CONFORMAL MEASURE AND DECAY OF CORRELATION FOR COVERING WEIGHTED SYSTEMS

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Abstract

We show that for a large class of piecewise monotonic transformations on a totally ordered, compact set one can construct conformal measures and obtain exponential mixing rate for the associated equilibrium state. The method is based on the study of the Perron-Frobenius operator. The conformal measure, the density of the invariant measure and the rate of mixing are deduced by using an appropriate Hilbert metric, without any compactness arguments, even in the case of a countable to one transformation.

1 INTRODUCTION

Invariant measures, absolutely continuous with respect to conformal measures, for 1-D maps should enjoy the same properties as Gibbs measures for Axiom-A systems [4, 23, 20]. One therefore expects that they verify strong statistical properties (exponential decay of correlations, central limit theorem, variational principles) and that their local behavior permits a complete fractal and multifractal description. Yet, the construction of conformal measures appears to be problematic. We refer to the introduction of [8] for a rather complete history of the various attempts to construct conformal measures. For dynamical systems considered in this paper two methods are available. Both of them look at the conformal measure as the fixed point of the adjoint of the Perron-Frobenius operator associated to the dynamics. The first method proposed in [27, 11, 12] consists in defining the transfer operator on some larger space where it acts on continuous functions and fixed points theorems are successively applied. The other approach has been developed in [8] for continuous, finite to one transformations, inspired by some previous work by Patterson [21] : the conformal measure turns out to be a weak accumulation point (computed at a transition parameter) of a sequence of measures constructed by weighting suitably the powers of the transfer operator. We also mention the work of V. Baladi [1], where a spectral gap of the Transfer operator implies both the existence of a conformal measure and the exponential decay of correlations.

Our contribution is the following. By iterating the Perron-Frobenius operator we obtain, at the same time, the conformal measure (see (3) below), the density of the invariant measure and a constructive estimate on the rate of decay of correlations. Such an unified approach works also in the case of countable to one maps. In addition, we obtain a variational characterization of the invariant measure.

Let $\widetilde{X}$ be an uncountable, totally ordered, order-complete set. We endow $\widetilde{X}$ with the topology given by the intervals, which makes $\widetilde{X}$ into a compact space [3]. Equipped with the σ-algebra $\mathcal{B}(\widetilde{X})$ of Borel sets, $\widetilde{X}$ becomes a measurable space. Let us call $B(\widetilde{X})$ the set of real bounded Borel measurable functions.

We call T a piecewise monotonic transformation on $\widetilde{X}$, if there is a finite or countable partition[1] $\mathcal{Z}$ of $\widetilde{X}$ in intervals Z such that T is strictly monotone and continuous on each $Z \in \mathcal{Z}$. $M_T(\widetilde{X})$ will

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[1] Note that here we allow countably many intervals of monotonicity, contrary to some other definition in the literature.

denote the set of invariant Borel measures of T. Next, we consider a potential $\varphi : \widetilde{X} \to I\!R \cup \{-\infty\}$ such that the weight $g = \exp(\varphi)$ is of bounded variation over $\widetilde{X}$ and $\sum_{Z\in\mathcal{Z}} \sup_Z g < \infty$. We can define the Perron-Frobenius (or transfer) operator given the map T and the weight g acting on bounded measurable functions $h \in B(\widetilde{X})$ as :

$$Ph(x) = \sum_{y\in T^{-1}\{x\}} g(y)h(y). \tag{1}$$

We assume the weighted system $(\widetilde{X}, T, \varphi)$ to be covering (some kind of topological mixing property, see definition 3.5).

The aim of this paper is to find a conformal measure ν and a positive eigenvector h_* of the transfer operator such that the measure $\mu = h_*\nu \in M_T(\widetilde{X})$ is an equilibrium state, and mixes exponentially fast for all the observable of bounded variation.
We briefly recall that a measure ν is said $\mathrm{e}^{-\tilde{\varphi}}$-conformal when

$$\nu(TA) = \int_A \mathrm{e}^{-\tilde{\varphi}} d\nu \quad \forall A \subset Z \in \mathcal{Z}. \tag{2}$$

In the sequel, we will refer to ν as a conformal measure, as it will be clear from the context what the potential is.

Traditionally, the above program is carried out in two steps :

First, the conformal measure is obtained via compactness arguments.

Second, one looks at the spectral properties of the transfer operator acting on a suitable Banach space (typically, Hölder continuous, Zygmund or bounded variation functions); using again compactness, or an Ergodic theorem by Ionescu-Tulcea and Marinescu, the density of the invariant measure is obtained (see [13] for an exhaustive review of these methods).

As already mentioned, our approach is different, and does not rely on compactness arguments: we construct a pseudo-metric (Hilbert metric) on a subset of functions of bounded variations, for which the transfer operator becomes a contraction. In addition, we can estimate explicitly the rate of contraction, which provides a bound on the decay of correlations.
We get the eigenvector of P and the conformal measure at once by the following limits :

$$\begin{aligned} \nu(h) &= \lim_{n\to\infty} \frac{P^n h}{P^n 1}, \\ h_* &= \lim_{n\to\infty} \frac{P^n 1}{\nu(P^n 1)}. \end{aligned} \tag{3}$$

Notice that, in the traditional constructions, first one obtains ν and then proves that it verifies the limit (3). On the contrary, we directly prove that the limit exists and defines a conformal measure, without studying explicitly the adjoint operator. An additional merit of our approach is the simple way in which we can deal with the discontinuities of the map (we simply ignore them) compared with other approaches in the literature (doubling these points or introducing equivalence classes), which become problematic in the case of countably many discontinuities. In conclusion, our method reveals itself particularly powerful in constructing conformal measures for countable to one maps, where only few, non exhaustive results, exists [2, 6, 5]. In particular, in [5] the shift on a infinite alphabet is considered; in [6] piecewise monotonic maps are studied with respect to Lebesgue measure only. In [2] spectral results are given (without looking at conformal measures), while in [24] spectral results are obtained assuming the existence of the conformal measure. Note that once the result on decay of correlations is obtained, it follows immediately the central limit theorem for bounded variation observable (see e.g., [18]). Moreover, we are able to establish a variational principle which is a new result in the infinite case.

The paper is organized as follows :

Section 2 : we briefly present the Hilbert metric and its properties.

Section 3 : we introduce covering weighted systems and state the main results of this paper.

Section 4 : we apply the technique presented in section 2 to the construction of conformal measures, and prove their statistical properties.

Section 5 : we show that the measures constructed enjoy a variational caracterization (equilibrium states).

2 HILBERT METRIC

In this section, we introduce a theory developed by G. Birkhoff [3], which is highly powerful to analyzing of the so called positive operators.

We will apply it to study the Perron-Frobenius operator for our maps. This strategy has been applied to other classes of dynamical systems, namely in dimension one in [10, 17] and for higher dimension in [16]; in the later case, it gives new results on the decay of correlations.

Definition 2.1 *Let $\mathcal{V}$ be a vector space. We will call convex cone a subset $\mathcal{C} \subset \mathcal{V}$ which enjoys the following properties*

(i) $\mathcal{C} \cap -\mathcal{C} = \emptyset$

(ii) $\forall \lambda > 0 \quad \lambda\mathcal{C} = \mathcal{C}$

(iii) $\mathcal{C}$ *is a convex set*

(iv) $\forall f, g \in \mathcal{C} \; \forall \alpha_n \in I\!\!R \;\; \alpha_n \to \alpha, \;\; g - \alpha_n f \in \mathcal{C} \Rightarrow g - \alpha f \in \mathcal{C} \cup \{0\}$.

Lemma 2.1 *The relation $\leq$ defined on $\mathcal{V}$ by*

$$f \leq g \Longleftrightarrow g - f \in \mathcal{C} \cup \{0\}$$

is a partial order relation, which is compatible with the algebraic structure of $\mathcal{V}$.

(i) $f \leq 0 \leq f \Rightarrow f = 0$

(ii) $\forall \lambda > 0 \quad 0 \leq f \Leftrightarrow 0 \leq \lambda f$

(iii) $f \leq g \Leftrightarrow \; 0 \leq g - f$

(iv) $\forall \alpha_n \in I\!\!R \;\; \alpha_n \to \alpha, \;\; \alpha_n f \leq g \Rightarrow \alpha f \leq g$

(v) $f \geq 0$ *and* $g \geq 0 \Rightarrow f + g \geq 0$.

We are now able to define the Hilbert metric on $\mathcal{C}$:

Definition 2.2 *The distance $\Theta(f, g)$ between two points f, g in $\mathcal{C}$ is given by*

$$\begin{aligned} \alpha(f,g) &= \sup\{\lambda > 0 | \lambda f \leq g\} \\ \beta(f,g) &= \inf\{\mu > 0 | g \leq \mu f\} \\ \Theta(f,g) &= \log \frac{\beta(f,g)}{\alpha(f,g)} \end{aligned}$$

where we take $\alpha = 0$ or $\beta = \infty$ when the corresponding sets are empty.

The distance Θ is a pseudo-metric, because two elements can be at an infinite distance from each others, and it is a projective metric because any two proportional elements have a null distance.

The next theorem, due to G. Birkhoff [3], will show that every positive linear operator is a contraction, provided that the diameter of the image is finite.

Theorem 2.1 *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two vector spaces, $\mathcal{C}_1 \subset \mathcal{V}_1$ and $\mathcal{C}_2 \subset \mathcal{V}_2$ two convex cone (see definition above) and $L : \mathcal{V}_1 \to \mathcal{V}_2$ a positive linear operator (which means $L(\mathcal{C}_1) \subset \mathcal{C}_2$). Let Θ_i be the Hilbert metric associated to the cone $\mathcal{C}_i$. If we denote*

$$\Delta = \sup_{f,g \in L(\mathcal{C}_1)} \Theta_2(f,g) \quad ,$$

then

$$\Theta_2(Lf, Lg) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta_1(f,g) \;\; \forall f, g \in \mathcal{C}_1$$

$(\tanh(\infty) = 1)$.

Theorem 2.1 alone is not completely satisfactory: given a cone $\mathcal{C}$ and its metric Θ, we don not know if $(\mathcal{C}, \Theta)$ is complete. This aspect is taken care by the following lemma, which allows to link the Hilbert metric to a suitable norm defined on $\mathcal{V}$.

Lemma 2.2 *Let $\|\cdot\|$ be a norm on $\mathcal{V}$ such that*

$$\forall f,g \in \mathcal{V} \quad -f \le g \le f \Rightarrow \|g\| \le \|f\|$$

and let $\rho : \mathcal{C} \to I\!\!R^+$ be a homogeneous and order preserving function, i.e.

$$\begin{aligned} \forall f \in \mathcal{C}, \forall \lambda \in I\!\!R^+ \qquad & \rho(\lambda f) = \lambda \rho(f) \\ \forall f,g \in \mathcal{C} \qquad & f \le g \Rightarrow \rho(f) \le \rho(g)\ , \end{aligned}$$

then

$$\forall f,g \in \mathcal{C} \;\; \rho(f) = \rho(g) > 0 \Rightarrow \|f-g\| \le (\mathrm{e}^{\Theta(f,g)} - 1)\min(\|f\|, \|g\|)$$

Proof :

Let $f,g \in \mathcal{C}$ with $\rho(f) = \rho(g) > 0$. If $\Theta(f,g) = \infty$ then the inequality is obvious. If not, we have

$$\Theta(f,g) = \log \frac{\beta}{\alpha}$$

where $\alpha f \le g \le \beta f$. Which yields, by the properties of ρ :

$$\alpha \rho(f) \le \rho(g) \le \beta \rho(f) \quad ,$$

hence $\alpha \le 1 \le \beta$. This implies

$$(\alpha - \beta) f \le (\alpha - 1) f \le g - f \le (\beta - 1) f \le (\beta - \alpha) f$$

that is

$$\begin{aligned} \|g-f\| &\le (\beta-\alpha)\|f\| \\ &\le \frac{\beta-\alpha}{\alpha}\|f\| \\ &\le (\mathrm{e}^{\Theta(f,g)} - 1)\|f\|. \end{aligned}$$

The proof is concluded by interchanging f and g. $\diamond$

Remark 2.1 *In the previous lemma, one can choose $\rho(\cdot) = \|\cdot\|$ which fulfill the hypothesis. An interesting case is also when ρ is a linear functional positive on $\mathcal{C}$. Nevertheless, we will use a nonlinear ρ in section 4.3.*

3 STATEMENTS OF THE MAIN RESULTS

The class of dynamical systems we intend to study will be formally defined in (3.5) below. We shortly call them "covering" because they satisfies a sort of topological dynamical covering reminiscent of Markov partitions. Dynamical systems with a (strong) covering property have been introduced in [7] (where they consist of expanding maps with a finite number of branches) and also investigated in [17] especially with respect to the decay of correlations for absolutely continuous measures. However the same analysis can be done for all others equilibrium states [25]. In this paper the covering property will be weakened to include countable to one maps and the hyperbolicity assumed in [7, 17] will be more generally stated in terms of a suitable assumption on the potential (see remark 3.1).
First of all, we need some basic definitions :

Definition 3.1 *A function $h \in B(\widetilde{X})$ is of bounded variation (or $h \in BV(\widetilde{X})$) if $\bigvee_{\widetilde{X}} h < \infty$. The variation $\bigvee_A h$ is defined for all subset $A \subset \widetilde{X}$ by*

$$\bigvee_A h = \sup \left\{ \sum_{i=0}^{k-1} |h(x_{i+1}) - h(x_i)| \;\Big|\; x_0 < x_1 < \cdots < x_k \;\; , x_i \in A, \forall i \le k \right\},$$

where the sup *is taken over all finite sets $\{x_i\} \subset A$, and we call $\bigvee_A h$ the variation of h over A.*

Definition 3.2 *$T : \widetilde{X} \to \widetilde{X}$ is a piecewise monotonic transformation if there exists a finite or countable partition $\mathcal{Z}$ of $\widetilde{X}$ into intervals such that for each $Z \in \mathcal{Z}$, $T(Z)$ is an interval and $T : Z \to T(Z)$ is continuous and strictly monotone.*

Definition 3.3 *Let D be the set of discontinuities of T, the union of the endpoints of Z, $Z \in \mathcal{Z}$. Let W be the singular set of T, defined by*

$$W = \bigcup_{k \geq 0} T^{-k} \bigcup_{j \geq 0} T^j D \, .$$

Note that W is countable and forward and backward invariant ($T^{-1}W = W = T(W)$). Let $X = \widetilde{X} \backslash W$. We endow X with the topology given by the interval (we put on X the restriction of the order of $\widetilde{X}$). Note that $T(X) \subset X$.

Notation. Let $\mathcal{Z}^{(1)} = \{Z \cap X, Z \in \mathcal{Z}\}$. For $n > 1$ we denote by $\mathcal{Z}^{(n)}$ the partition $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{Z}^{(1)}$, the partition of X on which T^n is monotone and continuous (in fact, T is continuous on X). Note that, by construction, $\mathcal{Z}^{(n)}$ consists of open (in the topology of X), nonempty intervals.

Definition 3.4 *We will call $\varphi : \widetilde{X} \to I\!R \cup \{-\infty\}$ a contracting potential if*

(i) g_1 is of bounded variation over $\widetilde{X}$,

(ii) $S_1 = \sum_{Z \in \mathcal{Z}} \sup_Z g_1 < \infty$,

(iii) $\exists n_0 \in I\!N \ \sup_{\widetilde{X}} g_{n_0} < \inf_X P^{n_0} 1$,

where, for all integer $n \geq 1$, $g_n = \exp\left(\varphi + \varphi \circ T + \cdots + \varphi \circ T^{n-1}\right)$.

Remark 3.1 *We will see later (section 4) that in our case, $p(\varphi)$, the pressure of φ, is given by $\lim_{n \to \infty} \frac{1}{n} \log P^n 1(x)$ for all $x \in X$. Our assumption on the potential is easier to verify than the usual one, $\sup \varphi < p(\varphi)$ (see [9, 8]).*

Proof :

$$\begin{aligned} \sup \varphi < p(\varphi) \quad &\Rightarrow \quad \exists n_0 \in I\!N \ \ \sup \varphi < \frac{1}{n_0} \log \inf_X P^{n_0} 1 \\ &\Rightarrow \quad \exists n_0 \in I\!N \ \ \exp(n_0 \sup \varphi) < \inf_X P^{n_0} 1 \\ &\Rightarrow \quad \exists n_0 \in I\!N \ \ \frac{\sup g_{n_0}}{\inf_X P^{n_0} 1} < 1. \end{aligned}$$

$\diamond$

Notation. We will denote by χ_B the characteristic function of a set $B \subset \widetilde{X}$.

Definition 3.5 (covering system) *We call the weighted system $(\widetilde{X}, T, \varphi)$ covering if T is a piecewise monotonic transformation and for each nonempty open interval I there exists an integer $N(I)$ and a constant $C(I) > 0$ such that $\inf_X P^N \chi_I \geq C(I)$.*

Remark 3.2 *$(\widetilde{X}, T, \varphi)$ covering implies in particular that for all intervals $I \subset \widetilde{X}$ there exists an N such that $T^N I \supset X$. Due to the discontinuities of T, it is natural to consider systems where an interval covers all the space but a countable set (W) after some iterations of the map. In fact, covering implies that for all interval $I \subset X \ \exists N : T^N I = X$. If the partition is finite and φ is bounded from below, then covering turns out to be equivalent to the above mentioned property : $\forall I \subset X \exists N : T^N I = X$ (which has been called "covering" in [7, 17]).*

If the system is covering each open interval must contain an uncountable number of points, and the partition $\mathcal{Z}$ is generating.

Notice that for countable to one maps, one cannot extend the space $\widetilde{X}$ by "doubling" the set W, making the new map T continuous[2] *(see [27, 12] for more details). Also the other approach found in the literature (introducing equivalence classes of functions [2]) to deal with discontinuities would lead to severe technical difficulties. Instead of doubling these points, we study the transfer operator on X (despite the lack of compactness of X no technical problem arises).*

[2]If x is an accumulation point of D, left or right limit of T in x may not exists.

The above definition of covering weighted systems has been inspired by the following interesting class of dynamical systems [25], which are a natural extension of the expanding systems quoted in [7, 17].

Example 3.1 *Let us consider a piecewise monotonic map $T : \widetilde{X} \to \widetilde{X}$ such that*
(i) $\inf_Z g_1 > 0$ *for all* $Z \in \mathcal{Z}$.
(ii) For all interval $I \subset \mathcal{Z}^{(n)}$, X *may be covered by a finite number of smooth pieces of* $T^N I$, *i.e.*

$$\forall n\ \forall I \in \mathcal{Z}^{(n)},\ \exists N\ \exists \text{ finite } \mathcal{J} \subset \mathcal{Z}^{(N)} \bigvee \{I\} \quad \bigcup_{J \in \mathcal{J}} T^N J = X$$

(iii) $\mathcal{Z}$ *is generating* ($\bigvee_{n=1}^{\infty} \mathcal{Z}^{(n)} = \mathcal{B}$)

Proposition 3.1 *Example 3.1 is a covering weighted systems.*

Proof :

Let $K \subset X$ be an interval. As $\mathcal{Z}$ generates, it exists an n such that there is an element $I \in \mathcal{Z}^{(n)}$ with $I \subset K$. Then by hypothesis, there is N and a finite $\mathcal{J} \subset \mathcal{Z}^{(N)}$ such that $\cup_{J \in \mathcal{J}} T^N J = X$. Hence

$$P^N \chi_K \geq P^N \chi_I \geq \inf_{\mathcal{J}} g_N.$$

To conclude we take $C(K) = \inf_{\mathcal{J}} g_N$.

◇

The following proposition states that under some circumstance, it is sufficient to have the covering for one partition in the case of expanding maps of $[0, 1]$.

Proposition 3.2 *Let $\widetilde{X} = [0, 1]$, T be a piecewise $C^{(1)}$ monotonic map with a partition $\mathcal{Z}$ finite*[3]*. If there exists an integer K such that $\inf |DT^K| \geq \gamma > 2$ then*[4]

$$\forall n\ \forall I \in \mathcal{Z}^{(n)} \exists N(I)\ T^{N(I)}(I) = X \iff \exists N\ \ \forall I \in \mathcal{Z}^{(K)}\ T^N I = X\ .$$

Proof :

We will denote by $|I|$ the length of an interval I.
Let $n > 0$ and $I \in \mathcal{Z}^{(n)}$. By definition, $J_0 = T^n I$ is an interval.
Then we have three possibility :
(1)- J_0 contains an element of $\mathcal{Z}^{(K)}$, hence $T^{N+n} I = X$.
(2)- J_0 is included in an element of $\mathcal{Z}^{(K)}$, hence $J_1 = T^K J_0$ is an interval of length $|J_1| \geq \gamma |J_0|$,
(3)-J_0 intersects two pieces of $\mathcal{Z}^{(K)}$. Let us denote J_0' the biggest intersection. T^K is monotonous on J_0' hence $J_1 = T^K J_0'$ is again an interval, of length $|J_1| \geq \frac{\gamma}{2} |J_0|$.
We can construct, by repeating the same arguments, a sequence J_k which satisfies $|J_k| \geq (\frac{\gamma}{2})^k |J_0|$ $\forall k$ until case (1) is true.

◇

Definition 3.6 *For $m \in M_T(\widetilde{X})$, the conditional information of the partition $\mathcal{Z}$ given $T^{-1}\mathcal{B}$ is given by $I_m = -\log g_m$, where $g_m = \sum_{Z \in \mathcal{Z}} \chi_Z E_m[\chi_Z | T^{-1}\mathcal{B}]$. (See Parry [19]).*

Definition 3.7 *The pressure of a covering system for the contracting potential φ is given by $p(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \sup_X P^n 1$. Note that $p(\varphi)$ is well defined since the sequence $\log \sup_X P^n 1$ is sub-additive.*

We now state the main result of the paper

[3] When $\mathcal{Z}$ is countable, we still have ($\Leftarrow$) but this is not sufficient to satisfy (ii) of example 3.1. Nevertheless, if for example $\{T(Z), Z \in \mathcal{Z}\}$ is composed by a finite number of intervals, one can still show that (ii) is true if the right hand of the following equivalence is satisfied.

[4] In particular, this implies that (iii) of example 3.1 is satisfied.

Theorem 3.1 *Let $(\widetilde{X}, T, \varphi)$ be a covering system for the contracting potential φ. Then there exists a probability measure μ_φ equivalent to an $\mathrm{e}^{p(\varphi)-\varphi}$-conformal measure ν without atoms, and the correlations decay exponentially fast*[5] *for bounded variation observable :*

$$\exists \Lambda < 1, C > 0 \quad \left| \int f \circ T^n h d\mu_\varphi - \int f d\mu_\varphi \int h d\mu_\varphi \right| \le C \Lambda^n \|f\|_{L^1_{\mu_\varphi}} \|h\|_{BV}.$$

Theorem 3.2 *With the same hypothesis of theorem 3.1 we have that μ_φ is the unique equilibrium state for φ :*

$$p(\varphi) = \int (I_{\mu_\varphi}[\mathcal{Z}|T^{-1}\mathcal{B}] + \varphi) d\mu_\varphi = \sup_{m \in M_T(\widetilde{X})} \int (I_m[\mathcal{Z}|T^{-1}\mathcal{B}] + \varphi) dm.$$

Where the supremum is obtained iff $m = \mu_\varphi$.

Remark 3.3 *Theorem 3.2 reduces to the usual variational principle if $H_{\mu_\varphi}(\mathcal{Z})$ is finite*[6].

$$p(\varphi) = h_{\mu_\varphi}(T) + \int \varphi d\mu_\varphi \ge h_m(T) + \int \varphi dm \quad \forall m \in M_T(\tilde{X}), H_m(\mathcal{Z}) < \infty.$$

With equality if and only if $m = \mu_\varphi$.

Remark 3.4 *In the proof we do not assume the existence of an atom free conformal measure ν with full topological support. It will be a consequence of the contraction of the Perron-Frobenius operator, as we mentioned in the introduction. Also, we do not use any extension of the space $\widetilde{X}$ (see remark 3.2), thanks to this we can deal with a countable partition (in this case, it is unclear to us how to get the conformal measure by classical arguments [1]).*

4 CONFORMAL MEASURE (Proof of Theorem 3.1)

The proof of theorem 3.1 will be divided in several steps. Our first goal is to exhibit a cone which is mapped inside itself by the transfer operator. Our cone is similar to the one considered in [17], for absolutely continuous invariant measures and in [25] for the other equilibrium states. In order to prove this, we adapt a lemma by Rychlik [24], which is a generalization of the initial work of Lasota & Yorke [14]. After, we show that the diameter of $P^N\mathcal{C}$ in $\mathcal{C}$ becomes finite for some N, hence P^N is a contraction by theorem 2.1. Thanks to lemma 2.2, we find that the projective limit of $P^n h$ exists for $h \in BV(X)$, $h \ge 0$, and is equal to a (projective) fixed point of P.

Notation Let $B(X)$ the set of real bounded functions from X to $\mathbb{R}$, and $BV(X) \subset B(X)$ the set of bounded variation functions over X, endowed with the norm $\|f\|_{BV} = \bigvee_X f + \|f\|_\infty$.

Proposition 4.1 *P is a well defined continuous operator on $B(X)$ and $BV(X)$.*

Proof :

Using the fact that φ is a contracting potential, we obtain by (i) for all f in $B(X)$

$$\|Pf\|_\infty = \left\| \sum_{Z \in \mathcal{Z}} g_1 \circ T_{|Z}^{-1} f \circ T_{|Z}^{-1} \chi_{TZ} \right\|_\infty \le S_1 \|f\|_\infty.$$

The continuous action of P on $BV(X)$ is an immediate consequence of sub-lemma 4.1.1.

◇

[5] Actually, we prove a bit more : $\left\| \mathrm{e}^{-np(\varphi)} P^n h - \nu(h) h_* \right\|_\infty < C\Lambda^n \|h\|_{BV(X)}$ for all $h \in BV(X)$, and $Ph_* = \mathrm{e}^{p(\varphi)} h_*$.

[6] For $m \in M_T(\widetilde{X})$, by $H_m(\mathcal{Z})$ we mean the entropy of the partition $\mathcal{Z}$, $H_m(\mathcal{Z}) = -\sum_{Z \in \mathcal{Z}} m(Z) \log m(Z)$. If $H_m(\mathcal{Z}) < \infty$ then $h_m(T) = H(\mathcal{Z}|T^{-1}\mathcal{B}) = \int I_m[\mathcal{Z}|T^{-1}\mathcal{B}] dm$ (see [19]). It is unclear if the conditions $h_\mu(T), h_m(T) < \infty$ suffice.

Lemma 4.1 *For all positive integers n, we have*
(i) $g_n \in BV(X)$
(ii) $S_n := \sum_{Z \in \mathcal{Z}^{(n)}} \sup_Z g_n < \infty$

Proof :

We prove (i) by induction. Let us assume that $g_n \in BV(X)$. Using $g_{n+1} = g_1 g_n \circ T$ we obtain

$$\begin{aligned}
\bigvee_X g_{n+1} &\leq \sum_{Z \in \mathcal{Z}^{(1)}} \bigvee_Z g_{n+1} + 2 \sup_Z g_{n+1} \\
&\leq \sum_{Z \in \mathcal{Z}^{(1)}} \bigvee_Z g_1 \sup_{TZ} g_n + \sup_Z g_1 \bigvee_{TZ} g_n + 2 \sup_Z g_1 \sup_{TZ} g_n \\
&\leq (2S_1 + \bigvee_X g_1) \|g_n\|_{BV}.
\end{aligned}$$

We prove (ii) by induction as well. Let us assume that $S_n < \infty$. For $g_{n+1} = g_n g_1 \circ T^n$

$$\begin{aligned}
S_{n+1} &= \sum_{Z \in \mathcal{Z}^{(n+1)}} \sup_Z g_n \sup_{T^n Z} g_1 \\
&\leq \sum_{Z' \in \mathcal{Z}^{(n)}} \sum_{Z \in \mathcal{Z}^{(n+1)}, Z \subset Z'} \sup_{Z'} g_n \sup_{T^n Z} g_1 \\
&\leq \sum_{Z' \in \mathcal{Z}^{(n)}} \sup_{Z'} g_n \sum_{Z'' \in \mathcal{Z}^{(1)}} \sup_{Z''} g_1 \\
&\leq S_n S_1.
\end{aligned}$$

$\diamond$

4.1 Cone of functions

We define for all function $h \in B(X)$ the quantity $\nu(h)$ by

$$\nu(h) = \lim_{n \to \infty} \inf_{x \in X} \frac{P^n h(x)}{P^n 1(x)}.$$

Note that $\nu(h)$ is well defined because the bounded sequence $\inf_{x \in X} \frac{P^n h(x)}{P^n 1(x)}$ is increasing. We emphasize the fact that, at this point, ν as nothing to do with a measure (it may be non linear), yet for all $\lambda \in I\!R^+$ we have $\nu(\lambda h) = \lambda \nu(h)$; $\nu(1) = 1$; for each $h_1, h_2 \in B(X)$ $\nu(h_1 + h_2) \geq \nu(h_1) + \nu(h_2)$; ν is order preserving and $\forall \lambda \in I\!R$ $\nu(h + \lambda) = \nu(h) + \lambda$.
Let us consider the following convex cone (a is a positive real number)

$$\mathcal{C}_a = \left\{ h \in BV(X) \;\middle|\; h \geq 0, \bigvee_X h \leq a\nu(h) \right\}.$$

Remark 4.1 *This cone is far from being empty, in fact, given $h \in BV(X)$ then $h_c = h + c \in \mathcal{C}_a$, provided $c \geq a^{-1} \bigvee_X h - \inf h$.*

We will need the following cone too

$$\mathcal{C}_+ = \{h \in BV(X) | h \geq 0\}.$$

In the sequel, Θ and Θ_+ will denote respectively the Hilbert metrics induced by the cones $\mathcal{C}_a$ and $\mathcal{C}_+$.

Lemma 4.2 *The distance between two functions $f, h \in \mathcal{C}_+$ is given by*

$$\Theta_+(f, h) = \log \sup_{x, y \in X} \frac{f(y)h(x)}{f(x)h(y)}.$$

Proof :

Let $f, h \in \mathcal{C}_+$. We have to find λ and μ with $\lambda f \leq h \leq \mu f$. So $\lambda \leq \frac{h(x)}{f(x)}$ and $\mu \geq \frac{h(y)}{f(y)}$ for all $x, y \in X$, whence the result.

$\diamond$

Lemma 4.3 *For all $f, h \in \mathcal{C}_a$ we have $\Theta_+(f,h) \leq \Theta(f,h)$.*

Proof :

It is an immediate consequence of theorem 2.1, since the identity I is a linear map from $\mathcal{C}_a$ to $\mathcal{C}_+$.

$\diamond$

Lemma 4.4 *For all $\sigma < 1$ there exists an integer N_σ and a real number $a > 0$ such that $P^m\mathcal{C}_a \subset \mathcal{C}_{\sigma a}$ for all $m \geq N_\sigma$.*

Proof :

We first need the following :

Sub-lemma 4.1.1 *For all positive integer m, $\exists B_m < \infty$ such that*

$$\bigvee_X P^m h \leq \eta_m \bigvee_X h + B_m \nu(h)$$

$\forall h \in BV(X), h \geq 0$ *and where* $\eta_m = 9 \sup g_m$.

Proof :

The proof closely follows Rychlik (corollary 3 in [24]).
Using (i) and (ii) of lemma 4.1 we know that g_m is in $BV(X)$ and $S_m < \infty$.

$$\begin{aligned}
\bigvee_X P^m h &\leq \bigvee_X \sum_{Z \in \mathcal{Z}^{(m)}} g_m \circ T_{|Z}^{-m} h \circ T_{|Z}^{-m} \chi_{T^m Z} \\
&\leq \sum_{Z \in \mathcal{Z}^{(m)}} \bigvee_X \left(g_m \circ T_{|Z}^{-m} h \circ T_{|Z}^{-m} \chi_{T^m Z} \right) \\
&\leq \sum_{Z \in \mathcal{Z}^{(m)}} \bigvee_X g_m h \chi_Z \\
&\leq \sum_{Z \in \mathcal{Z}^{(m)}} \bigvee_Z g_m h + 2 \sup_Z g_m h \\
&\leq \bigvee_X g_m h + 2 \sum_{Z \in \mathcal{Z}^{(m)}} \sup_Z g_m h.
\end{aligned}$$

Choose an increasing sequence of points $z_1 < \cdots < z_{q-1} \in X$ such that on $Z_1 = \{x \in X, x \leq z_1\}$, $Z_i = [z_{i-1}, z_i]$ for $i \in [2, \ldots, q-1]$ and $Z_q = \{x \in X, x \geq z_{q-1}\}$, the interior of Z_i is nonempty, $\bigvee_{Z_i} g_m \leq 4\|g_m\|_\infty$ and $\sum_{Z \in \mathcal{Z}^{(m)}} \sup_{Z \cap Z_i} g_m \leq 2\|g_m\|_\infty$.
This can be achieved in the following way : we know that g_m belongs to $BV(X)$. So there exists a sequence z_i such that $\bigvee_X g_m - \|g_m\|_\infty < \sum_{i=1}^q |g_m(z_{i-1}) - g_m(z_i)|$.
For these points z_i, we have $\bigvee_{Z_i} g_m \leq 2\|g_m\|_\infty$ (otherwise one can find a finer partition such that the sums are bigger than the variation, which is impossible). If Z_1 has empty interior, then it can be eliminated. If Z_2 has empty interior, then we consider the interval $Z_2 \cup Z_3$ instead of Z_2. Such an interval contains in its interior the points z_2 and hence uncountably many points (see remark 3.2). Then we look at Z_4 and so on. Finally, we eliminate Z_q if it has empty interior and has not been joined to Z_{q-1}. For simplicity, we call again $\{Z_i\}$ the partition so obtained. Notice, that the element of the new partition consists, at most, of

the union of two elements of the old, so $\bigvee_{Z_i} g_m \leq 4\|g_m\|_\infty$. *Then we refine the partition* $\{Z_i\}$ *so that* $\sum_{Z \in \mathcal{Z}^{(m)}} \sup_{Z \cap Z_i} g_m < 2\|g_m\|_\infty$ *(it is possible since* $S_m < \infty$*).*
Then we get

$$
\begin{aligned}
\bigvee_X P^m h &\leq \sum_{i=1}^{q} \bigvee_{Z_i} h g_m + 2 \sum_{Z \in \mathcal{Z}^{(m)}} \sup_{Z \cap Z_i} g_m h \\
&\leq \sum_{i=1}^{q} \|g_m\|_\infty \bigvee_{Z_i} h + \|h\chi_{Z_i}\|_\infty (\bigvee_{Z_i} g_m + 2 \sum_{Z \in \mathcal{Z}^{(m)}} \sup_{Z \cap Z_i} g_m) \\
&\leq \sum_{i=1}^{q} \|g_m\|_\infty \bigvee_{Z_i} h + 8\|g_m\|_\infty \|h\chi_{Z_i}\|_\infty \\
&\leq \sum_{i=1}^{q} 9\|g_m\|_\infty \bigvee_{Z_i} h + 8\|g_m\|_\infty \inf_{Z_i} h \\
&\leq \sum_{i=1}^{q} 9\|g_m\|_\infty \bigvee_{Z_i} h + 8\|g_m\|_\infty \frac{P^N 1}{P^N \chi_{Z_i}} \frac{P^N h \chi_{Z_i}}{P^N 1}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\bigvee_X P^m h &\leq 9\|g_m\|_\infty \bigvee_X h + B_m \inf_X \frac{P^N h}{P^N 1} \\
&\leq \eta_m \bigvee_X h + B_m \nu(h)
\end{aligned}
$$

where $N = \max_{i \in [1..q]} N(Z_i)$ *is given by the covering and* $B_m = 8\|g_m\|_\infty \sup_{i \in [1..q]} \sup_{Z_i} \frac{P^N 1}{P^N \chi_{Z_i}}$, *which is finite by the covering hypothesis.*

◇

We now return to the proof of the lemma 4.4. As φ *is a contracting potential, the sequence* $\sup g_m / \inf P^m 1$ *converges to zero when* m *tends to infinity. So it exists an* N_σ *such that for all* $m \geq N_\sigma$ *we have* $\sigma > 9 \sup g_m / \inf P^m 1$. *By the previous sub-lemma we know that for* $h \in \mathcal{C}_a$

$$\bigvee_X P^m h \leq (\eta_m a + B_m) \nu(h) .$$

To conclude, it suffices to compare $\nu(h)$ *with* $\nu(P^m h)$. *Let* $k \in \mathbb{N}$, *we have*

$$\frac{P^{k+m} h}{P^{k+m} 1} \inf P^m 1 \leq \frac{P^{k+m} h}{P^{k+m} 1} \frac{P^{k+m} 1}{P^k 1} = \frac{P^k P^m h}{P^k 1}.$$

Thus $\nu(h) \inf P^m 1 \leq \nu(P^m h)$. *This implies*

$$\bigvee_X P^m h \leq \frac{\eta_m a + B_m}{\inf P^m 1} \nu(P^m h) \leq \sigma a \nu(P^m h)$$

whenever

$$a \geq \frac{B_m}{\sigma \inf P^m 1 - \eta_m}.$$

We conclude that, for all $m \geq N_\sigma$, *exists* a_m *such that*
$P^m(\mathcal{C}_a) \subset C_{\sigma a} \subset \mathcal{C}_a$ *for all* $a \geq a_m$. *Define* $a = \max\{a_m\}, m \in [N_\sigma, \ldots, 2N_\sigma[$.
Then for all $m \geq N_\sigma$, *write* $m = kN_\sigma + r$ *where* $r \in [N_\sigma, \ldots, 2N_\sigma[$. *We get*

$$P^m \mathcal{C}_a = P^r P^{kN_\sigma} \mathcal{C}_a \subset P^r \mathcal{C}_a \subset \mathcal{C}_{\sigma a}.$$

◇

4.2 Diameter of the image

Lemma 4.5 *Let $\sigma < 1$. For all $h \in \mathcal{C}_{\sigma a}$ the hyperbolic distance with respect to the metric induced by the cone $\mathcal{C}_a$ between 1 and h is bounded by*

$$\Theta(1,h) \le \log \frac{\sup h + \sigma\nu(h)}{\min\{\inf h, (1-\sigma)\nu(h)\}}.$$

Proof :

Let $h \in C_{\sigma a}$. To compute the distance between h and 1, we must find λ and μ such that $\lambda \le h \le \mu$, where the ordering is the one given by the cone $\mathcal{C}_a$.

$$\begin{aligned}
\lambda \le h \quad &\Leftrightarrow \quad \left\{ \begin{array}{l} h-\lambda \ge 0 \\ \bigvee(h-\lambda) \le a\nu(h-\lambda) \end{array} \right. \\
&\Leftrightarrow \quad \left\{ \begin{array}{l} \lambda \le \inf h \\ a^{-1}\bigvee h \le \nu(h) - \lambda \end{array} \right. \\
&\Leftarrow \quad \left\{ \begin{array}{l} \lambda \le \inf h \\ \sigma\nu(h) \le \nu(h) - \lambda \end{array} \right. \\
&\Leftarrow \quad \lambda \le \min\{\inf h, (1-\sigma)\nu(h)\}.
\end{aligned}$$

For μ we proceed in a similar way

$$\begin{aligned}
h \le \mu \quad &\Leftrightarrow \quad \left\{ \begin{array}{l} \mu - h \ge 0 \\ \bigvee(\mu-h) \le a\nu(\mu-h) \end{array} \right. \\
&\Leftrightarrow \quad \left\{ \begin{array}{l} \mu \ge \sup h \\ a^{-1}\bigvee h \le \mu + \nu(-h) \end{array} \right. \\
&\Leftarrow \quad \left\{ \begin{array}{l} \mu \ge \sup h \\ \sigma\nu(h) \le \mu - \sup h \end{array} \right. \\
&\Leftarrow \quad \mu \ge \sup h + \sigma\nu(h).
\end{aligned}$$

So we conclude that $\Theta(1,h) \le \log \frac{\mu}{\lambda}$.

⋄

We will see in the next lemma ([17], adapted to ν) that the functions in the cone cannot be small too often. More precisely

Lemma 4.6 *Given a finite partition $\mathcal{P}$ of X, if each element $p \in \mathcal{P}$ is an interval such that $b \equiv \sup_{p\in\mathcal{P}} \left\| \frac{P^M(\chi_p)}{P^M 1} \right\|_\infty < \frac{1}{2a}$ for some M then, for all $h \in \mathcal{C}_a$ there exists $p_h \in \mathcal{P}$ such that*

$$h(x) \ge \frac{1}{2}\nu(h) \quad \forall x \in p_h.$$

Proof :

Let $\mathcal{P}_- = \left\{p \in \mathcal{P} \mid \exists x_p \in p : \ h(x_p) < \frac{1}{2}\nu(h)\right\}$. It suffices to show that $\mathcal{P}_- \ne \mathcal{P}$.
Let us suppose the contrary :

$$\forall p \in \mathcal{P}, \exists x_p \in p, \ h(x_p) < \frac{1}{2}\nu(h).$$

Given n larger than M,

$$P^n(h\chi_p) \le P^n(\chi_p)\left(h(x_p) + \bigvee_p h\right) < P^n(\chi_p)\frac{\nu(h)}{2} + bP^n 1 \bigvee_p h.$$

Which implies, by summing over all $p \in \mathcal{P}$ and dividing by $P^n 1$,

$$\frac{P^n h}{P^n 1} < \frac{\nu(h)}{2} + b\bigvee_X h \le (\frac{1}{2} + ab)\nu(h).$$

We obtain a contradiction by letting n go to infinity, since $\frac{1}{2} + ab < 1$.

⋄

Fix $\sigma < 1$ and take a given by lemma 4.4.

Lemma 4.7 *There exists an $N \geq N_\sigma$ such that the diameter of $P^N \mathcal{C}_a$ in $\mathcal{C}_a$ is finite.*

$$\sup_{f,h \in C_a} \Theta(P^N f, P^N h) \leq \Delta < \infty.$$

Proof :

Let $h \in \mathcal{C}_a$, and $N \geq N_\sigma$ be an integer (which will be fixed later). To show the finiteness of the diameter, it suffices to find an uniform upper bound (independent of h) for the ratio :

$$\frac{\sup P^N h + \sigma\nu(P^N h)}{\min\{\inf P^N h, (1-\sigma)\nu(P^N h)\}} .$$

As $P^N h \in \mathcal{C}_{\sigma a}$, we have

$$\sup P^N h \leq \nu(P^N h) + \bigvee_X P^N h \leq (1+\sigma a)\nu(P^N h) .$$

Consequently all we need is a lower bound for $\inf P^N h$ in terms of $\nu(P^N h)$.

Let M be such that $\dfrac{\sup g_M}{\inf P^M 1} < \dfrac{1}{2a}$, M exists because φ is a contracting potential. So for all $p \in \mathcal{Z}^{(M)}$, $\dfrac{P^M(\chi_p)}{P^M 1} < \dfrac{1}{2a}$.

If the partition is finite, say $\mathcal{Z}^{(M)} = \{p_0, \ldots, p_L\}$ then we can apply lemma 4.6 with h and $\mathcal{P} = \mathcal{Z}^{(M)}$.

If not, it is possible to extract from $\mathcal{Z}^{(M)}$ a finite partition $\mathcal{P}$ which satisfies the hypotheses of lemma 4.6 in the following way : We choose intervals $p_0 \ldots p_l$ of $\mathcal{Z}^{(M)}$ such that $\dfrac{P^M(1-\chi_{p_0\cup\cdots\cup p_l})}{P^M 1} < \dfrac{1}{2a}$.

(we can do it since $P^M 1 = \sum_{p\in\mathcal{Z}^{(M)}} P^M \chi_p \leq \sum_{p\in\mathcal{Z}^{(M)}} \left\|P^M \chi_p\right\|_\infty \leq S_m < \infty$, by lemma 4.1). The set $X - (p_0 \cup \cdots \cup p_l)$ consists of a finite union (at most $l+1$) of nonempty open intervals, let's call them $p_{l+1} \ldots p_L$. Then the partition $\mathcal{P} = \{p_0, \ldots, p_L\}$ fulfills the assumptions of the previous lemma.

We know that in both cases (finite and infinite) there exists a $p_h \in \mathcal{P}$ such that $h(y) \geq \frac{1}{2}\nu(h)$ for all $y \in p_h$.

By the covering hypothesis applied to $\mathcal{P}$ we can find an $N = N(\mathcal{P})$ and a constant $C(N, \mathcal{P})$ such that $P^N \chi_p > C$ for all $p \in \mathcal{P}$.

$$\begin{aligned} P^N h &\geq P^N(h\chi_{p_h}) \\ &\geq \frac{C}{2}\nu(h) \\ &\geq \frac{C}{2\sup P^N 1}\nu(P^N h) . \end{aligned}$$

Since $\nu(P^N h) \leq \sup P^N 1 \nu(h)$.

So we obtain

$$\Theta(1, P^N h) \leq \log\left(\frac{1+\sigma+\sigma a}{\min\left\{\frac{C}{2\sup P^N 1}, 1-\sigma\right\}}\right) .$$

The proof is concluded by setting

$$\Delta = 2\log\left(\frac{1+\sigma+\sigma a}{\min\left\{\frac{C}{2\sup P^N 1}, 1-\sigma\right\}}\right) < \infty.$$

◇

4.3 Conformal measure and density

Lemma 4.8 *There exist $h_* \in \mathcal{C}_{\sigma a}$ and $\lambda > 0$, with $Ph_* = \lambda h_*$, such that for all $f \in \mathcal{C}_a$,*

$$h_* = \lim_{n\to\infty} \frac{P^n f}{\nu(P^n f)}$$

$$\lambda = \lim_{n\to\infty} \frac{\nu(P^{n+1} f)}{\nu(P^n f)}.$$

Proof :

Obviously, the space $B(X)$ endowed with the norm $\|\cdot\|_\infty$ and $\rho = \nu : \mathcal{C}_a \to I\!R$ satisfies the hypothesis of lemma 2.2.

Let $f \in \mathcal{C}_a$. Then $\dfrac{P^n f}{\nu(P^n f)}$ is a Cauchy sequence in $B(X)$ because by lemma 2.2

$$\left\| \frac{P^n f}{\nu(P^n f)} - \frac{P^{n+k} f}{\nu(P^{n+k} f)} \right\|_\infty \leq \left(\mathrm{e}^{\Theta(P^n f, P^{n+k} f)} - 1 \right) \left\| \frac{P^n f}{\nu(P^n f)} \right\|_\infty .$$

If we write $n = (q+2)N + r$ where $q = [\frac{n}{N}] - 2$ and $r = n \bmod N$ then we see, using Birkhoff theorem 2.1, that

$$\begin{aligned} \Theta(P^n f, P^{n+k} f) &= \Theta((P^N)^q P^{2N+r} f, (P^N)^q P^{2N+r+k} f) \\ &\leq \Lambda_0^q \Theta(P^N P^{r+N} f, P^N P^{r+N+k} f) \\ &\leq \Lambda_0^q \Delta \end{aligned}$$

where $\Lambda_0 = \tanh(\Delta/4)$.

For $P^n f \in \mathcal{C}_{\sigma a}$, if n is big enough, $\|P^n f\|_\infty \leq \nu(P^n f) + \bigvee P^n f \leq (1 + \sigma a)\nu(P^n f)$.

If we set $\Lambda = \Lambda_0^{\frac{1}{N}} < 1$ and $K_0 = \Delta \Lambda_0^{-3}$ we obtain

$$\left\| \frac{P^n f}{\nu(P^n f)} - \frac{P^{n+k} f}{\nu(P^{n+k} f)} \right\|_\infty \leq \left(\mathrm{e}^{K_0 \Lambda^n} - 1 \right) (1 + \sigma a).$$

Which goes to zero when n goes to infinity.

As $B(X)$ is a Banach space, it exists a function $h_f \in B(X)$ with $\dfrac{P^n f}{\nu(P^n f)} \to h_f$. Clearly $h_f \in BV(X)$, $\nu(h_f) = 1$ and $h_f \in \mathcal{C}_{\sigma a}$. Moreover,

$$P(h_f) = \lim_{n\to\infty} \frac{P^{n+1} f}{\nu(P^n f)} = \lim_{n\to\infty} \frac{P^{n+1} f}{\nu(P^{n+1} f)} \frac{\nu(P^{n+1} f)}{\nu(P^n f)} = \lambda_f h_f.$$

We will show now that given $f, g \in C_a$ we have $h_f = h_g = h_*$

$$\begin{aligned} \|h_f - h_g\|_\infty &\leq \left(\mathrm{e}^{\Theta(h_f, h_g)} - 1 \right) \|h_f\|_\infty \\ &\leq \left(\mathrm{e}^{\Theta(P^n h_f, P^n h_g)} - 1 \right) \|h_f\|_\infty \end{aligned}$$

which goes to zero when n goes to infinity, this implies that $\lambda_f = \lambda_g = \lambda$.

◇

Lemma 4.9 *The functional ν (restricted to $BV(X)$) is linear, positive, $\nu(Pf) = \lambda \nu(f)$ for all $f \in BV(X)$ and $\lambda = \mathrm{e}^{p(\varphi)}$.*

Proof :

Let $f \in \mathcal{C}_a$. For all integer n, k

$$\begin{array}{ccccc} \frac{P^{n+k} f}{P^n f} & = & \frac{P^{n+k} f}{\nu(P^{n+k} f)} & \frac{\nu(P^{n+k} f)}{\nu(P^n f)} & \frac{\nu(P^n f)}{P^n f} \\ \downarrow & & \downarrow & \downarrow & \downarrow \\ \lim\limits_{n\to\infty} \dfrac{P^{n+k} f}{P^n f} & = & h_* & \lambda^k & h_*^{-1} \end{array}$$

so $\lim_{n\to\infty} \dfrac{P^{n+k}f}{P^n f} = \lambda^k$. *But*

$$\begin{aligned}\left\| \frac{P^n f}{P^n 1} - \frac{P^{n+k} f}{P^{n+k} 1}\right\|_\infty &\leq \sup \frac{P^{n+k} f}{P^{n+k}1}\left(\frac{P^n f}{P^{n+k} f}\frac{P^{n+k}1}{P^n 1} - 1\right) \\ &\leq \|f\|_\infty \left(\frac{P^n f}{P^{n+k} f}\frac{P^{n+k}1}{P^n 1} - 1\right)\end{aligned}$$

and since the sequences $\frac{P^{n+k}f}{P^n f}$ *and* $\frac{P^{n+k}1}{P^n 1}$ *have the same limit* λ^k, $\frac{P^n f}{P^n 1}$ *is a Cauchy sequence in* $B(X)$, *hence converges to a function* ν_f. *Moreover, if we take two point* $x, y \in X$, *we have*

$$\begin{aligned}|\nu_f(x) - \nu_f(y)| &= \lim_{n\to\infty} \left|\frac{P^n f}{P^n 1}(x) - \frac{P^n f}{P^n 1}(y)\right| \\ &= \lim_{n\to\infty} \left|\frac{P^n f}{P^n 1}(y)\right| \cdot \left|\frac{P^n f(x) P^n 1(y)}{P^n 1(x) P^n f(y)} - 1\right| \\ &\leq \|f\|_\infty \limsup_{n\to\infty} \left(\mathrm{e}^{\Theta_+(P^n f, P^n 1)} - 1\right) \\ &\leq \|f\|_\infty \lim_{n\to\infty} \left(\mathrm{e}^{\Theta(P^n f, P^n 1)} - 1\right) = 0.\end{aligned}$$

Therefore, $\nu_f(x) = \nu(f)$ *for all* $x \in X$. *Hence,* $\nu(f) = \lim \frac{P^n f}{P^n 1}$ *for all* $f \in \mathcal{C}_a$. *Nevertheless, if* $f \in BV(X)$, *the function* $(f + a^{-1}\bigvee_X f - \inf f) \in \mathcal{C}_a$, *so* $\nu(f) = \lim \frac{P^n f}{P^n 1}$ *for all* $f \in BV(X)$. *Clearly,* ν *is linear by the linearity of the limit.*
Next, as $Pf \in BV(X)$, *we know that*

$$\nu(Pf) = \lim_{n\to\infty} \frac{P^{n+1} f}{P^n 1} = \lim_{n\to\infty} \frac{P^{n+1} f}{P^{n+1} 1}\frac{P^{n+1}1}{P^n 1} = \nu(f)\nu(P1) = \lambda\nu(f)\ .$$

We now prove that the pressure of φ *is equal to* $\log\lambda$. *By lemma 4.8 we know that* $\dfrac{P^n 1}{\lambda^n} = \dfrac{P^n 1}{\nu(P^n 1)}$ *converges uniformly to the function* h_*. *In addition, for* $P^N h_* = \lambda^N h_*$, *with the same argument used in lemma 4.7 we deduce that* $\inf h_* > 0$. *Hence,*

$$\left\|\frac{1}{n}\log P^n 1 - \log\lambda\right\|_\infty \leq \frac{1}{n}\left\|\log\frac{P^n 1}{\lambda^n}\right\|_\infty$$

goes to zero when n *goes to infinity. This implies in particular* $p(\varphi) = \log\lambda$.
$\diamond$

Lemma 4.10 *The functional* ν *can be extended to a non-atomic conformal measure, i.e.*

$$\int Pf d\nu = \lambda \int f d\nu \quad \forall f \in L^1_\nu(\widetilde{X}).$$

In addition, the measure $\mu = h_*\nu$ *is* T*–invariant.*

Proof :

¿From now on $C(\widetilde{X})$ *will denote the space of continuous functions from* $\widetilde{X}$ *to* $\mathbb{R}$ *endowed with the uniform norm* $\|\cdot\|_\infty$. *We define, for* $h \in BV(\widetilde{X})$, $\nu(h) = \nu(h|_X)$ *(remark that* $h|_X \in BV(X)$*). Since* ν *is positive,* $|\nu(f)| \leq \|f\|_\infty$ *for all* $f \in BV(\widetilde{X})$. *By compactness of* $\widetilde{X}$, *we can approximate uniformly each continuous function by bounded variation functions. This allow us to extend* ν *to a positive functional on* $C(\widetilde{X})$. *By the Riesz theorem, there exists a Borel probability measure* m *on* $\widetilde{X}$ *which agrees with* ν *on* $C(\widetilde{X})$.
We first show that $m(I) = \nu(\chi_I)$ *for all interval* $I \subset \widetilde{X}$. *Let* I *be an interval such that* $\overline{I} = [a_1, a_2]$.
Let $\varepsilon > 0$. *For* $i = 1, 2$, *let* V_i *be an open interval which contains* a_i, *such that* $\nu(\chi_{V_i}) < \varepsilon/2$. *This can be achieved by the following : let* n *be such that* $\nu(Z) \leq \varepsilon/4$ *for all* $Z \in \mathcal{Z}^{(n)}$

(this is possible for the potential is contracting). Since $\sum_{Z\in\mathcal{Z}^{(n)}}\nu(\chi_Z)=1$ (for $S_n<\infty$), one can find a subset Q of $\mathcal{Z}^{(n)}$ such that[7] *$a_i\in int(\bigcup_{Z\in Q} Z)$ and $\sum_{Z\in Q}\nu(Z)<\varepsilon/2$.*

Let $f\in BV(\widetilde{X})\cap C(\widetilde{X})$ be a function[8] *such that $\chi_I\le f\le 1$ and $f=0$ outside $(V_1\cup I\cup V_2)$. We have*

$$m(I)\le\nu(f)\le\nu(\chi_I)+\nu(\chi_{V_1})+\nu(\chi_{V_2})\le\nu(\chi_I)+\varepsilon.$$

Since ε was arbitrary, $m(I)\le\nu(\chi_I)$ for all intervals I. Moreover, we have $\widetilde{X}=A\cup I\cup J$, where A,J,B are disjoint intervals.

Since $1=m(A)+m(I)+m(B)\le\nu(\chi_A)+\nu(\chi_I)+\nu(\chi_B)=1$, $m(I)=\nu(\chi_I)$ for all interval $I\subset\widetilde{X}$.

¿From this we deduce immediately that m has no atoms.

We show now that m and ν agree on $BV(\widetilde{X})$. Let $f\in BV(\widetilde{X})$. Let $\varepsilon>0$, since jumps of f can be bigger than ε at most on a finite set D_ε, we can approximate f by a piecewise constant function f_ε, where $\|f-f_\varepsilon\|_\infty<\varepsilon$. This implies

$$|m(f)-\nu(f)|\le|m(f-f_\varepsilon)|+|m(f_\varepsilon)-\nu(f_\varepsilon)|+|\nu(f-f_\varepsilon)|<2\varepsilon$$

because m and ν agree on piecewise constant functions.

¿From this we deduce that $m(Pf)=\nu(Pf)=\lambda\nu(f)=\lambda m(f)$ for all bounded variation function f. For simplicity, we consider ν as the measure m itself.

This yields

$$\int Pf d\nu=\lambda\int f d\nu\quad\forall f\in L^1_\nu(\widetilde{X}).$$

In other words, ν is a $\lambda e^{-\varphi}$–conformal measure[9]*.*

If we set $\mu=h_*\nu$, i.e.

$$\int f d\mu=\int f h_* d\nu\quad\forall f\in,$$

then

$$\int f\circ T d\mu=\int f\circ T h_* d\nu=\frac{1}{\lambda}\int P(f\circ T h_*)d\nu=\frac{1}{\lambda}\int fPh_* d\nu=\int f d\mu$$

hence μ is T–invariant.

◇

Remark 4.2 *One can prove that, for each function $f\in L^1_\nu(\widetilde{X})$, $\frac{P^n f}{P^n 1}$ is a Cauchy sequence in $L^1_\nu(\widetilde{X})$. Yet this does not imply that on $B(\widetilde{X})\cap L^1_\nu(\widetilde{X})$ the measure ν coincide with the original functional ν due to the presence of the* inf *in the former definition, which is meaningless for functions in L^1_ν.*

4.4 Decay of correlations

As we know that the Perron-Frobenius operator converges exponentially fast to the fixed point h_*, it is not difficult to deduce the exponential decay of correlations for bounded variation functions.

Lemma 4.11 *It exists $C>0$ such that for all $h\in BV(\widetilde{X})$*

$$\left|\int f\circ T^n h d\mu-\int f d\mu\int h d\mu\right|<C\|f\|_{L^1_\nu}\|h\|_{BV}\Lambda^n$$

Proof :

[7] To be completely precise this is the case only if $a_i\in X$, if not then the above relation must be interpreted as $\exists\alpha,\beta\in int(\bigcup_{Z\in Q} Z)$ such that $\alpha<a_i<\beta$ (where the ordering is the one in $\widetilde{X}$).

[8] Actually, following the proof of Urysohn's Lemma (using order structure rather than topological one, see [22]) one can show that if $a<b$, there exists a continuous increasing (hence BV) function from $[a,b]$ onto $[0,1]$.

[9] Consider the function $f=e^{-\varphi}\chi_A$ where $A\subset Z\in\mathcal{Z}$, and remark that $Pf=\chi_{TA}$.

Let $h \in BV(X)$. If we set $h_c = h + c$ then $h_c h_* \in \mathcal{C}_a$ with

$$c = \frac{1}{(1-\sigma)}\left[(1+\sigma)\|h\|_\infty + (a^{-1}+\sigma)\bigvee_X h\right].$$

Because

$$\begin{aligned}
\bigvee(h_c h_*) &\leq \bigvee h_c \|h_*\|_\infty + \bigvee h_* \|h_c\|_\infty \\
&\leq (1+\sigma a)\bigvee h + \sigma a\|h\|_\infty + \sigma a c = (c - \|h\|_\infty)a \\
&\leq a\mu(h+c) = a\nu(h_c h_*).
\end{aligned}$$

Next, if we assume that $\mu(h) = 0$, it suffices to show that $\mu(f \circ T^n h) \leq C\|f\|_\infty \|h\|_{BV}\Lambda^n$. In fact,

$$\begin{aligned}
\mu(f \circ T^n h) &= \nu(f \circ T^n h h_*) \\
&= \nu(f \lambda^{-n} P^n(h h_*)) \\
&= \nu(f(\lambda^{-n} P^n(h_c h_*) - c h_*)) \\
&\leq \|f\|_{L^1_\nu} \left\|\lambda^{-n} P^n(h_c h_*) - c h_*\right\|_\infty \\
&\leq \|f\|_{L^1_\nu} \|c h_*\|_\infty \left(\mathrm{e}^{\Theta(P^n h_c h_*, P^n h_*)} - 1\right).
\end{aligned}$$

The last inequality is given by lemma 2.2, since $\nu(\lambda^{-n} P^n(h_c h_*)) = c = \nu(c h_*)$. So

$$|\mu(f \circ T^n h)| \leq \|f\|_{L^1_\nu} c(1+\sigma a)(\mathrm{e}^{K_0 \Lambda^n} - 1)$$

setting $C_0 = \frac{2K_0(1+\sigma a)}{1-\sigma}$, we obtain

$$|\mu(f \circ T^n h)| \leq \|f\|_{L^1_\nu} C_0 \left[(1+\sigma)\|h\|_\infty + (a^{-1}+\sigma)\bigvee_X h\right]\Lambda^n \leq C_0(2+a^{-1})\|f\|_{L^1_\nu}\|h\|_{BV}\Lambda^n.$$

And if h is not of zero mean, we have $\mu(h - \mu(h)) = 0$ hence

$$\left|\int f \circ T^n h d\mu - \int f d\mu \int h d\mu\right| \leq C_0 \|f\|_{L^1_\nu} 2(2+a^{-1})\|h\|_{BV}\Lambda^n$$

which yields the result with $C = 2(2+a^{-1})C_0$. (We recall that if $h \in BV(\widetilde{X})$ then $h|_X \in BV(X)$).

$\diamond$

5 EQUILIBRIUM STATE (Proof of Theorem 3.2)

When the partition is countable, it is possible that both the entropy and the integral of the potential are infinite. That is why we give a variational principle in terms of conditional information, which avoids the problem of infinite entropy (the general strategy has been sketched in [15, 26, 13]).

Lemma 5.1 *The pressure of φ is equal to $p(\varphi) = \displaystyle\int (I_\mu[\mathcal{Z}|T^{-1}\mathcal{B}] + \varphi) d\mu$.*

Proof :

We first renormalizes the potential so that $P_\mu 1 = 1$ on X by setting

$$P_\mu f = \frac{P(f h_*)}{\lambda h_*}.$$

P_μ is the Transfer operator with the new weight

$$g_\mu = \exp(\varphi - p(\varphi))h_*/h_* \circ T = \sum_{Z\in\mathcal{Z}} \chi_Z E_\mu[\chi_Z|T^{-1}\mathcal{B}].$$

We recall that the conditional information of μ is given by $I_\mu[\mathcal{Z}|T^{-1}\mathcal{B}] = -\log g_\mu$.
For $p(\varphi) = -\log g_\mu + \varphi + \log h_* - \log h_* \circ T$, we get

$$p(\varphi) = \int (I_\mu[\mathcal{Z}|T^{-1}\mathcal{B}] + \varphi)d\mu.$$

$\diamond$

Lemma 5.2 *Variational principle: for all $m \in M_T(X)$*

$$p(\varphi) \geq \int (I_m[\mathcal{Z}|T^{-1}\mathcal{B}] + \varphi)dm$$

Proof :

Let $m \in M_T(X)$. We have

$$\begin{aligned} I_m[\mathcal{Z}|T^{-1}\mathcal{B}] + \varphi &= -\log g_m + \log g_\mu + p(\varphi) - \log h_* + \log h_* \circ T \\ &= \log \frac{g_\mu}{g_m} + \Phi \end{aligned}$$

Where Φ is a bounded function and $m(\Phi) = p(\varphi)$. We must prove that
(i) $\log_+ \dfrac{g_\mu}{g_m}$ is m-integrable and
(ii) $\displaystyle\int \log \frac{g_\mu}{g_m} \leq 0$.
Let us start by proving (i).
For all functions $f \in \mathcal{F}$, where $\mathcal{F}$ is the set of bounded functions with support in a finite number of intervals of $\mathcal{Z}$, we define

$$P_m f(x) = \sum_{y\in T^{-1}x} g_m(y)f(y) \quad \forall x \in X$$

as

$$E_m[\chi_Z|T^{-1}\mathcal{B}] = T_m^*(\chi_Z) \circ T$$

we have $P_m f = T_m^* f$, $m - a.e.$, for all functions $f \in \mathcal{F}$.
We remark that $g_m > 0$ $m - a.e.$, since $g_m = 0$ on $A \subset X$ implies

$$\sum_{Z\in\mathcal{Z}} \chi_Z T_m^*(\chi_Z) \circ T.\chi_A = 0$$

therefore

$$0 = \sum_{Z\in\mathcal{Z}} \int_{A\cap Z} T_m^*(\chi_Z) \circ T dm = \sum_{Z\in\mathcal{Z}} \int T_m^*(\chi_{A\cap Z}) T_m^*(\chi_Z) dm \geq \sum_{Z\in\mathcal{Z}} \int |T_m^*(\chi_{A\cap Z})|^2 dm$$

so $m(A \cap Z) = 0$ for all $Z \in \mathcal{Z}$.
For $\mathcal{Z}$ is countable, we can label the intervals $\mathcal{Z} = \{Z_i, i = 1, 2, ...\}$. If we set

$$F_n(x) = \begin{cases} 1 \text{ if } x \in Z_1 \cup \cdots \cup Z_n \text{ and } |\log \frac{g_\mu}{g_m}(x)| < n \\ 0 \text{ otherwise} \end{cases}$$

and $\chi_+ = \{x \in X | \log \frac{g_\mu}{g_m}(x) \geq 0\}$, *for* $\log g_m > -\infty$ $m - a.e.$ *and* $\sup g_\mu < \infty$, *we have*

$$\begin{aligned}
\int \chi_+ \log \frac{g_\mu}{g_m} dm &= \lim_{n\to\infty} \int F_n \chi_+ \log \frac{g_\mu}{g_m} dm \\
&= \lim_{n\to\infty} \int P_m \left(F_n \chi_+ \log \frac{g_\mu}{g_m} \right) dm \\
&\leq \lim_{n\to\infty} \int P_m \left(F_n \chi_+ (\frac{g_\mu}{g_m} - 1) \right) dm \\
&\leq \lim_{n\to\infty} \left\{ \int P_\mu (F_n \chi_+) dm - \int F_n \chi_+ dm \right\} \leq 1
\end{aligned}$$

since $P_m(f \frac{g_\mu}{g_m})(x) = P_\mu f(x)$ *for all* $f \in \mathcal{F}$.
Hence $\chi_+ \log \frac{g_\mu}{g_m} \in L^1_m$, *which proves (i).*
We now prove (ii).
If $\displaystyle\int \log \frac{g_\mu}{g_m} dm = -\infty$ *then (ii) is proven; otherwise* $\log \frac{g_\mu}{g_m} \in L^1_m$. *In this last case we can repeat the above computation without the characteristic function* χ_+, *and obtain*

$$\int \log \frac{g_\mu}{g_m} dm \leq 1 - \lim_{n\to\infty} \int F_n dm = 0$$

Moreover, the equality is true iff $g_\mu = g_m$ $m - a.e.$, *that is* $P_m f = P_\mu f$ $m - a.e.$ *for all* $f \in BV$. *So for* $P^n_\mu f \to \mu(f)$ *uniformly and* $m(f) = m(P^n_m f)$ *we obtain* $m(f) = \mu(f)$ *for all bounded variation function* f. *We conclude by density that* $m = \mu$.
$\diamond$

To conclude, we need to show that "reintroducing" the singular set W does not change the result of lemma 5.2.

Lemma 5.3 *Lemma 5.2 yields Theorem 3.2.*

Proof :

We already have proven the variational principle for invariant measures which do not give any mass to W. *We can write any given* $m \in M_T(\widetilde{X})$, *as the convex combination of two invariant probability measures* m_a *and* m_c, *where* m_a *is an atomic measure and* m_c *has no atoms (since* W *is countable, this implies* $m_c(W) = 0$*). We have* $m = cm_c + am_a$ *where* $a, c \geq 0$ *and* $a + c = 1$. *Since an atomic invariant measures has the following form (A is a set of periodic points* x_p, *of period* N_p*),*

$$m_a = \sum_{p\in A} \sigma_p \frac{1}{N_p} \sum_{i=1}^{N_p} \delta_{T^i x_p}\ , \quad \sigma_p > 0,\ \sum_{p\in A} \sigma_p = 1$$

we have $I_{m_a}[\mathcal{Z}|T^{-1}\mathcal{B}] = 0$ $m_a - a.e.$. *Since* m_c *and* m_a *are singular, we can choose* $I_{m_a} = 0$ $m_c - a.e.$ *and* $I_{m_c} = 0$ $m_a - a.e.$ *and obtain* $I_m = I_{m_a} + I_{m_c} = I_{m_c}$. *This yields*

$$\begin{aligned}
\int I_m + \varphi dm &= c \int I_{m_c} + \varphi dm + a \sum_{p\in A} \sigma_p \frac{1}{N_p} \sum_{i=1}^{N_p} \varphi(T^i x_p) \\
&\leq cp(\varphi) + a \sum_{p\in A} \sigma_p \frac{1}{N_p} \log g_{N_p}(x_p) \\
&\leq cp(\varphi) + a \sum_{p\in A} \sigma_p \frac{1}{nN_p} \log(\sup g_n)^{N_p}.
\end{aligned}$$

Since φ *is a contracting potential, there exists a real number* s, $0 < s < 1$ *such that* $\sup g_n \leq s^n \inf_X P^n 1$ *if* n *is big enough. Therefore, if* $a > 0$,

$$\int I_m + \varphi dm \leq cp(\varphi) + a \sum_{p\in A} \sigma_p \left(\log s + \frac{1}{n} \log \inf_X P^n 1 \right) < p(\varphi).$$

$\diamond$

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