Block Renormalization Group in a Formalism with Lattice Wavelets: Correlation Function Formulas for Interacting Fermions

Emmanuel Pereira[1] and Aldo Procacci[2]
[1]Dep. Física-ICEx, UFMG, CP 702, Belo Horizonte MG 30.161-970, Brazil
[2]Dep. Matemática-ICEx, UFMG, CP 702, Belo Horizonte MG 30.161-970, Brazil

April 25, 1997

Abstract

Searching for a general and technically simple multi-scale formalism to treat interacting fermions, we develop a (Wilson-Kadanoff) block renormalization group mechanism, which, due to the property of "orthogonality between scales", establishes a trivial link between the correlation functions and the effective potential flow, leading to simple expressions for the generating and correlation functions. Everything is based on the existence of "special configurations" (lattice wavelets) for multi-scale problems: using a simple linear change of variables relating the initial fields to these configurations we establish the formalism. The algebraic formulas show a perfect parallel with those obtained for bosonic problems, considered in previous works.

n. pages: 19

Running Head: BLOCK RG FOR INTERACTING FERMIONS
Mailing address: Emmanuel Pereira, Dep. Física-ICEx, UFMG, CP 702, Belo Horizonte MG 30.161-970, Brazil. **E-mail**: aldo@mat.ufmg.br

1 Introduction

For a long time Renormalization Group (RG) methods have been a successful guide to the study of problems with many scales of length. Roughly, the rigorous formalization of the RG ideas leads to the analysis of a Gaussian measure through some multi-scale structure of its covariance: the covariance is written as a sum of new ones in distinct scales (now massive interactions), which automatically leads to a decomposition of the initial fields; and the procedure follows with the study of the effective interactions resulting by integrating out, step by step, the field in each scale, where the massive covariance is expected to make easier the computation. See the refs. [1-5], and references therein. According to the problem, several techniques may be used: polymer and tree expansions, small and large field analysis, etc. But, in spite of all this formalism, the resulting mathematical problem generally involves an entanglement of variables and intricate propositions, becoming quite difficult to be solved. In short, it is very useful to develop ideas and structures in order to simplify this mechanism.

Recently [6,7], searching for such simplifications, in the study of the well known lattice dipole gas (bosonic models with a laplacian plus perturbation as interaction) it was noted that, specifically in the RG formalism already developed in ref. [1] (Wilson-Kadanoff transformation: RG with δ weight function - details in the next section), there was an orthogonality between several terms associated to the covariance decomposition into different scales. This property, named "orthogonality between scales", appeared gratuitously due to lattice wavelets, fortunately implicit into the RG structure used (in the present article, we make this relation more explicit). The use of such property allowed establishing exact and simple formulas for the correlation functions (showing a trivial link between the correlation and effective potential flows), with a good control of the dominant and subdominant terms. In a few words, the Wilson-Kadanoff RG appeared as a very good tool for those problems.

As a natural step in order to expand these results one may ask about the possibility of obtaining a similar lattice formalism for fermionic systems. Here, and throughout this paper, for such systems we mean fermions with interaction given by a derivative plus perturbations, such as in a wide class of models: Gross-Neveu, Luttinger, Fermi liquids. For many reasons, fermions have been mostly treated in a different approach, with smooth cutoffs (avoiding the lattice [2], [4], [8]), or even mapped on bosonic models [9], but working in the lattice, although facing obstacles such as the doubling of spectrum, one may start from a precise mathematical problem (e.g., without worrying about continuous Grassmann algebra) with a simplified solution once implemented this special orthogonal property. A "smooth" RG transformation for lattice fermions is presented in ref. [10], but the results show that the orthogonal property is not there, and its implementation is not a trivial fact, which draws a pessimistic scenario.

In spite of that, in a successive work [11] useful structures were established, towards an orthogonal multi-scale formalism for fermions. The free propagator (corresponding to the Wilson version of Dirac operator in a lattice) was decomposed into operators associated to different scales and with orthogonal relations, and the uniform exponential decay (locality) of the effective actions (actions after k steps of the RG transformation) was shown. For that, an unusual procedure (complex averages in the position space) was adopted, and the desired effective actions were obtained as limit of expressions due to smooth RG transformations (with exponential weight function - details in sec.2). That is, the RG transformation with the orthogonal property (which shall facilitate the treatment of interacting systems) was not directly defined.

In the present paper, besides reviewing previous works, now improving some results in order to make clear the origin and extent of the orthogonal property, we conclude the formalism for fermions. In other words, we redeem the Wilson-Kadanoff block RG (δ type RG transformation) for fermionic systems. We rigorously establish this multi-scale structure with orthogonality between scales, and use it to obtain simple formulas for the generating and correlation functions of interacting fermions. We are guided by the bosonic results, in particular by the fact that, from a certain covariance decomposition, we may get "special field configurations" related to wavelets (lattice wavelets). Now even though working with Grassmann variables, we prove that it is still possible, using a simple linear change of variables, to write the initial field in terms of those "special configurations", which have properties quite useful for a multi-scale analysis, such as localization, orthogonality between scales, etc.

The rest of the article is organized as follows: in the section 2, to make the paper essentially self-contained, we review some results and structures already known (some propositions are just listed - proofs in the references). We also introduce some modifications in order to show the generality of the method and to make clear the connexion with wavelets. Section 3 is devoted to the derivation of the desired multi-scale formalism for interacting fermions, and section 4 to concluding remarks.

2 Previous Results and First Generalizations

2.1 Bosonic Systems

Now, reviewing some results [6,7] we describe (without details) how to derive the bosonic multi-scale formalism, emphasizing (and sometimes improving) the main aspects.

Let us consider scalar field models on unitary finite lattices $\Lambda_N = \left[-\frac{L^N}{2}, \frac{L^N}{2}\right]^d \cap \mathbf{Z}^d$, L odd, $d \geq 3$, given by interactions such as

$$\mathcal{H}(\phi) = \frac{1}{2} b_0(\phi, \Delta\phi) + V(\phi), \tag{2.1}$$

where b_0 is a constant, $\phi \in \mathbf{R}^{|\Lambda_N|}$, $\Delta \equiv \partial^\dagger \partial$ (for Dirichlet boundary conditions, otherwise plus a regularizer), V a function of $\partial_\mu \phi(x)$ (such as in the dipole gas or $(\nabla\phi)^4$ models). To obtain the formalism we follow the flow of the generating function

$$Z(h) \equiv \int \exp[-\mathcal{H}(\phi) + (h, \phi)] D\phi, \qquad D\phi = \prod_{x \in \Lambda_N} d\phi(x), \tag{2.2}$$

via the Wilson-Kadanoff RG transformation (with δ weight function)

$$\exp[-H^1(\psi)] = \frac{\int \exp[-H(\phi)]\delta(C\phi - \psi) D\phi}{\text{numerator with } h, \psi = 0} \tag{2.3}$$

where $H(\phi) = \mathcal{H}(\phi) - (h, \phi)$, $\psi \in \mathbf{R}^{|\Lambda_{N-1}|}$, $\delta(C\phi - \psi) = \prod_{x \in \Lambda_{N-1}} \delta(C\phi(x) - \psi(x))$, with $C\phi(x)$ meaning the rescaled average (canonical scaling) over blocks b^L_{Lx} of size L, centered in $Lx \in \Lambda_N$,

$$C\phi(x) = L^{(d-2)/2} L^{-d} \sum_{y \in b^L_{Lx}} \phi(y), \tag{2.4}$$

(we maintain the notation C for averages from Λ_{N-j} to Λ_{N-j-1}).

After n steps of the RG transformation ($n \leq N$), minimizing at each step the effective action (after discarding the perturbative potential V and considering the constraint $\delta(C\phi^j - \phi^{j+1})$, where ϕ^j means the block field at the jth scale), changing the variables in order to expand around the minima, and also separating the marginal terms (quadratic part) of the potential, we obtain (all details in ref. [6])

$$Z(h) = c\exp\left[\frac{1}{2}(h, \tilde{P}_n h)\right] \times \int \exp\left\{-\tilde{V}^n(\partial_\mu[M_n\phi + \tilde{G}_n h]) - \frac{1}{2}b_n(\phi, \Delta_n\phi)\right\} D\phi, \tag{2.5}$$

where $\phi \in \mathbf{R}^{\Lambda_{N-n}}$; c does not depend on h; b_n is the wavefunction renormalization constant at step n; $\tilde{V}^n$ is the nth irrelevant perturbative potential (the potential without its marginal quadratic part); the propagators $\tilde{P}_n$ and $\tilde{G}_n$ given by

$$\tilde{P}_n = P_n + \frac{1}{b_n}\tilde{\Delta}_n^{-1}, \quad P_n = \sum_{j=0}^{n-1}\left(2 - \frac{b_n}{b_j}\right)\frac{1}{b_j}\tilde{\Gamma}_j, \tag{2.6}$$

$$\tilde{G}_n = G_n + \frac{1}{b_n}\tilde{\Delta}_n^{-1}, \quad G_n = \sum_{j=0}^{n-1}\frac{1}{b_j}\tilde{\Gamma}_j \tag{2.7}$$

with $\tilde{\Gamma}_j = M_j\Gamma_j M_j^\dagger$, $\quad \tilde{\Delta}_n^{-1} = M_n\Delta_n^{-1}M_n^\dagger$, and

$$\Gamma_j = \Delta_j^{-1} - \Delta_j^{-1}C^\dagger\Delta_{j+1}C\Delta_j^{-1}, \;\; \Delta_j = (C_j\Delta^{-1}C_j^\dagger)^{-1}, \;\; M_j = \Delta^{-1}C_j^\dagger\Delta_j, \tag{2.8}$$

where C_j is the rescaled average over blocks of side L^j, given by (2.4) changing L by L^j. It is interesting to note that the structure of the operators $\tilde{P}_n$ and $\tilde{G}_n$ is directly related to the free propagator decomposition

$$\Delta^{-1} = \sum_{j=0}^{n-1}\tilde{\Gamma}_j + \tilde{\Delta}_n^{-1}, \tag{2.9}$$

which is a decomposition into massive terms

$$|\tilde{\Gamma}_j(x,y)| \leq L^{-j(d-2)}\exp[-\alpha' L^{-j}|x-y|], \;\; \alpha' > 0, \tag{2.10}$$

(x and y in unitary lattices); Δ_j also with exponential decay, and $\tilde{\Delta}_n^{-1}$ bounded by $cL^{-n(d-2)}$ (vanishing as $n \to \infty$). Roughly, $\tilde{\Gamma}_j$ describes the interaction around the momentum scale L^{-j}.

In a few words, the expressions above say that, using a properly chosen RG transformation, it is possible to write the generating function (of several systems) in terms of a "local" effective action Δ_n (which goes, as $n \to \infty$, to the Gaussian fixed point), a "small" irrelevant perturbative potential $\tilde{V}_n$, and two propagators $\tilde{P}_n$ (which shall contain the dominant part of the two-point function) and $\tilde{G}_n$ written in terms of interactions $\tilde{\Gamma}_j$, living in different momentum scales. The long distance behavior of the correlations is determined by a sequence of wavefunction renormalization constants and by field derivatives of the effective action at zero field. We must emphasize the simplicity of the formulas: there is no mix between different momentum scales in the expressions for $\tilde{P}_n$ and $\tilde{G}_n$, fact due to the orthogonal property

$$\tilde{\Gamma}_j\Delta\tilde{\Gamma}_k = \delta_{ij}\tilde{\Gamma}_k, \;\; \tilde{\Gamma}_j\Delta\tilde{\Delta}_n^{-1} = \tilde{\Delta}_n^{-1}\Delta\tilde{\Gamma}_j = 0, \;\; \tilde{\Delta}_n^{-1}\Delta\tilde{\Delta}_n^{-1} = \tilde{\Delta}_n^{-1}. \tag{2.11}$$

It is also important to note that this property is due to the type of the RG transformation considered here (with δ weight function - more comments in the part 2 of this section), which, say, "abruptly separates the scales": RG with exponential weight function, for instance, does not present this orthogonality.

Now we derive some additional results which will later guide us in the construction of the fermionic formalism. From the free propagator decomposition (2.9) we get

$$I \;=\; \sum_{j=0}^{n-1} \tilde{\Gamma}_j \, \Delta \;+\; \tilde{\Delta}_n^{-1} \, \Delta \;\;\equiv\; \sum_{j=0}^{n} \mathcal{P}_j \tag{2.12}$$

with, from (2.11), for $j = 0, 1, \ldots, n$

$$\mathcal{P}_j \mathcal{P}_k = \delta_{jk} \mathcal{P}_k, \quad \mathcal{P}_j^\dagger \Delta \mathcal{P}_k = \delta_{jk} \Delta \mathcal{P}_k, \tag{2.13}$$

where $\dagger$ means conjugate transposte (i.e. adjoint in $\mathbf{C}^{|\Lambda_N|}$ with the canonical scalar product).

In ref. [6], $\mathcal{P}_j$ and $\mathcal{P}_n$ (defined there as $\Delta^{1/2}\tilde{\Gamma}_j\Delta^{1/2}$ and $\Delta^{1/2}\tilde{\Gamma}_n\Delta^{1/2}$) are self-adjoint in the canonical Hilbert space $\mathbf{R}^{|\Lambda_N|}$, and useful properties follows. Here we use the definitions above in order to obtain results extendable to fermions (see next section). However, defining a new inner product in $\mathbf{C}^{|\Lambda_N|}$: $< f, g >= (f, \Delta g)$, where $(\cdot, \cdot)$ indicates the canonical inner product (note that Δ is strictly positive), the adjoints (respect to $< \cdot, \cdot >$) become

$$\mathcal{P}_j^* = \Delta^{-1} \mathcal{P}_j^\dagger \Delta = \Delta^{-1} \Delta \tilde{\Gamma}_j \Delta = \mathcal{P}_j, \quad \mathcal{P}_n^* = \Delta^{-1} \mathcal{P}_n^\dagger \Delta = \Delta^{-1} \Delta \tilde{\Delta}_n \Delta = \mathcal{P}_n.$$

Thus we can still view $\mathcal{P}_j$ as orthogonal projections and, from (2.12), (2.13), it follows that the eigenfunctions of $\mathcal{P}_j$ $(j = 1, 2, \ldots, n)$ furnish a basis for $\mathbf{C}^{|\Lambda_N|}$. Let us derive the eigenfunctions. From $\mathcal{P} f = f$ we get $\tilde{\Gamma}_j \Delta f = M_j \Gamma_j \Delta_j C_j f$, and so, for

$$f = M_j v_j, \qquad\qquad C v_j = 0, \qquad v_j \in \mathbf{C}^{|\Lambda_{N-j}|} \tag{2.14}$$

i.e., $f = \Delta^{-1} C_j^\dagger \Delta_j$, we have $\mathcal{P}_j f = f$. The eigenfunctions of $\mathcal{P}_n$ are given by

$$g = M_n w \;, \quad w \in \mathbf{C}^{|\Lambda_{N-n}|} \tag{2.15}$$

(since $\mathcal{P}_n M_n w = M_n \Delta_n^{-1} M_n^\dagger \Delta M_n w = M_n w$, once $M_n^\dagger \Delta M_n = \Delta_n$). It may be checked that the total number of eigenfunctions described above is $|\Lambda_N|$, and so we have a basis for $\mathbf{C}^{|\Lambda_N|}$.

Suitable translations of eigenfunctions are still eigenfunctions, but not dilations. However, using operators that go from ε-lattices to $L\varepsilon$-lattices, we may pass to the continuum, taking $\varepsilon \to 0$, and in this limit the eigenfunctions above, toghether with their translations and dilations (details in [12]), generate a basis of continuum wavelets. For this reason we baptize these eigenfunctions as lattice wavelets.

¿From (2.12-15) we may write any field configuration in terms of lattice wavelets, which we claim to be "special configurations" for problems with many scale of length. Actually, for any field configuration $\phi \in \mathbf{C}^{|\Lambda_N|}$,

$$\phi = \sum_{j=0}^{n-1} M_j v_j + M_n w, \qquad v_j \in \mathbf{C}^{|\Lambda_{N-j}|}, \quad C v_j = 0, \qquad w \in \mathbf{C}^{|\Lambda_{N-n}|} \tag{2.16}$$

where the vectors v_j, w are uniquely determinated by ϕ.

The special feature of this decomposition is that lattice-wavelets bring out the ideas of multi-scale, orthogonality and localization present in the RG procedure. These properties are related to algebraic and analytic behavior of the structures present in the lattice wavelets. The analytic behavior, for instance, depends sensibly on the choice of the averaging operator C.

The decomposition above, interpreted as a linear change of variables, is a simple way to get the generating functional formula (2.5) in the multi-scale decomposition. For example, taking $n = 1$, we pose (introducing a shift to adjust the external field h) $\phi = M_1\psi + Q\zeta + b_0^{-1}\Gamma_0 h$, with $CQ = 0$, Q injection from $\mathbf{C}^{|\Lambda_N|-|\Lambda_{N-1}|}$ to $\mathbf{C}^{|\Lambda_N|}$ (i.e., a parametrization of Ker C), $\psi \in \mathbf{C}^{|\Lambda_{N-1}|}$, $\zeta \in \mathbf{C}^{|\Lambda_N|-|\Lambda_{N-1}|}$; and thus, in terms of the variables $\{\psi, \zeta\}$, we may rewrite (2.2) as

$$\begin{aligned} Z(h) &= const.\ \exp[\frac{1}{2}b_0^{-1}(h, \Gamma_0 h)] \int D\psi \exp[-\frac{1}{2}b_0(\psi, \Delta_1\psi) + (h, M_1\psi)] \\ & \int D\zeta\ \exp[-\frac{1}{2}b_0(Q\zeta, \Delta Q\zeta) - V(M_1\psi + Q\zeta + b_0^{-1}\Gamma h)], \end{aligned}$$

where we used $CQ = 0$, the definitions (2.8) and the orthogonal relations (2.11). ¿From the effective potential appearing in formula above

$$\exp[-V_1(\chi = M_1\psi + b_0^{-1}\Gamma h)] = \frac{\int D\zeta \exp[-\frac{1}{2}b_0(Q\zeta, \Delta Q\zeta) - V(M_1\psi + Q\zeta + b_0^{-1}\Gamma h)]}{\text{numerator with } \zeta, h = 0},$$

extracting the marginal quadratic part proportional to $(\chi, \Delta\chi)$, and using the orthogonal properties (2.11) we finally obtain

$$Z(h) = const. \exp[\frac{1}{2}(h, P_1 h)] \int D\psi\ \exp[-\frac{1}{2}b_1(\psi, \Delta_1\psi) + (h, M_1\psi) - \tilde{V}_1(M_1\psi + G_1 h)]$$

where $\tilde{V}_1$ is the irrelevant effective potential ($\tilde{V}_1(\chi) = V_1(\chi) - \frac{1}{2}\delta b_0(\chi, \Delta\chi)$), and $b_1 = b_0 + \delta b_0$.

The usual definition of the effective potential using the standard definition of the RG transformation (eq. (2.3)), with the δ-function made explicit, is given by

$$\exp[-V_1(\chi = M_1\psi + b_0^{-1}\Gamma h)] = \frac{\int D\eta\delta(C\eta)exp[-\frac{1}{2}b_0(\eta, \Delta\eta) - V(M_1\psi + \eta + b_0^{-1}\Gamma h)]}{\text{numerator with } \psi,\ h = 0}$$

which, up to a constant, is the same as in the formulas above. As said, the injection Q is just a parametrization of Ker C with parameters ζ (a choice of Q is found in ref. [1]).

As a final comment, note that, specifying the lattice wavelets in each scale

$$M_j Q\eta_j(x) = \sum_i a_j^i \eta_j^i, \qquad a_j^i \in \mathbf{R},$$

supposing η_j^i a basis for the jth scale with $(\eta_j^i, \Delta\eta_j^i) = \delta_{ij}$, writing $\phi = \sum_{(k)} a_{(k)}\eta_{(k)}(x)$, $a_{(k)} \in \mathbf{R}$ (where the index (k) include the scale index j and the internal index i at a fixed scale), one can obtain the Wilson-Kadanoff block RG (for $h = 0$) in a expression similar to eq. (0.1) in ref. [13], there the starting point for a phase cell cluster expansion for Euclidean field theories. We need also to mention that in ref. [14] wavelets are related to the Gaussian fixed point of a block spin RG.

2.2 Fermionic Systems

Now we recall some results and structures present in fermionic systems in order to understand their specificness. We will see that it is still possible, after suitable manipulations, to obtain multi-scale structures with the orthogonal property which will lead (next section) to simple formulas for the correlation functions.

In [10] *free* lattice Euclidean fermions are carefully studied via an RG transformation with a Gaussian weight function. The authors consider, in lattices with spacing ε (initially), the Wilson version of the Dirac operator

$$D = \sum_{\mu=1}^{d} \gamma_\mu \left(\frac{\partial^\varepsilon_\mu - \partial^{\varepsilon\,\dagger}_\mu}{2} \right) - \frac{1}{2}\varepsilon\Delta^\varepsilon, \quad \Delta^\varepsilon = \sum_{\mu=1}^{d} \frac{1}{\varepsilon}(\partial^\varepsilon_\mu + \partial^{\varepsilon\,\dagger}_\mu), \tag{2.17}$$

(for finite lattices, depending on boundary conditions, it may be necessary to introduce another regularizer in order to make D invertible) where ∂^ε_μ is the ε-lattice forward derivative ($\partial^{\varepsilon\,\dagger}_\mu$ the canonical adjoint), and γ_μ anti-hermitian Dirac matrices. The extra term breaking chiral symmetry, introduced to supress the doubling of spectrum, vanishes in the continuous limit ($\varepsilon \to 0$) and is subdominant in relation to the infrared behavior of the free propagator. The RG transformation $T^\varepsilon_{a,L}$ with a "smooth" Gaussian weight function is defined as

$$\begin{aligned} \exp(\bar{\chi}, D_1\chi) &\equiv [T^\varepsilon_{a,L} \exp(\cdot, D\cdot)](\bar{\chi}, \chi) \\ &= N \int d\bar{\psi} d\psi \exp[a(L\varepsilon)^{-1}(\bar{\chi} - C\bar{\psi}, \chi - C\psi)] \exp(\bar{\psi}, D\psi) \end{aligned} \tag{2.18}$$

where the ε ($L\varepsilon$) lattice fields $\bar{\psi}$, ψ ($\bar{\chi}, \chi$) are independent Grassmann algebra generators (with supressed spinor and lattice indices); C is the usual arithmetic averaging operator over a block of side size $L\varepsilon$; a is a real positive parameter; N a normalization constant such that

$$\int \exp[\bar{\chi}, D_1\chi] d\bar{\chi} d\chi = \int \exp[\bar{\psi}, D\psi] d\bar{\psi} d\psi. \tag{2.19}$$

Successive RG transformations are introduced according to the semi-group property

$$T^{L^{k-1}\varepsilon}_{a,L}\, T^{L^{k-2}\varepsilon}_{a,L} \ldots T^\varepsilon_{a,L} = T^\varepsilon_{a_k, L^k},$$

$T^\varepsilon_{a_k,L^k}$ defined as in (2.18) with $a_k = \frac{1-L^{-1}}{1-L^{-k}}a$, L^k and C_k (arithmetic averaging operators over blocks of side $L^k\varepsilon$) replacing a, L and C. The same symbol is used for the arithmetic averages over L^{kd} points irrespective of the domain lattice (lattice spacement).

After some manipulations, it is obtained the telescopic decomposition of the free propagator as in the bosonic formula (2.9), now just replacing Δ by D, and taking

$$D_k = a_k(I + a_k C_k D^{-1} C_k^\dagger)^{-1}, \tag{2.20}$$

with the decomposition still involving massive terms. In fact, it is shown that (Th. III.1 in ref. [10]), rescaling the operators after k steps to the unitary lattice, $D_k(x - x')$ (and other operators in the decomposition of D^{-1}) admit a bound with uniform exponential decay (independent on k). But the results hold only for a sufficiently small.

Hence, for the RG with smooth weight (finite a) there is no orthogonal property (2.11), and consequently, no hope to obtain simple correlation function formulas for interacting systems. In the limit $a \to \infty$, D_k^{-1} becomes $C_k D^{-1} C_k^\dagger$ (as for bosons) and the orthogonality is recovered, but the uniform exponential decay of effective actions is invalidated, which destroys the usefulness of the multi-scale decomposition. In a few words, ref. [10] says how one may obtain a decomposition of the free propagator in massive terms without the orthogonal property, or with orhogonality but non massive terms (without uniform exponential decay).

In ref. [11] the theorem about uniform exponential decay was extended for infinite a (i.e., when $D_k^{-1} = C_k D^{-1} C_k^\dagger$) by taking suitable complex averages C (the consideration of complex averages is a must - lemma 3.1 in ref. [11]). Roughly, this necessity appears because the main term in D is proportional to ∂_μ, an odd function in the momentum space ($\sin p_\mu$), and using real averages the effective actions, due to cancellations between positive and negative terms, may acquire a nasty behavior (spoiling the fixed point [10], [11]). In order to differentiate p_μ of $-p_\mu$ one is forced to use different weights, which means complex averages in the position space. It is easy to see that this procedure works also for other fermionic actions D with main part given by a derivative.

However, in ref. [11] the RG transformation with the orthogonal property (able to treat a perturbative potential) is not directly presented: all the structures (related to the free propagator) are studied as limit of those coming from the smooth Gaussian RG transformation. Surely, it is not a good idea to introduce a perturbative potential in the Gaussian RG transformation, study the correlation formulas carrying out enormous expressions, and then take the limit $a \to \infty$ in order to recover orthogonality, hoping for drastic simplifications. So, in the next section, guided by the idea of "special configurations" for muti-scale problems, we develop this expected formalism and show how to get simple formulas for correlation functions of interacting fermions.

3 Fermionic RG Formalism

In this section, we present the general multi-scale structure (orthogonal RG transformation) which we hope to be useful for a large class of fermionic systems (in particular, the final formulas are directly applicable to asymptotically free models). We omit some technical details dependent on the specificness of each model but irrelevant for the properties of the formalism (details such as regularizers in the interaction, subdominant terms, etc.). The results of ref. [11], although carefully proved for a specific model (Wilson version of the Dirac operator), are still considered since they are trivially extendable for the class of fermionic systems considered here.

We take models in $\Lambda_N \subset \mathbf{Z}^d$, finite unitary lattice, $|\Lambda_N| = L^{Nd}$, with the inverse free propagator of the Fermi theory given by D (always ruled by a derivative), a linear, self-adjoint, inversible operator acting on $\mathbf{C}^{|\Lambda_N|}$ (with suitable periodic conditions, regulators, etc.). We deliberately ignore the internal degree of freedom, since they do not play any role in the algebraic contruction of our RG. The matricial notation $\bar\psi D \psi$ will denote the product of a row vector $\bar\psi$, a matrix D and a column vector ψ (entries of $\bar\psi$, ψ given by $\bar\psi_x$, ψ_x, $x \in \Lambda_N$, independent generators of a finite Grassmann algebra). Newly introduced fields in sequel (unless stated otherwise) are Grassmann generators, anticommuting with all the others.

The generating function now is given by

$$Z(\bar h, h) = \int d\psi d\bar\psi \, \exp[-b_0 \bar\psi D \psi - V_0(\bar\psi, \psi) + \bar h \psi + \bar\psi h], \tag{3.1}$$

where b_0 is a constant, and V_0, depending on $\bar\psi$, ψ, the perturbative potential. The multiscale decomposition starts with the definition of the averaging operator C. We consider rescaled operators $C\ :\ \mathbf{C}^{|\Lambda_N|} \to \mathbf{C}^{|\Lambda_{N-1}|}$ (or generally from $\mathbf{C}^{|\Lambda_{N-j}|}$ to $\mathbf{C}^{|\Lambda_{N-j-1}|}$) defined as

$$(C\psi)_u = \sum_{x\in\Lambda_N} W(x)\psi_{Lu+x}, \qquad\qquad u \in \Lambda_{N-1} \quad (or \quad x \in \Lambda_{N-j}, u \in \Lambda_{n-j-1}). \tag{3.2}$$

In the bosonic case of sec.2 (where Laplacian Δ replaces D) $W(x)$ was taken as the canonical average: constant for x inside a block of size L centered in u, and zero outside it. For D the Wilson-Dirac operator (properly adjusted to the finite lattice) $W(x)$ may be taken as in ref. [11], leading to a multi-scale decomposition of D^{-1} with nice properties (as discussed in last section). Actually, the general contitions that we need for C are $C^\dagger$ injection and C surjection such that the effetive actions (in the expression related to the orthogonal property) $D_j = (CD_{j-1}^{-1}C^\dagger)^{-1}$ make sense (i.e. the inverses exist), and also that these effective actions admit a uniform exponential bound (theorem (2.1) in [11]). Admiting that C satisfies the conditions above, we construct the operators D_j, Γ_j, M_j as in the bosonic case:

$$D_j = (CD_{j-1}^{-1}C^\dagger)^{-1} = (C_j D_0^{-1} C_j^\dagger)^{-1}, \qquad D_0 = D, \ \ j \geq 1, \tag{3.3}$$

here, C_j is just the composition of C j times,

$$\begin{aligned} \Gamma_j &= D_j^{-1} - D_j^{-1}C^\dagger D_{j+1}CD_j^{-1}, \qquad j \geq 0 \\ M_j &= m_1 m_2 \ldots m_j = D_0^{-1}C^\dagger D_j, \qquad m_j = D_{j-1}^{-1}C^\dagger D_j, \quad m_0 = M_0 = 1 \end{aligned} \tag{3.4}$$

D_j and Γ_j operators in $\mathbf{C}^{|\Lambda_{N-j}|}$; M_j from $\mathbf{C}^{|\Lambda_{N-j}|}$ to $\mathbf{C}^{|\Lambda_N|}$; m_j from $\mathbf{C}^{|\Lambda_{N-j}|}$ to $\mathbf{C}^{|\Lambda_{N-j+1}|}$. As in the previous section, a telescopic decomposition of the free propagator follows

$$D^{-1} = \sum_{j=0}^{n-1}\tilde{\Gamma}_j + \tilde{D}_n^{-1} = \sum_{j=0}^{n-1}[D^{-1}C_j^\dagger D_j C_j D^{-1} - D^{-1}C_{j+1}^\dagger D_{j+1}C_{j+1}D^{-1}] + D^{-1}C_n^\dagger D_n C_n D^{-1} \tag{3.5}$$

(just an algebraic fact) where $\tilde{\Gamma}_j \equiv M_j\Gamma_j M_j^\dagger$, $\tilde{D}_n^{-1} = M_n D_n^{-1} M_n^\dagger$; and it is also immediate the desired orthogonal properties:

$$\tilde{\Gamma}_j D\tilde{\Gamma}_k = \delta_{jk}\tilde{\Gamma}_k, \quad \tilde{\Gamma}_j D\tilde{D}_n^{-1} = \tilde{D}_n^{-1}D\tilde{\Gamma}_j = 0, \quad \tilde{D}_n^{-1}D\tilde{D}_n^{-1} = \tilde{D}_n^{-1}. \tag{3.6}$$

Our strategy, on a parallel with the derivation of the bosonic formalism (sec.2), is to exploit the decomposition above in order to find the "special configurations", and use them to contruct the multi-scale structure, i.e., to define the othogonal RG for fermions. ¿From (3.5) we have

$$I = \sum_{j=0}^{n-1}\tilde{\Gamma}_j D + \tilde{D}_n^{-1}D = \sum_{j=0}^{n}\mathcal{P}_j, \tag{3.7}$$

and from (3.6),

$$\mathcal{P}_j\mathcal{P}_k = \delta_{jk}\mathcal{P}_k, \qquad \mathcal{P}_j^\dagger D\mathcal{P}_k = \delta_{jk}D\mathcal{P}_k, \qquad (A^\dagger = \bar{A}^T) \tag{3.8}$$

Comparing to the bosonic case of sec. 2, the difference is that D now is not positive, and so we cannot define $(\cdot, D\cdot)$ as a scalar product in $\mathbf{C}^{|\Lambda_N|}$ respect to which $\mathcal{P}_j$ would be self-adjoint and $\{\mathcal{P}_j\}$ a set of orthogonal projections. Anyway, we will see that the eigenfunctions of $\mathcal{P}_j$ are all that we need to reach our aim.

Let us first make some remarks. Considering C as an operator from $\mathbf{C}^{|\Lambda_{N-j}|}$ to $\mathbf{C}^{|\Lambda_{N-j-1}|}$, since C is a surjection, it follows that dim $Ker\ C = |\Lambda_{N-j}| - |\Lambda_{N-j-1}|$. Thus for $u \in Ker\ C$, we may write $u = Qv$, with Q any injection from $\mathbf{C}^{|\Lambda_{N-j}|-|\Lambda_{N-j-1}|}$ to $\mathbf{C}^{|\Lambda_{N-j-1}|}$, with $CQ = 0$ (i.e., Q is a parametrization of $Ker\ C$), and with v uniquely determinated by u (and by the choice of Q). Now turning to the eigenfunctions of $\mathcal{P}_j$ and $\mathcal{P}_n$, $j \leq n-1$, we have

$$\mathcal{P}_j M_j Q v_j = M_j Q v_j, \qquad \mathcal{P}_n M_n w_n = M_n w_n, \tag{3.9}$$

where $v_j \in \mathbf{C}^{|\Lambda_{N-j}|-|\Lambda_{N-j-1}|}$ and $w_n \in \mathbf{C}^{|\Lambda_{N-n}|}$. Clearly, if v_j, v_j' are linearly independent in $\mathbf{C}^{|\Lambda_{N-j}|-|\Lambda_{N-j-1}|}$, then $M_j Q v_j$, $M_j Q v_j'$ are linearly independent in $\mathbf{C}^{|\Lambda_N|}$ (since $M_j Q$, as composition of injections, is an injection), and so, the correspondent eigenfunctions are also linearly independent. For different j's the same follows since $\mathcal{P}_j \mathcal{P}_k = 0$ if $j \neq k$. Thus, any vector f in $\mathbf{C}^{|\Lambda_N|}$ may be decomposed in a unique way as

$$f = \sum_{j=0}^{n-1} M_j Q v_j + M_n w_n \ , \tag{3.10}$$

where $v_j \in \mathbf{C}^{|\Lambda_{N-j}|-|\Lambda_{N-j-1}|}$ and $w_n \in \mathbf{C}^{|\Lambda_{N-n}|}$. For bosons, a similar decomposition (2.16) relates the new variables to wavelets. So, we maintain here the name *lattice wavelets* for these eigenfunctions (special configurations with properties of localization, orthogonality, etc.).

Turning to the Grassmann algebra, the procedure above suggests constructing the block RG transformation via a change of *Grassmann* variables following (3.10). Thus, we take

$$\psi = \sum_{j=0}^{n-1} M_j Q \zeta_j + M_n \xi_n \quad , \quad \bar{\psi} = \sum_{j=0}^{n-1} \bar{\zeta}_j Q^\dagger M_j^\dagger + \bar{\xi}_n M_n^\dagger. \tag{3.11}$$

This is a genuine change of variables: $2|\Lambda_N|$ variables ψ, $\bar{\psi}$ related to others $2|\Lambda_N|$ given by $\{\zeta_0, \bar{\zeta}_0, \zeta_1, \bar{\zeta}_1, \cdots, \zeta_{n-1}, \bar{\zeta}_{n-1}, \xi_n, \bar{\xi}_n\}$. Now everything follows quite similarly to what we have done for bosons in the end of sec. 2. To study the flow of the effective potential, all that we need is (3.11) (perform the change of variables (3.11) in (3.1) with $h = 0$, $\bar{h} = 0$, then integrate step by step the fields in each scale). To study the generating function, it is again necessary to make some shifts in order to adjust the external fields h and $\bar{h}$.

Let us see the first step of the RG tranformation in details. We replace in r.h.s. of (3.1)

$$\psi = M_1 \xi_1 + Q\zeta_0 + b_0^{-1}\Gamma_0 h \quad , \quad \bar{\psi} = \bar{\zeta}_0 Q^\dagger + \bar{\xi}_1 M_1^\dagger + b_0^{-1}\bar{h}\Gamma_0$$

i.e., (3.11) with $n = 1$ plus a shift; we use then elementary properties of Grassmann integrals, in particular: if $\psi = B\xi$, $\bar{\psi} = \bar{\xi}B^\dagger$ such that $\det B \neq 0$, then $\int d\psi d\bar{\psi} \rho(\psi, \bar{\psi}) = \int d\xi d\bar{\xi}\ \rho(B\xi, \bar{\xi}B^\dagger) \big/ \det[B^\dagger B]$ (change of variables); if h, $\bar{h}$, ψ, $\bar{\psi}$ are independent generators, then $\int d\psi d\bar{\psi} \rho(\psi, \bar{\psi}) = \int d\psi d\bar{\psi} \rho(\psi + h, \bar{\psi} + \bar{h})$ (translation formula). Also using $CQ = 0$ and orthogonal properties, we finally rewrite r.h.s. of (3.1) as

$$Z(h, \bar{h}) = \exp[b_0^{-1}\bar{h}\Gamma_0 h] N_0 \int d\xi_1 d\bar{\xi}_1 \exp[-b_0 \bar{\xi}_1 D_1 \xi_1 + \bar{h} m_1 \xi_1 + \bar{\xi}_1 m_1^\dagger h]$$
$$\times \int d\zeta_0 d\bar{\zeta}_0 \exp[-b_0 \bar{\zeta}_0 Q^\dagger D_0 Q \zeta_0] \exp[-V(m_1\xi_1 + Q\zeta_0 + b_0^{-1}\Gamma_0 h,\ \ \bar{\xi}_1 m_1^\dagger + \bar{\zeta}_0 Q^\dagger + b_0^{-1}\bar{h}\Gamma_0)]$$

where N_0 is a constant, and the last expression in r.h.s. define the effective potential at scale 1

$$\exp[-V_1(\chi, \bar{\chi})] = \frac{\int d\zeta_0 d\bar{\zeta}_0\ \exp[-b_0 \bar{\zeta}_0 Q^\dagger D_0 Q \zeta_0 - V(\chi + Q\zeta_0\ ,\ \bar{\chi} + \bar{\zeta}_0 Q^\dagger)]}{\text{numerator with } \chi, \bar{\chi} = 0}$$

with $\chi = m_1\xi_1 + b_0^{-1}\Gamma_0 h$, $\bar\chi = \bar\xi_1 m_1^\dagger + b_0^{-1}\bar h\Gamma_0$.

Now we separate out the main part (respect to the scaling properties) in V_1 in order to isolate the dominant term. Here we will only take care of the quadratic marginal term, i.e. the term in $V_1(\chi,\bar\chi)$ which is proportional to $\bar\chi D_0\chi$ (depending on the specificness of the model a more detailed analysis in the "main part" of the effective potential may be necessary, e.g. massive and quartic terms may be controlled in separate). We define the "irrelevant" potential $\tilde V_1$ by $V_1(\chi,\bar\chi) = \tilde V_1(\chi,\bar\chi) + \delta b_0\bar\chi D_0\chi$, where, due to orthogonal relations: $\delta b_0\bar\chi D_0\chi = \delta b_0\bar\xi_1 D_1\xi_1 + \frac{\delta b_0}{b_0^2}\bar h\Gamma_0 h$. So, writing $b_1 = b_0 + \delta b_0$, $\gamma_0^{(1)} = b_0^{-1} - (b_1 - b_0)b_0^{-2}$, $P_1 = \gamma_0^{(1)}\Gamma_0$ and $G_1 = b_0^{-1}\Gamma_0$ we get

$$
\begin{aligned}
Z(h,\bar h) &= N_1\ \exp[\bar h P_1 h] \\
&\quad\times\int d\xi_1 d\bar\xi_1 \exp[-b_1\bar\xi_1 D_1\xi_1 + \bar h m_1\xi_1 + \bar\xi_1 m_1^\dagger h]\exp[-\tilde V_1(M_1\xi_1 + G_1 h\ ,\ \bar\xi_1 M_1^\dagger + \bar h G_1)].
\end{aligned}
$$

The expression above corresponds to the first step of RG tranformation. The second step follows with the change of variables $\xi_1 = m_2\xi_2 + Q\zeta_1 + b_1^{-1}\Gamma_1 M_1^\dagger h$, $\bar\xi_1 = \bar\xi_2 m_2^\dagger + \bar\zeta_1 Q^\dagger + b_1^{-1}\bar h M_1\Gamma_1$ and similar procedures. Iterating we obtain (as in the bosonic case)

$$
\begin{aligned}
Z(h,\bar h) &= N\ \exp[\bar h(P_n + b_n^{-1}M_n D_n^{-1}M_n^\dagger)h]\int d\xi_n d\bar\xi_n \exp[-b_n\bar\xi_n D_n\xi_n] \\
&\quad\times\exp[-\tilde V_n(M_n\xi_n + b_n^{-1}M_n D_n^{-1}M_n^\dagger h + G_n h\ ,\ \bar\xi_n M_n^\dagger + b_n^{-1}\bar h M_n D_n^{-1}M_n^\dagger + \bar h G_n)]
\end{aligned}
$$

with

$$
\begin{aligned}
b_j &= b_{j-1} + \delta b_{j-1}, \qquad \gamma_j^{(n)} = b_j^{-1} - (b_n - b_j)b_j^{-2}, \\
G_n &= \sum_{j=0}^{n-1} b_j^{-1}\tilde\Gamma_j, \qquad P_n = \sum_{j=0}^{n-1}\gamma_j^{(n)}\tilde\Gamma_j
\end{aligned}
\tag{3.12}
$$

Once more we emphasize that the final formulas are simple due to the orthogonal properties (3.6). Using Grassmann derivative the correlation function formulas are immediate. For example, for the two point function $S_2(x,y)$

$$
S_2(x,y) = \tilde P_n(x,y) - \sum_{u,v}\tilde G_n(x,u)\left[\frac{\partial}{\partial\bar\chi}W_n\frac{\partial}{\partial\chi}(u,v)\right]_{\chi,\bar\chi=0}\tilde G_n(v,y) \tag{3.13}
$$

where

$$
\tilde P_n(x,y) = P_n(x,y) + b_n^{-1}M_n D_n^{-1}M_n^\dagger(x,y), \qquad \tilde G_n(x,y) = G_n(x,y) + b_n^{-1}M_n D_n^{-1}M_n^\dagger(x,y);
$$

$$
\exp[-W_n(\chi,\bar\chi)] = \int d\xi_n d\bar\xi_n\ \exp[-b_n\bar\xi_n D_n\xi_n]\exp[-\tilde V_n(M_n\xi_n + \chi, \bar\xi_n M_n^\dagger + \bar\chi)].
$$

The usefulness of the representation above is that its analysis poses no difficulty: the dominant term is isolated in $\tilde P_n$ and the subdominant contribution in the correlations are given by the field derivatives of the "irrelevant" effective potential at zero field (in a combination with $\tilde G_n$). The behaviour of $\tilde P_n$ and $\tilde G_n$ depends only on the running coupling constants $b_0,\ b_1,\cdots,\ b_n$. And since this sequence is given by the effective potential flow (remind that, such as in the bosonic case, the fermionic RG is also expected to present suitable multiscale properties), we see that our formalism provides a trivial link between the effective potential theory and the correlation function theory. This triviality is a consequence of the particular block RG here constructed, which does not mix the scales (orthogonal property). We want to remark that the key idea to build up the block RG transformation is the *linear* change of the initial Grassmann variables by the "special fields" configurations (our lattice-wavelets). We also give emphasis to the perfect parallel between the fermionic and bosonic RG tranformation, which is not such a surprise, since everything is based on a linear change of variables, and in linear combinations Grassmann variables behave just like trivial complex numbers.

4 Conclusion

Although making only general considerations without using the proposed orthogonal multi-scale formalism in the study of a particular fermionic system, that is, without carrying out the calculation of the running coupling constants and the effective potential flow for a precise model, we strongly believe that now one may be optmistic about the usefulness of the Wilson-Kadanoff (δ-type) RG transformation in the study of fermions. Besides the comments at the end of last section concerning the simplicity of the generating and correlation functions (due to the orthogonal property) which facilitates their analysis, we point out the "good" behavior of fermionic perturbation to say that the computation of these parameters in the RG flow may be even easier than for bosonic models (already successfully considered). Fermions, for instance, are free of the unpleasant large field regions of bosons, responsible for intrincate technical problems.

We intend to use the RG formalism developed here to study also systems controlled by a non trivial fixed point: with new rescaling in the averaging operators, i.e., carefully redefining the effective potential theory, we hope to adapt the formulas deduced here to face anomalous scaling problems (see [15] as an example for such systems).

Finally, we also emphasize the promising aspect of the connexion between multi-scale structures (RG transformations) and the lattice wavelets. It has been proved [12] that the simplest structure of lattice wavelets (with canonical averaging and laplacian operator in the expressions) leads to true wavelets in the continuum limit (in this case, Lemarie functions). Thus, it comes to mind questions such as what to learn about wavelet construction with RG techniques, and vice-versa, which other properties of these functions may improve the multi-scale analysis.

Acknowledgements: This work was partially supported by CNPq and FAPEMIG (Brazil).

References

1. K. Gawedzki and A. Kupiainen: "Block spin renormalization group for dipole gas and $(\nabla\phi)^4$", *Ann. Phys.* **147**, 198-243 (1983).

2. G. Benfatto and G. Gallavotti: "Perturbation theory of the Fermi surface in a quantum liquid", *J. Stat. Phys.* **59**, Nos 3/4, 541-664 (1990).

3. T. Balaban: "$(\mathrm{Higgs})_{2,3}$ quantum fields in a finite volume", *Comm. Math Phys.* **89**, 411-445 (1983).

4. J. Feldman, J. Magnen, V. Rivasseau, E. Trubowitz: "An infinite volume expansion for many fermion Green's functions" *Helvetica Physica Acta*, **65**, 679- 721, (1992)

5. D. Brydges and H. T. Yau: "Grad ϕ perturbations of massless Gaussian fields", *Comm. Math Phys.* **129**, 351-392 (1990)

6. M. O'Carroll and E. Pereira: "A representation for the generating and correlation functions in the block field renormalization group formalism and asymptotic freedom", *Ann. Phys.*, **218**, 139-159 (1992).

7. E. Pereira and M. O'Carroll: "Orthogonality between scales and wavelets in a representation for correlation functions. The lattice dipole gas and $(\nabla\phi)^4$ models", *J. Stat. Phys.*, **73**, Nos 3/4, 695-721 (1993).

8. K. Gawedzki and A. Kupiainen: "Gross-Neveu model through convergent perturbation expansions", *Commun. Math. Phys.*, **102**, 1-30 (1985).

9. C. de Calan, P. A. Faria da Veiga, J. Magnen and R. Sénéor: "Constructing the three-dimensional Gross-Neveu model with a large number of flavor components", *Phys. Rev. Lett.* **66**, 3233-3236 (1991).

10. T. Balaban, M. O'Carroll and R. Schor: "Block renormalization group for Euclidean fermions", *Commun. Math. Phys.*, **122**, 233-247 (1989).

11. E. Pereira: "Orthogonality Between Scales in a Renormalization Group for Fermions", *J. Stat. Phys.*, **78**, Nos. 3/4, 1067-1082 (1995).

12. M. O'Carroll: "Lattice and continuum wavelets and the block renormalization group", *J. Stat. Phys.*, **71**, Nos 3/4, 415-423 (1993).

13. G. A. Battle and P. Federbush: "A Phase Cell Cluster Expansion for Euclidean Field Theories", *Ann. Phys.*, **142**, 95-139 (1982)

14. G. Battle: "Wavelets: a renormalization group point of view", in *Wavelets and their applications*, A. Bartlett and A. Jones, eds. M. B. Ruskai (1992)

15. G. Benfatto, G. Gallavotti, A. Procacci, B. Scoppola: "Beta function and Schwinger functions for a many fermions system in one dimension. Anomaly of the Fermi surface" *Comm. Math. Phys.*, **160**, 91-171 (1994)