Surfing Arnold's Web

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Dedicated to Gerald E. Brown on the Occasion of his Seventieth Birthday

Abstract

The free motion of a rigid one-dimensional stick colliding elastically within an infinitely massive circular wall is first shown to be equivalent to the three-dimensional motion of a billiard ball within a spiral column and then mapped onto a two-dimensional billiard problem with a rotating billiard wall. Indications that such a system has chaotic orbits and can possess integrable orbits is provided through the use of projected Poincaré sections. When chaotic and integrable orbits co-exist, the chaotic trajectories appear in the form of Arnold's web. We also consider the limit of a stick of zero length in which the system becomes integrable.

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I. INTRODUCTION

The last ten years of research in classical and quantum chaos has rekindled interest in a variety of systems in classical and quantum mechanics which had previously been regarded as elementary. Complexity in these two theories has often been associated with large numbers of interacting particles while simplicity or integrability is connected to systems with small numbers of weakly interacting particles. This philosophy has, in turn, defined the traditional scope of statistical mechanics as being the realm of many-body systems for which the use of such concepts as the postulate of equal *a priori* probability, thermodynamic equilibrium, and the ergodic hypothesis are natural and useful. However, the ergodic hypothesis, for instance, is not limited to complex systems but appears in integrable systems such as the simple pendulum which is ergodic on its energy manifold.

With the introduction of Sinai's billiard in the early seventies [1], it was realized that even a non-interacting one particle system in a closed two-dimensional domain can exhibit many features which are more frequently encountered in statistical mechanical systems [2], [3]. In fact, Sinai's interest in this simple billiard model was motivated by his desire to prove certain ergodic properties of a hard sphere Bose gas. The traditional billiard problem consists of a point particle moving in a rectilinear manner and undergoing specular reflection at the billiard wall. Depending upon the choice of the boundary wall, the system may be either integrable, KAM-like, or chaotic (K-flow, C-flow, or Bernoulli). One cannot use initial conditions of finite accuracy to predict the long-term description of chaotic billiard systems any more than one can follow the motion of individual particles in a statistical mechanical system. Rather, physicists turn to probabilistic concepts such as entropy (thermodynamic entropy in statistical mechanics or the Kolmorogov-Sinai entropy for billiards) in order to describe the complex behavior found in these systems.

Recently, it was realized that the free motion of any rigid body in a confined domain is equivalent to a well-defined problem of point particle motion [4], [5]. This generalized class of billiard problems has provided a simple arena for exploring chaos in dynamical systems.

While their immediate physical origin renders these intuitively appealing systems appealing intuitive, they have been of specific value, *e.g.*, in proving the long-standing conjecture that the tossing of a coin is a (completely chaotic) Bernoulli system [4]. The equivalence of rigid body motion and billiards is of considerable value in the analysis of numerical simulations using the Poincaré surface of section technique. The relative abundance of analytic results available for billiards can simplify the rigorous examination of rigid body systems significantly.[1] Further, the appearance of chaotic or integrable orbits can be adjusted with easily tunable parameters (such as the shape of the boundary wall). This makes billiards appealing dynamical systems to study.

In this paper, we continue our investigation of billiards equivalent to rigid body motion. Specifically, we investigate the free motion of a one-dimensional stick making elastic collisions within a circle. We shall show that such a system is equivalent to a three-dimensional billiard problem of a point particle moving inside a spiral column. (This equivalence is more general and holds for arbitrary two-dimensional rigid bodies and confining shapes.) The specific example of a stick inside a circle considered here is further equivalent to a two-dimensional billiard with a freely-rotating boundary wall. This problem differs from those investigated previously in two main aspects. First, the stick here is placed within a closed domain of constant non-zero curvature as opposed to the earlier papers where the stick bounced between two infinite lines or infinite planes. Second, the appearance of a rotating billiard wall is a new phenomena not previously encountered. Our earlier extension of the stick problem to three dimensions in [5] led to an interacting billiard system. Again, a relatively simple change of the original rigid body problem produces a new and interesting features in the mapped billiard system. Here, the almost surgical separation of rectilinear and rotational motion offers new ways of thinking about the old problem of rigid body motion.

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[1]For example, the demonstration that coin tossing is a Bernoulli system is made elementary once it is recognized that this corresponds to a billiard which is everywhere convex.

We begin this paper with a brief review of earlier results on the billiard problems related to the motion of a stick in two and three dimensions. In section III, we introduce the spiral column billiard, and in section IV, we describe the equivalent two-dimensional rotating billiard system. We shall present a variety of numerical results. These will illustrate that this system displays KAM-type motion (*i.e.*, a mixture of chaotic and integrable orbits) when the length of the stick is larger than the radius of the circle. Shorter sticks appear to lead to resonance islands and motion which seems to be more chaotic. In general, the chaotic trajectories exhibit the phenomenon of Arnold diffusion. We discuss the limiting case of an integrable circular billiard obtained for a stick of length zero and an associated limiting problem in which the inertial parameter of the wall is allowed to become infinite. Finally, some suggestions for future work are mentioned in section V.

II. WHAT'S THE STICK?

We begin by considering the case of a one-dimensional stick of total mass M which is composed of two equal point masses separated by a rigid rod of length 2ℓ and which makes elastic collisions between two flat parallel walls separated by a distance h. Let us recall some results from [4] for the case when the stick moves in two dimensions. The coordinates of the stick will be z, the height of its center of mass above the lower wall, and the angle of rotation θ from the vertical. Scaling z with the radius of gyration, κ, as $\eta = z/\kappa$, the (scaled) energy of the stick becomes

$$E = \frac{M}{2}(\dot{\theta}^2 + \dot{\eta}^2) \quad . \tag{1}$$

This looks formally like the energy of a free point particle moving in an Euclidean plane parameterized by the dimensionless coordinates $x_1 = \theta$ and $x_2 = \eta$. The distance of closest approach of the center of mass to the plane is $\eta_{\min} = (\ell/\kappa)|\cos(\theta)$ so that this point particle moves between boundaries at the bottom, $b(\theta)$, and the top, $t(\theta)$, with

$$b(\theta) = \frac{\ell}{\kappa}|\cos(\theta)|,$$

$$t(\theta) = \frac{1}{\kappa}\left(h - \ell|\cos(\theta)|\right). \tag{2}$$

A collision between the stick and the wall is described by

$$\begin{aligned}(M\kappa)\Delta\dot{\eta} &= f_n\tau \ , \\ (M\kappa^2)\Delta\dot{\theta} &= \ell\sin(\theta)f_n\tau \ .\end{aligned} \tag{3}$$

The impulse can be eliminated to obtain a relation between $\Delta\dot{\eta}$ and $\Delta\dot{\theta}$,

$$\Delta\dot{\theta} = \frac{\ell}{\kappa}\sin\theta\Delta\dot{\eta}, \tag{4}$$

and conservation of energy can be used to show that

$$\Delta\dot{\eta} = \frac{-2(\dot{\eta} + (\ell/\kappa)\dot{\theta}\sin\theta)}{1 + (\ell/\kappa)^2\sin^2\theta}. \tag{5}$$

The proof of specular reflection is as follows. The tangent to the lower wall is

$$\vec{T} = (1, -\frac{\ell}{\kappa}\sin\theta). \tag{6}$$

Using the condition that $\vec{N}\cdot\vec{T} = 0$, the normal vector is

$$\vec{N} = (\frac{\ell}{\kappa}\sin\theta, 1). \tag{7}$$

Denoting the velocity of the point particle as $\vec{v} = (\dot{\theta}, \dot{\eta})$, the conditions for specular reflection are

$$\vec{T}\cdot(\vec{v} + \Delta\vec{v}) = \vec{T}\cdot\vec{v}, \tag{8}$$

and

$$\vec{N}\cdot(\vec{v} + \Delta\vec{v}) = -\vec{N}\cdot\vec{v}. \tag{9}$$

Using equation (4), one has

$$\vec{T}\cdot(\Delta\vec{v}) = \Delta\dot{\theta} - \frac{\ell}{\kappa}\sin\theta\Delta\dot{\eta} \ = \ 0, \tag{10}$$

whereas the normal constraint gives

$$\vec{N} \cdot (2\vec{v} + \Delta\vec{v}) = (2\dot{\theta} + \Delta\dot{\theta})(\frac{\ell}{\kappa}\sin\theta) + (2\dot{\eta} + \Delta\dot{\eta}). \tag{11}$$

¿From the conservation of energy and (4), one finds that

$$\begin{aligned} &\Delta\dot{\theta}(2\dot{\theta} + \Delta\dot{\theta}) + \Delta\dot{\eta}(2\dot{\eta} + \Delta\dot{\eta}) \\ &= \Delta\dot{\eta}[(2\dot{\theta} + \Delta\dot{\theta})(\frac{\ell}{\kappa}\sin\theta) + (2\dot{\eta} + \Delta\dot{\eta})] \;=\; 0, \end{aligned} \tag{12}$$

and thus, (11) is satisfied.

We now allow the stick to move in three dimensions and make elastic collisions with two flat walls which are the planes $z = 0$ and $z = h$. We orient the stick using the usual polar angles, θ and ϕ. The corresponding energy can be written as

$$E = \frac{1}{2}M\ell^2[\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2] + \frac{1}{2}M\dot{z}^2 \quad , \tag{13}$$

and the angular momentum of the stick by

$$\begin{aligned} L_x &= M\ell^2[-\sin\phi\ \dot{\theta} - \sin\theta\ \cos\theta\ \cos\phi\ \dot{\phi}] \\ L_y &= M\ell^2[\cos\phi\ \dot{\theta} - \sin\theta\ \cos\theta\ \sin\phi\ \dot{\phi}] \\ L_z &= M\ell^2[\sin^2\theta\ \dot{\phi}] \quad . \end{aligned} \tag{14}$$

This problem initially appears to be five-dimensional. However, it is clear that the x and y motion of the center of mass of the stick are trivial and can be ignored. It is also clear that the force-free motion of the stick does not result in θ and ϕ being linear functions of time except for geometrical accidents. Now consider a collision with the wall which imparts some impulse, $f\tau$, in the z-direction.

$$\Delta(M\dot{z}) = f\tau \quad . \tag{15}$$

There is a corresponding change in the angular momenta:

$$\begin{aligned} \Delta L_x &= -\ell\sin\theta\ \sin\phi\ (f\tau) \\ \Delta L_y &= \ell\sin\theta\ \cos\phi\ (f\tau) \\ \Delta L_z &= 0 \quad . \end{aligned} \tag{16}$$

As a consequence of the third of these equations, we see that

$$\Delta\dot{\phi} = 0 \quad . \tag{17}$$

Using this fact, the equations for L_x and L_y, and the equations for ΔL_x and ΔL_y, we find that

$$\Delta\dot{\theta} = \frac{1}{\ell}\sin\theta\ \Delta\dot{z} \quad . \tag{18}$$

Finally, we can use the fact that the collision is strictly elastic and equate kinetic energies before and after the collision. This leads us to a quadratic equation with a trivial solution $\Delta\dot{z} = 0$ and a non-trivial solution of

$$\Delta\dot{z} = \frac{-2(\dot{z} + \ell\dot{\theta}\sin\theta)}{1 + \sin^2\theta} \quad . \tag{19}$$

We are now in a position to draw all desired conclusions about this special problem. Since $\dot{\phi}$ does not change during the collisions, the coordinate ϕ is quite passive. It serves only to "complicate" the motion in θ. Equation (19) is *identical* to what was found in the above two-dimensional problem. There is no ϕ-dependence in this equation. Furthermore, there can be no ϕ-dependence in the wall function. The wall function is also exactly what we had in the two-dimensional case. Thus, with the scaling of variables described previously, we again find that we have specular reflection in the $(\ell\theta, z)$ plane for every collision. This apparently three-dimensional problem is really a two-dimensional problem in the $(\ell\theta, z)$ plane. The only difference is that, as a consequence of the more complicated equations of motion, the trajectories between consecutive wall hits are no longer straight lines. Although the time-dependence of θ is not linear, it is not complicated. We simply consider the free motion in a rotated coordinate system such that the angular momentum vector lies along the z'-axis. In this frame the angular velocity $\omega_{z'}$ will be a constant. It is then easy to transform back to the original θz-coordinates.

The energy (13) can be rewritten as

$$E = \frac{1}{2}M[(\ell\dot{\theta})^2 + \dot{z}^2] + \frac{L_z^2}{2M\ell^2\sin^2\theta} \quad . \tag{20}$$

Since all reference to ϕ has disappeared, this is the total energy of the billiard ball in the reduced $(\ell\theta, z)$ plane. The third term, $L_z^2/2M\ell^2\sin^2\theta$, can be interpreted as the potential energy for the two-dimensional billiard system. This explains the non-linear time dependence of the θ variable. Thus, a one-dimensional stick bouncing elastically between two flat walls is equivalent to an interacting billiard problem (with suitable walls) on a flat two-dimensional manifold (with a specific form of the interacting field).

III. THE SPIRAL COLUMN BILLIARD

Consider the usual stick of length 2ℓ with point masses of $M/2$ at the ends. This time, it will move inside a circle of radius R making elastic collisions as usual. Let us mark the two ends of the stick as P_1 and P_2. Locate the center of mass of the stick with polar angles (r, θ) and orient the stick according to the angle α which P_1 makes with the x-axis. The symmetry between P_1 and P_2 is simply $\alpha \to \alpha + \pi$. Thus, we will derive all the necessary equations for P_1 where the P_2 equations are obtained easily through this symmetry.

The energy of this system is

$$E = \frac{M}{2}(\ell^2\dot{\alpha}^2 + \dot{x}^2 + \dot{y}^2) \ . \tag{21}$$

The contact point of the stick with the circle is given by

$$\begin{aligned} x_c &= r_c\cos\theta + \ell\cos\alpha, \\ y_c &= r_c\sin\theta + \ell\sin\alpha, \end{aligned} \tag{22}$$

with the constraint that

$$x_c^2 + y_c^2 = R^2 = r_c^2 + \ell^2 + 2r_c\ell\cos(\theta - \alpha) \ . \tag{23}$$

Solving for r_c, the wall function for the billiard is

$$b(\theta, \alpha) \equiv r_c(\theta, \alpha) = -\ell|\cos(\theta - \alpha)| + (R^2 - \ell^2\sin^2(\theta - \alpha))^{1/2}. \tag{24}$$

We are now considering a configuration space of $(x, y, \ell\alpha)$ for our billiard system (*i.e.*, the location of the center of mass of the stick plus its orientation). The boundary wall is given by

$$(b(\theta,\alpha)\cos\theta,\ b(\theta,\alpha)\ \ .\sin\theta,\ \ell\alpha) \tag{25}$$

This is a spiral column.

Let us now construct the dynamics of this problem. The polar angle locating the physical point of contact will be denoted as Φ, *i.e.*,

$$\begin{aligned} R\cos\Phi &= b(\theta,\alpha)\cos\theta + \ell\cos\alpha\ , \\ R\sin\Phi &= b(\theta,\alpha)\sin\theta + \ell\sin\alpha\ . \end{aligned} \tag{26}$$

The normal at this point is strictly in the radial direction,

$$\hat{n} = (\cos\Phi, \sin\Phi)\ , \tag{27}$$

and the corresponding tangent vector is

$$\hat{t} = (-\sin\Phi, \cos\Phi)\ . \tag{28}$$

Since the impulsive force is assumed to be normal to the surface at the physical point of contact, the linear impulse equations are

$$\begin{aligned} M\Delta\vec{v}\cdot\hat{n} &= f\tau\ , \\ M\Delta\vec{v}\cdot\hat{t} &= 0\ , \end{aligned} \tag{29}$$

or

$$M(\cos\Phi\Delta\dot{x} + \sin\Phi\Delta\dot{y}) = f\tau\ , \tag{30}$$

and

$$-\sin\Phi\Delta\dot{x} + \cos\Phi\Delta\dot{y} = 0\ . \tag{31}$$

The angular impulse equation

$$I\Delta\omega = |\vec{\ell}||\vec{f}|\sin(\chi)\tau \tag{32}$$

gives

$$M\ell^2\Delta\dot{\alpha} = \ell f \sin(\Phi-\alpha)\tau \ . \tag{33}$$

Equating (30) and (33) appropriately, one finds

$$\cos\Phi\Delta\dot{x} + \sin\Phi\Delta\dot{y} = \frac{\ell\Delta\dot{\alpha}}{\sin(\Phi-\alpha)} \tag{34}$$

and using (31),

$$\Delta\dot{\alpha} = \left(\frac{\sin(\Phi-\alpha)}{\cos\Phi}\right)\frac{\Delta\dot{x}}{\ell} \ . \tag{35}$$

Eliminating the angle Φ in favour of θ and α *via* equations (26), we have

$$\Delta\dot{\alpha} = \left(\frac{b(\theta,\alpha)\sin(\theta-\alpha)}{b(\theta,\alpha)\cos\theta + \ell\cos\alpha}\right)\frac{\Delta\dot{x}}{\ell} \ , \tag{36}$$

whereas (31) gives

$$\Delta\dot{y} = \left(\frac{b(\theta,\alpha)\sin\theta + \ell\sin\alpha}{b(\theta,\alpha)\cos\theta + \ell\cos\alpha}\right)\Delta\dot{x} \ . \tag{37}$$

The value of $\Delta\dot{x}$ is obtained from energy conversation,

$$\ell^2\dot{\alpha}^2 + \dot{x}^2 + \dot{y}^2 = \ell^2(\dot{\alpha}+\Delta\dot{\alpha})^2 + (\dot{x}+\Delta\dot{x})^2 + (\dot{y}+\Delta\dot{y})^2 \tag{38}$$

resulting in

$$\Delta\dot{x} = \frac{[\dot{y}(b\sin\theta+\ell\sin\alpha) + \dot{x}(b\cos\theta+\ell\cos\alpha) + \ell\dot{\alpha}(b\sin(\theta-\alpha))](b\cos\theta+\ell\cos\alpha)}{R^2 + b^2\sin^2(\theta-\alpha)} \ . \tag{39}$$

The analogous equations which apply when the other end of the stick, P_2, collides with the circle are obtained by letting $\alpha \to \alpha + \pi$.

We now have all the information needed in order to construct the proof of specular reflection ((8) and (9)). The two (unnormalized) tangents to the billiard wall (25) are

$$\vec{T}_\theta = (b_\theta \cos\theta - b\sin\theta,\ \ b_\theta \sin\theta + b\cos\theta, 0) \tag{40}$$

and

$$\vec{T}_\alpha = (b_\theta \cos\theta,\ \ b_\theta \sin\theta, -\ell)\ , \tag{41}$$

where

$$b_\theta = \frac{\partial b}{\partial \theta} = \frac{b\ell \sin(\theta - \alpha)}{b + \ell\cos(\theta - \alpha)}\ . \tag{42}$$

Simple manipulations will show that

$$\vec{T}_\theta \cdot \Delta\vec{v} = 0 \tag{43}$$

produces (37) whereas

$$\vec{T}_\alpha \cdot \Delta\vec{v} = 0 \tag{44}$$

results in (36). Finally, the (unnormalized) surface $\vec{N}$ derived from $\vec{T}_\theta \cdot \vec{N} = 0$ and $\vec{T}_\alpha \cdot \vec{N} = 0$ is

$$\vec{N} = (b_\theta \sin\theta + b\cos\theta, -b_\theta\cos\theta + b\sin\theta, bb_\theta/\ell). \tag{45}$$

It can be shown that using this vector,

$$\vec{N} \cdot (2\vec{v} + \Delta\vec{v}) = 0 \tag{46}$$

is equivalent to the energy conservation condition (38).

The results of this section have been obtained for the special case of a stick moving inside a circle. They are, however, of materially greater generality. It is possible to show that the motion of any rigid body inside a two-dimensional confining wall of arbitrary shape is also equivalent to a spiral billiard in three dimensions.

IV. THE ROTATING BILLIARD WALL

Our goal here is to reinterpret the previous billiard problem in such a way that we can reduce it to a two-dimensional system. This can be achieved as follows. In the present special case of the motion of a rigid body inside a circle, the cross-section of the spiral column, (25), has a fixed shape which rotates uniformly with α. This cross-section can be promoted to a rigid body which is allowed to rotate about its (fixed) center with an angular orientation given by α. Furthermore, we associate the inertial parameter $M\ell^2$ with this new rotating rigid wall. (In distinction to most other billiard problems, the wall is no longer infinitely massive.) The motion of the center of mass of the original stick is now described by the motion of a point particle with a mass M which moves inside the new rotating wall. The billiard ball therefore sees an (instantaneous) cross-section of the spiral column wall at every moment in time. Our configuration space has been reduced from $(x, y, \ell\alpha)$ to (x, y) by creating a rotating billiard wall of angular velocity $\dot{\alpha}$ and moment of inertia $I = M\ell^2$.

Once again, the point particle makes collisions with the rotating wall in which the impulsive force is perpendicular to the wall surface at the point of impact. However, since the wall is *not* infinitely massive, it will recoil at every collision such that the total energy of the point particle $M(\dot{x}^2 + \dot{y}^2)/2$ plus the rotating wall $M\ell\dot{\alpha}^2/2$ will be conserved. Therefore, energy conservation is exactly as before (38). The final step is to show that with every collision, the billiard ball velocity changes by equations (37) and (39) while the walls angular momentum is shifted by (36).

The local tangent and local normal vectors at the collision point are

$$\vec{T} = (t_x, t_y)/(t_x^2 + t_y^2)^{1/2} \tag{47}$$

and

$$\vec{N} = (-t_y, t_x)/(t_x^2 + t_y^2)^{1/2} \tag{48}$$

where

$$t_x = b_\theta \cos\theta - b\sin\theta \ , \tag{49}$$

and

$$t_y = b_\theta \sin\theta + b\cos\theta \ . \tag{50}$$

Since the impulsive force is strictly normal, we have

$$\vec{T}\cdot\Delta\vec{v} = \frac{(t_x\Delta\dot{x} + t_y\Delta\dot{y})}{(t_x^2+t_y^2)^{1/2}} = 0 \ , \tag{51}$$

which is equivalent to the tangential condition (43) of section III. Thus, equation (37) is satisfied. Now, the linear impulsive force equation is

$$\Delta\vec{p} = M(\Delta\dot{x},\Delta\dot{y}) = -f\tau\hat{n} \ , \tag{52}$$

while the angular impulse is

$$M\ell^2\Delta\dot{\alpha} = \frac{f\tau b b_\theta}{(t_x^2+t_y^2)^{1/2}} \ . \tag{53}$$

¿From the two equations above, we have

$$(t_y\Delta\dot{x} - t_x\Delta\dot{y}) = \frac{\ell^2}{bb_\theta}\Delta\dot{\alpha}(t_x^2+t_y^2) \ , \tag{54}$$

which results in (36) after some simple manipulations.

The advantage of this billiard problem over the equivalent problem of section III is two-fold. We have accomplished a very clean separation of linear and rotational motion. Further, the problem has effectively been reduced to two spatial dimensions and lends itself to convenient numerical analysis. The present argument permits some generalization. Specifically, similar results can be obtained for a rigid body of arbitrary shape. However, as the arguments above indicate, the original static wall must be circular if the equivalent rotating wall is to be rigid with a shape which is independent of its orientation.

A sample trajectory is depicted in Fig. 1 along with the rotating boundary wall. Even though the phase space of the billiard ball alone is three-dimensional ((x, y, p_x, p_y) plus energy conservation), the phase space of the total billiard system is actually five-dimensional

($(x, y, p_x, p_y, \ell\alpha, p_\alpha)$ plus energy conservation). It is evidently not possible to depict the complete, four-dimensional Poincaré section for this billiard problem. Rather, we have obtained a projected Poincaré section of the actual orbits in phase space in order to indicate the stochastic behaviour of the system under consideration. This type of section is simply a projection of the complete four-dimensional Poincaré manifold onto a two-dimensional plane in phase space. On this two-dimensional cross-section, orbits will appear to overlap each other even though they are spatially distinct.

Fig. 2 shows such a section for the billiard with $R = 2$ and $\ell = 1.9$. We first note that we have made a canonical transformation from the coordinates $(x, y, p_x, p_y, \ell\alpha, p_\alpha)$ to $(L_z, xp_x + yp_y, \tan^{-1}(p_y/p_x), (1/2)\ln(p_x^2 + p_y^2), \ell\alpha, p_\alpha)$ where $L_z = yp_x - xp_y$. At each collision with the wall, the angular momentum L_z was plotted along with the angle of incidence $\phi_c = \tan^{-1}(p_y/p_x)$ at the point of contact. Fig. 2 contains two such orbits: one integrable and one chaotic. The chaotic orbit is an example of Arnold diffusion [6], [7]. This diffusive process for chaotic orbits is not trapped by integrable KAM tori as is the case in lower-dimensional systems (*i.e.*, systems with two degrees of freedom) and thus, the whole of phase space is permeated by a network of stochastic trajectories. It was shown by Arnold that chaotic diffusion proceeds along a 'web' of dense overlapping resonances and, therefore, the chaotic structure of Fig. 2 is called an *Arnold Web*.[2]

We have followed the stability of these structures for fixed $R = 2$ as a function of the length of the rod, ℓ. The KAM tori shown in Fig. 2 deform as ℓ is reduced. When $\ell = 1.1$, these same tori are transformed into five double resonant islands located roughly in the area between the original tori shown in Fig. 2. For even small values of ℓ, these resonant islands seem to disappear leaving apparently chaotic motion.[3] The qualitative result of this

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[2]The chaotic orbit in Fig. 2 was terminated after 10^4 collisions in order to make the web structure apparent.

[3]It is always possible to demonstrate the absence of chaos by offering a numerically determined

investigation is the motion becomes "more chaotic" as the stick grows shorter with KAM tori first evolving into resonance islands which subsequently dissolve into chaos. On the other hand, the motion of a stick with length exactly zero is an integrable system. (This limit is simply a point particle moving in a circle, which is obviously integrable.) This provides an indication that the $\ell \to 0$ limit is somewhat delicate and might even suggest that the transition from the non-integrable motion found for $\ell \neq 0$ to the integrable circle at $\ell = 0$ is, in some sense, a phase transition. The mathematical puzzle can be resolved by physical thinking. Consider a plot of the motion using the present canonical co-ordinates, L_z and ϕ_c. For fixed energy the system will explore a decreasing range of L_z as $\ell \to 0$. Motion within this range will become increasingly chaotic, as indicated. However, for sufficiently small ℓ, the width of this range will become comparable to our ability to resolve the details of the structure within it. We will feel comfortable in declaring the particle a point particle, and we will not be bothered by an underlying chaotic motion which we cannot actually observe. A similar observation could be made regarding our earlier work [4] on an ellipse bouncing between two parallel walls. As the distance between the walls is increased, the system becomes more chaotic and, in fact, undergoes a transition from a KAM-system to a K-system. (Increasing the distance between the walls is equivalent to making the ellipse smaller and entirely analogous to letting $\ell \to 0$ as above).

In the context of the equivalent problem of point particle and rotating wall, there are two consequences of changing the length of the stick. Both the shape of the rotating wall and its moment of inertia, $I = M\ell^2$, change with the length. Thus, it is in some sense more natural to consider changes in the inertial parameter of the wall while maintaining its shape. This can be realized by regarding the stick as a rigid, massless rod of length 2ℓ with point masses $M/2$ located at some adjustable point along its length (or extension). Specifically, it is useful

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periodic trajectory as a counter example. Evidently, positive proof of chaotic motion cannot be made numerically.

to consider the somewhat artificial limit $I \to \infty$ in which the wall is stationary. In this limit, the phase space of the system is reduced to $(L_z, xp_x + yp_y, \tan^{-1}(p_y/p_x), (1/2)\ln(2ME))$. Comparing these with the canonical coordinates for the general case given above, we see that the final coordinate is now reduced to an obvious integral of the motion. The phase space structure of such a system (with $R = 2$ and $\ell = 1.5$) is depicted in Fig. 3. Since the dimensionality of this system is so low ($2N = 4$), Fig. 3 describes a complete Poincaré section rather than the projection of such a section. As a result, tori and chaotic regions are cleanly separated in a familiar fashion, and the phenomena of Arnold diffusion cannot appear. The chaotic regions can never penetrate the areas bounded by KAM tori. This would suggest that the system is KAM for all $I \geq M\ell^2$. Numerical studies confirm this expectation. If more chaotic motion is to be found, it must require $I < M\ell^2$. Tori and the web seem to persist if the original moment of inertia is reduced by a factor of 100. All remnants of these indicators of a KAM-system vanish when the original moment of inertia is reduced by a factor of 200.

The important observation is that we can control the qualitative nature of the motion — from integrable to KAM to (probably) K — by the tuning of simple physical parameters (*i.e.*, the length of the rod and the distribution of its mass).

V. CONCLUSION

This work represents a natural extension of our earlier work [4], [5] by investigating rigid body systems in closed rather than open domains. This has led to the new phenomena of a rotating billiard wall with a finite moment of inertia. This feature arose naturally from the mapping of the rigid body problem and was not put in by hand in some *ad hoc* manner. This is in the same spirit as in our previous work where the introduction of curvilinear motion was exclusively dictated by the three-dimensional rigid body under consideration and was not constructed *a priori* as is usually the case in the standard billiards possessing curvilinear trajectories such as the gravitational billiard or the Aharonov-Bohm billiard.

There are positive consequences of having such complications arise from the introduction of new physics rather than by artifice. Here, we have constructed three distinct physical systems which can be mapped onto each other and which have precisely the same content. Each physical system comes with its own intuitive understanding and complementary "natural" limits, which lead to a richer understanding of them all. The variant described in section IV of a point particle and a rotating wall is particularly appealing since it makes an exceptionally clean separation between the translational and rotational aspects of the problem.

The appearance of Arnold's web is extremely satisfying since virtually all aspects of Hamiltonian chaotic systems have now appeared in our rigid body billiard models. We have therefore introduced a rich class of two- and three-dimensional dynamical systems in which there exist both integrable and chaotic phenomena and in which the interesting features of curvilinear motion, rotating billiard walls, and Arnold diffusion can be probed by tuning physically meaningful parameters.

It is both a professional and personal pleasure to dedicate this work to Gerald E. Brown on the occasion of his seventieth birthday. Through more than three decades, Gerry Brown has continued to be a primary source of new and exciting ideas in nuclear physics. Time and again, he displays the courage to make major changes in his intellectual focus in order to follow new insights and enthusiasms. Life with Gerry is never dull. We await what will come next with admiration and affection.

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FIGURES

FIG. 1. A trajectory in configuration space for the stick system with $R = 2$ and $\ell = 1$. The rotating billiard wall at the point of contact is explicitly shown.

FIG. 2. A projected Poincaré section for the stick system with $R = 2$ and $\ell = 1.9$ where ϕ_c is plotted in units of π.

FIG. 3. A projected Poincaré section for the billiard system with $M_{wall} \to \infty$, $R = 2$ and $\ell = 1.5$.