Commuting-flow symmetries and common solutions to differential equations with common symmetries

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Abstract: We point out that in certain cases, all the differential equations (for given indipendent and dependent variables) possessing a given symmetry necessarily share a common solution. We characterize this situation and the common solution, and give concrete examples.

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Introduction

We are all familiar with elementary occurences of a remarkable phenomenon: in some cases, *all* the differential equations possessing a given symmetry share a common solution.

As trivial examples of this fact it suffices to think of reflection-invariant ODEs in $\mathcal{R}^1$, $\dot{x} = f(x,t)$ [these are identified by $f(x,t) = -f(-x,t)$], which all share the common solution $x(t) \equiv 0$; or also reflection and/or rotation invariant ODEs in $\mathcal{R}^n$, which again share the common (trivial) solution $x(t) \equiv 0$.

The purpose of this short note is to point out that this phenomenon can be present in far less trivial cases, for both ODEs and PDEs; and to give a way to characterize the common solutions, at least for a relevant class of symmetries, i.e. the "commuting-flow symmetries" defined below (which are natural in a dynamical systems geometric approach).

This question can be tackled making use of Michel's theory (see below), and indeed our result can be seen as a simple corollary – maybe reaching unexpected fields – of the original Michel theorem.

A classical result of L. Michel [1] ensures that, given a finite dimensional smooth manifold M with an action of a compact Lie group G, we can identify the points $x \in M$ which are critical for *all* the G-invariant potentials $V(x)$. The investigation of Michel was motivated principally by fundamental problems in Physics (the theory of strong interactions) [2], but the theorem in itself is of a geometric nature, and of a much wider applicability to a range of problems involving *spontaneous symmetry breaking*.

The Michel's result can be suitably generalized to variational problems in function spaces [3], and it has been recently extended to consider gauge invariant functionals [4].

On the other side, the proof given by Michel in [1] for his theorem immediately generalizes to consider equivariant dynamics defined by a vector field on M, and its fixed points; it also guarantees, more generally, the invariance under the equivariant dynamics of a family of subspaces of M (identified by sharing a common isotropy subgroup for the G-action). In this way, one could also readily recover a number of fundamental results in equivariant dynamics and equivariant bifurcation theory, such as the reduction lemma of Golubitsky and Stewart [5] and the equivariant branching lemma of Cicogna and Vanderbauwhede [6], originally obtained with no use of Michel's theory [7].

It is thus quite natural to ask if, beside the extension of the variational aspects of Michel theory to infinite dimensional spaces [4], one could not obtain also a similar extension for its equivariant dynamics aspects. The purpose of this note is to show that this is indeed the case, and namely that we can identify subsets of a given function space $\mathcal{F}$ which are invariant under any dynamics on $\mathcal{F}$ which satisfy certain symmetry requirements with respect to an algebra $\mathcal{G}$ of vector fields; moreover, proceeding pretty much as in the finite dimensional case, we can identify functions $f_0 \in \mathcal{F}$ which are solutions for *any* differential equation Δ which is $\mathcal{G}$-invariant (in a sense to be precisely defined in the following).

Our discussion makes use of ideas and formalisms from the symmetry theory of differential equations [8-11], which we assume is known – at least in its basics – to the reader. Also, our discussion will be conducted at a rather formal level; thus, in particular, we will assume the existence of flows generated by the (generalized) vector fields to be considered in the following.

1. Evolutionary vector fields and symmetry

Let us consider a smooth manifold $M = \mathcal{X} \times \mathcal{Y}$, where the $x \in \mathcal{X} \subseteq \mathcal{R}^q$ and the $u \in \mathcal{Y} \subseteq \mathcal{R}^p$ will be thought as, respectively, the independent and dependent variables for the differential equations to be considered in a moment.

We consider a vector field Y on M, which we write as

$$Y \ = \ \sum_{i=1}^{q} \xi^i(x,u) \frac{\partial}{\partial x^i} \ + \ \sum_{\alpha=1}^{p} \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} \tag{1}$$

(with ξ, φ smooth functions on M); as it is well known [8], to any such vector field we can associate an *evolutionary representative*; this will be denoted by X and is given by

$$X \ = \ \left[\varphi^\alpha(x,u) - \sum_i \xi^i(x,u) u^\alpha_i \right] \partial_\alpha \ \equiv Q^\alpha[u] \partial_\alpha \ . \tag{2}$$

Here we write ∂_α for $\partial / \partial u^\alpha$, and u^α_i for $\partial u^\alpha / \partial x^i$ (these notations will be understood throughout in the following; from now on we will also understand summation over repeated indices). The notation $Q[u]$ – used extensively in the following – means that Q is a function of x, u, and u derivatives of any order (although in the present case these are only of first order). We will also write X as X_Q, to stress the association with the vector $Q[u]$; more generally, X_P will denote the evolutionary vector field $X_P = P^\alpha[u]\partial_\alpha$.

It should be recalled that, while Y is a legitimate vector field on M, its evolutionary representative X should be considered with some extra care; we can either consider it as a notation for a vector field in the function space $\mathcal{M}$ (the space of smooth functions from $\mathcal{R}^q$ to $\mathcal{R}^p$), such that $(I + \varepsilon X_Q)$ transforms $f(x)$ into the function

$$\widetilde{f}^\alpha(x) = f^\alpha(x) + \varepsilon \left[\varphi^\alpha(x, f(x)) - \xi^i(x, f(x)) f^\alpha_i(x) \right] \ , \tag{3}$$

either as a generalized vector field acting in the jet space associated to M; in this case, one should more precisely consider the infinite order prolongation X^* of X, acting in the infinite order jet space $J^*(M)$, in order to have a proper vector field.

It should also be stressed that, for the argument to be considered below, one could start directly from X^*, i.e. consider generalized vector fields rather than limit to consider only representative of geometric vector fields.

Let us now consider a system of p differential equations for functions $u^\alpha = u^\alpha(x) \in \mathcal{M}$; this will be written as $\Delta[u] = 0$ (this will be a vector Δ^α, with $\alpha = 1, ..., p$; we will say for simplicity "the equation" Δ to refer to this system). With the equation Δ we associate an evolutionary generalized vector field X_Δ given by

$$X_\Delta \ = \ \Delta^\alpha[u] \partial_\alpha \ ; \tag{4}$$

as it was the case for X_Q, this can be seen either as a vector field on $\mathcal{M}$, either as denoting the infinite prolongation X^*_Δ acting in $J^*(M)$.

We recall [8] that X_Q is a *symmetry* of Δ if

$$X^*_Q(\Delta) \ = \ 0 \ , \tag{5}$$

is satisfied whenever $\Delta = 0$, and a *strong symmetry* of Δ if (5) is identically satisfied.

2. CF symmetries

Given two evolutionary generalized vector fields (EGVF), $X_P = P^\alpha[u]\partial_\alpha$ and $X_Q = Q^\alpha[u]\partial_\alpha$, we can consider their commutator $[X_P, X_Q]$; this will be again an EGVF $Y_R = R^\alpha[u]\partial_\alpha$, with

$$R^\alpha[u] = X_P^* (Q^\alpha) - X_Q^* (P^\alpha) \ . \tag{6}$$

It is easy to check [8] that this gives indeed a proper commutator, i.e. the usual properties of the commutator are satisfied.

Given a differential equation Δ, and an EGVF X_Q, we say that X_Q is a **Commuting Flow (CF) symmetry** for Δ if the associated EGVF commute (in the above sense), i.e. if

$$[X_Q, X_\Delta] \ = \ 0 \ . \tag{7}$$

For a given equation Δ, we denote by $\mathcal{G}_\Delta$ the set of EGVF which commute with X_Δ (this is easily seen to be a Lie algebra, with X_Δ belonging to its center). Conversely, for a given EGVF X_Q we denote by $\mathcal{I}_Q$ the set of differential equations which have X_Q as a CF symmetry; again, the X_Δ corresponding to $\Delta \in \mathcal{I}_Q$ form a Lie algebra.

Lemma 1. *If X_Q is a CF symmetry for Δ, then it is also a symmetry of Δ in the ordinary sense.*

`Proof.` Indeed, on $\Delta = 0$ we have that both X_Δ and X_Δ^* vanish, as it is clear since $X_\Delta = \Delta^\alpha[u]\partial_\alpha$ and $X_\Delta^* = \sum_J (D_J \Delta^\alpha[u])(\partial/\partial u_J^\alpha)$ (here J is again a multiindex). Thus, $X_\Delta^*(Q) = 0$ on $\Delta = 0$ for any Q, and the vanishing of $[X_Q, X_\Delta]$ is equivalent to the vanishing of $X_Q^*(\Delta)$, which is the condition (4) for X_Q to be a symmetry of Δ in the ordinary sense. $\odot$

3. Invariance of fixed space and common solutions to symmetric equations

With any $X_Q : \mathcal{F} \to \mathrm{T}\mathcal{F}$ we associate the set $F(Q) \subseteq \mathcal{F}$ of functions which are fixed points for the vector field X_Q; equivalently, this means that $Q[f] = 0$ for $f \in F(Q)$.

Lemma 2. *If X_Q is a CF symmetry of Δ, then X_Δ leaves $F(Q)$ globally invariant, i.e. $X_\Delta : F(Q) \to \mathrm{T}F(Q)$.*

`Proof.` Let us consider, for the sake of clarity, the integrated version of the commutation relations (assuming suitable existence and unicity conditions for the flows of the vector fields); we have – with λ, μ real parameters – that

$$e^{\lambda Q[u]} e^{\mu \Delta[u]} (f) \ = \ e^{\mu \Delta[u]} e^{\lambda Q[u]} (f) \ ; \tag{8}$$

from $f \in F(Q)$ it follows that $e^{\lambda Q}(f) = f$, and thus we have that

$$e^{\mu \Delta[u]} \, (f) \in F(Q) \quad \forall \mu \ , \tag{9}$$

from which the lemma follows immediately. $\odot$

Corollary. *If there exists an $f_0 \in F(Q)$ isolated in $F(Q)$, then necessarily f_0 is solution to any $\Delta \in \mathcal{I}_Q$.*

`Proof.` In this case, f_0 is a fixed point of $\Delta[u]$; but this is equivalent to being a solution to Δ. $\odot$

`Remark.`When we speak of an "isolated" function, we are implicitely assuming $\mathcal{M}$ is equipped with some natural topology; in the examples below, in which we consider a space of L^2 functions, this will be the L^2 norm. Actually, we could consider any topology in which the flows of X_Q and X_Δ are continuous.

The above lemma and corollary are immediately generalized to the case of a whole algebra of CF symmetries; we denote by $\mathcal{G} \equiv \mathcal{G}_\Delta$ this (possibly, but not necessarily, abelian) algebra, and say that $\mathcal{G}$ is a CF symmetry algebra for Δ if $[X_Q, X_\Delta] = 0$ for any $X_Q \in \mathcal{G}$. In this case we denote by $F(\mathcal{G})$ the set of functions invariant under all the $X_Q \in \mathcal{G}$, and by $\mathcal{I}_\mathcal{G}$ the set of equations for which $\mathcal{G}$ is a CF symmetry algebra. We have then immediately the

Lemma 3. *If $\mathcal{G}$ is a CF symmetry algebra of Δ, then X_Δ leaves $F(\mathcal{G})$ globally invariant, i.e. $X_\Delta : F(\mathcal{G}) \to \mathrm{T}F(\mathcal{G})$.*

Corollary. *If there exists a $f_0 \in F(\mathcal{G})$ isolated in $F(\mathcal{G})$, then necessarily f_0 is solution of any $\Delta \in \mathcal{I}_\mathcal{G}$.*

It should be noted that it is not easy to have such an isolated $f_0 \in F(Q) \subseteq \mathcal{M}$; e.g., in the case where $Q[u]$ is a homogeneous linear function of u, we have that necessarily $F(Q)$ is a linear subspace of $\mathcal{M}$, and we cannot have isolated points (unless $F[Q]$ reduces to the zero function alone).

However, we have so far considered the whole function space $\mathcal{M}$; when we deal with differential equations, we usually have to look for solutions in some given function space, less general than $\mathcal{M}$; restricting to a smaller function space makes it easier to find isolated fixed points f_0.

Let us consider a closed subspace $\mathcal{F} \subseteq \mathcal{M}$, which we think as fixed once and for all. We have now to consider only those Δ such that X_Δ are vector fields on $\mathcal{F}$. We can then obtain again the equivalent of the lemma 3 and its corollary given above, provided we consider now the intersection of $F(\mathcal{G}) \subseteq \mathcal{M}$ with $\mathcal{F}$.

We summarize our discussion in the following form.

Proposition. *Let $\mathcal{G}$ be an algebra of EGVF on $\mathcal{M}$, and $F(\mathcal{G})$ the space of fixed points for $\mathcal{G}$; let $\mathcal{F} \subseteq \mathcal{M}$ be a subspace of $\mathcal{M}$, such that there exists a f_0 isolated in $F(\mathcal{G}) \cap \mathcal{F}$. Then, any differential equation $\Delta \in \mathcal{I}_\mathcal{G}$ such that $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ admits $f_0(x) \in \mathcal{F}$ as a solution.*

4. Examples

Example 1. Let us consider $\mathcal{X} = S^1$, $\mathcal{Y} = R$, so that $\mathcal{M}$ is the space of $\mathcal{C}^\infty$ function $f : S^1 \to R$; in this we select the space $\mathcal{F}$ of functions which satisfy

$$||f||^2 \ \equiv \ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \mathrm{d}x \ = \ 1 \ . \tag{10}$$

We consider $Y = -\partial/\partial x$, and correspondingly

$$X_Q = u_x \partial_u \quad ; \quad Q[u] = u_x \tag{11}$$

Let us now look for $F(Q) \cap \mathcal{F}$; $F(Q)$ is given by constant functions $f(x) = c$, and the constraint (10) selects the two functions $f_\pm(x) = \pm 1$. These are obviously isolated in $F(Q) \cap \mathcal{F}$.

Let us now determine the $\Delta \in \mathcal{I}_Q$; the condition $[X_Q, X_\Delta] = 0$ amounts to $\partial\Delta/\partial x = 0$. Indeed, $X_Q^* = \sum_j (D_j u_x) \partial_{u_j}$, while the first prolongation of X_Δ is given by $X_\Delta^{(1)} = \Delta \partial_u + (D_x \Delta) \partial_{u_x}$. Thus,

$$X_Q^*(\Delta) - X_\Delta^*(Q) = \sum_j (D_j u_x) \frac{\partial \Delta}{\partial u_j} - D_x \Delta \ . \tag{12}$$

Expanding $D_x \Delta = \Delta_x + \sum_j (\partial\Delta/\partial u_j) u_{x,j}$ (where $u_{x,j} = D_j u_x$), we get that (12) reduces to Δ_x.

Thus, $\Delta \in \mathcal{I}_Q$ are written as

$$\Delta \ = \ \Delta \left(u, u_x, u_{xx}, ... \right) = 0$$

Now we have to check that all such Δ which moreover have X_Δ leaving $\mathcal{F}$ invariant admit $f_\pm(x)$ as solutions.

The requirement $X_\Delta : \mathcal{F} \to \mathcal{F}$ amounts to asking that

$$||e^{\lambda X_\Delta}(f)|| = 1 \tag{13}$$

whenever $||f|| = 1$. In the limit $\lambda \to 0^+$,

$$e^{\varepsilon X_\Delta}(f) \ \simeq \ (I + \varepsilon X_\Delta)(f) \simeq f + \varepsilon \Delta[f] \ ,$$

and thus

$$||e^{\lambda X_\Delta}(f)||^2 = ||f||^2 + 2\varepsilon \langle f \cdot \Delta[f] \rangle + o(\varepsilon) \ . \tag{14}$$

As (13) must hold for any $f \in \mathcal{F}$, it must in particular hold on $f_\pm$; evaluating (14) in $f_\pm$ we obtain then $\Delta(\pm 1, 0, 0, 0, ...) = 0$, i.e. that $f_\pm$ is a solution to Δ.

Example 2. We consider now an example with $u = u(x_1, x_2)$ and an algebra of CF symmetries spanned by two vector fields, i.e.

$$\begin{aligned} Y_1 \ &= \ - x_1 \partial_{x_2} + x_2 \partial_{x_1} = - \partial_\theta \\ Y_2 \ &= \ x_1 \partial_{x_1} + x_2 \partial_{x_2} + k u \partial_u = r \partial_r + k u \partial_u \ , \end{aligned} \tag{15}$$

where (r, θ) are polar coordinates in the (x_1, x_2) plane; clearly, Y_1 represents a rotation in the independent variables, and Y_2 a scale transformation.

Correspondingly, we have

$$X_1 = u_\theta \partial_u \quad , \quad X_2 = (ku - u u_r) \partial_u \ . \tag{16}$$

and it is easy to see that

$$F(Q_1) = \{ u = u(r) \} \quad , \quad F(Q_2) = \{ u(x_1, x_2) = \sum_{b=0}^{k} c_b x_1^b x_2^{k-b} \} \tag{17}$$

(with c_b real constants) and therefore

$$F(\mathcal{G}) \ = \ \{ u = \alpha r^k \} \qquad (\alpha \in \mathcal{R}) \ . \tag{18}$$

We will consider, for the sake of clarity, only first order polynomial PDEs, i.e. we assume

$$\Delta \ = \ \Delta \left(r, \theta; u, u_r, u_\theta \right) \ ; \tag{19}$$

this also means that

$$X_\Delta^{(1)} \ = \ \Delta \partial_u + \left(D_r \Delta \right) \partial_{u_r} + \left(D_\theta \Delta \right) \partial u_\theta \tag{20}$$

(in the following computations, we only need this first order prolongation of X_Δ). With this, it follows from straightforward computations that

$$[X_1, X_\Delta] \ = \ - \frac{\partial \Delta}{\partial \theta} \tag{21}$$

$$[X_2, X_\Delta] \ = \ - k\Delta + r \frac{\partial \Delta}{\partial r} + k u \frac{\partial \Delta}{\partial u} + k u_\theta \frac{\partial \Delta}{\partial u_\theta} + (k-1) u_r \frac{\partial \Delta}{\partial u_r} \tag{22}$$

The (21), and the polynomial assumption on Δ, guarantees that $\Delta \in \mathcal{I}_{Q_1}$ implies

$$\Delta \ = \ \gamma_{abcd} r^a u^b u_r^c u_\theta^d \ ; \tag{23}$$

moreover, (22) requires that, in order to have $\Delta \in \mathcal{I}_{\mathcal{G}}$, only the γ_{abcd} with

$$k \ (b+c+d-1) \ = \ c-a \tag{24}$$

can be nonzero.

We can now check that indeed X_Δ leaves the space of functions of the form $u = \alpha r^k$ invariant. In facts,

$$X_\Delta(\alpha r^k) \ = \ \gamma_{abcd} r^a \alpha^b r^{kb} (\alpha k)^c r^{c(k-1)} \delta_{d,0} \ = \ \beta \gamma_{abc0} r^{a+kb+c(k-1)} \ , \tag{25}$$

and it follows from (24) that the exponent of r in the final expression is just k. This shows that indeed $X_\Delta : F(\mathcal{G}) \to \mathrm{T}F(\mathcal{G})$, as claimed.

Let us now consider again $\mathcal{F}$ given by the functions of unit norm, where we define

$$||f||^2 = \langle f, f \rangle \equiv \int_D |f(x_1,x_2)|^2 \mathrm{d}x_1 \mathrm{d}x_2 \tag{26}$$

(here D is the unit disk $r \le 1$); in this case the only functions of $F(\mathcal{G})$ which are also in $\mathcal{F}$ are

$$f_\pm(x_1,x_2) = \pm \alpha_0 r^k \qquad [\alpha_0 = \sqrt{(k+1)/\pi}] \ . \tag{27}$$

We rewrite the action of X_Δ as

$$(I + \varepsilon X_\Delta)\,(f) = f + \varepsilon \Delta[f] \ ; \tag{28}$$

thus – for $||f|| = 1$ – the norm of this is one, i.e. X_Δ leaves $\mathcal{F}$ invariant, if and only if

$$\langle f, \Delta[f] \rangle \ = \ 0 \ . \tag{29}$$

We notice that $F(\mathcal{G})$ is a one dimensional linear space, so that $X_\Delta : F(\mathcal{G}) \to \mathrm{T}F(\mathcal{G})$ means that necessarily $X_\Delta(f) = \lambda f$, for some real number λ, for any $f \in F(\mathcal{G})$. With this, (29) means

$$\langle f, \lambda f \rangle \equiv \lambda ||f||^2 = 0 \ , \tag{30}$$

which can be true only for $\lambda = 0$. This shows that the condition that $X_\Delta : \mathcal{F} \to \mathrm{T}\mathcal{F}$ guarantees also that the isolated points in $F(\mathcal{G}) \cap \mathcal{F}$ are solutions to Δ, as claimed by our theorem.

Example 3. Finally, let us consider $\mathcal{X} = R \times R_+$ and $\mathcal{Y} = R^1$; then $\mathcal{M}$ is the space of $\mathcal{C}^\infty$ functions $f : R^2 \to R$, and in this we select the space $\mathcal{F}$ of functions which satisfy, with (x,t) coordinates in $\mathcal{X}$,

$$\int_{-\infty}^{+\infty} f(x,t) \mathrm{d}x \ = \ 1 \tag{31}$$

and such that $f_x(x,t)$ go to zero when x go to infinity.

We consider $Y = 2t\partial_x - xu\partial_u$, and correspondingly

$$X \ = \ -(xu + 2tu_x)\,\partial_u \ ; \tag{32}$$

notice that Y is one of the Lie-point symmetries of the heat equation [8-11], and it can be checked that X is a CF symmetry of it.

It can also be checked easily that the functions in $F(Q)$ are those of the form

$$f(x,t) \ = \ k(t)\; e^{-x^2/4t} \ , \tag{33}$$

and the intersection of this $F(Q)$ with $\mathcal{F}$ yields precisely

$$f_0(x,t) \ = \ \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \ , \tag{34}$$

i.e. the fundamental solution of the heat equation.

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