda + i0) \in B(\mathcal{H}_{s,1}; \mathcal{H}_{-s,\infty})$ is of class C^ℓ on $\mu^A(H)$ in the weak* topology if $s = \ell + 1/2$. Moreover its weak* derivatives are given by

$$\frac{d^k}{d\lambda^k} R(\lambda + i0) = \lim_{\mu\to+0} k! R(\lambda + i\mu)^{k+1} \equiv k! R^{k+1}(\lambda + i0) \tag{5.7}$$

where the limit exists in the weak* topology of $B(\mathcal{H}_{s,1}; \mathcal{H}_{-s,\infty})$, locally uniformly in $\lambda \in \mu^A(H)$.

Our next purpose is to describe the regularity properties of the function $\lambda \mapsto R(\lambda + i0)$ in terms of the classes Λ^α. For the proof of the next result we need the following lemma (proved in [BG3]):

Lemma 5.3. *Let $J \subset \mathbb{R}$ be an open set, $\varepsilon_0 > 0$ a real number and $\tilde{J} = \{(\lambda, \varepsilon) \in \mathbb{R}^2 \mid \lambda \in J, 0 < \varepsilon < \varepsilon_0\}$. Let $F : \tilde{J} \to \mathbb{C}$ be a function of class C^m for some integer $m \geq 1$ and assume that there are real numbers σ, M, with $0 < \sigma < m$ and $M > 0$, such that $\sum_{\ell+k=m} |\partial_\lambda^\ell \partial_\varepsilon^k F(\lambda, \varepsilon)| \leq M\varepsilon^{\sigma-m}$ on $\tilde{J}$. Then the limit $\lim_{\varepsilon\to 0} F(\lambda, \varepsilon) \equiv F_0(\lambda)$ exists uniformly in $\lambda \in J$ and the function $F_0 : J \to \mathbb{C}$ is locally of class Λ^σ. Moreover, there is a constant C_m (depending only on m) such that*

$$|[(T_\nu - 1)^m F_0](\lambda)| \leq C_m M \sigma^{-1} |\nu|^\sigma \tag{5.8}$$

if $\lambda \in J$ and $\nu \in \mathbb{R}$ have the properties $|\nu| < \varepsilon_0$ and $\lambda + t\nu \in J$ for all $t \in [0, m]$. In (5.8) the translation operator T_ν acts according to $(T_\nu g)(\lambda) = g(\lambda + \nu)$.

Moreover, we shall need the following particular case of the Theorem 2.1: *if $\chi : \mathbb{R} \to \mathbb{C}$ is a bounded Borel function and $s = \alpha + 1/2$ is a real number $> 1/2$, and if χ has a zero of order $> \alpha$ at the origin (i.e. $|\chi(x)| \leq c|x|^\beta$ for some $\beta > \alpha$), then there is a constant $C < \infty$ such that for all $\varepsilon > 0$:*

$$||\chi(\varepsilon A)||_{\mathcal{H}_{s,\infty} \to \mathcal{H}_{1/2,1}} + ||\chi(\varepsilon A)||_{\mathcal{H}_{-1/2,\infty} \to \mathcal{H}_{-s,1}} \leq C\varepsilon^\alpha. \tag{5.9}$$

Theorem 5.4. *Let H be of class $\mathcal{C}^{1+\alpha}(A)$ for some real $\alpha > 0$ and let us set $s = \alpha + 1/2$. Then the function*

$$\mu^A(H) \ni \lambda \mapsto R(\lambda + i0) \in B(\mathcal{H}_{s,\infty}; \mathcal{H}_{-s,1}) \tag{5.10}$$

is locally of class Λ^α.

Proof. (i) As explained in the first part of the proof of Theorem 5.1 it is sufficient to consider the case when H is a bounded (everywhere defined) operator. From now on we keep the notations and assumptions of the part (ii) of the proof of Theorem 5.1. We first prove that for each $\ell, m \in \mathbb{N}$ with $m > 2\alpha$ we have

$$||L_\varepsilon^{(\ell,m)}||_{\mathcal{H}_{s,\infty} \to \mathcal{H}_{-s,1}} \leq C(\ell, m)\varepsilon^{\alpha - \ell - m} \tag{5.11}$$

for a number $C(\ell, m) < \infty$ independent of $\varepsilon \in (0, \varepsilon_0)$, $\lambda \in J$ and $\mu > 0$. For this purpose we use the Proposition 4.5. Note that for each term of the first sum on the r.h.s. of (4.17) we have either $a > \alpha$ or $b > \alpha$. If, for example $a > \alpha$, we use the estimate (5.9) with $c = \varphi_{i,a}$ and get that the corresponding term is bounded by a constant times $\varepsilon^\alpha ||g||_{s,\infty} |||f|||$, and this is better than needed (because $s > 1/2$). A typical term of the second sum on the r.h.s. of (4.17) is dominated by $\text{const} \cdot |||g||| \cdot |||f||| \cdot ||\varepsilon^c K_\varepsilon^{(c)}||$ and now we may use Proposition 4.2 (d).

(ii) Now let $f \in \mathcal{H}_{s,\infty}$ and $F(\lambda, \varepsilon) = \langle f, L_\varepsilon(\lambda + i\mu) f\rangle$. Then (5.11) gives

$$|\partial_\lambda^\ell \partial_\varepsilon^m F(\lambda, \varepsilon)| \leq C(\ell, m)||f||_{s,\infty}^2 \varepsilon^{\alpha - \ell - m}. \tag{5.12}$$

This implies the hypothesis of Lemma 5.3, namely $|\partial_\lambda^\ell \partial_\varepsilon^k F(\lambda, \varepsilon)| \leq M\varepsilon^{\alpha - m}$ if $\ell + k = m$, with $M = \text{const.}||f||_{s,\infty}^2$. Indeed, if $\ell = 0$ this is a particular case of (5.12). If $\ell \geq 1$ we integrate (5.12) ℓ times with respect to ε over an interval of the form (τ, ε_0) with $0 < \tau < \varepsilon_0$; since $\alpha - m < 0$ we shall get $|\partial_\lambda^\ell \partial_\tau^{m-\ell} F(\lambda, \tau)| \leq M\tau^{\alpha - m}$, which is the estimate we were looking for. Now we use Lemma 5.3. Since $F_0 = \langle f, R(z) f\rangle$ and $\mathcal{H}_{s,\infty} = (\mathcal{H}_{-s,1})^*$, the estimate (5.8) implies the assertion of the theorem. $\square$

5.2. $\mathcal{K}^* = \mathcal{H}_{-1/2,\infty}$ is the smallest space in the Besov scale associated to A which contains the set $R(\lambda + i0)\mathcal{H}_\infty$ (if $\lambda \in J$ is a spectral value of H). We show now that the operator $\Pi_- R(\lambda + i0)$ behaves much better (similar assertions hold for $\Pi_+ R(\lambda - i0)$). Here $\Pi_- = E_A((-\infty, 0])$ extends to a continuous operator in $\mathcal{H}_{-\infty}$ which leaves invariant each $\mathcal{H}_{s,p}$; hence the product $\Pi_- R(\lambda + i0)$ is well defined and belongs to $B(\mathcal{K}; \mathcal{K}^*)$. Note that under the conditions of Theorem 5.5 we have $R(z)\mathcal{H}_{s,p} \subset \mathcal{H}_{s,p}$, hence the r.h.s. of (5.13) makes sense.

Theorem 5.5. *Assume that H is of class $\mathcal{C}^{s+1/2,p}(A)$ for some real number $s > 1/2$ and some $p \in [1,\infty]$. Then for all $\lambda \in \mu^A(H)$ one has $\Pi_- R(\lambda + i0)\mathcal{H}_{s,p} \subset \mathcal{H}_{s-1,p}$. Let $\ell \geq 0$ be an integer such that $\ell < s - 1/2$. Then for each $f \in \mathcal{H}_{s,p}$ and $g \in \mathcal{H}_{\ell+1-s,p'}$ the function $\lambda \mapsto \langle g, \Pi_- R(\lambda + i0)f\rangle$ is of class C^ℓ on $\mu^A(H)$ and one has*

$$\frac{d^\ell}{d\lambda^\ell}\langle g, \Pi_- R(\lambda + i0)f\rangle = \lim_{\mu\to+0}\langle \Pi_- g, \ell! R(\lambda + i\mu)^{\ell+1} f\rangle \tag{5.13}$$

where the limit exists locally uniformly in $\lambda \in \mu^A(H)$.

Proof. Exactly as in the proof of Theorem 5.1 it suffices to consider the case where H is a bounded operator. Then, J being chosen as in (ii) of the proof of Theorem 5.1, we may assume that the assumptions of Section 4 are satisfied. Let L_ε be the operator introduced in Proposition 4.6, where ψ is assumed to have the property $\psi(0) = 1$. We set $h(\varepsilon) = \langle g, L_\varepsilon^{(\ell,0)} f\rangle$ for $0 < \varepsilon \leq \varepsilon_0$ and some given vectors $f \in \mathcal{H}_{s,p}$ and $g \in \mathcal{H}_{1+\ell-s,p'}$. Here p' is defined by $1/p + 1/p' = 1$. Then (4.19) gives

$$\begin{aligned} |\varepsilon^m h^{(m)}(\varepsilon)| \leq{} & C\varepsilon^{-\ell}|||\Pi_- e^{\varepsilon A} g||| \cdot |||\zeta(\varepsilon A) f||| \\ & + C|||f||| \sum_{i+j\leq m-1} \varepsilon^{-\ell}|||\Pi_-(\varepsilon A)^i e^{\varepsilon A} g||| \cdot ||\varepsilon^j K_\varepsilon^{(j)}|| \end{aligned} \tag{5.14}$$

where C is a constant independent of ε, λ, μ, f and g. We choose $m > \alpha \equiv s - 1/2$. Then the integral over the interval $(0,1)$ with respect to the measure $\varepsilon^{-1}d\varepsilon$ of the first term on the r.h.s. of (5.14) is bounded by

$$\begin{aligned} & C\left[\int_0^1 |||\varepsilon^{\alpha-\ell}\Pi_- e^{\varepsilon A} g|||^{p'}\varepsilon^{-1}d\varepsilon\right]^{1/p'} \cdot \left[\int_0^1 |||\varepsilon^{-\alpha}\zeta(\varepsilon A) f|||^p \varepsilon^{-1} d\varepsilon\right]^{1/p} \leq \\ & \leq C'||g||_{1/2-\alpha+\ell,p'}||f||_{1/2+\alpha,p}. \end{aligned}$$

We have used the Theorem 2.1 which is allowed by the fact that $\alpha - \ell > 0$, $0 < \alpha < m$ and $\zeta(x) = O(x^m)$ as $x \to 0$. The integral over $(0,1)$ with respect to $\varepsilon^{-1}d\varepsilon$ of a typical term of the sum in the r.h.s. of (5.14) is similarly bounded by

$$\begin{aligned} & C|||f|||\left[\int_0^1 |||\varepsilon^{\alpha-\ell}\Pi_-(\varepsilon A)^i e^{\varepsilon A} g|||^{p'}\varepsilon^{-1}d\varepsilon\right]^{1/p'} \left[\int_0^1 ||\varepsilon^{-\alpha+j}K_\varepsilon^{(j)}||^p \varepsilon^{-1}d\varepsilon\right]^{1/p} \leq \\ & \leq C'|||f||| \cdot ||g||_{1/2-\alpha+\ell,p'}. \end{aligned}$$

The rest of the proof is similar to that of Theorem 5.1. The conclusion is that the function $z \mapsto \langle \Pi_- g, R(z)f\rangle$, which is holomorphic on $\mathbb{C}_+$, extends to a function of class C^ℓ on $\mathbb{C}_+ \cup J$ (in a sense explained in the statement of Theorem 5.1). In particular, if we take $\ell = 0$ we see that $\lim_{\mu\to+0}\langle \Pi_- g, R(\lambda+i\mu)f\rangle$ exists (uniformly in $\lambda \in J$) for each $f \in \mathcal{H}_{s,p}$ and $g \in \mathcal{H}_{1-s,p'}$. Now recall that $\mathcal{H}_{s-1,p} = (\mathcal{H}_{1-s,p'})^*$ if $1 < p \leq \infty$ and $\mathcal{H}_{s-1,1} = (\mathcal{H}^0_{1-s,\infty})^*$. Hence for each $f \in \mathcal{H}_{s,p}$ the limit $\lim_{\mu\to+0}\Pi_- R(\lambda + i\mu)f$ exists in the weak* topology of $\mathcal{H}_{s-1,p}$ so $\Pi_- R(\lambda + i0)\mathcal{H}_{s,p} \subset \mathcal{H}_{s-1,p}$. □

We define the w-topology on the space $B(\mathcal{H}_{s,p};\mathcal{H}_{t,q})$ as the topology associated to the family of seminorms $S \mapsto |\langle g, Sf\rangle|$ with $f \in \mathcal{H}_{s,p}$ and

$g \in \mathcal{H}_{-t,q'}$. Then the second part of Theorem 5.5 can be expressed as follows: the map $\lambda \mapsto \Pi_- R(\lambda + i0) \in B(\mathcal{H}_{s,p}; \mathcal{H}_{s-\ell-1,p})$ is of class C^ℓ in the w-topology.

Theorem 5.6. *Let s, α be real numbers such that $0 < \alpha < s - 1/2$ and assume that H is of class $\mathcal{C}^{s+1/2}(A)$. Then the map*

$$\mu^A(H) \ni \lambda \mapsto \Pi_- R(\lambda + i0) \in B(\mathcal{H}_{s,\infty}; \mathcal{H}_{s-1-\alpha,1}) \tag{5.15}$$

is locally of class Λ^α.

Proof. (i) We keep the assumptions and notations of the proof of Theorem 5.5. We first show that the operator L_ε satisfies the following estimates: for each $\ell, m \in \mathbb{N}$ with $m > s - 1/2 \equiv \beta$ there is a number $C(\ell, m)$, independent of $\varepsilon, \lambda, \mu$, such that

$$||L_\varepsilon^{(\ell,m)}||_{\mathcal{H}_{s,\infty} \to \mathcal{H}_{s-1-\alpha,1}} \leq C(\ell, m)\varepsilon^{\alpha-\ell-m}. \tag{5.16}$$

We use Proposition 4.6. Then (5.9) with $\chi = \zeta$ (which vanishes of order $m > \beta$ at the origin, see (4.18)) implies $|||\zeta(\varepsilon A)f||| \leq C'\varepsilon^\beta ||f||_{s,\infty}$. On the other hand the Theorem 2.1 implies for $\beta - \alpha > 0$:

$$\varepsilon^{\beta-\alpha}|||\Pi_- e^{\varepsilon A} g||| \leq C''||g||_{1/2-\beta+\alpha,\infty} = C''||g||_{1-s+\alpha,\infty}. \tag{5.17}$$

Hence the first term on the r.h.s. of (4.19) is bounded by a constant times $\varepsilon^\alpha ||g||_{1-s+\alpha,\infty}||f||_{s,\infty}$. Now we bound the terms of the sum from (4.19) by using $|||(\varepsilon A)^b \psi^{(j)}(\varepsilon A)f||| \leq C'|||f||| \leq C''||f||_{s,\infty}$ and Proposition 4.2 (d). We shall get terms of the form $C'''\varepsilon^\beta |||\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot ||f||_{s,\infty}$. By an estimate similar to (5.17) (use the Theorem 2.1 again) we finally obtain

$$|\langle g, \varepsilon^{\ell+m} L_\varepsilon^{(\ell,m)} f\rangle| \leq C\varepsilon^\alpha ||g||_{1-s+\alpha,\infty}||f||_{s,\infty}.$$

This implies (5.16) because $\mathcal{H}_{1-s+\alpha,\infty} = (\mathcal{H}_{s-1-\alpha,1})^*$.

(ii) Let $F(\lambda, \varepsilon) = \langle g, L_\varepsilon(\lambda + i\mu)f\rangle$ with $f \in \mathcal{H}_{s,\infty}$ and $g \in \mathcal{H}_{1+\alpha-s,\infty}$. If $\ell, m \geq 0$ are integers and $m > \beta$ then (5.16) gives

$$|\partial_\lambda^\ell \partial_\varepsilon^m F(\lambda, \varepsilon)| \leq C(\ell, m)||f||_{s,\infty}||g||_{1+\alpha-s,\infty}\varepsilon^{\alpha-\ell-m}.$$

Now the proof can be finished as in the case of Theorem 5.4. ∎

5.3. If $f \notin \mathcal{H}_{1/2,1}$ then $\Pi_- R(\lambda + i0)f$ has no meaning in general. However, one can give a sense to this expression if $E_A((-\infty, a))f = 0$ for some $a \in \mathbb{R}$.

Theorem 5.7. *Assume that H is of class $\mathcal{C}^{1+\alpha,r}(A)$ with $\alpha > 0$ real and $r \in [1, \infty]$. Let $\ell \in \mathbb{N}$ with $\ell < \alpha$, let s be a real number such that $1/2 - (\alpha - \ell) \leq s \leq 1/2$, and let us denote $t = s - 1 + (\alpha - \ell)$, so that $-1/2 \leq t \leq -1/2 + (\alpha - \ell)$. Finally, let $f \in \mathcal{H}_{s,p}$ and $g \in \mathcal{H}_{-t,q'}$ where $p, q \in [1, \infty]$ are such that*

(i) *if $s = 1/2 - (\alpha - \ell)$ then $p = r'$ and $q = \infty$;*
(ii) *if $s = 1/2$ then $p = 1$ and $q = r$;*
(iii) *if $1/2 - (\alpha - \ell) < s < 1/2$ then p, q are related by $1/q = 1/p + 1/r$.*

Then the holomorphic map $\mathbb{C}_+ \ni z \mapsto \langle \Pi_- g, R(z)\Pi_+ f\rangle$ extends to a function of class C^ℓ on $\mathbb{C}_+ \cup \mu^A(H)$ and one has

$$\frac{d^\ell}{d\lambda^\ell}\langle \Pi_- g, R(\lambda + i0)\Pi_+ f\rangle = \lim_{\mu \to +0}\langle \Pi_- g, \ell! R(\lambda + i\mu)^{\ell+1}\Pi_+ f\rangle \tag{5.18}$$

where the limit exists locally uniformly in $\lambda \in \mu^A(H)$.

Proof. (i) As usual we reduce ourselves to the case when H is a bounded operator, but this time the argument is slightly more involved. With the notations of part (i) of the proof of Theorem 5.1, we write

$$\begin{aligned}\langle \Pi_- g, R(z)\Pi_+ f\rangle &= \zeta\langle \Pi_- g, (R-\zeta)^{-1}R\Pi_+ f\rangle \\ &= \zeta\langle \Pi_- g, (R-\zeta)^{-1}\Pi_+ R\Pi_+ f\rangle + \zeta\langle \Pi_- g, (R-\zeta)^{-1}\Pi_- R\Pi_+ f\rangle.\end{aligned}$$

The first term in the last member here is easy to treat (because $R\Pi_+ f \in \mathcal{H}_{s,\infty}$ if $f \in \mathcal{H}_{s,\infty}$). For the last term we first use Theorem 2.2, which gives $\Pi_- R\Pi_+ f \in \mathcal{H}_{t+3/2}$ if $f \in \mathcal{H}_{s,\infty}$. In conclusion, for the rest of the proof we can assume that H is bounded and that the hypotheses of Section 4 are fulfilled.

(ii) Let L_ε be as in Proposition 4.7 and let us set $h(\varepsilon) = \langle g, L_\varepsilon^{(\ell,0)} f\rangle$ with $f \in \mathcal{H}_{s,p}$ and $g \in \mathcal{H}_{-t,q'}$. Then, according to (4.20), for each integer $m \geq 1$ (in fact this time one can take $m = 1$, which simplifies the proof, but not significantly) we have

(5.19)
$$\begin{aligned}&|\varepsilon^m h^{(m)}(\varepsilon)| \leq \\ &\leq C \sum_{a+b+c\leq m-1} |||\varepsilon^{\alpha-\ell-\sigma}\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot |||\varepsilon^{\sigma}\Pi_+(\varepsilon A)^b e^{-\varepsilon A} f||| \cdot ||\varepsilon^{-\alpha+c} K_\varepsilon^{(c)}||,\end{aligned}$$

where σ is the real number defined by $s = 1/2 - \sigma$, so that $0 \leq \sigma \leq \alpha - \ell$. If $\sigma = 0$ (case (ii) of the theorem) then we bound a term in the sum from (5.19) by

$$C'|||\varepsilon^{\alpha-\ell}\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot |||f||| \cdot ||\varepsilon^{-\alpha+c} K_\varepsilon^{(c)}||.$$

Then by using Theorem 2.1 we obtain $|\varepsilon^m h^{(m)}(\varepsilon)| \leq \chi(\varepsilon)$, where $\chi(\varepsilon)$ is independent of λ and μ, and

$$\int_0^{\varepsilon_0} |\chi(\varepsilon)|\varepsilon^{-1} d\varepsilon \leq C''|||f||| \cdot ||g||_{1/2-\alpha+\ell,r'}.$$

If $\sigma = \alpha - \ell$ (case (i) of the theorem) we estimate a typical term in the r.h.s. of (5.19) by

$$C'|||g||| \cdot |||\varepsilon^{\alpha-\ell}\Pi_+(\varepsilon A)^b e^{-\varepsilon A} f||| \cdot ||\varepsilon^{-\alpha+c} K_\varepsilon^{(c)}||.$$

Then as above we get

$$\int_0^{\varepsilon_0} |\chi(\varepsilon)|\varepsilon^{-1} d\varepsilon \leq C''|||g||| \cdot ||g||_{1/2-\alpha+\ell,r'}.$$

Finally, if $0 < \sigma < \alpha - \ell$ we can use Theorem 2.1 for each of the factors in (5.19) which contains f or g. So the relation $1/q' + 1/p + 1/r = 1$ and the Hölder inequality with three factors will give

$$\int_0^{\varepsilon_0} |\chi(\varepsilon)|\varepsilon^{-1} d\varepsilon \leq C''||g||_{1/2-\alpha+\ell+\sigma,q'}||f||_{1/2-\sigma,p}.$$

Since $1/2 - \alpha + \ell + \sigma = 1 - s - (\alpha - \ell) = -t$, we can finish the proof as usual (see the proof of Theorem 5.1). □

We shall reformulate Theorem 5.7 as follows. We know that for each real s with $|s| < 1 + \alpha$ and each $p \in [1, \infty]$ the operator $R(z)$ has a canonical extension to a bounded operator in $\mathcal{H}_{s,p}$ if $z \in \mathbb{C}_+$, and the map $z \mapsto R(z) \in$

$B(\mathcal{H}_{s,p})$ is holomorphic. If s, p, t, q are as in Theorem 5.7 then $t < s$, so we have a holomorphic map

$$\mathbb{C}_+ \ni \mapsto \Pi_- R(z)\Pi_+ \in B(\mathcal{H}_{s,p}; \mathcal{H}_{t,q}). \tag{5.20}$$

Now the theorem says that if we equip $B(\mathcal{H}_{s,p}; \mathcal{H}_{t,q})$ with the w-topology then (5.20) extends to a map of class C^ℓ on $\mathbb{C}_+ \cup \mu^A(H)$. If $\lambda \in \mu^A(H)$ then the operator $\Pi_- R(\lambda + i0)\Pi_+$ is a well defined element of $B(\mathcal{K}; \mathcal{K}^*)$ (by Theorem 5.1). Hence, according to Theorem 5.7, this operator sends $\mathcal{H}_{1/2,1}$ into $\mathcal{H}_{-1/2+\alpha,r}$ and, more generally, induces a continuous operator $\mathcal{H}_{s,p} \to \mathcal{H}_{t,q}$ (with s, p, t, q as in the theorem). Moreover, the map

$$\mu^A(H) \ni \lambda \mapsto \Pi_- R(\lambda + i0)\Pi_+ \in B(\mathcal{H}_{s,p}; \mathcal{H}_{t,q}) \tag{5.21}$$

is of class C^ℓ in the w-topology.

Theorem 5.8. *Assume that H is of class $\mathcal{C}^{1+\alpha}(A)$ for some $\alpha > 0$. Let β, s, t be real numbers such that $0 < \beta < \alpha$, $1/2 - (\alpha - \beta) \le s \le 1/2$ and $t = s - 1 + (\alpha - \beta)$, so that $-1/2 \le t \le -1/2 + (\alpha - \beta)$. Finally, let $p, q \in [1, \infty]$ be such that*
(i) *if $s = 1/2 - (\alpha - \beta)$ then $p = q = \infty$;*
(ii) *if $s = 1/2$ then $p = q = 1$;*
(iii) *if $1/2 - (\alpha - \beta) < s < 1/2$ then $p = \infty$, $q = 1$.*
Then the map (5.21) is locally of class Λ^β.

Proof. As usual, we may assume that we are in the context of part (ii) of the proof of Theorem 5.7. We use Lemma 5.3 with $F(\lambda, \varepsilon) = \langle g, L_\varepsilon(\lambda + i\mu)f\rangle$ where L_ε is as in Proposition 4.7 and $f \in \mathcal{H}_{s,p}$, $g \in \mathcal{H}_{-t,q'}$. As in the proof of Theorems 5.4 and 5.6 we shall need the following estimate: for $\ell, m \in \mathbb{N}$ there is a constant $C(\ell, m)$, independent of $\varepsilon \in (0, \varepsilon_0)$, $\lambda \in J$ and $\mu > 0$, such that

$$||L_\varepsilon^{(\ell,m)}||_{\mathcal{H}_{s,p} \to \mathcal{H}_{t,q}} \le C(\ell, m)\varepsilon^{\beta - \ell - m}. \tag{5.22}$$

In order to prove this we use the inequality established in Proposition 4.7. Each term in the r.h.s. of (4.20) is of the form $|||\varphi(\varepsilon A)g||| \cdot |||\psi(\varepsilon A)f||| \cdot ||\varepsilon^c K_\varepsilon^{(c)}||$ where $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ but do not vanish at zero in general. By Proposition 4.2 (d) such a term is bounded by a constant times

$$\varepsilon^\alpha |||\varphi(\varepsilon A)g||| \cdot |||\psi(\varepsilon A)f||| = \varepsilon^\beta |||\varepsilon^{\alpha - \beta - \sigma}\varphi(\varepsilon A)g||| \cdot |||\varepsilon^\sigma \psi(\varepsilon A)f||| \tag{5.23}$$

where σ could be an arbitrary real number. If $0 < \sigma < \alpha - \beta$ then the r.h.s. of (5.23) can be estimated with the help of Theorem 2.1. We clearly get a bound of the form $c\varepsilon^\beta ||g||_{1/2-\alpha+\beta+\sigma,\infty} ||f||_{1/2-\sigma,\infty}$. We set $s = 1/2 - \sigma$ and we obtain (5.22) by a simple argument. The limit cases $\sigma = 0$ and $\sigma = \alpha - \beta$ are treated similarly. ☐

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