1 Introduction

In a recent paper ([1]) we discussed the L^2 limit of solutions $u^{(NS)}$ of the Navier-Stokes equations in the case of vortex patch initial data. We proved that, if the initial vorticity is a vortex patch with smooth boundary, then the difference $u^{(NS)} - u^{(E)}$ between the the Navier-Stokes and Euler velocities corresponding to this initial datum is in L^2 and converges to zero at a rate proportional to $\sqrt{\nu}$. This is a slower rate of convergence than the rate $(\mathrm{O}(\nu))$ of the inviscid limit for smooth solutions ([2], [3], [4], [5]). The fact that there is a drop in the rate of convergence when one passes from the smooth to the non-smooth regime is not an artifact: there are elementary examples providing lower bounds.

In the present work we investigate the L^p inviscid limit for vorticities. We are motivated in our study by the statistical equilibrium theory of vortices ([6], [7]). The initial vorticities are taken in the phase space $\mathbf{Y}$ of bounded functions that vanish outside a compact set. We are mostly interested in long time, uniform bounds, i.e., bounds that are valid for many turnover times and that have an explicit rate of vanishing. The smoothing effect that is present in the Navier-Stokes equations is absent in the Euler equations. Because of this, internal transition layers prevent *uniform* L^p bounds for the difference between vorticities of solutions with the same non-smooth initial data. Therefore, it seems that a pathwise uniform Eulerian inviscid limit in this phase space is not possible. The term pathwise refers here to the comparison of individual solutions, paths that start from the same initial data. We find that in order to obtain uniform bounds we need to consider non-pathwise bounds: the most convenient close companion to a solution of the Navier-Stokes equation might be a mollified Euler solution. To be more precise, if $S^{NS}(t)b$ represents the solution (vorticity) of the Navier-Stokes equation with initial vorticity $b \in \mathbf{Y}$, if $S^E(t)b$ represents the solution of the Euler equation and if we denote by $f_\delta = f * \phi_\delta$ the convolution with a mollifier ϕ_δ, then a pathwise estimate concerns the difference $S^{NS}(t)b - S^E(t)b$ and non pathwise estimates concern differences $S^{NS}(t)b - S^E(t)b_\delta$ and $S^{NS}(t)b - \left(S^E(t)b\right)_\delta$. We find that the latter is better suited for long time estimates. While $S^E(t)b_\delta$ solves the Euler equations, $\left(S^E(t)b\right)_\delta$ solves suitably modified Euler equations.

We prove uniform L^p bounds that vanish as $\nu^{\frac{s}{2p}}$ for the difference between Navier-Stokes and modified Euler solutions corresponding to initial data in Besov spaces $b \in \mathbf{Y} \cap B_2^{s,\infty}(R^2)$. We find that the optimal mollification is over a distance of order $\delta \sim \sqrt{\nu}$, a fact that is consistent with the estimate for the smallest length scales in two dimensional turbulence. In order to obtain a short time result it is enough to mollify the initial datum for the Euler evolution. However, in order to obtain a long time result we have to mollify the solution. Thus, the long time approximation follows slightly modified Eulerian dynamics. The assumptions we require for the long time results are satisfied by vortex patches with smooth boundaries.

The main difficulty is due to the fact that one needs to estimate gradients of the Eulerian vorticity. We use the method of ([1]) to show that velocity differences are small and we obtain estimates for the gradients of the Eulerian vorticity; the smallness of velocity differences counterbalances the large vorticity gradients. The *non pathwise uniform* results can be used to obtain *non-uniform pathwise* results (that is, pathwise results without rates of convergence). In particular we prove the strong pathwise convergence in L^p, $1 < p < \infty$.

2 Previous Results

The Navier-Stokes equations and the Euler equations in R^2 are

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu \Delta u$$

$$\nabla \cdot u = 0$$

where $\nu > 0$ in the case of the Navier-Stokes equations, $\nu = 0$ in that of the Euler equation. The corresponding vorticity

$$\omega = \nabla^{\perp} \cdot u$$

satisfies

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \Delta \omega$$

and u can be recovered from ω via

$$u = \frac{1}{2\pi} \left(\nabla^{\perp} \log(| \cdot |) \right) * \omega.$$

The notation $\nabla^{\perp}$ refers to the gradient rotated by 90 degrees.

We consider the evolution in the vorticity space $\mathbf{Y}$

$$\mathbf{Y} = L^1(R^2) \cap L_c^{\infty}(R^2)$$

of bounded functions with compact support; the norm is the sum of the L^1 and L^{∞} norms. The solutions

$$S^{NS}(t)a_0 = \omega^{NS}(x,t)$$

and

$$S^E(t)a_0 = \omega^E(x,t)$$

of the Navier-Stokes and, respectively Euler equation, corresponding to initial datum $\omega(x,0) = a_0 \in \mathbf{Y}$, exist for all $t \geq 0$, $(t \in R)$ and are unique.

A much studied class of examples of $a \in \mathbf{Y}$ is that of vortex patches: the initial vorticity $a_0(x)$ is a simple function

$$a_0 = \sum_{j=1}^{N} \omega_0^{(j)} \chi_{D_j}$$

where $\omega_0^{(j)}$ are real constants and χ_{D_j} are characteristic functions of bounded, simply connected domains in R^2.

We associate to any $a \in \mathbf{Y}$ certain basic objects: two functions and four numbers. The functions are a stream function ψ_a and a velocity field u_a:

$$\psi_a(x) = \frac{1}{2\pi} \int \log(|x-y|)a(y)dy,$$

and

$$u_a = \nabla^{\perp}\psi_a.$$

The numbers are a length scale L_a, a time scale T_a, a velocity scale U_a, and a kinetic energy E_a:

$$L_a = \sqrt{\frac{\|a\|_{L^1(R^2)}}{\|a\|_{L^\infty(R^2)}}},$$

$$T_a = \frac{1}{\|a\|_{L^\infty(R^2)}},$$

$$U_a = \sqrt{\|a\|_{L^1(R^2)}\|a\|_{L^\infty(R^2)}},$$

and

$$E_a = -\frac{1}{2}\int \psi_a(y)a(y)dy.$$

We also associate to $a \in \mathbf{Y}$ a distribution $\pi_a(dy)$ defined by

$$\int f(y)\pi_a(dy) = \int_{spt\ a} f(a(x))dx.$$

If the initial vorticity is in $\mathbf{Y}$ then the fundamental existence result, due to Yudovich ([9]) is For every $a \in \mathbf{Y}$ there exists a unique weak solution

$$\omega^E(x,t) = S^E(t)a$$

of the Euler equations that satisfies $\omega^E(x,0) = a(x)$.

The quantities L_a, T_a, U_a, E_a and the distribution π_a are conserved by the Eulerian flow, i.e.

$$C_{S^E(t)a} = C_a$$

if C_a stands for any of these quantities. The velocity

$$u^E(x,t) = u_{S^E(t)a}$$

satisfies

$$\|u^E(\cdot,t)\|_{L^\infty} \le U_a$$

for all $t \in R$. We denote by S the strain matrix – the symmetric part of the gradient of velocity:

$$S(x,t) = \frac{1}{2}\left((\nabla u) + (\nabla u)^*\right).$$

If the initial vorticity is smooth then the solution is a classical solution. The following precise estimates will be used in the sequel:

Let $a \in Y \cap W^{1,\infty}$ be a smooth initial vorticity. Then the strain matrix

$$S(x,t) = \frac{1}{2}\left((\nabla u^E) + (\nabla u^E)^*\right)$$

satisfies

$$\|S(\cdot,t)\|_{L^\infty} \le \|a\|_{L^\infty}\left[\left(2+\frac{1}{\pi}\right) + 2\log_+\left(L_a\frac{\|\nabla a\|_{L^\infty}}{\|a\|_{L^\infty}}\right)\right]\exp\left(2\|a\|_{L^\infty}t\right).$$

The gradient of the vorticity $\omega^E = S^E(t)a$ satisfies

$$\|\nabla\omega^E(\cdot,t)\|_{L^p} \le \|\nabla a\|_{L^p}\exp\left(\int_0^t \|S(\cdot,s)\|_{L^\infty}ds\right).$$

for all time $t \in R$ and all p including infinity: $1 \le p \le \infty$.

The Lagrangian trajectory map $X(x,t)$ defined by

$$\frac{d}{dt}(X(x,t)) = u^E(X(x,t),t), \quad X(x,0) = x$$

satisfies

$$\|\nabla X(\cdot,t)\|_{L^\infty} \le \exp\left(\int_0^t \|S(\cdot,s)\|_{L^\infty}ds\right).$$

The quantity

$$\mathcal{A}(t) = \int_0^t \|S(\cdot,s)\|_{L^\infty}ds$$

plays an important role. It controls not only the growth of the Lipschitz norm of particle trajectories and of the L^p norms of gradients of vorticity but also the L^2 operator norm of the Gateaux derivative of the velocity solution map. The class of initial vorticities for which the quantity $\mathcal{A}(t)$ is finite for all time is therefore included in the class of initial vorticities for which the velocity map $u_a \mapsto u^E(\cdot, t)$ is continuous in L^2. The only class of non-smooth functions $a \in \mathbf{Y}$ that are known to have $\mathcal{A}(t)$ finite for all time are vortex patches with *smooth* boundaries ([8]) or minor variations thereof.

We start by estimating the difference between velocities of solutions of the Navier-Stokes equations and Euler equations. Assume that $a \in \mathbf{Y}$ and $b \in \mathbf{Y}$ are initial vorticities for the Euler and respectively Navier-Stokes equation. The difference

$$u(x,t) = u_{S^{NS}(t)b} - u_{S^E(t)a}$$

satisfies

$$\left(\partial_t + u^{NS} \cdot \nabla - \nu\Delta\right) u + \nabla q = \nu \Delta u^E - u \cdot \nabla u^E.$$

Using the method of [1] one obtains Let $a \in \mathbf{Y}$ be the initial vorticity for a solution of the Euler equations and $b \in \mathbf{Y}$ the initial vorticity for a solution of the Navier-Stokes equations with kinematic viscosity ν. If the corresponding velocities, u_a and u_b are such that $u_b - u_a$ is square integrable then

$$\|u^{NS}(\cdot,t) - u^E(\cdot,t)\|_{L^2(R^2)} \le \left(\|u_b - u_a\|_{L^2(R^2)} + \|a\|_{L^2(R^2)}\sqrt{\nu t}\right) \exp\left(\mathcal{A}(t)\right)$$

holds for all $t \ge 0$ with

$$\mathcal{A}(t) = \int_0^t \|\frac{1}{2}\left(\left(\nabla u^E\right) + \left(\nabla u^E\right)^*\right)\|_{L^\infty} ds.$$

For general a, $b \in \mathbf{Y}$, $u_a - u_b$ is not square integrable. We have, however, quite obviously Assume that $b \in \mathbf{Y}$ and that $a = b_\delta$ where

$$b_\delta = b * \phi_\delta$$

with $\phi_\delta(x) = \delta^{-2}\phi(\frac{x}{\delta})$ a standard mollifier. Then

$$\|u_a - u_b\|_{L^2(R^2)} \le C\delta\|b\|_{L^2(R^2)}.$$

3 Further Results

If $a = b_\delta$ and $b \in \mathbf{Y}$ we have so far a L^2 bound

$$\|u^{NS}(\cdot,t) - u^E(\cdot,t)\|_{L^2(R^2)} \le C\|b\|_{L^2(R^2)}\left[\delta + \sqrt{\nu t}\right] \exp\left(\mathcal{A}_\delta(t)\right).$$

where $\mathcal{A}_\delta$ is computed on the Euler solution $S^E(t)b_\delta$. We will keep the notation b for the initial vorticity for the Navier-Stokes evolution and a for that of the

Euler evolution. A direct consequence of Theorem 2 is: Let $b \in \mathbf{Y}$ and let $a = b_\delta$. Then there exists a nondimensional constant C depending only on the mollifier ϕ such that

$$\mathcal{A}_\delta(t) \le \left[C + \log_+\left(\frac{L_b}{\delta}\right)\right]\left[\exp\left(2\|b\|_{L^\infty(R^2)}t\right) - 1\right]$$

As a consequence of this inequality it follows that the exponential $\exp(\mathcal{A}_\delta(t))$ is bounded by a small power of δ^{-1} for times that are short compared to T_b. In order to continue the estimates we will make an additional assumption regarding b: we will assume a certain degree of regularity:

$$b \in \mathbf{Y} \cap \left(\cup_{0<s<1} B_2^{s,\infty}(R^2)\right)$$

where $B_2^{s,\infty}(R^2)$ is the Besov space (see for instance, [10]) of vorticities in $L^2(R^2)$ that have an L^2 modulus of continuity that is Hölder continuous of order s. If $b \in B_2^{s,\infty}(R^2)$ then

$$\delta\|\nabla b_\delta\|_{L^2(R^2)} \le C\left(\frac{\delta}{\rho_b}\right)^s \|b\|_{L^2(R^2)}$$

where

$$\rho_b = \left(\frac{\|b\|_{L^2(R^2)}}{N_s(b)}\right)^{\frac{1}{s}}$$

and

$$N_s(b) = \sup_{y\in R^2} \frac{\|b(\cdot + y) - b(\cdot)\|_{L^2(R^2)}}{|y|^s}.$$

Note that, if b is a vortex patch initial datum and if the boundary of the patch is smooth then $b \in B_2^{\frac{1}{2},\infty}$. More generally,

Let $b = \chi_D$ be the characteristic function of a bounded domain whose boundary has box-counting (fractal) dimension not larger than $d < 2$:

$$d_F(\partial D) \le d.$$

Then

$$b \in B_p^{\frac{2-d}{p},\infty}(R^2)$$

for $1 \le p < \infty$.

We start with a short time result: Assume that $b \in \mathbf{Y} \cap B_2^{s,\infty}$ for some $0 < s < 1$. Consider

$$\omega^{NS}(\cdot, t) = S^{NS}(t)b,$$

and

$$\omega^E(\cdot, t) = S^E(t)b_\delta$$

with $\delta = \sqrt{\nu T_b}$.

For every $\epsilon > 0$, there exists an absolute constant γ depending only on s, ϵ such that, if

$$0 \le \frac{t}{T_b} \le \gamma$$

then, for every $p \ge 2$ there exists a constant K_b depending on p and b alone, such that

$$\|\omega^{NS}(\cdot, t) - \omega^E(\cdot, t)\|_{L^p(R^2)} \le K_b \nu^{\frac{s-\epsilon}{2p}}$$

holds for all ν small enough.

In order to obtain a long time result we need to know that

$$\limsup_{\delta \to 0} \mathcal{A}_\delta(t) < \infty.$$

Recall that this quantity is computed by solving a family of Euler equations. The map $\delta \mapsto \mathcal{A}_\delta(t)$ is not known to be upper semicontinuous. In other words, even in the class of vortex patches with smooth boundaries, we can not rule out the possibility that there exists b, a time t and a sequence $\delta \to 0$ such that $\mathcal{A}(t) < \infty$ for the solution starting from b but $\lim_{\delta \to 0} \mathcal{A}_\delta(t) = \infty$. If this does not happen then the result above holds without loss of ϵ and without restriction on time. Remarkably though, the global estimates can be obtained if one reverses the order of operations and, instead of mollifying the initial datum and then solving the Euler equations, one rather solves first the Euler equations and then mollifies. Let us consider thus $b \in \mathbf{Y} \cap B_2^{s,\infty}$ and assume that

$$\mathcal{A}(t) < \infty$$

for $0 \le t \le T$. In view of the results of [8] this is the case if b represents a vortex patch with smooth boundaries. Now we consider

$$\omega^E(x, t) = S^E(t)b$$

and mollify it, i.e. we consider the function

$$\omega_\delta(x, t) = \left(S^E(t)b\right) * \phi_\delta.$$

The equation obeyed by the mollified vorticity $\omega_\delta = \left(S^E(t)b\right)_\delta$ is

$$(\partial_t + u_\delta \cdot \nabla)\, \omega_\delta = \nabla \cdot \tau_\delta(u^E, \omega^E).$$

where

$$\tau_\delta(v, w) = (v - v_\delta)\,(w - w_\delta) - r_\delta(v, w)$$

with

$$r_\delta(v, w)(x) = \int \phi(y)\,(v(x - \delta y) - v(x))\,(w(x - \delta y) - w(x))\, dy.$$

The three dimensional analogues of these formulae were first used in a proof of the Onsager conjecture ([11]).

We will choose $\delta = \sqrt{\nu T_b}$ and compare ω_δ to $\omega^{NS}(x,t) = S^{NS}(t)b$. Let $b \in \mathbf{Y} \cap B_2^{s,\infty} \cap B_4^{\frac{s}{2},\infty}$ with $0 < s < 1$. Consider the solution

$$\omega^E(\cdot,t) = S^E(t)b$$

and assume that

$$\mathcal{A}(T) = \int_0^T \|\frac{1}{2}\left((\nabla u^E) + (\nabla u^E)^*\right)\|_{L^\infty} dt < \infty$$

and that the solution satisfies

$$\omega^E \in L^2(0,T; B_4^{\frac{s}{2},\infty} \cap B_2^{s,\infty}).$$

Consider $\delta = \sqrt{\nu T_b}$ and set

$$\omega_\delta = \omega^E * \phi_\delta$$

and

$$u_\delta = u^E * \phi_\delta$$

where ϕ_δ is an appropriate mollifier. Consider the solution

$$\omega^{NS}(\cdot,t) = S^{NS}(t)b.$$

Then there exists K (depending on b only) such that

$$\|u^{NS} - u_\delta\|_{L^2} \le K\sqrt{\nu}$$

holds for all $0 \le t \le T$ and ν small. Moreover, for every $p \ge 2$ there exists a constant K_p (depending only on p and and the solution $S^E(t)b$ for $0 \le t \le T$) such that

$$\|\omega^{NS}(\cdot,t) - \omega_\delta(\cdot,t)\|_{L^p} \le K_p \nu^{\frac{s}{2p}}$$

holds for all $0 \le t \le T$. In particular, if b is a vortex patch with smooth $\left(\mathcal{C}^{1,\gamma}\right)$ boundaries then

$$\|S^{NS}(t)b - \left(S^E(t)b\right)_{\sqrt{\nu T_b}}\|_{L^p} \le K_p \nu^{\frac{1}{4p}}$$

holds for all $0 \le t \le T$. As a consequence of the *non-pathwise, uniform* results one can obtain *pathwise, non-uniform* results. We recall a theorem from [1]: Assume that the initial vorticity $b \in L^1 \cap L^\infty$ has compact support, included in the disk

$$\{x; |x| \le L\}.$$

Then

$$\|\omega^{NS}(\cdot,t)\|_{L^\infty(Q_t)} \le \|b\|_{L^\infty} e^{-\frac{U_b L}{\nu}}.$$

where

$$Q_t = \{x; |x| \geq C\left(L + \frac{\nu}{U_b} + U_b t\right)\}$$

and C is an absolute constant. Moreover, if we set

$$[[\frac{x}{\delta}]] = \sqrt{1 + \frac{|x|^2}{\delta^2}},$$

then, for any $\delta > 0$

$$\|\omega^{NS}(\cdot, t) e^{[[\frac{\cdot}{\delta}]]}\|_{L^p} \leq \|b(\cdot) e^{[[\frac{\cdot}{\delta}]]}\|_{L^p} e^{\left(\frac{U_b t}{\delta} + \frac{7\nu t}{\delta^2}\right)}$$

holds for any p, $1 \leq p \leq \infty$. We will now state the pathwise results: Let $b \in \mathbf{Y} \cap B_2^{s,\infty}$, $0 < s < 1$, be an initial vorticity and consider $\omega^{NS} = S^{NS}(t)b$. There exists an absolute constant γ such that, for all $t \in [0, \gamma T_b]$ and all Lipschitz functions f that vanish at the origin

$$\lim_{\nu \to 0} \int f(\omega^{NS}(x,t) dx = \int f(b(x)) dx$$

holds. Consequently the weak limit of distributions is

$$\lim_{\nu \to 0} \pi_{[S^{NS}(t)b]}(dy) = \pi_b(dy).$$

The same result holds for arbitrary any time interval $[0, T]$ provided the solution of the Euler equation $\omega^E = S^E(t)b$ satisfies the assumptions

$$b \in \mathbf{Y} \cap B_2^{s,\infty} \cap B_4^{\frac{s}{2},\infty},$$

$$\mathcal{A}(T) = \int_0^T \|\frac{1}{2}\left((\nabla u^E) + (\nabla u^E)^*\right)\|_{L^\infty} dt < \infty$$

and

$$\omega^E \in L^2(0, T; B_4^{\frac{s}{2},\infty} \cap B_2^{s,\infty}).$$

As a consequence, the strong limit

$$\lim_{\nu \to 0} S^{NS}(t)b = S^E(t)b$$

holds in the L^p norm for vorticities, for all $1 < p < \infty$ and the time intervals corresponding to the two situations considered above.

4 Proofs

The proof of Theorem 2 follows from a few well-known observations. First a classical inequality for Calderon-Zygmund singular integrals yields in this case

$$\|S\|_{L^\infty} \le \|\omega^E\|_{L^\infty}\left[\left(2+\frac{1}{\pi}\right)+2\log_+\left(L_{\omega^E}\frac{\|\nabla\omega^E\|_{L^\infty}}{\|\omega^E\|_{L^\infty}}\right)\right].$$

Using the conservation laws this becomes

$$\|S\|_{L^\infty} \le \|a\|_{L^\infty}\left[\left(2+\frac{1}{\pi}\right)+2\log_+\left(L_a\frac{\|\nabla\omega^E\|_{L^\infty}}{\|a\|_{L^\infty}}\right)\right].$$

The equation obeyed by $\nabla^\perp\omega^E$ is

$$\left(\partial_t + u^E\cdot\nabla\right)\nabla^\perp\omega^E = (\nabla u^E)\nabla^\perp\omega^E.$$

Consequently

$$\|\nabla^\perp\omega^E\|_{L^\infty} \le \|\nabla^\perp a\|_{L^\infty}\exp\left(\int_0^t \|S\|_{L^\infty}ds\right)$$

and the theorem follows by combining these facts and a straightforward Gronwall inequality argument.

The proof of Theorem 3 follows closely the proof of the similar pathwise result in [1] and will not be repeated here. Proposition 1 is elementary and Lemma 1 is just a direct application of Theorem 2 for the case of $a = b_\delta$.

We sketch the proof of Lemma 2. Consider $b = \chi_D$ and consider a vector y of small length δ. The function $b(\cdot + y) - b(\cdot)$ is supported in a thin open neighborhood of ∂D of width δ. The two dimensional area of this open set vanishes as δ^{2-d}: Indeed one can cover ∂D with $N(\delta)\sim\delta^{-d}$ balls of radii less or equal to δ. Any point that is at distance at most δ from ∂D is at distance at most 2δ from the center of at least one of these balls. So the union of the balls with the same centers but with radii 2δ covers the δ neighborhood of the boundary and has the advertised area. The L^p norm of the difference is bounded by the $\frac{1}{p}$ power of this area. Note that the argument fails if we replace box-counting dimension by Hausdorff dimension.

Now we turn to the ideas for the proof of Theorem 4. We note first that, in view of the fact that $b \in \mathbf{Y}\cap B_2^{s,\infty}$, it follows that

$$\|b - b_\delta\|_{L^p} \le C\|b\|_{L^\infty}^{1-\frac{2}{p}}\|b\|_{L^2}^{\frac{2}{p}}\left(\frac{\delta}{\rho_b}\right)^{\frac{2s}{p}}.$$

The equation obeyed by the difference of vorticities

$$\omega = \omega^{NS} - \omega^E$$

between solutions of the Navier-Stokes equations with initial data b and solutions of the Euler equation with initial data $a = b_\delta$ is

$$\left(\partial_t + u^{NS} \cdot \nabla - \nu\Delta\right)\omega = \nu\Delta\omega^E - u \cdot \nabla\omega^E$$

where

$$u = u_\omega$$

is the corresponding velocity. We take $p \geq 2$, multiply by $|\omega|^{p-2}\omega$ and integrate. The first term on the right hand side is integrated by parts and a straightforward Holder inequality is applied.

Using the bounds in Theorem 2 and Theorem 3 one can estimate

$$\int |u||\nabla\omega^E||\omega|^{p-1}dx \leq C\|b\|_{L^\infty}^{p-1}\|b\|_{L^2}\left[\delta + \sqrt{\nu t}\right]\|\nabla b_\delta\|_{L^2}\exp\left(2\mathcal{A}_\delta(t)\right).$$

and

$$\nu\int |\nabla\omega^E|^2|\omega|^{p-2}dx \leq C\nu\|b\|_{L^\infty}^{p-2}\|\nabla b_\delta\|_{L^2}^2\exp\left(2\mathcal{A}_\delta(t)\right).$$

Now, in view of the fact that $b \in B_2^{s,\infty}$ these would be respectively δ^s and $\nu\delta^{-2+2s}$ estimates if $\exp\left(2\mathcal{A}_\delta(t)\right)$ would be bounded uniformly as $\delta \to 0$. For general $b \in \mathbf{Y} \cap B_2^{s,\infty}$ we can use Lemma 1 to estimate this exponential by a small power $\delta^{-\epsilon}$ for times that are short by comparison to T_b. We omit further details of the proof of Theorem 4.

For the long time estimates of Theorem 5 we mollify the Euler solution. If u^E solves the Euler equation then $u_\delta = \left(u^E\right)_\delta$ solves ([11])

$$\left(\partial_t + u_\delta \cdot \nabla\right)u_\delta + \nabla p_\delta = \nabla \cdot \tau_\delta(u^E, u^E)$$

where

$$\tau_\delta(v,w) = (v - v_\delta) \otimes (w - w_\delta) - r_\delta(v,w)$$

with

$$r_\delta(v,w)(x) = \int \phi(y)\left(v(x-\delta y) - v(x)\right) \otimes \left(w(x-\delta y) - w(x)\right)dy.$$

If u^{NS} solves the Navier-Stokes equations with initial velocity u_b for $b \in \mathbf{Y}$ then the difference $u = u^{NS} - u_\delta$ solves

$$\left(\partial_t + u^{NS} \cdot -\nu\Delta\right)u + \nabla q = \nu\Delta u_\delta - u \cdot \nabla u_\delta - \nabla \cdot \tau_\delta(u^E, u^E).$$

Note that, because we assumed (or proved) that

$$\int_0^T \|\frac{1}{2}\left(\left(\nabla u^E\right) + \left(\nabla u^E\right)^*\right)\|_{L^\infty}ds$$

is finite then it follows immediately that the same is true for

$$\int_0^T \|\frac{1}{2}\left((\nabla u_\delta^E) + (\nabla u_\delta^E)^*\right)\|_{L^\infty} ds.$$

The term involving τ_δ is handled in the following manner. One integrates by parts and after using the viscosity one has to estimate

$$\frac{1}{\nu}\|\tau_\delta(u^E, u^E)\|_{L^2}^2.$$

In view of the fact that

$$\|u_\omega(\cdot - y) - u_\omega(\cdot)\|_{L^4} \le C|y|\|\omega\|_{L^4}$$

it follows that

$$\|\tau_\delta(u^E, u^E)\|_{L^2} \le C\delta^2\|b\|_{L^4}^2.$$

But $\nu \sim \delta^2$ so we conclude that

$$\|u\|_{L^2} = O(\delta).$$

The equation obeyed by the mollified vorticity $\omega_\delta = \left(S^E(t)b\right)_\delta$ is

$$\left(\partial_t + u_\delta \cdot \nabla\right)\omega_\delta = \nabla \cdot \tau_\delta(u^E, \omega^E).$$

Consequently the equation for the difference

$$w = S^{NS}(t)_b - \left(S^E(t)b\right)_\delta$$

is

$$\left(\partial_t + u^{NS}\cdot\nabla - \nu\Delta\right) w = \nu\Delta\omega_\delta - u\cdot\nabla\omega_\delta - \nabla\cdot\tau_\delta(u^E, \omega^E).$$

Using the fact that $\omega_\delta \in B_2^{s,\infty}$ we obtain

$$\|\nabla\omega_\delta\|_{L^2} = O(\delta^{-1+s})$$

and together with the estimate $\|u\|_{L^2} = O(\delta)$ it follows that the estimates for the first two terms on the right hand side of the equation obeyed by $\int |w|^p dx$ are similar to the corresponding ones in the proof of Theorem 4. The estimate for the term involving τ_δ uses the viscosity, integration by parts and the estimate

$$\frac{1}{\nu}\|\tau_\delta(u^E, \omega^E)\|_{L^2}^2 \le K\nu^{-1}\delta^{2+s}$$

where

$$K = C\|b\|_{L^4}^2\|S^E(t)b\|_{B_4^{\frac{s}{2},\infty}}^2.$$

One obtains thus estimates of the type $\|w\|_{L^p} = O(\delta^{\frac{s}{p}}) = O(\nu^{\frac{s}{2p}})$, and this concludes our presentation of the ideas for the proof of Theorem 5.

We show now hints for the proof of Theorem 7. In view of the result of Theorem 6 that was proved in [1] we have

$$\int_{R^2} f(\omega^{NS}(x,t))dx \simeq \int_{R^2 \setminus Q_t} f(\omega^{NS}(x,t))dx.$$

(We use $\simeq$ to denote quantities that become equal in the limit $\nu \to 0$.) In view of the fact that

$$\int_{R^2 \setminus Q_t} f\left(\left(\omega^E(x,t)\right)_\delta\right) dx \simeq \int_{R^2 \setminus Q_t} f(\omega^E(x,t))dx = \int_{R^2} f(\omega^E(x,t))dx$$

one needs to show that

$$\int_{R^2 \setminus Q_t} f\left(\left(\omega^E(x,t)\right)_\delta\right) dx \simeq \int_{R^2 \setminus Q_t} f(\omega^{NS}(x,t))dx.$$

Using the Lipschitz condition for f and the fact that $R^2 \setminus Q_t$ is compact it is enough to have

$$\int_{R^2 \setminus Q_t} \left|\omega^{NS}(x,t) - \left(\omega^E(x,t)\right)_\delta\right|^2 dx \simeq 0.$$

But this follows from the non-pathwise estimates

$$\|\omega^{NS} - \omega_\delta\|_{L^2} \le K\nu^{\frac{s}{4}}.$$

The proof of the L^p convergence follows from the fact that the norms converge ($f(y) = |y|^p$)

$$\lim_{\nu \to 0} \|S^{NS}(t)b\|_{L^p} = \|S^E(t)b\|_{L^p}$$

and the strong convergence of velocities in L^2 (Theorem 3). This last fact implies that any weak limit as $\nu \to 0$ of a sequence $S^{NS}(t)b$ for fixed b and t must be $S^E(t)b$. Strong convergence in L^p for $1 < p < \infty$ follows from weak convergence and convergence of the norms because these spaces are uniformly convex.

References

[1] P. Constantin, J. Wu, Inviscid limit for vortex patches. Report No. **7**, Institut Mittag-Leffler 1994/95

[2] F. McGrath, Nonstationary plane flow of viscous and ideal fluids, Arch. Rational Mech. Anal., **27**, (1968), 329-348.

[3] T. Kato, Nonstationary flows of viscous and ideal fluids in R3, J. Funct. anal, **9**, (1972), 296-305.

[4] J. T. Beale, A. Majda, Rates of convergence for viscous splitting of the Navier-Stokes equations, Math. Comp. **37**, (1981), 243-259.

[5] P. Constantin, Note on loss of regularity for solutions of the 3D incompressible Euler and related equations, Commun. Math. Phys., **104**, (1986), 311-326.

[6] J. Miller, Statistical mechanics of Euler equations in two dimensions, Phys. Rev. Lett., **65**, (1990), 2137-2140.

[7] R. Robert, A maximum-entropy principle for two dimensional perfect fluid dynamics, J. Stat. Phys. **65**, (1991), 531-551.

[8] A. Bertozzi, P. Constantin, Global regularity for vortex patches. Comm. Math. Phys. **152**, 19-28 (1993)

[9] V.I. Yudovich, Non-stationary flow of an ideal incompressible liquid. Zh. Vych. Mat. **3**, 1032-1066(1963) (in Russian)

[10] J. Peetre, *New Thoughts on Besov Spaces*, Duke University, N.C., 1986.

[11] P. Constantin, W. E, E. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equation, Commun. Math. Phys., **165** (1994), 207-209.