Renormalization group around a sphere for interacting fermion systems in $d > 1$.

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Abstract: We will sketch, see [N] for full details, a momentum-space renormalization group approach suited to a first-principles analysis, in perturbation theory, of interacting Fermi systems in more than one dimension. Our techniques are a momentum-space version of the ones introduced by Benfatto and Gallavotti [BG1], [BG2], incorporating some important ideas by Shankar [S]. A central role is played by bounds on contributions of certain Feynman graphs, substantially improving power counting ones. These bounds implement the important idea that renormalization is to be done around a surface in momentum space, rather than around a point, as usual. As a consequence of the improved bounds we can simplify the beta functional. Despite simplifications, the beta functional is still complicated and we cannot yet prove normality (Fermi liquid) or anomaly (non-Fermi liquid) of the system, but we think the formalism is a good starting point for further simplifications or numerical

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1 Introduction

Since Landau's work in the 50's [L], understanding of interacting Fermi systems in $d > 1$, d being spatial dimension, has been unsatisfactory. In fact, there are still many mechanisms being currently investigated which could account for high-temperature superconductivity, and much of the work is being carried out on simplified models, such as Hubbard, or using uncontrolled approximations as mean-field, infinite dimension, *etc.* On the other hand, the case $d = 1$ and its non-Fermi liquid behavior are very well understood from a rigorous point of view [BGPS] by using renormalization group (RG) techniques.

The purpose of this work is to study interacting fermions in more than 1 dimension, understanding why they are more difficult to study than their 1-dimensional counterparts. The answer is that the source of the differences is the fact that, if the dimension d is greater than 1, the fermion propagator is singular on a surface (we are going to analyze the simpler case of spherical Fermi surface) rather than on a single point (as in Field Theory or in the theory of Critical Phenomena), or on 2 points as in the 1-dimensional interacting fermions case. Consequently, we are going to see that naive power counting or dimensional arguments employed in usual RG problems fail. Furthermore, this failure has a simple geometric explanation and important concepts such as Landau quasiparticles and Cooper pairs arise rather naturally in this geometric context. In this paper we intend to concentrate the discussion more on the *ideas* and less on the technical details. The more interested reader should refer to our long

Our formalism is similar to the one developed for fermions by Benfatto and Gallavotti, see [BG1], [BG2], in the sense that it is based on a multiscale decomposition of the fermion propagator that works only after introduction of degrees of freedom on the Fermi surface, identified with quasiparticle momenta, not present in the original problem. One difference is that the decomposition of Benfatto and Gallavotti works only for *odd* spatial dimensions, thus excluding the case $d = 2$ believed to be important for high-temperature superconductivity. Another difference is that their decomposition is performed in real space, whereas we work in momentum space, which is a more natural setting for fundamental concepts in the theory, such as Fermi surface, Fermi momentum and Cooper pairs.

The most important difference of our approach with respect to the work of Benfatto and Gallavotti is that they rely only on dimensional arguments in classifying as relevant or irrelevant, in the RG sense, the various couplings that appear. As a consequence, their beta functional contains not only an infinite number of running couplings (so is ours), consequence of the infinite number of points on the Fermi surface (if $d > 1$), but it contains also complicated terms in these couplings, failing to distinguish important momentum configurations as Cooper pairs and forward scattering. On the contrary, in our formalism we show by improved bounds that these configurations appear naturally as the relevant ones and a renormalization scheme is devised in order to distinguish them.

The distinguished role played by Cooper pairs and forward scattering configurations, as a consequence of the unusual fact that the RG is being performed around a surface, was also noticed by Shankar, [S]. The difference between his formalism and ours is that his is non-rigorous, neglecting all irrelevant couplings and thus obtaining an oversimplified beta functional, perhaps hiding some important complications.

Our result is a new rigorous RG formalism which can be used to study interacting Fermi systems in $d > 1$ to any order in perturbation theory. The beta functional we obtain is more complicated than Shankar's, but simpler than Benfatto and Gallavotti's. Up to now it has not yet been completely analyzed, even at only second order in perturbation theory, so we can not conclude yet about normality (Fermi liquid) or anomaly (non-Fermi liquid) of the system. Nevertheless, we think that our approach can be useful for a further study of the beta functional either by numerical methods, or in some kind of approximation.

We came to know recently that the same results of ours concerning the improved bounds with respect to power counting were obtained by Feldman *et al.* in [FST], in the more general case of non necessarily spherical, non-nested Fermi surfaces. But they use a different renormalization scheme and, as far as we can see, they cannot conclude about normality or anomaly, either.

The plan of our exposition is as follows. In the next section we sketch, for the sake of comparison, the "usual" RG formalism for the infrared problem of bosonic massless scalar fields together with the corresponding formalism for fermions. Finally, in section 3 we explain the different renormalization scheme for the case of fermions and present the beta functional to second order in perturbation theory.

2 Comparing the RG for fermions in $d > 1$ with the "usual" case

Consider a system of *spinless* fermions (for simplicity) in d spatial dimensions at zero temperature interacting through a rotational invariant pair potential $\lambda_0(\vec{x}-\vec{y})$. The system is studied in the grand-canonical ensemble. The (grand-canonical) Hamiltonian for the non-interacting system is $H_0 = T - \mu N$, where $T = \frac{1}{2m}\int d\vec{x}\, \vec{\partial}\psi^+_{\vec{x}} \cdot \vec{\partial}\psi^-_{\vec{x}}$ is the total kinetic energy, μ is the chemical potential and $N = \int d\vec{x}\, \psi^+_{\vec{x}}\psi^-_{\vec{x}}$ is the number of fermions operator. In the above formulae $\psi^\pm_{\vec{x}}$ are respectively creation and annihilation operators for fermions at position $\vec{x}$. The interaction part to be summed to H_0 is $\overline{\nu}_0 N + V$, where V, given by $V = \frac{1}{2}\int d\vec{x}\, d\vec{y}\, \lambda_0(\vec{x}-\vec{y})\, \psi^+_{\vec{x}}\psi^+_{\vec{y}}\psi^-_{\vec{y}}\psi^-_{\vec{x}}$, is the pair potential interaction and $\overline{\nu}_0$ is the chemical potential counterterm. The Fermi momentum p_F is given by $p_F = \sqrt{2m\mu}$.

The free propagator for fermions is [BG1]

$$\langle \psi^-_x \psi^+_y \rangle \;\equiv\; C(x-y) \;=\; \int \frac{dp}{(2\pi)^{d+1}} \, \frac{e^{ip_0(x_0-y_0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{ip_0+e(\vec{p})} \;,$$

where $e(\vec{p}) = (\vec{p}^2 - p_F^2)/2m$.

The propagator $\hat{C}(p) = \frac{1}{ip_0+e(\vec{p})}$ in momentum space is singular at $p = \infty$ (ultraviolet) and at the *Fermi sphere* $p_0 = 0$, $|\vec{p}| = p_F$ (infrared). If we drop the UV part, which is not interesting to our low energy problem, the remaining IR part may be decomposed, see [N], as

$$C_{IR}(x) \;=\; \sum_{n=-\infty}^{0} \int \frac{dk d\vec{\omega}}{(2\pi)^{d+1}} \, 2^{-n} \, g_n(2^{-n}k) \, e^{i[k^0 t + (k^1+p_F)\vec{\omega}\cdot\vec{x}]} \;,$$

with

$$g_n(k) = \Omega_d p_F^{d-1} \int_{1/4}^{1} \; d\alpha \, \theta\left(k^1 + 2^{-n} p_F\right) \left[-ik^0 + \beta k^1 (1 + \frac{2^{n-1}}{p_F} k^1)\right]$$
$$(1 + \frac{2^n}{p_F} k^1)^{d-1} e^{-\alpha\left[(k^0)^2 + \beta^2 (k^1)^2 \left(1+\frac{2^{n-1}}{p_F}k^1\right)^2\right]} \;,$$

being $\beta = p_F/m$ the Fermi velocity, Ω_d the surface of the d-dimensional unit sphere, integration in k means integration over real variables k^0 and k^1, integration over $\vec{\omega}$ means averaging over the d-dimensional unit sphere and θ is the step function. In these equations, $k^0 = p_0$ and k^1 is not a momentum, but the algebraic distance to the Fermi sphere (that is why we have also a step function). The unit vector $\vec{\omega}$ gives the direction of momentum. Accordingly, the spatial part of the momentum is written as $\vec{p} = (k^1 + p_F)\vec{\omega}$.

Notice that $g_n(k)$ depends weakly on n in the asymptotic limit $n \to -\infty$ and that it decays exponentially at scale 1 for all $n \leq 0$. Our decomposition is then a multiscale decomposition appropriate for a RG, being $C_n(k) = 2^{-n} g_n(2^{-n}k)$ the propagator responsible for a region in momentum space with $k = O(2^n)$, *i.e.* a shell of momentum scale $O(2^n)$ around the Fermi sphere.

The corresponding decomposition of the fields is

$$\psi^{\pm}_x \;=\; \sum_{n=-\infty}^{0} \int \frac{dk d\vec{\omega}}{(2\pi)^{d+1}} \, \psi^{\pm(n)}_{k,\vec{\omega}} \, e^{\pm i[k^0 t + (k^1+p_F)\vec{\omega}\cdot\vec{x}]} \theta(k^1 + p_F)$$

and the new independent fermionic fields $\psi^{(n)}_{k,\vec{\omega}}$ have as propagator

$$\langle \psi^{-(m)}_{k,\vec{\omega}} \psi^{+(n)}_{k',\vec{\omega}'} \rangle \;=\; \delta_{mn} \, \delta(\vec{\omega} - \vec{\omega}') \, \delta(2^{-n}(k-k')) \, 2^{-3n} g_n((2^{-n}k)) \;.$$

When $n \to -\infty$, then $k = O(2^n) \to 0$ and the fields $\psi^{\pm(n)}_{k,\vec{\omega}}$ represent fermionic objects with momentum $p_F\vec{\omega}$ *living on the Fermi surface*, which are naturally interpreted as the quasiparticles in Landau's model, [L].

Take now the IR problem for massless scalar fields as an example of "usual" RG. The corresponding propagator in momentum space is $\hat{C}(k) = \frac{1}{k^2}$, singular again both at $k = 0$ (IR) and $k = \infty$ (UV). Its IR part may be scale decomposed as $\hat{C}_{IR}(k) = \sum_{n=-\infty}^{0} 2^{-2n} C(2^{-n}k)$, being

$$C(k) = \frac{e^{-1/4k^2}}{k^2}(1 - e^{-3/4k^2})$$

As $C(k)$ decays exponentially at scale 1, then $2^{-2n}C(2^{-n}k)$ is very small for $|k| \gg 2^n$ and can be interpreted as the propagator of the fields at momentum scale 2^n. This is the multiscale decomposition of the IR part of the propagator in the "usual" case, analogous to the previous decomposition for fermions. Notice the difference that the scales in the fermion case are built around the *surface* $k = 0$, whereas in the "usual" case they are built around the *point* $k = 0$.

Consider as an example the graph in the figure below, where the lines may represent either fermionic or "usual" propagators.

Figure 1

Its form factor is

$$f_h(p_1, \ldots, p_4) \;=\; \int \frac{dp_5}{(2\pi)^{d+1}}\, \lambda_h(1,2,5,6)\, \lambda_h(5,6,3,4)\, 2^{-h} g_h(2^{-h}k_5)\, 2^{-h} g_h(2^{-h}k_6)\ .$$

Let us describe the meaning of the symbols in both cases. The $\lambda_h(i_1, \ldots, i_4)$ are the running couplings associated with the vertices at scale h, where the i's are a notation to show the momenta on which they depend in the fermion case. In the "usual" case, the λ_h are constants, *i.e.* they do not depend on any momenta. We will see ahead why there is such a difference. In both cases, integration on p_5 means integrating over all possible momenta of the line 5. In the "usual" case, the k-variables are then plain momenta, whereas in the fermion case, they are the distance to the Fermi surface and dp_5 means $dk_5 d\vec{\omega}_5$. In both cases, too, there is a momentum-conservation δ-function at each vertex. One of these guarantees overall momentum conservation in the graph; the other has been already integrated, so that the momentum of line 6 is to be seen as a function of momenta of lines 1, 2 and 5. In the "usual" case, this means only $k_6 = k_1 + k_2 - k_5$. In the fermion case we have

$$\begin{aligned} k_6^0 &= k_1^0 + k_2^0 - k_5^0 \\ k_6^1 &= |(k_1^1 + p_F)\vec{\omega}_1 + (k_2^1 + p_F)\vec{\omega}_2 - (k_5^1 + p_F)\vec{\omega}_5| - p_F \\ \vec{\omega}_6 &= \frac{(k_1^1 + p_F)\vec{\omega}_1 + (k_2^1 + p_F)\vec{\omega}_2 - (k_5^1 + p_F)\vec{\omega}_5}{k_6^1 + p_F}\ . \end{aligned}$$

Divergences and renormalization in the "usual" case:

The form factor above can be easily estimated by dimensional arguments. As most of the contribution to the integral comes from momentum $k_5 = O(2^h)$ (otherwise $g(2^{-h}k_5)$ is extremely small), define the dimensionless variable $k_5' = 2^{-h}k_5$. Then

$$f_h(k_1, k_2) \;=\; 2^{(d-4)h} \lambda^2 \int dk_5'\, g(k_5')\, g(2^{-h}(k_1 + k_2) - k_5'))\ .$$

If k_1 and k_2 are $O(2^h)$ or less, then both g-functions inside the integral are maximum approximately simultaneously and the integral is $O(1)$. Otherwise, the integral is less than $O(1)$. We have then

$$|f_h(k_1, k_2)| < C\, 2^{(d-4)h} \;=\; C\, 2^{(d-4)h'}\, 2^{(d-4)(h-h')}\ ,$$

where C is some $O(1)$ constant. This result is the *power counting* estimate for the form factor considered. If we sum this estimate from $h = h' + 1$ up to $h = 0$, in order to recover the contribution of all scales $h > h'$ to the effective potential at scale h', we get a divergence as $h' \to -\infty$ if $d \geq 4$.

Of course, this power counting procedure may be generalized to any graph.

We define the *renormalized form factor* for the graph considered as

$$f_h^R(k_1, k_2) \;=\; f_h(k_1, k_2) - f_h(0, 0)\;.$$

Repeating power counting bounds for this renormalized form factor, we see that the subtraction of $f_h(0,0)$ produces an extra $2^{-h}(k_1+k_2)$ factor, which can be rewritten as $2^{-h'}(k_1+k_2)2^{-(h-h')}$. As $2^{-h'}(k_1+k_2)$ is dimensionless, we have

$$|f_h^R(k_1, k_2)| \sim C\, 2^{(d-4)h'}\, 2^{(d-5)(h-h')}\;,$$

showing that the sum of $f_h^R(k_1, k_2)$ from $h = h'+1$ up to $h = 0$ does not diverge anymore if $d = 4$. The subtracted parts $f_h(0,0)$ at scale h will contribute to the running couplings λ_h at scale $h-1$. As they do not depend on momenta, we have the usual running coupling *constants*.

Divergences in the fermion case:

If it were not for the extra integration on $\vec{\omega}$, then the interacting-fermions problem would be a standard RG. In fact, if we bound, as in the "usual" case, each of the λ's and the function $g_h(2^{-h}k_6)$ by a constant, then every dependence on the $\vec{\omega}$'s is lost (recall that k_6^1 depends on $\vec{\omega}_5$) and, rescaling $k_5' = 2^{-h}k_5$, we obtain

$$|f_h(k_1, \ldots, k_4, \vec{\omega}_1, \ldots, \vec{\omega}_4)| \;<\; C\;.$$

This kind of bound can be generalized to any graph. It was obtained in [BG1] using real-space techniques and is the fermion version of the *power counting*. By summing it over h we would have a divergent bound for the graph considered, *regardless of the momentum configuration of external legs.*

Benfatto and Gallavotti proceeded by renormalizing this graph and all others according to their degree of divergence obtained by power counting. In the present case, in our momentum space language, they would renormalize the form factor as

$$f_h^R(k_1, \ldots, k_4, \vec{\omega}_1, \ldots, \vec{\omega}_4) \;=\; f_h(k_1, \ldots, k_4, \vec{\omega}_1, \ldots, \vec{\omega}_4) - f_h(0, \ldots, 0, \vec{\omega}_1, \ldots, \vec{\omega}_4)\;.$$

Of course their procedure works, in the sense that it removes all divergences of the theory. In the case $d = 1$, as the Fermi sphere is only 2 points, one loses nothing by throwing away integration on $\vec{\omega}$ as in power counting, and their formalism produced great results. [3] On the other hand, if $d > 1$, their renormalization procedure removes

[3] We are not claiming that the case $d = 1$ is easy. We are saying only that it is "usual" in the RG sense. In spite of its "usualness", it is a difficult problem as many other difficulties, such as anomalous scaling, have to be overcome. This was accomplished in [BGPS] up to the point of completely understanding this interesting case by means of a convergent perturbation theory.

the divergences at the cost of subtracting from the bare form factor more than the necessary, complicating thus the beta functional.

Here is where our formalism comes in. The fact that the running couplings depend on the $\vec{\omega}$'s, is unavoidable, also in the formalism of Benfatto and Gallavotti, and we are thus forced to deal with running coupling *functions* in our beta functional, instead of the *finite* number of running coupling *constants* of the "usual" cases. We have now an infinite number of couplings to keep track of. It is then important to simplify the beta functional as much as possible. That can be done if we get bounds better than power counting and then choose a renormalization procedure according to these improved bounds. The main technical result on which our different renormalization procedure is based is the following theorem, see [N] for the proof and a more general and complete formulation:

Main theorem *Using the same numbering as in figure 1, let*

$$\begin{aligned}\Gamma_0 &= k_1^0 + k_2^0 \\ \vec{\Gamma} &= (k_1^1 + p_F)\vec{\omega}_1 + (k_2^1 + p_F)\vec{\omega}_2 \\ \Gamma &= |\Gamma_0| + |\vec{\Gamma}| \,.\end{aligned}$$

Then, if $d > 1$, there exist constants C and N_0 independent of h such that for $2^h N_0 < \Gamma < p_F$ we have

$$|f_h| < C\,\frac{2^h}{\Gamma}\,.$$

This result shows that if $0 \neq \Gamma < p_F$ and h is small enough, then power counting is improved by a 2^h factor, which is enough to turn divergence into convergence. We turn now to heuristic arguments showing that it is consequence of the "unusual" fact that the singularity of the propagator is on a surface, rather than on a point.

Recall the analysis of the divergence of the same graph in the scalar-field case: regardless of the direction of external momenta k_1 and k_2, if they are both $O(2^h)$ then k_6 is $O(2^h)$ whenever k_5 is also $O(2^h)$. That does not happen in the fermions case, where the k's are not momenta anymore, but distances to the Fermi surface; we may even put $k_1, k_2, k_5 = 0$ (momenta $\vec{p}_1, \vec{p}_2, \vec{p}_5$ on the Fermi surface) and have k_6 far away from 0, as the example in the figure below shows.

Figure 2

Suppose $k_5^1 = O(2^h)$ (dominant values in the integral as $g_h(2^{-h}k_5^1)$ is close to its maximum) and $|\vec{\Gamma}| \gg 2^h$. Draw $\vec{\Gamma}$ and three circles of radius p_F and $p_F \pm 2^h$, respectively, with center at the origin of $\vec{\Gamma}$, and another circle with center at the extremity of $\vec{\Gamma}$ and radius p_F, see figure below.

Figure 3

Then the points on the right circle in the figure are the extremities of all possible vectors $\vec{p}_6 \approx \vec{\Gamma} - p_F\vec{\omega}_5$ when $\vec{\omega}_5$ is varied. The intersections with the left circle of radius p_F are the values of $\vec{p}_6$ when its modulus is p_F, thus $k_6^1 = 0$. The regions of the right circle between the circles of radii $p_F \pm 2^h$ represent the regions where $\vec{\omega}_5$ can vary and such that k_6^1 is $O(2^h)$, meaning that the propagator of line 6 is not too small. Of course the size of these regions is also $O(2^h)$ provided that $|\vec{\Gamma}| \gg 2^h$. If we neglect the dependence of k_6 on $\vec{\omega}_5$, as was done in power counting, then integration on $\vec{\omega}_5$ yields an $O(1)$ constant. If we see, as in figure 3, that the bound $|g_h(k_6)| < C$ is too big an overestimate for all values of $\vec{\omega}_5$ except for those in the shaded region of size $O(2^h)$, then integration on $\vec{\omega}_5$ yields instead the 2^h improvement of the theorem. Of course the argument breaks down if $|\vec{\Gamma}|$ is $O(2^h)$ a

Consequences of the Main theorem:

The theorem shows that, opposedly to what one would expect by power counting, the form factor of the graph in figure 1 (when summed on h) does not diverge for most external-momenta configurations. No matter how small it is, if $\Gamma \neq 0$, then there will be some small enough h such that condition $2^h N_0 < \Gamma$ is fulfilled and the result holds. This shows that the only configurations of external momenta for which the graph may diverge are the ones with $\Gamma = 0$. But $\Gamma = 0$ implies that $k_1^0 = -k_2^0$, $k_1^1 = k_2^1$, $\vec{\omega}_1 = -\vec{\omega}_2$, *i.e.* the momenta $\vec{p}_1$ and $\vec{p}_2$ of the particles created by the scattering process represented by the graph of figure 1 are such that $\vec{p}_1 = -\vec{p}_2$. In other words, particles 1 and 2 form a Cooper pair and we call a $\Gamma = 0$ configuration a *Cooper pairs configuration.*

Of course, Γ has the dimension of a momentum, so any given value of Γ should be compared to the momenta at scale h of the RG. If $\Gamma \ll 2^h$, where 2^h is a typical value at scale h, then the external-momenta configuration with parameter Γ can be considered at scale h as an *approximate Cooper pair.* Given any small $\Gamma \neq 0$, a configuration with such Γ is initially ($h = 0$) an approximate Cooper pair; after the RG evolves and h is decreased, it becomes less and less so, until h will be small enough and it will not be anymore an approximate Cooper pair. By this point of view, condition $2^h N_0 < \Gamma$ in the main theorem says that, as long as the configuration considered is an approximate Cooper pair, *i.e.* the condition is *not* fulfilled, we may not use the theorem and the best we can do is estimate the form factor by power counting. As h is decreased, the condition is eventually fulfilled and the theorem can be used to improve power counting.

Using more RG terminology, the main theorem states that the contribution of the graph in figure 1 is *relevant* (*marginal*, to be more exact) while h is not small enough and the configuration of external momenta is an approximate Cooper pair;

it becomes then *irrelevant* when h is sufficiently small for the configuration not to be anymore an approximate Cooper pair. This is to be contrasted with the "usual" scalar case in which the contribution of the same graph is relevant (marginal, to be more exact) regardless of the configuration of external momenta.

The same result obtained for the special graph in figure 1 holds for a whole class of graphs, which we call *direct*, with constants C and N_0 being the same for any of these graphs. The definition is the following: a graph is called *direct* if it has 4 external legs, if these external legs originate as pairs from only 2 vertices, if both external legs at the same vertex have arrows pointing in the same direction and if it has no loops sharing an internal line. In figure 4 we show some examples of direct graphs.

Figure 4

It is also obvious that an analogous result holds for the so-called *exchange graphs*. A graph is called *exchange*, see figure 5 for the prototype of them, if it has 4 external legs, if these legs originate as pairs from only 2 vertices, if the external legs originating from the same vertex have arrows pointing in opposite directions and if it has no loops sharing an internal line. Other examples may be obtained from direct graphs by reversing some arrows. Notice the numbering of the external lines in direct and exchange graphs; the convention is that outgoing lines are 1 and 2 and incoming are 3 and 4 for both types.

The result for exchange graphs is the same (with the same constants C and N_0), but with Γ substituted by Δ, defined as

$$\begin{aligned}
\Delta_0 &= k_1^0 - k_3^0 \\
\vec{\Delta} &= (k_1^1 + p_F)\vec{\omega}_1 - (k_3^1 + p_F)\vec{\omega}_3 \\
\Delta &= |\Delta_0| + |\vec{\Delta}| \;.
\end{aligned}$$

The only configurations for which the exchange graphs are divergent are the ones with $\Delta = 0$, which are *forward scattering* configurations.

There exist also 4-legged graphs which are neither direct, nor exchange. See figure 6 below for the simplest example of them.

Figure 5

Figure 6

An important feature of these graphs is that they must have at least two loops sharing an internal line. We have proved, as a consequence again of the main theorem, see theorem 5 on loop improvements in [N], that the condition of having 2 loops sharing an internal line is sufficient to guarantee an improvement beyond power counting *independent* of momentum configuration. As 4-legged graphs are already marginal by power counting, this improvement proves irrelevance in the RG sense of 4-legged graphs which are neither direct, nor exchange.

3 Renormalization and beta functional

Graphs with 6 or more external legs can be shown by pure power counting to be irrelevant, and thus need no renormalization. As we have already seen at the end of the last section, also 4-legged graphs which are neither direct, nor exchange need not be renormalized. It remains then to renormalize direct and exchange graphs and 2-legged graphs.

Let us first consider the direct ones. We should invent a renormalization procedure that acts only on approximate Cooper pairs, considering that *approximate* is a scale-dependent concept, and such that the renormalized form factor does not diverge anymore when summed over the scales. Our definition is

$$f_h^R = (1 - e^{-2^{-(h-1)}\Gamma})\, f_h \ ,$$

which passes smoothly from $f_h^R \approx 0$ when $\Gamma \ll 2^h$ (Cooper pairs) to no renormalization at all, *i.e.* $f_h^R \approx f_h$ when $\Gamma \gg 2^h$. It can be shown, using the bound in the main theorem, see proposition 6 in [N], that if m is the number of vertices in some direct graph and M is a bound for the running couplings, then $|\sum_{h=h'+1}^{0} f_h^R| < C^m M^m$, where C is a constant independent of h', Γ and of the particular direct graph considered.

Analogously, exchange graphs are renormalized as $f_h^R = (1 - e^{-2^{-(h-1)}\Delta})\, f_h$.

The relevant parts $f_h^L = f_h - f_h^R$ subtracted at scale h by renormalization are added together to build the running coupling functions at scale $h-1$. As direct and exchange graphs behave differently, it is interesting not to unite them in a single running coupling function for 4-field terms. We define then one running coupling λ_{h-1} formed by contributions of the direct graphs and one μ_{h-1} formed by the exchange graphs.

As f_h^L for each direct graph is proportional to $e^{-2^{-(h-1)}}\Gamma$, it seems reasonable to define λ_{h-1} in a *dimensionless* fashion as the sum of the f_h^L for all direct graphs (up to some perturbative order) at scale h *divided by* $e^{-2^{-(h-1)}}\Gamma$. Analogously for exchange graphs. The contribution of a vertex at scale h is then either $e^{-2^{-h}\Gamma}\lambda_h$, or $e^{-2^{-h}\Delta}\mu_h$.

As there are two different types of running couplings associated with 4-legged vertices, we have now to identify each of these vertices as a λ or a μ. As the factors $e^{-2^{-h}\Gamma}$ and $e^{-2^{-h}\Delta}$ appended respectively to λ_h and μ_h are less than 1, then all preceding bounds remain valid, but can be still improved in certain cases, simplifying further the beta functional. Consider for example the graph in figure 1 when one of the vertices is a μ and the other a λ. As the Δ for this vertex depends on the integration variables, and it is forced to be $O(2^h)$ so that the factor $e^{-2^{-h}\Delta}$ is not too small, the effective size of the region of integration on $\vec{\omega}$ is $O(2^h)$. As we gain an extra 2^h in the bound, this graph need not be renormalized. This reasoning shows that the only direct graphs which must be renormalized are the ones in which all vertices are λ and the only exchange graphs to be renormalized are the ones in which all vertices are μ.

It remains now to renormalize the graphs with 2 external legs. Power counting gives them dimension -1, see [N], so they seem to be relevant for any momentum $\vec{p}$. A simple argument, see again [N], shows that they are relevant only for momentum close to the Fermi surface, *i.e.* $k \approx 0$. We propose a smooth prescription in which k is compared to 2^h at scale h:

$$f_h^R(k) \;=\; \left(1 - e^{-2^{-(h-1)}|k|}\right)^2 \; f_h(k) \;,$$

where $|k| \equiv |k_0| + |k_1|$. Of course it can be shown that $\sum_{h=h'+1}^{0} f_h^R$ does not diverge when $h' \to -\infty$.

Finally, we define the dimensionless running coupling functions $\nu_h(k)$ related to the 2-legged graphs. As each f_h^L is proportional to 2^h (power counting dimension equal to -1) and to $\chi(2^{-(h-1)}|k|) \equiv 1 - (1 - e^{-2^{-(h-1)}|k|})^2$, then it is natural to define $\nu_{h-1}(k)$ as the sum of the f_h^L of all the 2-legged graphs at scale h divided by $2^{h-1}\,\chi(2^{-(h-1)}|k|)$. Using the rotational invariance of the potential $\lambda_0(\vec{x}-\vec{y})$ we can show that the $\nu_{h-1}(k)$ do not depend on $\vec{\omega}$.

Having already renormalized the theory, we may turn to the beta functional. To first order in perturbation theory it is

$$\begin{aligned}
\nu_{h-1}(k) &= 2\,\chi(2^{-h}|k|)\,\nu_h(k) \\
\lambda_{h-1}(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) &= e^{-2^{-h}\Gamma}\,\lambda_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) \\
\mu_{h-1}(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) &= e^{-2^{-h}\Delta}\,\mu_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4)
\end{aligned}$$

These equations show that, to first order, running couplings in configurations with non-zero k, Γ, Δ, respectively, are strongly depleted to zero as $h \to -\infty$. This shows to first order in perturbation theory that all interacting Fermi systems are (normal) Fermi liquids. See section 5 in [N] for the definition of a normal Fermi liquid in terms of the beta functional.

We believe that this picture may still hold to higher orders, only if the interaction potential is repulsive. Let us take a look at the second order to see why. The equation for the flow of the λ_h at second order is

$$\begin{aligned}\lambda_{h-1}(1,\ldots,4) &= e^{-2^{-h}\Gamma}\lambda_h(1,\ldots,4)\\ &- \int dk_5\, d\vec{\omega}_5\, G_h(k_5,\vec{\omega}_5)\,\Lambda_h(1,2,5,6)\,\Lambda_h(5,6,3,4)\ ,\end{aligned}$$

with

$$G_h(k_5,\vec{\omega}_5) = \frac{4}{(2\pi)^{d+1}}\, 2^{-2h}\,\overline{g}_h(2^{-h}k_6)\,\Big[g_h(2^{-h}k_5) + 2g_h^s(2^{-h}k_5)\Big] \tag{1}$$

and

$$\Lambda_h = e^{-2^{-h}\Gamma}\lambda_h + \sum_{l=h+1}^{0}\Big[(1-e^{-2^{-(l-1)}\Gamma})e^{-2^{-l}\Gamma}\lambda_l + (1-e^{-2^{-(l-1)}\Delta})e^{-2^{-l}\Delta}\mu_l\Big]\ . \tag{2}$$

In these equations, k_6 and $\vec{\omega}_6$ are functions given by momentum conservation in terms of momenta of lines 1, 2 and 5. In the formula for $G_h(k_5,\vec{\omega}_5)$, g_h^s is the propagator of a *soft* line, see appendix C in [G]. The various contributions to Λ_h come from the various possibilities for each vertex in the graph of figure 1, which is the only direct one at second order. The contributions in the terms in λ and μ inside the sum come from the irrelevant part of the first order vertices λ and μ, respectively, at preceding scales.

If $\Gamma \neq 0$, by the main theorem, the integral in the second order term goes as $\frac{2^h}{\Gamma}$ times an estimate for the Λ_h, when h is small enough. So it tends to be small as $h \to -\infty$, reinforcing the conjecture that the λ_h with $\Gamma \neq 0$ tend to zero as $h \to -\infty$, provided that these couplings have not grown too much while h is not small enough. What we mean by h not small enough is $2^h \gg \Gamma$. While this condition holds, we may expect that the evolution should be more or less that of exact Cooper pairs $\Gamma = 0$. If the λ_h do not grow unbounded at $\Gamma = 0$, we may expect that the same will be true while $2^h \gg \Gamma$.

In [N] we examine the case $\Gamma = 0$ in more detail and show that the evolution equation for the λ with $\Gamma = 0$ has interesting properties. An angular momentum decomposition of the λ_h in Cooper pairs already present in [BG1] is possible, in which positivity and real-valuedness properties of the equation allow us to conjecture that the λ_h at Cooper pairs also tend to zero *if the interaction potential is repulsive.* If the potential is attractive the λ seem to grow unbounded and the perturbative approach becomes meaningless.

The evolution of the μ-couplings is similar, in the sense that the configurations $\Delta \neq 0$ seem to go to zero quickly also at second order. The evolution at $\Delta = 0$ is also remarkably simplified and the same angular momentum decomposition can be performed, but it lacks the important positivity property of the λ's. In spite of that, a bounded evolution of the μ's does not seem to be ruled out. Concluding, to second order in perturbation theory, it seems reasonable that interacting Fermi systems in $d > 1$ are (normal) Fermi liquids if the potential is repulsive.[4].

About further orders, we can say that the angular momentum decomposition can still be used to simplify the evolution at any order. This should be contrasted with the formalism of [BG1], in which already at third order the decomposition fails. The reason for this difference is the fact that in our approach only direct and exchange graphs contribute to the beta functional, while the same is not true in the other. As the first 4-legged graph which is neither direct, nor exchange appears at third order, it is then very difficult to proceed with the analysis in the formalism of [BG1].

References

[BG1] G. Benfatto, G. Gallavotti: Perturbation theory of the Fermi surface in a quantum liquid. A general quasi-particle formalism and one-dimensional systems, *J. Stat. Phys.* **59**: 541 (1990).

[BG2] G. Benfatto, G. Gallavotti: Renormalization group approach to the theory of the Fermi surface, *Phys. Rev. B* **42**: 9967 (1990).

[BGPS] G. Benfatto, G. Gallavotti, A. Procacci, B. Scoppola: Beta function and Schwinger functions for a many fermion system in one dimension. Anomaly of the Fermi surface, *Comm. Math. Phys.***160**: 93(1994).

[FKST] J. Feldman, H. Knörrer, R. Sinclair, E. Trubowitz: Superconductivity in a repulsive model, *preprint*.

[FST] J. Feldman, M. Salmhofer, E. Trubowitz: Perturbation theory around non-nested Fermi surfaces I. Keeping the Fermi surface fixed, to appear in the September 1996 issue of *J. Stat. Phys.*.

[G] G. Gallavotti: Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods, *Rev. Mod. Phys.* **57**: 471 (1985).

[KL] W. Kohn and J. Luttinger, *Phys. Rev. Lett.* **15**, 524 (1965).

[L] L. Landau: Soviet Phys. JETP **3**: 920 (1957), Soviet Phys. JETP **5**: 101 (1957), Soviet Phys. JETP **8**: 70 (1958).

[4]This conjecture is in contrast with the non-rigorous result of Kohn and Luttinger for $d = 3$, [KL]. On the other hand, it is confirmed to second order and refuted at third in $d = 2$ for the particular model in [FKST]. Within our methods we do not see why there should be some difference between $d = 2$ and $d = 3$.

[N] A. G. M. Neves: Perturbation theory for the Fermi liquids in $d > 1$, *submitted to Braz. J. Phys.*.

[S] R. Shankar: Renormalization group approach to interacting fermions, *Rev. Mod. Phys.* **66**, no. 1 (1994).