METRICS AND PAIRS OF LEFT AND RIGHT CONNECTIONS ON BIMODULES

Ludwik Dąbrowski
SISSA, Via Beirut 2-4, Trieste, Italy.
E-MAIL: DABROW@SISSA.IT

Piotr M. Hajac*
ICTP, Strada Costiera 11, Trieste, Italy.
E-MAIL: PMH@ICTP.TRIESTE.IT

Giovanni Landi
Dipartimento di Scienze Matematiche, Universitá di Trieste, P.le Europa 1, Trieste, Italy
and INFN, Sezione di Napoli, Napoli, Italy.
E-MAIL: LANDI@UNIV.TRIESTE.IT

Pasquale Siniscalco
SISSA, Via Beirut 2-4, Trieste, Italy.
E-MAIL: SINIS@SISSA.IT

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Abstract

Properties of metrics and pairs consisting of left and right connections are studied on the bimodules of differential 1-forms. Those bimodules are obtained from the derivation based calculus of an algebra of matrix valued functions, and an $SL_q(2,\mathbb{C})$-covariant calculus of the quantum plane plane at a generic q and the cubic root of unity. It is shown that, in the aforementioned examples, giving up the middle-linearity of metrics significantly enlarges the space of metrics. A metric compatibility condition for the pairs of left and right connections is defined. Also, a compatibility condition between a left and right connection is discussed. Consequences entailed by reducing to the centre of a bimodule the domain of those conditions are investigated in detail. Alternative ways of relating left and right connections are considered.

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*Partially supported by a visiting fellowship at the International Centre for Theoretical Physics in Trieste and by the KBN grant 2 P301 020 07. On leave from: Department of Mathematical Methods in Physics, Warsaw University, ul. Hoża 74, Warsaw, 00–682 Poland. E-MAIL: PMH@FUW.EDU.PL

Introduction

Motivated to a great extent by the need to reconcile the geometric theory of gravity with the (noncommutative) operator algebraic theory of quantum physics, there is considerable interest in generalising the formalism of General Relativity to the realm of Noncommutative Differential Geometry [2]. In this paper, we study three concepts that are apparently needed for such a generalisation: metric, linear connection and metric compatibility condition.

We define a metric $g : E \times E \to A$ as a τ-symmetric A-bilinear pairing on an A-bimodule E, where τ is some generalised permutation. Then we argue that giving up the (often postulated) requirement that a metric factor to a map defined on $E \otimes_A E$ one can obtain an essentially bigger space of metrics. In particular, we provide an example with an ample supply of τ-symmetric metrics but where the requirement that a τ-symmetric metric g descend to $E \otimes_A E$ amounts to demanding that $g = 0$ (see Proposition 3.1).

Inspired by [4] on the one hand and by [3] on the other, we consider a pair of mutually compatible connections on a bimodule. First connection of such a compatible pair is a left connection in the sense that it satisfies the Leibniz rule with respect to the left module structure. Similarly, the second connection is a right connection in the sense that it fulfills the Leibniz rule on the right. The compatibility condition is simply a requirement that the left and the right connection agree on the centre of a bimodule up to a bimodule isomorphism σ (e.g., braiding). This bimodule isomorphism is, again, a generalised permutation. Restricting the domain of the aforementioned left-right compatibility condition to the centre of a bimodule permits, at least in the considered examples, a significantly bigger space of solutions to this condition. (This seems desired at least from the point of view of developing variational calculus on the space of connections.)

As to the compatibility between metrics and pairs of left and right connections, we define a 2-parameter family of compatibility conditions, but then restrict ourselves to the one that seems the most natural.

In the first section, we provide the general formalism and fix the notation. Then we proceed to the first example, where A is the algebra of matrix valued functions on a parallelizable manifold, and $E = A^1$ is the bimodule of 1-forms of the derivation based differential calculus equipped with the pullback-of-permutation automorphism. Next we pass on to the quantum plane and the differential calculus with the braiding employed in [4]. First we consider the case of a generic q, and then the case of the cubic root of unity. To obtain a non-zero τ-symmetric metric on the quantum plane we have to 'rescale' the braiding σ used in [4] by q^2. The thus obtained automorphism τ appears to be in a better agreement with the theory presented in [15]. (Even though in this case both automorphisms can be considered over the same domain, they are different deformations of the usual tensor product permutation.)

In what follows, Einstein's convention of summing over repeating indices is assumed, and the unadorned tensor product stands for the tensor product over a field.

1 General Definitions

Let A be a unital associative algebra over a field k, and E be a left and right projective A-bimodule. We begin with a definition of a linear pairing $g : E \times E \to A$, which, for the sake of simplicity (neglecting the nondegeneracy and reality conditions), we call a metric on E (cf. Section 8 and Section 9 in [7]).

Definition 1.1 *Let $\tau : E \otimes E \to E \otimes E$ be a bimodule automorphism. A linear map g from $E \otimes E$ to A is called a* τ-symmetric metric *on E (or simply* metric*, if no confusion arises) iff it is:*
1) bilinear over A, i.e. $g(a\zeta, \rho b) = ag(\zeta, \rho)b\,,\ \forall\, \zeta, \rho \in E,\ a, b \in A\,;$
2) τ-symmetric, i.e. $g \circ \tau = g$.

Note that if E is a central bimodule [7], that is, if the left and right multiplications by the elements of $Z(A)$ coincide on any element of E (which is always the case in the examples considered in this article), then any metric g can be regarded as a map from $E \otimes_{Z(A)} E$ to A. We would like to emphasize here that, contrary to many other papers (e.g., see (10) in [13]), we do not require g to be well-defined on $E \otimes_A E$. As we show in our three examples, a requirement like this (which goes under the name of middle-linearity) can be considered too restrictive (see Proposition 2.1, Proposition 3.1 and Proposition 4.2). Giving up the middle-linearity condition allows us to get rid of those restrictions.

Another structure on a bimodule E that we wish to discuss is a pair of compatible left and right connections.

Definition 1.2 *Let (A^1, d) be a first order differential calculus on A, and*

$$\sigma : E \otimes_A A^1 \longrightarrow A^1 \otimes_A E$$

be a bimodule isomorphism. Also, let ∇^L be a left connection, i.e. a linear map from E to $A^1 \otimes_A E$ satisfying the left Leibniz rule

$$\nabla^L(a\zeta) = da \otimes_A \zeta + a\nabla^L\zeta\,,\ \forall\, a \in A,\, \zeta \in E,$$

and let ∇^R be a right connection, i.e. a linear map from E to $E \otimes_A A^1$ fulfilling the right Leibniz rule

$$\nabla^R(\zeta a) = (\nabla^R\zeta)a + \zeta \otimes_A da\,,\ \forall\, a \in A,\, \zeta \in E.$$

A pair (∇^L, ∇^R) *is called* σ-compatible *iff*

$$\forall\, \zeta \in Z(E):\ \nabla^L \zeta = (\sigma \circ \nabla^R)\zeta\,, \tag{1.1}$$

where $Z(E) := \{\zeta \in E \,|\, a\zeta = \zeta a,\, \forall\, a \in A\}$ *is the centre of* E*.*

Let us recall that in Section 8 of [3] a connection on a bimodule is also defined as a pair consisting of a left and right connection. There, however, instead of σ-compatibility condition (1.1), the condition of ∇^L being a right A-homomorphism and ∇^R being a left A-homomorphism is imposed. The latter condition, albeit it permits for an interesting algebraic theory, cannot be directly transferred to the commutative case $Z(E) = E$. On the other hand, defining a connection on a bimodule by requiring, much as in Definition 3.2 in [8], that the σ-compatibility condition is fulfilled on the whole bimodule E rather than just its centre $Z(E)$, automatically yields, for the appropriate σ (i.e. the usual tensor product flip), the standard definition of a connection in the classical case, but entails essential restrictions on the space of connections in noncommutative examples (see (2.10), (3.19), (4.32), [9] and Theorem 5.6 in [8]; cf. Lemma 1 in [7]). If we demand that the equality $\nabla^L = \sigma \circ \nabla^R$ be satisfied on the whole bimodule E, we can equivalently think of a pair (∇^L, ∇^R) as a left connection ∇^L fulfilling an additional (right) Leibniz rule of the form

$$\nabla^L(\zeta a) = (\nabla^L \zeta)a + \sigma(\zeta \otimes_A da)\,,\ \forall\, a \in A,\, \zeta \in E,$$

(cf. the Introduction in [10]). In the classical differential geometry, with the help of the tensor product flip, any left connection uniquely determines the corresponding right connection, and vice-versa, without imposing any limitations on either of the connections. As we demonstrate in the next section, this is precisely what happens with pairs of σ-compatible connections in the noncommutative example of a 'matrix geometry' (see Proposition 2.2). A very similar result (Proposition 4.3) is obtained for a bimodule of differential 1-forms on the quantum plane (see Proposition 2 in [12]) at the cubic root of unity. (In fact, in all examples presented in this paper, we put $E = A^1$, so that the pairs of σ-compatible connections studied here can be thought of as linear connections.) When dealing with connections and metrics, it seems that in both cases we have the same mechanism at work: solving constraints over just the commutative part allows solutions to be parametrized by a whole noncommutative algebra, whereas solving them over an entire noncommutative space renders the solutions parametrized by the centre of an algebra. The examples of subsequent sections allow us to explore this mechanism in some detail.

Let us now consider possible compatibility conditions between a pair of connections (∇^L, ∇^R) and a metric g. In order to do so, first we must define two extensions of g :

$$\check{g} : A^1 \otimes_A E \otimes E \longrightarrow A^1 \ \text{ and } \ \hat{g} : E \otimes E \otimes_A A^1 \longrightarrow A^1\,.$$

It appears natural to choose:

$$\check{g}(\alpha \otimes_A \zeta, \rho) = \alpha g(\zeta, \rho)\,, \qquad \forall \alpha \in A^1,\ \zeta, \rho \in E; \tag{1.2}$$

$$\hat{g}(\zeta, \rho \otimes_A \alpha) = g(\zeta, \rho)\alpha\,, \qquad \forall \alpha \in A^1,\ \zeta, \rho \in E. \tag{1.3}$$

In principle, one can formulate the class of metric compatibility conditions by requiring the diagram

$$\begin{array}{ccccc} E \otimes E & \xrightarrow{f_L(t)\otimes id \oplus id \otimes f_R(s)} & & & (A^1 \otimes_A E \otimes E) \oplus (E \otimes E \otimes_A A^1) \\ \Big\downarrow g & & & & \Big\downarrow \check{g} \oplus \hat{g} \\ A & \xrightarrow{\quad d \quad} & A^1 & \xleftarrow{\quad + \quad} & A^1 \oplus A^1, \end{array} \tag{1.4}$$

where $f_L(t) := (1-t)\nabla^L + t(\sigma \circ \nabla^R)$, $f_R(s) := s(\sigma^{-1} \circ \nabla^L) + (1-s)\nabla^R$, $t, s \in k$, to commute. Here, however, in order to ensure that $f_L(t)$ and $f_R(s)$ are connections, we settle for the particular case $t = 0 = s$:

Definition 1.3 *We say that a pair of connections* (∇^L, ∇^R) is compatible with g *iff*

$$dg(\zeta, \rho) = \check{g}(\nabla^L \zeta, \rho) + \hat{g}(\zeta, \nabla^R \rho), \quad \forall\, \zeta, \rho \in E. \tag{1.5}$$

As we show in Proposition 4.6, formula (1.5) is not always sensitive to whether we consider it over E or only over $Z(E)$. Observe that, if (1.1) is satisfied, then on the centre of a bimodule we have $f_L(t) = \nabla^L$ and $f_R(s) = \nabla^R$ for any values of $t, s \in k$, and, consequently, all metric compatibility conditions for a pair of σ-compatible connections are equivalent when considered over $Z(E)$. Finally, let us remark that the metric compatibility condition for a pair of connections related by $\nabla^L = \sigma \circ \nabla^R$ (on the whole bimodule) that is given by (1.21) and (1.26) in [4] seems inappropriate for the cases when the metric is not middle-linear.

Next, we apply the above definitions in some quantum geometric models.

2 Algebras of Matrix Valued Functions

Let us choose $A = C^\infty(M) \otimes M_n(\mathbb{C}),\ E = A^1 = \mathrm{Hom}_{Z(A)}(\mathrm{Der}A, A)$,

$$\tau(\alpha \otimes \beta)(X, Y) = (\alpha \otimes \beta)(Y, X),\ \forall\, X, Y \in \mathrm{Der}A,$$

and σ equal to τ factored to an automorphism of $A^1 \otimes_A A^1$. Here M is a parallelizable manifold of dimension m, and the ground field of A is the field of complex numbers. Now, let $\{\theta^i\}_{i \in \{1,\ldots,m+n^2-1\}}$ be the basis of A^1 as defined in Section 3 of [10]. An important property of this basis is that

$$a\theta^i = \theta^i a, \quad \forall\, a \in A,\ i \in \{1, ..., m + n^2 - 1\}\,, \tag{2.6}$$

$$\tau(\theta^i \otimes \theta^j) = \theta^j \otimes \theta^i, \quad \forall\, a \in A,\ i, j \in \{1, ..., m + n^2 - 1\}\,. \tag{2.7}$$

In this setting, one can immediately verify the following claim (cf. Section 9 in [7] and p.5861 in [10]):

Proposition 2.1 *Let g^{ij} denote $g(\theta^i \otimes \theta^j)$, where $i,j \in \{1, ..., m+n^2-1\}$. The map $\psi : g \mapsto (g^{ij})$ provides a one-to-one correspondence between the metrics (the middle-linear metrics) on A^1 and the symmetric matrices of $M_{m+n^2-1}(A)$ (the symmetric matrices of $M_{m+n^2-1}(Z(A))$ respectively).*

In the same basis, let us define the Christoffel symbols of ∇^L and ∇^R by

$$\begin{aligned} \nabla^L \theta^i &= \theta^j \otimes_A \theta^k \Gamma^i_{jk} \,, \\ \nabla^R \theta^i &= \theta^j \otimes_A \theta^k \widetilde{\Gamma}^i_{jk} \,. \end{aligned} \tag{2.8}$$

Taking into account (2.6) and (2.7) and noticing that $\{\theta^i\}_{i\in\{1,...,m+n^2-1\}}$ is also a basis of the $Z(A)$-module $Z(A^1)$, it is straightforward to prove:

Proposition 2.2 *A pair of connections (∇^L, ∇^R) is σ-compatible if and only if its Christoffel symbols satisfy the equation*

$$\widetilde{\Gamma}^i_{kj} = \Gamma^i_{jk} \,. \tag{2.9}$$

Similarly, a pair of connections is σ-compatible on the whole bimodule if and only if ((3.9) in [10])

$$\widetilde{\Gamma}^i_{kj} = \Gamma^i_{jk} \in C^\infty(M) \,. \tag{2.10}$$

Thus the σ-compatibility condition (over $Z(E)$) allows ∇^L to uniquely determine ∇^R and vice-versa. This is not unexpected since (in this set-up)

$$A^1 = AZ(A^1) = Z(A^1)A \tag{2.11}$$

and ∇^L and ∇^R satisfy the left and right Leibniz rule respectively. On the other hand, as we shall see in Section 4 (Remark 4.5) ∇^L and ∇^R can mutually determine each other even if (2.11) is not satisfied.

Concerning the metric compatibility of (∇^L, ∇^R), we can again take an advantage of (2.6) to show that (1.5) is equivalent to

$$dg^{ij} = (\Gamma^i_{kl} g^{lj} + g^{il} \widetilde{\Gamma}^j_{lk}) \theta^k \,, \quad \forall\, i,j \in \{1, ..., m+n^2-1\} \,. \tag{2.12}$$

To end this section, let us observe that, if (∇^L, ∇^R) is σ-compatible, then (2.12) coincides with (3.13) in [10]. One should bear in mind, however, that the latter has been obtained from a different starting point ((1.9) in [10]) and only for middle-linear metrics.

3 Generic Quantum Plane

The next example that we study regards a bimodule of differential 1-forms ($E = A^1$) on the quantum plane. We choose as our space of differential 1-forms the grade one of the differential algebra $\Omega(A) = A \oplus A^1 \oplus A^2$ (e.g., see [4]) that is given by the generators $1, x, y, \xi, \eta$, where $\xi = \theta^1 = dx$, $\eta = \theta^2 = dy$, and relations

$$
\begin{aligned}
&xy = qyx \ , \\
&x\xi = q^2\xi x \ , \quad x\eta = q\eta x + (q^2-1)\xi y \ , \quad y\xi = q\xi y \ , \quad y\eta = q^2\eta y \ , \\
&\eta\xi + q\xi\eta = 0 \ , \quad \xi^2 = 0 \ , \quad \eta^2 = 0 \ .
\end{aligned}
\tag{3.13}
$$

For our bimodule automorphism σ it is natural (see the paragraph between (2.11) and (2.12) in [4]) to take the map defined by

$$
\begin{array}{ll}
\sigma(\xi \otimes_A \xi) = q^{-2}\xi \otimes_A \xi \ , & \sigma(\xi \otimes_A \eta) = q^{-1}\eta \otimes_A \xi \ , \\
\sigma(\eta \otimes_A \xi) = q^{-1}\xi \otimes_A \eta - (1-q^{-2})\eta \otimes_A \xi \ , & \sigma(\eta \otimes_A \eta) = q^{-2}\eta \otimes_A \eta \ .
\end{array}
\tag{3.14}
$$

First we consider the case of a generic q. Then the centre of A is $\mathbb{C}$ and, as can be checked by a direct computation, the centre of A^1 is zero. If we choose τ equal to σ (modulo the tensor product over A, as was done in the previous section), we can immediately see that, unless $q = 1$, there exists no non-zero metric [4]. To remedy this problem, we 'rescale' σ by q^2. More precisely, we put

$$
\begin{array}{ll}
\tau(\xi \otimes \xi) = \xi \otimes \xi \ , & \tau(\xi \otimes \eta) = q\eta \otimes \xi \ , \\
\tau(\eta \otimes \xi) = q\xi \otimes \eta - (q^2-1)\eta \otimes \xi \ , & \tau(\eta \otimes \eta) = \eta \otimes \eta \ .
\end{array}
\tag{3.15}
$$

It turns out that this τ is quite natural from the point of view of [15] — it factors to an automorphism of $A^1 \otimes_A A^1$ and preserves $\theta \otimes_A \theta$, where $\theta = x\eta - qy\xi$ is the only (up to a multiplication by a complex number) left and right $SL_q(2, \mathbb{C})$-coinvariant 1-form (see Section 2 in [4]). (Recall that if A is a Hopf algebra, then there exists a unique bimodule homomorphism τ such that $\tau(\alpha_L \otimes_A \alpha_R) = \alpha_R \otimes_A \alpha_L$ for any left coinvariant 1-form α_L and right coinvariant 1-form α_R — see Proposition 3.1 in [15].)

It is obvious that with τ specified as above, the only constraints that the coefficients of a metric g have to satisfy is

$$
g(\xi, \eta) = qg(\eta, \xi) \ . \tag{3.16}
$$

On the other hand, it can be computed that the middle-linearity of g is equivalent to the following equations:

$$
\begin{aligned}
xg(\xi,\xi) &= q^4 g(\xi,\xi)x \ , \\
yg(\xi,\xi) &= q^2 g(\xi,\xi)y \ , \\
xg(\xi,\eta) &= q^3 g(\xi,\eta)x + q^2(q^2-1)g(\xi,\xi)y \ , \\
yg(\xi,\eta) &= q^3 g(\xi,\eta)y \ , \\
xg(\eta,\xi) &= q^3 g(\eta,\xi)x + q(q^2-1)g(\xi,\xi)y \ , \\
yg(\eta,\xi) &= q^3 g(\eta,\xi)y \ , \\
xg(\eta,\eta) &= q^2 g(\eta,\eta)x + q^2(q^2-1)g(\xi,\eta)y + q(q^2-1)g(\eta,\xi)y \ , \\
yg(\eta,\eta) &= q^4 g(\eta,\eta)y \ .
\end{aligned}
\tag{3.17}
$$

Notice that, due to the commutation relation $xy = qyx$, as long as q is generic and no negative powers of x and y are allowed, there is no non-zero solution to (3.17) (look at the equations with y). Consequently, we obtain:

Proposition 3.1 *If q is not a root of unity, there is no (τ-symmetric) middle-linear metric on A^1.*

Remark 3.2 If we admit negative powers of x and y, the solutions of (3.17) form the following three-parameter family:

$$\begin{aligned} g(\xi,\xi) &= ax^{-2}y^4 \ , \\ g(\xi,\eta) &= qx^{-3}\left(by^3 + q^3ay^5\right) \ , \\ g(\eta,\xi) &= x^{-3}\left(by^3 + q^3ay^5\right) \ , \\ g(\eta,\eta) &= x^{-4}\left(cy^2 + q^3(q^2+1)by^4 + q^8ay^6\right) \ , \end{aligned} \tag{3.18}$$

where a, b, c are complex parameters. $\diamond$

Since the centre of A^1 is zero, we can immediately conclude that

Proposition 3.3 *If q is not a root of unity, any pair (∇^L, ∇^R) of σ-compatible connections on A^1 is a pair of independent and unrestricted left and right connections.*

On the other hand, there exists only a one-parameter family of solutions of the σ-compatibility condition considered over the whole A^1 (see (2.13) in [4]). Defining Christoffel symbols $\{F^i_{jk},\, \widetilde{F}^i_{jk}\}_{i,j,k\in\{1,2\}}$ of such a compatible pair of connections as in (2.8), we can write the aforementioned solutions in the following way:

$$\begin{aligned} &F^1_{11} = \nu qxy^2 \ , \quad F^1_{12} = -\nu q^3x^2y \ , \quad F^1_{21} = -\nu q^2x^2y \ , \quad F^1_{22} = \nu q^5x^3 \ , \\ &F^2_{11} = \nu q^3y^3 \ , \quad F^2_{12} = -\nu q^4xy^2 \ , \quad F^2_{21} = -\nu q^3xy^2 \ , \quad F^2_{22} = \nu q^5x^2y \ , \end{aligned} \tag{3.19}$$

where ν is a complex parameter. The Christoffel symbols $\{\widetilde{F}^i_{jk}\}_{i,j,k\in\{1,2\}}$ can be expressed in a similar fashion.

As to the metric compatibility condition, formula (1.5) reads

$$a_i\big(dg(\theta^i,\theta^j) - \theta^k g(\theta^l\Gamma^i_{kl},\theta^j) - g(\theta^i,\theta^k)\theta^l\widetilde{\Gamma}^j_{kl}\big)\tilde{a}_j = 0, \quad \forall\, a_i, \tilde{a}_j \in A,\ i,j\in\{1,2\}. \tag{3.20}$$

Clearly, (3.20) is satisfied if and only if the expression in the large parentheses vanishes for any i and j.

4 Quantum Plane at the Cubic Root of Unity

The setting of this section is identical with the setting of the previous one except that now we take $q = e^{\frac{2\pi i}{3}}$ rather than generic q. (For the sake of simplicity, we call $e^{\frac{2\pi i}{3}}$ *the* cubic root of unity.) Long but rather straightforward reasoning enables one to prove the following lemma:

Lemma 4.1 *Let q be the cubic root of unity. Then*

$$\begin{aligned} & Z(A) = \{a_{ij}x^{3i}y^{3j} \mid a_{ij} \in \mathbb{C}\}, && (4.21)\\ & dZ(A) = 0, && (4.22)\\ & \forall\, a \in Z(A),\, \alpha \in A^1 :\ a\alpha = \alpha a, && (4.23)\\ & Z(A^1) = \{c_i\theta^i \in A^1 \mid c_1 = axy - bxy^3,\, c_2 = bx^2y^2,\, a, b \in Z(A)\}, && (4.24)\\ & c_i\theta^i = \theta^j\tilde{c}_j,\ \textit{where}\ \ c_1 = axy - bxy^3,\, c_2 = bx^2y^2, && \\ & \tilde{c}_1 = axy - qbxy^3,\, \tilde{c}_2 = c_2,\, a, b \in Z(A)\ \textit{(see (A.38))}. && (4.25) \end{aligned}$$

Changing q from generic to $q = e^{\frac{2\pi i}{3}}$ entails no consequence as far as the (general τ-symmetric) metric is concerned. However, regarding middle-linear metrics, with the help of commutation formulas provided in the appendix, one can prove:

Proposition 4.2 *If q is the cubic root of unity and g is middle-linear (but not necessarily τ-symmetric), then (3.17) is equivalent to:*

$$\begin{aligned} g(\xi,\xi) &= x^3Zxy\ , \\ g(\xi,\eta) &= qx^3Zy^2 + x^3Y\ , \\ g(\eta,\xi) &= x^3Zy^2 + x^3W\ , \\ g(\eta,\eta) &= Ux^2y^2 + (qY+W)x^2y + q^2Zx^2y^3\ , \end{aligned} \qquad (4.26)$$

where Z, Y, W, U are arbitrary elements of $Z(A)$. Furthermore, g is τ-symmetric if and only if $Y = qW$.

Thus, much as in Proposition 2.1, the space of middle linear metrics is three-dimensional over $Z(A)$. This is not unexpected, if one remembers that the quantum plane at the n-th root of unity is nothing but $\mathbb{C}[x, y] \otimes M_n(\mathbb{C})$ (cf. Section IV.D.15 of [14]).

Our next step is to determine the space of pairs of σ-compatible connections. In order to make our reasoning more transparent, we introduce *formal* inverses of x and y. (It is simply more convenient, for instance, to write $\tilde{\Gamma} = xy\Gamma x^{-1}y^{-1}$ as the solution of the equation $\tilde{\Gamma}xy = xy\Gamma$ rather than consider $\tilde{\Gamma}$ and Γ as power series in x and y and then express the complex coefficients of $\tilde{\Gamma}$ in terms of the complex coefficients of Γ. However, we neither need nor assume the existence of x^{-1} and y^{-1} in our algebra.) Treating $\{\theta^i \otimes_A \theta^j\}_{i,j\in\{1,2\}}$ as a basis of the right A-module

$A^1 \otimes_A A^1$ and taking advantage of Lemma 4.1, one can carry out lengthy but straightforward calculations that show that the σ-compatibility condition (1.1) is equivalent to:

$$\begin{aligned}
\widetilde{\Gamma}^1_{11} &= q^2xy\Gamma^1_{11}y^{-1}x^{-1} + (1-q)y^2\Gamma^1_{12}y^{-1}x^{-1} + (q^2-1)y^2\Gamma^1_{21}y^{-1}x^{-1} \ , \\
\widetilde{\Gamma}^1_{12} &= (q^2-1)xy\Gamma^1_{12}y^{-1}x^{-1} + qxy\Gamma^1_{21}y^{-1}x^{-1} + (q-q^2)y^2\Gamma^1_{22}y^{-1}x^{-1} \ , \\
\widetilde{\Gamma}^1_{21} &= qxy\Gamma^1_{12}y^{-1}x^{-1} + (1-q)y^2\Gamma^1_{22}y^{-1}x^{-1} \ , \\
\widetilde{\Gamma}^1_{22} &= q^2xy\Gamma^1_{22}y^{-1}x^{-1} \ , \\
\widetilde{\Gamma}^2_{11} &= xy\Gamma^1_{11}x^{-2} - x\Gamma^1_{11}yx^{-2} + (q-1)y\Gamma^1_{12}yx^{-2} + (q-q^2)y^2\Gamma^1_{12}x^{-2} \\
&\quad +(1-q^2)y\Gamma^1_{21}yx^{-2} + (1-q)y^2\Gamma^1_{21}x^{-2} + q^2x^2y^2\Gamma^2_{11}y^{-2}x^{-2} \\
&\quad +(q-1)x\Gamma^2_{12}yx^{-2} + (1-q^2)x\Gamma^2_{21}yx^{-2} + 3y\Gamma^2_{22}yx^{-2} \ , \\
\widetilde{\Gamma}^2_{12} &= (1-q^2)x\Gamma^1_{12}yx^{-2} + (q^2-1)y^2\Gamma^1_{22}x^{-2} + (1-q)xy\Gamma^1_{12}x^{-2} \\
&\quad +q^2xy\Gamma^1_{21}x^{-2} - qx\Gamma^1_{21}yx^{-2} + (q-1)y\Gamma^1_{22}yx^{-2} \\
&\quad +(q^2-1)x^2y^2\Gamma^2_{12}y^{-2}x^{-2} + qx^2y^2\Gamma^2_{21}y^{-2}x^{-2} + (q^2-q)x\Gamma^2_{22}yx^{-2} \ , \\
\widetilde{\Gamma}^2_{21} &= q^2xy\Gamma^1_{12}x^{-2} - qx\Gamma^1_{12}yx^{-2} + (1-q^2)y\Gamma^1_{22}yx^{-2} \\
&\quad +(q-q^2)y^2\Gamma^1_{22}x^{-2} + qx^2y^2\Gamma^2_{12}y^{-2}x^{-2} + (q-1)x\Gamma^2_{22}yx^{-2} \ , \\
\widetilde{\Gamma}^2_{22} &= xy\Gamma^1_{22}x^{-2} - qx\Gamma^1_{22}yx^{-2} + q^2x^2y^2\Gamma^2_{22}y^{-2}x^{-2} \ ,
\end{aligned} \tag{4.27}$$

where the Christoffel symbols are defined as in (2.8). The above system of equations allows one to determine uniquely ∇^R through ∇^L, but, as can be seen from the powers of x and y, it cannot be done for an arbitrary left connection ∇^L. (The total power of x and the total power of y in each term of the right hand side of (4.27) have to be non-negative in order for (4.27) to make sense.) Since only total powers of x turn negative in (4.27), it is convenient to think of an element of A as a polynomial in x with coefficients in polynomials in y:

$$\Gamma^i_{jk} = x^l\Gamma^i_{jkl} \, , \quad i,j,k \in \{1,2\}. \tag{4.28}$$

To determine the necessary and sufficient conditions that $\{\Gamma^i_{jk}\}_{i,j,k\in\{1,2\}}$ have to satisfy in order to make (4.27) well-defined on the quantum plane, we substitute (4.28) to (4.27) and conclude that the necessary and sufficient conditions for $\{\Gamma^i_{jk}\}_{i,j,k\in\{1,2\}}$ are fully given by:

$$\begin{aligned}
&(1-q)y^2\Gamma^1_{120} + (q^2-1)y^2\Gamma^1_{210} = 0 \ , \\
&(q-q^2)y^2\Gamma^1_{220} = 0 \ , \\
&(1-q)y^2\Gamma^1_{220} = 0 \ ,
\end{aligned}$$

$$
\begin{aligned}
& xy\Gamma^1_{110} - x\Gamma^1_{110}y + (q-1)y\Gamma^1_{120}y + (q-1)yx\Gamma^1_{121}y + (q-q^2)y^2\Gamma^1_{120} \\
& +(q-q^2)y^2x\Gamma^1_{121} + (1-q^2)y\Gamma^1_{210}y + (1-q^2)yx\Gamma^1_{211}y + (1-q)y^2\Gamma^1_{210} \\
& +(1-q)y^2x\Gamma^1_{211} + (q-1)x\Gamma^2_{120}y + (1-q^2)x\Gamma^2_{210}y + 3y\Gamma^2_{220}y + 3yx\Gamma^2_{221}y = 0 \ , \\
& (1-q^2)x\Gamma^1_{120}y + (q^2-1)y^2\Gamma^1_{220} + (q^2-1)y^2x\Gamma^1_{221} + (1-q)xy\Gamma^1_{120} + q^2xy\Gamma^1_{210} \\
& -qx\Gamma^1_{210}y + (q-1)y\Gamma^1_{220}y + (q-1)yx\Gamma^1_{221}y + (q^2-q)x\Gamma^2_{220}y = 0 \ , \\
& q^2xy\Gamma^1_{120} - qx\Gamma^1_{120}y + (1-q^2)y\Gamma^1_{220}y + (1-q^2)yx\Gamma^1_{221}y \\
& +(q-q^2)y^2\Gamma^1_{220} + (q-q^2)y^2x\Gamma^1_{221} + (q-1)x\Gamma^2_{220}y = 0 \ , \\
& xy\Gamma^1_{220} - qx\Gamma^1_{220}y = 0 \ . \qquad (4.29)
\end{aligned}
$$

Those equations are equivalent to:

$$
\begin{aligned}
& \Gamma^1_{210} = \Gamma^1_{120} = \Gamma^1_{220} = \Gamma^1_{221} = \Gamma^2_{220} = 0 \\
& (q-1)\Gamma^2_{120} + (1-q^2)\Gamma^2_{210} + 3q^2y\Gamma^2_{221} = 0 \qquad (4.30)
\end{aligned}
$$

Thus we have obtained:

Proposition 4.3 *The σ-compatibility condition (1.1) has a solution if and only if*

$$
\begin{array}{llll}
\Gamma^1_{11} \in A \ , & \Gamma^1_{12} \in xA \ , & \Gamma^1_{21} \in xA \ , & \Gamma^1_{22} \in x^2A \ , \\
\Gamma^2_{11} \in A \ , & (q-1)\Gamma^2_{12} + (1-q^2)\Gamma^2_{21} + 3q^2yx^{-1}\Gamma^2_{22} \in xA \ , & \Gamma^2_{22} \in xA \ . & \qquad (4.31)
\end{array}
$$

Moreover, if (4.31) is satisfied, then the general solution of (1.1) is given by (4.27).

Remark 4.4 Expressing $\{\Gamma^i_{jk}\}_{i,j,k\in\{1,2\}}$ in terms of $\{\widetilde{\Gamma}^i_{jk}\}_{i,j,k\in\{1,2\}}$ would yield conditions for $\{\widetilde{\Gamma}^i_{jk}\}_{i,j,k\in\{1,2\}}$ similar to these in Proposition 4.3. Only this time y would play the role of x. Let us also observe that, if we replaced σ by $q^2\sigma$ in (1.1), then (1.1) would have no solutions whatsoever as long as the negative powers of x and y are disallowed. $\diamond$

Remark 4.5 As in Section 2, equations (4.27) uniquely determine ∇^L from ∇^R, and vice-versa. Here, however, it happens despite the fact that formula (2.11) is not fulfilled. (It follows from (4.24) that $AZ(A^1) \subseteq xyA^1$.) $\diamond$

Notice that, had we required the σ-compatibility condition to be satisfied on the whole bimodule A^1, then, in contrast with Proposition 4.3, we would obtain that the Christoffel symbols of a left connection have to fulfill the following equations:

$$
\begin{aligned}
F^1_{11} &= x\left(y^3(-qf^1_{12} + f^1_{21}) + yf^1_{11} - qy^2f^1_{22}\right) \ , \\
F^1_{12} &= x^2(yf^1_{22} + y^2f^1_{12}) \ ,
\end{aligned}
$$

$$
\begin{aligned}
F^1_{21} &= x^2(q^2 y f^1_{22} + y^2 f^1_{21})\ ,\\
F^1_{22} &= -q^2 x^3 f^1_{22}\ ,\\
F^2_{11} &= f^2_{11} + qy^4(f^2_{22} - f^1_{12} - \tfrac{3q}{1-q} f^1_{21}) + qy^2(f^1_{11} - f^2_{21} - qf^2_{12})\ ,\\
F^2_{12} &= x\left(q^2y^3(-f^2_{22} + f^1_{12}) + yf^2_{12} + qy^2 f^1_{22}\right)\\
F^2_{21} &= x\left(qy^3(-f^2_{22} + qf^1_{21}) + yf^2_{21} + y^2 f^1_{22}\right)\ ,\\
F^2_{22} &= x^2(-q^2 y f^1_{22} + y^2 f^2_{22})\ ,
\end{aligned}
\tag{4.32}
$$

where $f^i_{jk} \in Z(A)$, $i, j, k \in \{1, 2\}$.

The metric compatibility condition (1.5) can again be written in the form of formula (3.20). This time, however, the centre of A^1 is non-trivial and it makes sense to ask what would happen if we imposed the metric compatibility condition only over $Z(A^1)$. It turns out that we have:

Proposition 4.6 *Requiring that the metric compatibility condition (1.5) be satisfied only over $Z(A^1)$ is equivalent to demanding that it be fulfilled over the whole A^1.*

Proof. Let $\mathbf{P}^{ij}$ denote the expression in the large parentheses in (3.20). Furthermore, let us put $\mathbf{P}^{ij} = \theta^k \mathbf{P}^{ij}_k$, $a_1 = axy - bxy^3$, $a_2 = bx^2y^2$, $\tilde{a}_1 = a'xy - qb'xy^3$, $\tilde{a}_2 = b'x^2y^2$, where a, b, a', b' are arbitrary elements of $Z(A)$. Thanks to (4.24), substituting those terms to (3.20) yields an equation that expresses the metric compatibility condition over $Z(A^1)$:

$$
\begin{aligned}
&(axy - bxy^3)\theta^k \mathbf{P}^{11}_k (a'xy - qb'xy^3) + (axy - bxy^3)\theta^k \mathbf{P}^{12}_k b'x^2y^2\\
&+bx^2y^2\theta^k \mathbf{P}^{21}_k (a'xy - qb'xy^3) + bx^2y^2\theta^k \mathbf{P}^{21}_k b'x^2y^2 = 0
\end{aligned}
\tag{4.33}
$$

Commuting everything to the right of $\{\theta^i\}_{i\in\{1,2\}}$ and taking advantage of the fact that $\{\theta^i\}_{i\in\{1,2\}}$ is a basis, one can reduce (4.33) to:

$$
\begin{aligned}
&\Big(\big(xy\mathbf{P}^{11}_1 + (q-q^2)y^2\mathbf{P}^{11}_2\big)a\\
&+\big(-q^2x\mathbf{P}^{11}_1 y^3 - (q^2-1)y\mathbf{P}^{11}_2 y^3 + x^2y^2\mathbf{P}^{21}_1 + (q^2-q)xy^3\mathbf{P}^{21}_2\big)b\Big)(a'xy - qb'xy^3)\\
&+\Big(\big(xy\mathbf{P}^{12}_1 + (q-q^2)y^2\mathbf{P}^{12}_2\big)a\\
&+\big(-q^2x\mathbf{P}^{12}_1 y^3 - (q^2-1)y\mathbf{P}^{12}_2 y^3 + x^2y^2\mathbf{P}^{22}_1 + (q^2-q)xy^3\mathbf{P}^{22}_2\big)b\Big)b'x^2y^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&\left(xy\mathbf{P}^{11}_2 a - (qx\mathbf{P}^{11}_2 y^3 - x^2y^2\mathbf{P}^{21}_2)b\right)(a'xy - qb'xy^3)\\
&+\left(xy\mathbf{P}^{12}_2 a - (qx\mathbf{P}^{12}_2 y^3 - x^2y^2\mathbf{P}^{22}_2)b\right)b'x^2y^2 = 0
\end{aligned}
\tag{4.34}
$$

Now, since a, b, a', b' are arbitrary elements of $Z(A)$, the above two equations boil down to:

$$xy\mathbf{P}^{11}_2 xy = 0$$

$$xy\mathbf{P}^{11}_1 xy + (q-q^2)y^2\mathbf{P}^{11}_2 xy = 0$$

$$-qxy\mathbf{P}_2^{11}xy^3 + xy\mathbf{P}_2^{12}x^2y^2 = 0$$

$$-qxy\mathbf{P}_1^{11}xy^3 + (1-q^2)y^2\mathbf{P}_2^{11}xy^3 + xy\mathbf{P}_1^{12}x^2y^2 + (q-q^2)y^2\mathbf{P}_2^{12}x^2y^2 = 0$$

$$-qx\mathbf{P}_2^{11}xy^4 + x^2y^2\mathbf{P}_2^{21}xy = 0$$

$$-q^2x\mathbf{P}_1^{11}xy^4 - (q^2-1)y\mathbf{P}_2^{11}xy^4 + x^2y^2\mathbf{P}_1^{21}xy + (q^2-q)xy^3\mathbf{P}_2^{21}xy = 0$$

$$q^2x\mathbf{P}_2^{11}xy^4 - qx^2y^2\mathbf{P}_2^{21}xy^3 - qx\mathbf{P}_2^{12}x^2y^5 + x^2y^2\mathbf{P}_2^{22}x^2y^2 = 0$$

$$\begin{aligned} &x\mathbf{P}_1^{11}xy^6 + (1-q)y\mathbf{P}_2^{11}xy^6 - qx^2y^2\mathbf{P}_1^{21}xy^3 - (1-q^2)xy^3\mathbf{P}_2^{21}xy^3 - q^2x\mathbf{P}_1^{12}x^2y^5 \\ &+(1-q^2)y\mathbf{P}_2^{12}x^2y^5 + x^2y^2\mathbf{P}_1^{22}x^2y^2 + (q^2-q)xy^3\mathbf{P}_2^{22}x^2y^2 = 0 \end{aligned} \tag{4.35}$$

Hence $\mathbf{P}_k^{ij} = 0$ for any $i, j, k \in \{1, 2\}$. Consequently, $\mathbf{P}^{ij} = 0$ for any $i, j \in \{1, 2\}$, and (1.5) follows, as claimed. □

The same effect, though in a more trivial way, occurs in the setting of Section 2.

5 Conclusions

As we have demonstrated, restricting the domain of the σ-compatibility condition $\nabla^L = \sigma \circ \nabla^R$ to $Z(A^1)$ yields a theory of noncommutative linear connections that coincides with the classical theory in the commutative case, does not discriminate against the left or right structure of a bimodule and appears to be rich in the noncommutative set-up. It is easy to check, however, that this σ-compatibility equation is not, in general, gauge covariant, either when considering it over the whole bimodule [4], or when considering it only over the centre of a bimodule. To provide a simple example, let us assume the setting of Section 4 and choose our gauge transformation to be $\theta^i \mapsto U^i{}_j\theta^j$, $U = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. Its action on ∇^L is given by the formula (cf. p.547 and p.559 in [2]) $N \mapsto dU.U^{-1} + UNU^{-1}$, where $\nabla^L\theta^i = N^i{}_k \otimes_A \theta^k$, $N = (N^i{}_k) \in M_2(A^1)$. It is clear that, although the pure gauge connections ($\Gamma^i_{jk} = 0 = \widetilde{\Gamma}^i_{jk}$, $i, j, k \in \{1, 2\}$) satisfy $\nabla^L = \sigma \circ \nabla^R$, the Christoffel symbol Γ^1_{12} of the connection obtained by the action of U on the left pure gauge connection does not fulfill (4.31). More precisely, $\Gamma^1_{12} = 1 \notin xA$. (This can be obtained from the equation $(dU.U^{-1})^i{}_k \otimes_A \theta^k = \theta^j \otimes_A \theta^k\Gamma^i_{jk}$, $i \in \{1, 2\}$.)

A different approach to the generalized permutation σ has been suggested in [5]. In the set-up of Section 5 in [5], σ is a function of the connection. Consequently, to determine the space of connections one needs to take into account all possible bimodule homomorphisms σ. Allowing σ to vary makes it possible to change it under the action of a gauge transformation. If we take as a gauge transformation a bimodule automorphism $f : E \to E$ and define its action by the formula

$$\left(\nabla^L, \nabla^R, \sigma\right) \mapsto \left((id \otimes_A f^{-1}) \circ \nabla^L \circ f,\ (f^{-1} \otimes_A id) \circ \nabla^R \circ f,\ (id \otimes_A f^{-1}) \circ \sigma \circ (f \otimes_A id)\right), \quad (5.36)$$

then the σ-compatibility condition $\nabla^L = \sigma \circ \nabla^R$ is gauge covariant. Furthermore, since the centre of a bimodule is preserved by the bimodule automorphisms, the σ-compatibility condition is also gauge covariant when considered only over $Z(E)$. [1] One should bear in mind, however, that, roughly speaking, the bimodule automorphisms correspond to the 'commutative sector' of the space of gauge transformations.

A more radical point of view that might deserve a detailed investigation relies on employing the metric compatibility condition (1.5) rather than the equation $\nabla^L = \sigma \circ \nabla^R$ to relate ∇^L and ∇^R uniquely and without (undesirable) restrictions on (∇^L, ∇^R). Clearly, for nondegenerate metrics, formulas (2.12) and (3.20) provide this kind of mutual dependance of ∇^L and ∇^R. (In those cases, the nondegeneracy of a metric g simply means that (g^{ij}) is an invertible matrix.)

Finally, let us remark that it is plausible that in order to obtain a satisfactory definition of a bimodule connection, one needs to use the language of quantum principal bundles, and, having understood and thoroughly worked out the left-right relationship in this context (see Theorem 4.13 and Remark 4.14 in [11] and Appendix B in [1]), translate the solution(s) to bimodule terms.

[1] We owe noticing this point to Michel Dubois-Violette.

A Appendix: Commutation Formulas

Here we provide commutation formulas for the differential algebra $\Omega(A)$ that is defined with the help of (3.13). Let $Q_{-1} = Q_0 = 0$ and, for $n > 0$, let $Q_n = \sum_{k=1}^{n} q^{2(k-1)}$. For any natural $n \geq 0$, we have:

$$x^n\xi = q^{2n}\xi x^n, \quad x^n\eta = q^n\eta x^n + (q^2-1)Q_n\xi x^{n-1}y, \quad y^n\xi = q^n\xi y^n, \quad y^n\eta = q^{2n}\eta y^n;$$

$$\begin{aligned}
&x^n\xi \otimes_A \xi = q^{4n}\xi \otimes_A \xi x^n\ ,\\
&x^n\xi \otimes_A \eta = q^{3n}\xi \otimes_A \eta x^n + (q^2-1)q^{2n}Q_n\xi \otimes_A \xi x^{n-1}y\ ,\\
&x^n\eta \otimes_A \xi = q^{3n}\eta \otimes_A \xi x^n + (q^2-1)q^{2n-1}Q_n\xi \otimes_A \xi x^{n-1}y\ ,\\
&x^n\eta \otimes_A \eta = q^{2n}\eta \otimes_A \eta x^n + (q^2-1)q^nQ_n\eta \otimes_A \xi x^{n-1}y\\
&+(q^2-1)q^{n+1}Q_n\xi \otimes_A \eta x^{n-1}y + q^2(q^2-1)^2Q_nQ_{n-1}\xi \otimes_A \xi x^{n-2}y^2\ ;
\end{aligned}$$

$$\begin{aligned}
&y^n\xi \otimes_A \xi = q^{2n}\xi \otimes_A \xi y^n\ , \quad y^n\xi \otimes_A \eta = q^{3n}\xi \otimes_A \eta y^n\ ,\\
&y^n\eta \otimes_A \xi = q^{3n}\eta \otimes_A \xi y^n\ , \quad y^n\eta \otimes_A \eta = q^{4n}\eta \otimes_A \eta y^n\ ;
\end{aligned} \tag{A.37}$$

One also has:

$$\begin{aligned}
&a\xi + b\eta := a_{pr}x^py^r\xi + b_{st}x^sy^t\eta\\
&= \xi\big(q^{2p+r}a_{pr}x^py^r + q^{2t}(q^2-1)Q_sb_{st}x^{s-1}y^{t+1}\big) + \eta\big(q^{2t+s}b_{st}x^sy^t\big)\\
&= \xi\big(\big(q^{2p+r}a_{pr} + q^{2r-2}(q^2-1)Q_{p+1}b_{p+1\ r-1}\big)x^py^r + q^{2p}a_{p0}x^p\big) + \eta\big(q^{2t+s}b_{st}x^sy^t\big)\\
&=: \xi\tilde{a} + \eta\tilde{b}\ ,
\end{aligned} \tag{A.38}$$

where $a_{pr}, b_{st} \in \mathbb{C}$, $p, r, s, t \in \{0, 1, ...\}$.

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References

[1] T.Brzeziński & S.Majid "Quantum Group Gauge Theory on Quantum Spaces" *Commun. Math. Phys.* 1993 **157** 591–638 (hep-th/9208007)

[2] A.Connes *Noncommutative Geometry* Academic Press, 1994

[3] J.Cuntz & D.Quillen "Algebra Extensions and Nonsingularity" *J. Amer. Math. Soc.* 1995 **8** (2) 251–89

[4] M.Dubois-Violette, J.Madore, T.Masson, J.Mourad "Linear Connections on the Quantum Plane" *Lett. Math. Phys.* 1995 **35** 352–8 (hep-th/9410199)

[5] M.Dubois-Violette & T.Masson "On the First Order Operators in Bimodules" (q-alg/9507028)

[6] M.Dubois-Violette & P.W.Michor "Dérivations et calcul différentiel non commutatif II" *C. R. Acad. Sci. Paris* 1994 **319** (I) 927–31 (hep-th/9406166)

[7] M.Dubois-Violette & P.W.Michor "Connections on Central Bimodules" (q-alg/9503020)

[8] Y.Georgelin, J.Madore, T.Masson, J.Mourad "On the noncommutative Riemannian geometry of $GL_q(n)$" (q-alg/9507002)

[9] Y.Georgelin, T.Masson, J.Wallet "Linear Connections on the Two Parameter Quantum Plane" (q-alg/9507032)

[10] A.Kehagias, J.Madore, J.Mourad, G.Zoupanos "Linear Connections on Extended Space-Time" *J. Math. Phys.* 1995 **36** (10) 5855-67 (hep-th/9502017)

[11] M.Pflaum & P.Schauenburg "Differential Calculi on Noncommutative Bundles" preprint gk-mp-9407/7

[12] K.Schmüdgen & A.Schüler "Covariant differential calculi on quantum spaces and on quantum groups" *C. R. Acad. Sci. Paris* 1993 **316** (I) 1155–60

[13] A.Sitarz "Gravity from noncommutative geometry" *Class. Quantum Grav.* 1994 **11** (8) 2127–34 (hep-th/9401145)

[14] H.Weyl *The Theory of Groups and Quantum Mechanics* Dover, 1931

[15] S.L.Woronowicz "Differential Calculus on Compact Quantum Pseudogroups (Quantum Groups)" *Commun. Math. Phys.* 1989 **122** 125–70