Inverse Two Cluster Scattering *

Volker Enss
Ricardo Weder †
Institut für Reine und Angewandte Mathematik
Rheinisch–Westfälische Technische Hochschule Aachen
D–52056 Aachen, Germany
email: enss@rwth–aachen.de and weder@servidor.unam.mx

Abstract

We prove that the effective potential for the interaction between two clusters of quantum mechanical particles is uniquely determined by the high velocity limit of any one of the modified Dollard scattering operators. We give a formula with error term for the reconstruction of the effective potential which may be of long range.

Available by anonymous ftp from work1.iram.rwth-aachen.de (134.130.161.65)
in the directory /pub/papers/enss/ as en-95-2.tex, en-95-2.dvi, and en-95-2.ps;
or with WWW through URL:
http://www.iram.rwth-aachen.de/~enss/publEnss.html
or as Preprint No. 38, Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas, Febr. 1996, Universidad Nacional Autonoma de Mexico;
submitted for publication.

* AMS Classification 35P, 35R, 81U; PACS number 03.65 Nk

† In leave of abscence from IIMAS-UNAM. Apartado Postal 20–726. México, D.F. 01000. Fellow Sistema Nacional de Investigadores. Partially supported by Deutsche Forschungsgemeinschaft under grant Nr. Aa 2/94-1.

Introduction

We study in this paper an inverse two cluster scattering problem for the N–body Schrödinger equation where a cluster consisting of M particles in a bound state scatters off the other cluster that is a bound state of $N-M$ particles. This is a typical situation in scattering experiments, for example in atom–atom, atom–molecule, or ion–ion collisions. The interactions between the particles are given by pair potentials of short and long range. We prove that the high velocity limit of any one of the modified Dollard scattering operators, S^D, determines uniquely the effective multiparticle potential wich describes the interaction between the two clusters. We also obtain a formula with bound of the error term for the reconstruction of the effective potential from the high velocity limit of any one of the S^D's. See Theorem 1.1 at the end of the next section.

See e.g. [7] as a general reference for N–body (direct) scattering.

1 Notation and Main Results

We denote by m_j and $\tilde{\mathbf{x}}_j \in \mathbf{R}^n, n \geq 2,\ j = 1, 2, \ldots$ respectively, the masses and the positions of the particles. The free Hamiltonian

$$\tilde{H}_0 = \sum_{j=1}^{N} (2m_j)^{-1} \tilde{\mathbf{p}}_j^2,\ \ \tilde{\mathbf{p}}_j = -i\nabla_{\tilde{\mathbf{x}}_j},$$

generates the free time evolution. As usual we study the N–body system in the center of mass frame and we take off the kinetic energy of the center of mass

$$H_{CM} = (2\sum_{j=1}^{N} m_j)^{-1} (\sum_{j=1}^{N} \tilde{\mathbf{p}}_j)^2.$$

The free Hamiltonian in the center of mass frame is given by $H_0 := \tilde{H}_0 - H_{CM}$. The Hilbert space, $\mathcal{H}$, of states Φ in the center of mass frame is represented in the configuration representation by wave functions in

$$L^2(\mathbf{X}),\ \mathbf{X} = \{\, (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_N) : \sum_{j=1}^{N} m_j \tilde{\mathbf{x}}_j = 0 \,\} \cong \mathbf{R}^{n(N-1)},$$

with the measure induced on $\mathbf{X}$ by the norm $\left[\sum_{j=1}^{N} m_j \tilde{\mathbf{x}}_j^2 \right]^{\frac{1}{2}}$ on $\mathbf{R}^{nN}$.

The set of momentum space wave functions is the space

$$L^2(\hat{\mathbf{X}}),\ \hat{\mathbf{X}} = \{\, (\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_N) : \sum_{j=1}^{N} \tilde{\mathbf{p}}_j = 0 \,\} \cong \mathbf{R}^{N(n-1)},$$

where $\hat{X}$ is equipped with the dual metric induced by $[\sum_{j=1}^{N} (m_j)^{-1} \tilde{\mathbf{p}}_j^2]^{\frac{1}{2}}$ on $\mathbf{R}^{nN}$.

The measures on $\mathbf{X}$ and $\hat{\mathbf{X}}$ are equivalent to Lebesgue measure. The free Hamiltonian H_0 is self–adjoint on its domain $D(H_0) = W^{2,2}(\mathbf{X})$. The potential is a sum of pair potentials that are multiplication operators

$$V = \sum_{j<k} V_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j).$$

We split each pair potential into parts of short or long range depending on their differentiability and decay rate towards infinity.

$$V_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) = V^s_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V^\ell_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j). \tag{1.1}$$

An operator is said to be Kato–small in $L^2(\mathbf{R}^n)$ if it is bounded with respect to the Laplacian with relative bound zero (see [6]). Our class of short–range potentials is given by

$$\mathcal{V}_{SR} = \{\, V^s = \sum_{j<k} V^s_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) : V^s_{jk} \text{ is Kato–small in } L^2(\mathbf{R}^n), \text{ and for some}$$

$$0 \le \rho \le 1, \int_0^\infty dR\,(1+R)^\rho \|V^s_{jk}(\mathbf{y})(-\Delta + I)^{-1} F(|\mathbf{y}| \ge R)\| < \infty \,\}, \tag{1.2}$$

where for any set $O \subset \mathbf{R}^n$ we denote by $F(\mathbf{x} \in O)$ the multiplication operator by the characteristic function of O. Note that the decay condition in (1.2) is equivalent to the more intuitive condition that

$$(1+R)^\rho \|F(|\mathbf{y}| \ge R)\, V^s_{jk}(\mathbf{y})(-\Delta + I)^{-1}\|$$

decays integrably. The same implies integrability of

$$(1+R)^\rho \|V^s_{jk}(\mathbf{y}) g(\mathbf{p_y}) F(|\mathbf{y}| \ge R)\|$$

for any $C_0^\infty(\mathbf{R}^n)$ function g of the momentum $\mathbf{p_y}$ corresponding to $\mathbf{y}$.

Our class of long–range potentials is defined as follows.

$$\begin{aligned} \mathcal{V}_{LR} = \Big\{\, & V^\ell = \sum_{j<k} V^\ell_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) : V^\ell_{jk} \in C^1_\infty(\mathbf{R}^n), \\ & |\nabla V^\ell_{jk}(\mathbf{y})| \le C(1+|\mathbf{y}|)^{-\gamma}, \gamma > \frac{3}{2} \Big\}. \end{aligned} \tag{1.3}$$

C^1_∞ is the space of all continuously differentiable functions that tend to zero towards infinity. To simplify the notation we will only use $\gamma < 2$. It is clear that the splitting into short– and long–range parts is not unique. Without loss of generality [5] we can make it e.g. in such a way that furthermore

$$V^\ell_{jk} \in C^\infty(\mathbf{R}^n), |D^\alpha V^\ell_{jk}(\mathbf{y})| \le C(1+|\mathbf{y}|)^{-1-|\alpha|(\varepsilon+\frac{1}{2})} \tag{1.4}$$

for $1 \le |\alpha|, 0 < \varepsilon < \gamma - \frac{3}{2}$, where D^α denotes the derivatives with the usual multi–index notation.

The total Hamiltonian is defined as

$$H = H_0 + V = H_0 + V^s + V^\ell,$$

and it is a self–adjoint operator on $D(H) = D(H_0)$. The operator e^{-itH} generates the time evolution.

We consider a decomposition of our system into a cluster $C^{(1)}$ consisting of particles $1, 2, \ldots, M$, and a cluster $C^{(2)}$ that contains particles $M+1, M+2, \ldots, N$, where $1 \le M \le N-1$. We denote by $\mathbf{x}^{(j)}_{_{CM}}$, $j = 1, 2$, respectively, the coordinates of the centers of mass of the clusters $C^{(j)}$,

$$\mathbf{x}^{(1)}_{_{CM}} = (\sum_{j=1}^{M} m_j \mathbf{x}_j)(\sum_{j=1}^{M} m_j)^{-1}, \tag{1.5}$$

$$\mathbf{x}^{(2)}_{_{CM}} = (\sum_{j=M+1}^{N} m_j \mathbf{x}_j)(\sum_{j=M+1}^{N} m_j)^{-1}. \tag{1.6}$$

Let us denote by $\mathbf{x}$ and $\mathbf{p}$ the distance between the centers of mass of the two clusters, and the corresponding relative momentum, respectively,

$$\mathbf{x} = \mathbf{x}^{(2)}_{_{CM}} - \mathbf{x}^{(1)}_{_{CM}}, \ \mathbf{p} = -i\nabla_{\mathbf{x}}. \tag{1.7}$$

Using these coordinates the Hilbert space of states decomposes as follows

$$\mathcal{H} = \mathcal{H}^{(12)} \otimes \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}, \tag{1.8}$$

where $\mathcal{H}^{(12)} \cong L^2(\mathbf{R}^n)$ is the space of the states for the relative motion of the centers of mass of the two clusters, and $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ are the internal state spaces of the clusters $C^{(1)}$ and $C^{(2)}$ in their respective center of mass frames. We define the cluster Hamiltonian

$$H_c := H - \sum\nolimits^{(int)} V_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) = T + H^{(1)} + H^{(2)}, \tag{1.9}$$

with

$$T = \mathbf{p}^2/2m, \ m = \Big(\sum_{j=1}^{M} m_j\Big)\Big(\sum_{j=M+1}^{N} m_j\Big)\Big/\Big(\sum_{j=1}^{N} m_j\Big), \tag{1.10}$$

where $\sum^{(int)}$ means the sum over all pairs (jk) with $1 \le j \le M$, and $M+1 \le k \le N$ (intercluster potentials). T is the kinetic energy of the relative motion of the centers of mass of the two clusters, with m their reduced mass. $H^{(1)}$ and $H^{(2)}$ are, respectively, the internal Hamiltonians of the clusters $C^{(1)}$ and $C^{(2)}$.

In this two cluster channel the Dollard modified free time evolution is generated by the time–dependent Hamiltonian

$$H^D(t) := H_c + \sum\nolimits^{(int)} V_{jk}^{\ell}(t\mathbf{p}/m). \tag{1.11}$$

Note that $\mathbf{p}/m$ is the relative velocity of the centers of mass of the two clusters. The Dollard propagator for fixed initial time 0 is given by

$$U^D(t) := \exp\left\{ -itH_c - i\int_0^t ds \sum\nolimits^{(int)} V_{jk}^{\ell}(s\mathbf{p}/m) \right\} = e^{-it(H^{(1)}+H^{(2)})} U^D_{CM}(t), \tag{1.12}$$

where

$$U^D_{CM}(t) = e^{-itT} \exp\left\{ -i\int_0^t ds \sum\nolimits^{(int)} V_{jk}^{\ell}(s\mathbf{p}/m) \right\}$$

is the Dollard propagator for the relative motion of the centers of mass of the two clusters.

To specify a scattering channel we assume that the two clusters are, respectively, in normalized bound states $\Phi_b^{(1)} \in \mathcal{H}^{(1)}, \Phi_b^{(2)} \in \mathcal{H}^{(2)}$, of energies $E^{(1)}, E^{(2)}$, and we denote by P the orthogonal projector onto

$$\mathcal{H}^{(12)} \otimes \{ \Phi_b^{(1)} \} \otimes \{ \Phi_b^{(2)} \}.$$

Then

$$\begin{aligned} &H_c P = (T+E)P, \ E = E^{(1)} + E^{(2)}, \\ &U^D(t)P = e^{-iEt} U^D_{CM}(t)P. \end{aligned} \tag{1.13}$$

If one of the clusters consists of only one particle the corresponding internal space is just not introduced (or $\mathbf{C}$).

The two cluster channel wave operators are given by

$$\begin{aligned} \Omega^D_{\pm} :=& s - \lim_{t\to\pm\infty} e^{itH} U^D(t)P \\ =& s - \lim_{t\to\pm\infty} e^{itH} e^{-itE} U^D_{CM}(t)P. \end{aligned} \tag{1.14}$$

The existence of the strong limits is well known and it also follows from our estimates below, see (2.16). The two cluster Dollard modified channel scattering operator is given by

$$S^D := (\Omega^D_+)^* \, \Omega^D_-. \tag{1.15}$$

Note that since in the high velocity limit the interaction between the clusters goes to zero it is natural to consider the case where the clusters are in the same bound state before and after the scattering takes place.

The appropriate high velocity states are defined as follows. Let us take states

$$\Phi_0 = \Phi_0^{(12)} \otimes \Phi_b^{(1)} \otimes \Phi_b^{(2)}, \quad \Psi_0 = \Psi_0^{(12)} \otimes \Phi_b^{(1)} \otimes \Phi_b^{(2)}, \tag{1.16}$$

with $\hat{\phi}_0^{(12)}(\cdot) \in C_0^\infty(\mathbf{R}^n)$ and all factors are normalized. We define

$$\begin{aligned} \Phi_{\mathbf{v}}^{(12)} &= e^{im\mathbf{v}\cdot\mathbf{x}}\Phi_0^{(12)} \iff \hat{\phi}_{\mathbf{v}}^{(12)}(\mathbf{p}) = \hat{\phi}_0^{(12)}(\mathbf{p} - m\mathbf{v}), && (1.17)\\ \Phi_{\mathbf{v}} &= e^{im\mathbf{v}\cdot\mathbf{x}}\Phi_0 \iff \hat{\phi}_{\mathbf{v}}(\mathbf{p}) = \hat{\phi}_0^{(12)}(\mathbf{p} - m\mathbf{v}) \otimes \Phi_b^{(1)} \otimes \Phi_b^{(2)}, && (1.18)\end{aligned}$$

where by $\hat{}$ we denote the momentum representation (Fourier transform) only for the relative motion of the centers of mass of the two clusters. Analogously for $\Psi_{\mathbf{v}}$ with the same bound state content.

The effective potential for the interaction between the two clusters is the following continuously differentiable function (see Lemma 2.1) of the relative distance between the centers of mass of the clusters

$$\begin{aligned} V_{\text{eff}}(\mathbf{x}) &= V_{\text{eff}}^s(\mathbf{x}) + V_{\text{eff}}^\ell(\mathbf{x}), \quad \text{with} && (1.19)\\ V_{\text{eff}}^s(\mathbf{x}) &= \sum\nolimits^{(int)} \left(V_{jk}^s(\mathbf{x} + \mathbf{y}_k - \mathbf{y}_j)\Phi_b^{(1)} \otimes \Phi_b^{(2)}, \Phi_b^{(1)} \otimes \Phi_b^{(2)}\right)_{\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}}, && (1.20)\\ V_{\text{eff}}^\ell(\mathbf{x}) &= \sum\nolimits^{(int)} \left(V_{jk}^\ell(\mathbf{x} + \mathbf{y}_k - \mathbf{y}_j)\Phi_b^{(1)} \otimes \Phi_b^{(2)}, \Phi_b^{(1)} \otimes \Phi_b^{(2)}\right)_{\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}}, && (1.21)\end{aligned}$$

where $\mathbf{y}_i, 1 \le i \le N$, is the distance of the particle i relative to the center of mass of the corresponding cluster, i.e.

$$\begin{aligned} \mathbf{y}_j &= \tilde{\mathbf{x}}_j - \mathbf{x}_{CM}^{(1)}, 1 \le j \le M, && (1.22)\\ \mathbf{y}_k &= \tilde{\mathbf{x}}_k - \mathbf{x}_{CM}^{(2)}, M + 1 \le k \le N. && (1.23)\end{aligned}$$

Then $\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j = \mathbf{x} + \mathbf{y}_k - \mathbf{y}_j$, for $1 \le j \le M$, and $M + 1 \le k \le N$.

In what follows we assume that the bound state eigenfunctions have the following decay

$$\begin{aligned} &(1 + R)^\sigma \left\{ \left\|F(|\mathbf{y}_j| \ge R)\Phi_b^{(1)}\right\|_{\mathcal{H}^{(1)}} + \left\|F(|\mathbf{y}_k| \ge R)\Phi_b^{(2)}\right\|_{\mathcal{H}^{(2)}} \right\} \\ &\qquad \in L^1\big((0,\infty), dR\big) \text{ for } 1 \le j \le M < k \le N \text{ and some } \sigma \ge 0. \end{aligned} \qquad (1.24)$$

Condition (1.24) will be true if the bound state eigenfunctions $\Phi_b^{(j)}$ have enough decay. Typically they decay exponentially which is more than enough. We use these assumptions in Section II to estimate the effective potential, see Lemma 2.1.

In the following theorem we state our formulae for the reconstruction of the effective potential. We denote $\hat{\mathbf{v}} := \mathbf{v}/|\mathbf{v}|$, and $v := |\mathbf{v}|$.

THEOREM 1.1. (Reconstruction Formulae) *Suppose that $V^s \in \mathcal{V}_{SR}$, $V^\ell \in \mathcal{V}_{LR}$, and that (1.24) is satisfied. Then for all $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ as in (1.18)*

$$
\begin{aligned}
& v\Big[\Big(i(S^D - I)\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\Big) \\
& \quad - \Big(\int_{-\infty}^{+\infty} ds \sum\nolimits^{(int)} \Big(V^\ell_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j) - V^\ell_{jk}(t\mathbf{p}/2m)\Big)U^D(t)\Phi_{\mathbf{v}}, U^D(t)\Psi_{\mathbf{v}}\Big)\Big] \\
& = \Big(\int_{-\infty}^{\infty} d\tau V^S_{\text{eff}}(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi^{(12)}, \Psi^{(12)}\Big)_{\mathcal{H}^{(12)}} + \begin{cases} o(v^{-\rho'}), & 0 \le \rho' < 1, \\ O(v^{-1}), & \rho' = 1, \end{cases}
\end{aligned} \tag{1.25}
$$

where $\rho' \le \min\{\rho, \sigma\}$, and in particular

$$
\begin{aligned}
& \lim_{v\to\infty} v\Big(i(S^D - I)\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\Big) \\
& = \Big(\int_{-\infty}^{+\infty} d\tau \Big[V^S_{\text{eff}}(\mathbf{x}+\tau\hat{\mathbf{v}}) + V^\ell_{\text{eff}}(\mathbf{x}+\tau\hat{\mathbf{v}}) - \sum\nolimits^{(int)} V^\ell_{jk}(\tau\hat{\mathbf{v}})\Big]\Phi^{(12)}, \Psi^{(12)}\Big)_{\mathcal{H}^{(12)}}.
\end{aligned} \tag{1.26}
$$

For p_1 (or any other component of the momentum)

$$
v\Big(i[S^D, p_1]\Phi_{\mathbf{v}},\ \Psi_{\mathbf{v}}\Big) = i\int_{-\infty}^{+\infty} d\tau \Big((\partial_1 V_{\text{eff}})(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi^{(12)}, \Psi^{(12)}\Big)_{\mathcal{H}^{(12)}} + o(v^{-\rho'}) \tag{1.27}
$$

with $\rho' \le$ min$\{\rho,\ \sigma\}$ and $\rho' < \gamma - 1$.

If the long–range part of the potential, V^ℓ, is a priori known then using formula (1.26) we obtain from the high velocity limit of $(S^D - I)$ the X-ray transform of $V^S_{\text{eff}}(\mathbf{x})$. Inverting this transform we reconstruct $V^S_{\text{eff}}(\mathbf{x})$. Using (1.27) we obtain from the high velocity limit of the commutator of S^D with any component of the relative momentum of the centers of mass of the two clusters the X-ray transform of the corresponding derivative of the full effective potential $V_{\text{eff}}(\mathbf{x})$. Inverting this transform and integrating we reconstruct the effective potential. See the proof of Corollary 1.2 at the end of Section II for details.

COROLLARY 1.2. *Suppose that the conditions of Theorem 1.1 are satisfied with $\rho = 0$. Then (the high speed limit of) any one of the Dollard scattering operators, S^D, determines uniquely the effective potential $V_{\text{eff}}(\mathbf{x})$.*

In the particular case of $M = 1$ and $V^\ell \equiv 0$, the reconstruction formula (1.26) was obtained in Section 5 of Enss and Weder (1995) [1].

We prove Theorem 1.1 and Corollary 1.2 in Section II by extending to this case the simple time–dependent geometrical method of [1], [3]. For an easy introduction see [2] and for a generalization of this method to scattering in a constant external electric field see [8].

2 The Proofs

By definition of the modified wave and scattering operators $(\Omega_-^D)^* \Omega_-^D = I$ and $S^D - I = (\Omega_+^D - \Omega_-^D)^* \Omega_-^D$. Representing $(\Omega_+^D - \Omega_-^D)$ as an integral we get

$$i\Big(S^D - I\Big)\Phi_{\mathbf{v}} = \int_{-\infty}^{+\infty} dt \Big(U^D(t)\Big)^* \sum\nolimits^{(int)} \Big(V_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^\ell(\mathbf{p}t/m)\Big) e^{-itH}\Omega_-^D \Phi_{\mathbf{v}}, \tag{2.1}$$

and we obtain that

$$\begin{aligned} & v\Big[i\Big((S^D - I)\Phi_{\mathbf{v}},\, \Psi_{\mathbf{v}}\Big) \\ & \quad - \Big(\int_{-\infty}^{+\infty} dt \sum\nolimits^{(int)} \Big(V_{jk}^\ell(\mathbf{x} + \mathbf{y}_k - \mathbf{y}_j) - V_{jk}^\ell(t\mathbf{p}/m)\Big) U^D(t)\Phi_{\mathbf{v}},\, U^D(t)\Psi_{\mathbf{v}}\Big)\Big] \\ & = \int_{-\infty}^{+\infty} d\tau\ \ell_{\mathbf{v}}(\tau) + R(\mathbf{v}), \end{aligned} \tag{2.2}$$

with $\tau = vt$,

$$\ell_{\mathbf{v}}(\tau) := \Big(V_{\text{eff}}^S(\mathbf{x}) U_{CM}^D(\tau/v)\Phi_{\mathbf{v}}^{(12)}, U_{CM}^D(\tau/v)\Psi_{\mathbf{v}}^{(12)}\Big)_{\mathcal{H}^{(12)}}, \tag{2.3}$$

and

$$R(\mathbf{v}) := v \int_{-\infty}^{+\infty} \Big((e^{itH}\Omega_-^D - U^D(t))\Phi_{\mathbf{v}}, \sum\nolimits^{(int)} \Big(V_{jk}(\mathbf{x}+\mathbf{y}_k - \mathbf{y}_j) - V_{jk}^\ell(t\mathbf{p}/m)\Big) U^D(t)\Psi_{\mathbf{v}}\Big)\, dt. \tag{2.4}$$

$$\begin{aligned} \Big|vR(\mathbf{v})\Big| \leq & \Big(\sup_{t\in\mathbf{R}} v \Big\|(e^{itH}\Omega_-^D - U^D(t))\Phi_{\mathbf{v}}\Big\|\Big) \\ & \times \int_{-\infty}^{+\infty} d(|vt|) \sum\nolimits^{(int)} \Big\|\Big(V_{jk}(\mathbf{x} + \mathbf{y}_k - \mathbf{y}_j) - V_{jk}^\ell(t\mathbf{p}/m)\Big) U^D(t)\Psi_{\mathbf{v}}\Big\|. \end{aligned} \tag{2.5}$$

We prove below that all integrals exist and that (2.5) is bounded uniformly for large v. Thus

$$R(\mathbf{v}) = O(\, v^{-1}\,). \tag{2.6}$$

Under translation in configuration or momentum space generated, respectively, by $\mathbf{p}$ or $\mathbf{x}$ we obtain

$$e^{i\mathbf{p}\cdot\mathbf{v}t} f(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{v}t} = f(\mathbf{x} + \mathbf{v}t), \tag{2.7}$$

$$e^{-im\mathbf{v}\cdot\mathbf{x}} f(\mathbf{p}) e^{im\mathbf{v}\cdot\mathbf{x}} = f(\mathbf{p} + m\mathbf{v}), \tag{2.8}$$

and in particular

$$e^{-im\mathbf{v}\cdot\mathbf{x}} e^{-itH_0} e^{im\mathbf{v}\cdot\mathbf{x}} = e^{-i\mathbf{p}\cdot\mathbf{v}t} e^{-itH_0} e^{-imv^2t/2}. \tag{2.9}$$

Using (2.7) and (2.9) $\ell_{\mathbf{v}}(\tau)$ can be rewritten (omitting the subscript $\mathcal{H}^{(12)}$)

$$\ell_{\mathbf{v}}(\tau) = \Big(V^S_{\text{eff}}(\mathbf{x} + \tau\hat{\mathbf{v}})\, e^{-iT\tau/v}\tilde{U}(\mathbf{v}, \tau/v)\, \Phi_0^{(12)},\ e^{-iT\tau/v}\tilde{U}(\mathbf{v}, \tau/v)\, \Psi_0^{(12)}\Big), \tag{2.10}$$

where

$$\tilde{U}(\mathbf{v}, t) := \exp\Big[-i\int_0^t ds \sum\nolimits^{(int)} V^\ell_{jk}(s\mathbf{v} + s\mathbf{p}/m)\Big]. \tag{2.11}$$

Since $s - \lim\limits_{v\to\infty} e^{-iT\tau/v}\tilde{U}(\mathbf{v}, \tau/v) = I$ for each fixed τ, we have that

$$\lim_{v\to\infty} \ell_{\mathbf{v}}(\tau) = \Big(V^S_{\text{eff}}(\mathbf{x} + \tau\hat{\mathbf{v}})\, \Phi_0^{(12)}, \Psi_0^{(12)}\Big) \tag{2.12}$$

pointwise in τ.

To justify (1.25) we use the dominated convergence theorem and show that $\ell_{\mathbf{v}}(\tau)$ is bounded uniformly in v by an integrable function of τ. We collect first some properties of the effective potential in the following lemma.

LEMMA 2.1. *Under the decay assumptions on the pair potentials (1.2), (1.3) and the eigenfunctions (1.24) the effective potentials (1.19)–(1.21) are bounded continuously differentiable functions with the following decay properties:*

$$(1+R)^{\rho'} \sup_{|\mathbf{x}|\geq R} \{|V^S_{\text{eff}}(\mathbf{x})| + |\nabla V^S_{\text{eff}}(\mathbf{x})|\} \in L^1\Big((0,\infty), dR\Big), 0 \leq \rho' \leq \min\{\rho, \sigma\}, \tag{2.13}$$

$$(1+R)^{\gamma'} \sup_{|\mathbf{x}|\geq R} |\nabla V^\ell_{\text{eff}}(\mathbf{x})| \in L^1\Big((0,\infty), dR\Big), 0 \leq \gamma' \leq \sigma, \gamma' < \gamma - 1. \tag{2.14}$$

Proof: $V^S_{\text{eff}}(\mathbf{x})$ is a finite sum of terms

$$\begin{aligned}&\Big(V^S_{jk}(\mathbf{x} + \mathbf{y}_k - \mathbf{y}_j)\Phi_b^{(1)} \otimes \Phi_b^{(2)}, \Phi_b^{(1)} \otimes \Phi_b^{(2)}\Big)_{\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}} \\ &= (E^{(1)} - i)\Big((H^{(1)} - i)^{-1} V^S_{jk}(-\mathbf{y}_j)\Phi_b^{(1)} \otimes e^{-i\mathbf{p}_k(\mathbf{x}+\mathbf{y}_k)}\Phi_b^{(2)},\ \Phi_b^{(1)} \otimes e^{-i\mathbf{p}_k(\mathbf{x}+\mathbf{y}_k)}\Phi_b^{(2)}\Big)_{\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}}\end{aligned}$$

which are bounded and twice continuously differentiable functions of $\mathbf{x}$ because $\Phi_b^{(2)} \in D(H^{(2)}) \subset D(\mathbf{p}_k^2)$. We use $(E^{(1)} - i)(H^{(1)} - i)^{-1}\Phi_b^{(1)} = \Phi_b^{(1)}$ to insert or remove regularizations where needed. For the long–range part the regularization of the potential is not needed. To estimate (2.13) we have a sum of terms

$$
\begin{aligned}
&\sup_{|\mathbf{x}|\geq R}\left|\left(V^{S}_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)\Phi_b^{(1)}\otimes\Phi_b^{(2)},\Phi_b^{(1)}\otimes\Phi_b^{(2)}\right)\right| \\
&\leq |E^{(1)}-i|\Big\{\sup_{|\mathbf{x}|\geq R}\left\|(H^{(1)}-i)^{-1}V^{S}_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)F\Big(|\mathbf{y}_k|+|\mathbf{y}_j|\leq\frac{|\mathbf{x}|}{2}\Big)\right\| \\
&\qquad+\left\|(H^{(1)}-i)^{-1}V^{S}_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)\right\|\left\|F\Big(|\mathbf{y}_k|+|\mathbf{y}_j|\geq\frac{R}{2}\Big)\Phi_b^{(1)}\otimes\Phi_b^{(2)}\right\|\Big\} \\
&\leq \mathrm{C}\Big\{\left\|(-\Delta_{\mathbf{z}}-i)^{-1}V^{S}_{jk}(\mathbf{z})F(|\mathbf{z}|\geq R/2)\right\|+\left\|F(|\mathbf{y}_j|\geq R/4)\Phi_b^{(1)}\right\|+\left\|F(|\mathbf{y}_k|\geq R/4)\Phi_b^{(2)}\right\|\Big\}.
\end{aligned}
$$

The gradient terms in $(\nabla V^{S}_{\text{eff}})(\mathbf{x})$ are

$$
\begin{aligned}
&\nabla_{\mathbf{x}}\Big(V^{S}_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)\Phi_b^{(1)}\otimes\Phi_b^{(2)},\ \Phi_b^{(1)}\otimes\Phi_b^{(2)}\Big) \\
&=(E^{(1)}-i)\Big((H^{(1)}-i)^{-1}V^{S}_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)\Phi_b^{(1)}\otimes\Phi_b^{(2)},\ \Phi_b^{(1)}\otimes(-i\mathbf{p}_k)\Phi_b^{(2)}\Big) \\
&\quad+\overline{(E^{(1)}-i)}\Big(\Phi_b^{(1)}\otimes(-i\mathbf{p}_k)\Phi_b^{(2)},\ (H^{(1)}-i)^{-1}V^{S}_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)\Phi_b^{(1)}\otimes\Phi_b^{(2)}\Big).
\end{aligned}
$$

The two terms can be estimated individually as before. This shows (2.13). For (2.14) observe that $(\nabla V^{\ell}_{jk})(\mathbf{z})$ satisfies the short–range condition (1.2) for all $\rho<\gamma-1$.

∎

Knowing the integrable decay (2.13) of V^{S}_{eff} with $\rho=0$ the bound $|\ell_{\mathbf{v}}(\tau)|\leq h(|\tau|)$ with $h\in L^1(0,\infty)$ for all large enough $\mathbf{v}$ follows as in the two–body case by Lemma 3.2 of [1]. The latter relies on propagation properties of wave packets under the (modified) free time evolution. In the notation of [1] $v=|\mathbf{v}|\geq 4\eta$ is sufficient. This condition is always meant when we state uniform estimates for large $\mathbf{v}$ in the sequel. With the pointwise limit (2.12) this proves for $\rho=0$ that

$$
\lim_{v\to\infty}\int_{-\infty}^{\infty}d\tau\ \ell_{\mathbf{v}}(\tau)=\int_{-\infty}^{\infty}d\tau\Big(V^{S}_{\text{eff}}(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_0^{(12)},\ \Psi_0^{(12)}\Big). \tag{2.15}
$$

The rate of convergence in (2.15) for $0<\rho\leq\min\{\sigma,1\}$ gives the error terms in (1.25) exactly as in the two–body case, see Theorems 3.5 and 2.4 of [1].

To complete the proof of (1.25) it remains to show that $|vR(v)|$ is bounded. The difference in the first factor of (2.5)

$$
\begin{aligned}
&e^{-iTH}\Omega^{D}_{-}-U^{D}(T) \\
&=\lim_{t\to-\infty}e^{itH}U^{D}(t+T)-U^{D}(T) \\
&=-i\int_{-\infty}^{0}dt\ e^{itH}\Big[H-H^{D}(t+T)\Big]U^{D}(t+T)
\end{aligned}
$$

can be estimated by

$$
\begin{aligned}
&\sup_{T\in\mathbf{R}} v\left\|\left(e^{-iTH}\Omega_-^D - U^D(T)\right)\Phi_{\mathbf{v}}\right\| \\
&\leq \int_{-\infty}^{\infty} d(|vt|)\ \sum^{(int)} \left\|\left(V_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j) - V_{jk}^{\ell}(t\mathbf{p}/m)\right)U^D(t)\Phi_{\mathbf{v}}\right\|
\end{aligned}
\tag{2.16}
$$

which coincides with the second factor in (2.5). All we need is integrable decay in $(|vt|)$ of the norms in (2.16).

$$
\begin{aligned}
&\left(V_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j) - V_{jk}^{\ell}(t\mathbf{p}/m)\right) \\
&= V_{jk}^{S}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j) + \left[V_{jk}^{\ell}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j) - V_{jk}^{\ell}(\mathbf{x})\right] + \left[V_{jk}^{\ell}(\mathbf{x}) - V_{jk}^{\ell}(t\mathbf{p}/m)\right].
\end{aligned}
\tag{2.17}
$$

The contributions to (2.16) from the three summands in (2.17) are estimated individually. Choose some $g \in C_0^\infty(\mathbf{R}^n)$ with $g(\mathbf{p}) = 1$ for $\mathbf{p} \in \mathrm{supp}\hat{\phi}_0^{(12)}$. Then $\Phi_0^{(12)} = g(\mathbf{p})\Phi_0^{(12)}$ and

$$U^D(t)\Phi_{\mathbf{v}} = g(\mathbf{p}-m\mathbf{v})U^D(t)\Phi_{\mathbf{v}}.$$

This serves to regularize possible singularities of the short–range potentials.

$$
\begin{aligned}
&\|V_{jk}^{S}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)U^D(t)\Phi_{\mathbf{v}}\| \\
&\leq \|V_{jk}^{S}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)\,g(\mathbf{p}-m\mathbf{v})\|\cdot\|F(|\mathbf{x}|\leq|t|v3/8)\,U^D(t)\,\Phi_{\mathbf{v}}\| \\
&+ \|V_{jk}^{S}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)\,g(\mathbf{p}-m\mathbf{v})\|\cdot\|F(|\mathbf{y}_k|+|\mathbf{y}_j|\geq|t|v/4)\,U^D(t)\,\Phi_{\mathbf{v}}\| \\
&+ \|V_{jk}^{S}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)\,g(\mathbf{p}-m\mathbf{v})\,F(|\mathbf{x}|\geq|t|v3/8)\,F(|\mathbf{y}_k|+|\mathbf{y}_j|\leq|t|v/4)\|\cdot\|U^D(t)\Phi_{\mathbf{v}}\|.
\end{aligned}
\tag{2.18}
$$

In the first summand the second factor decays rapidly in $v|t|$ by standard propagation estimates, because the clusters separate with speed of order v. For an explicit proof of decay like $(v|t|)^{-2-\varepsilon}$, see e.g. (3.6), (3.7) in [1]. In the second term

$$\|F(|\mathbf{y}_k|+|\mathbf{y}_j|\geq|t|v/4)\,U^D(t)\,\Phi_{\mathbf{v}}\| = \|F(|\mathbf{y}_k|+|\mathbf{y}_j|\geq|t|v/4)\,\Phi_b^{(1)}\otimes\Phi_b^{(2)}\|$$

decays as assumed in (1.24). In the third term the argument of the potential is bounded below by $|t|v/8$ and the norm decays as in (1.2). Thus in particular

$$\|V_{jk}^{S}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)U^D(t)\Phi_{\mathbf{v}}\| \in L^1\Big((0,\infty), d(|t|v)\Big). \tag{2.19}$$

For the second contribution to (2.17) we mimic the estimate (2.18) replacing $V_{jk}^{S}\,g$ by

$$[V_{jk}^{\ell}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j) - V_{jk}^{\ell}(\mathbf{x})].$$

The first and second summands are then (up to a constant) the same, in the third

$$\begin{aligned}&\|[V^\ell_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)-V^\ell_{jk}(\mathbf{x})]\,F(|\mathbf{x}|\geq|t|v3/8)\,F(|\mathbf{y}_k|+|\mathbf{y}_j|\leq|t|v/4)\,U^D(t)\Phi_v\|\\&\leq\sup_{|\mathbf{z}|\geq|t|v/8}|\nabla V^\ell_{jk}(\mathbf{z})|\;\||\mathbf{y}_k-\mathbf{y}_j|\,\Phi^{(1)}_b\otimes\Phi^{(2)}_b\|.\end{aligned}\tag{2.20}$$

The decay (1.24) implies boundedness of $\|\,|\mathbf{y}_j|\,\Phi^{(1)}_b\|$ and $\|\,|\mathbf{y}_k|\,\Phi^{(2)}_b\|$ and thus (2.20) inherits decay like $(1+|t|v)^{-\gamma}$ from (1.3).

It remains to treat the last difference in (2.17) which is purely a two–body estimate. Lemma 3.3 in [1] applies literally and yields

$$\|[V^\ell_{jk}(\mathbf{x})-V^\ell_{jk}(t\mathbf{p}/m)]U^D(t)\Phi_\mathbf{v}\|\leq C\,(1+v|t|)^{-1-\varepsilon},\varepsilon>0.$$

Thus the integral (2.16) is uniformly bounded in v for large enough v. This finishes the proof of (2.6) and of (1.25). We derive (1.26) from (1.25) as in Theorem 3.5 of [1]. The crucial observation is that

$$s-\lim_{v\to\infty}V^\ell_{jk}(\tau(\mathbf{p}/mv)+\tau\hat{\mathbf{v}})=V^\ell_{jk}(\tau\hat{\mathbf{v}})$$

pointwise in τ. The integrability of the $r.h.s$ of (1.26) can be directly seen from the fact that

$$V^\ell_{\text{eff}}(\mathbf{z})-\sum\nolimits^{(int)}V^\ell_{jk}(\mathbf{z}),\quad \mathbf{z}\in\mathbf{R}^n,$$

is of short range, which follows from (2.20).

We now prove (1.27). Noting that $p_1=-i\partial/\partial x_1$ satisfies $(p_1-mv_1)\Phi_\mathbf{v}=(p_1\Phi)_\mathbf{v}$, and that $[S^D,p_1]=[S^D,p_1-mv_1]$ we obtain from (2.1) – (2.4) that

$$v(i[S^D,p_1]\Phi_\mathbf{v},\Psi_\mathbf{v})=\int_{-\infty}^{+\infty}\hat{\ell}_\mathbf{v}(\tau)d\tau+\hat{R}(\mathbf{v}),\tag{2.21}$$

where

$$\hat{\ell}_\mathbf{v}(\tau):=i\Big((\partial_1V_{\text{eff}})(\mathbf{x})\,U^D(\tau/v)\,\Phi_\mathbf{v},\;U^D(\tau/v)\,\Psi_\mathbf{v}\Big),$$

$$\begin{aligned}\hat{R}(\mathbf{v}):=v\int_{-\infty}^{+\infty}dt\,\Big\{&\Big((e^{-itH}\Omega^D_--U^D(t))(p_1\Phi)_\mathbf{v},\\&\sum\nolimits^{(int)}(V^S_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)+V^\ell_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)-V^\ell_{jk}(\mathbf{p}t/m))U^D(t)\Psi_\mathbf{v}\Big)\\-&\Big((e^{-itH}\Omega^D_--U^D(t))\Phi_\mathbf{v},\\&\sum\nolimits^{(int)}(V^S_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)+V^\ell_{jk}(\mathbf{x}+\mathbf{y}_k-\mathbf{y}_j)-V^\ell_{jk}(\mathbf{p}t/m))U^D(t)(p_1\Psi)_\mathbf{v}\Big)\Big\}.\end{aligned}\tag{2.22}$$

Then as above

$$\hat{R}(v) = O(v^{-1}), \tag{2.23}$$

and we complete the proof of (1.27) as in the case of (1.25) using (2.14).

■

Proof of Corollary 1.2: For all normalized $\Phi^{(12)}$, $\Psi^{(12)} \in \mathcal{H}^{(12)}$ with $\hat{\varphi}^{(12)}$, $\hat{\psi}^{(12)} \in C_0^\infty(\mathbf{R}^n)$ the high speed limits of the scattering operator

$$\lim_{v\to\infty} \Big(i[S^D, p_1]\, \Phi_{\mathbf{v}},\, \Psi_{\mathbf{v}} \Big)$$
$$= i \int d^n x\, \overline{\varphi^{(12)}(\mathbf{x})}\, \psi^{(12)}(\mathbf{x}) \int_{-\infty}^{\infty} d\tau\, (\partial_1 V_{\text{eff}})(\mathbf{x} + \tau\, \hat{\mathbf{v}}) \tag{2.24}$$

converge by (1.27) and thus determine uniquely the continuous function

$$W_1(\mathbf{x}, \hat{\mathbf{v}}) := \int_{-\infty}^{\infty} d\tau\, (\partial_1 V_{\text{eff}})(\mathbf{x} + \tau\, \hat{\mathbf{v}}), \tag{2.25}$$

the X-ray transform of $(\partial_1 V_{\text{eff}})$.

In dimension $n = 2$ the X-ray transform coincides with the Radon transform and $(\partial_1 V_{\text{eff}}) \in L^2(\mathbf{R}^2)$ by (2.13), (2.14). By Theorem 2.17 in Chapter I of Helgason [4] the Radon transform $W_1(\cdot, \hat{\mathbf{v}})$ for all $\hat{\mathbf{v}} \in S^1$ determines $(\partial_1 V_{\text{eff}})$ uniquely. In higher dimension keep $n-2$ components, e.g. $x_3, \cdots, x_n$, fixed and it is sufficient to vary $\mathbf{v}$ in a plane. Then

$$V_{\text{eff}}(\mathbf{x}) = \int_{-\infty}^{x_1} d\xi\, (\partial_1 V_{\text{eff}})(\xi, x_2, \cdots, x_n).$$

Similarly, if the V_{jk}^{ℓ} are known, one can reconstruct V_{eff}^{s} directly from (1.25).

■

For an alternative proof of Corollary 1.2 which does not use continuity of V_{eff} or $(\partial_1 V_{\text{eff}})$ see the proof of Theorem 3.5 in [1].

Acknowledgment

This work was done while the second author was visiting the Institut für Reine und Angewandte Mathematik of the RWTH Aachen. Ricardo Weder thanks Volker Enss for his kind hospitality, and the Deutsche Forschungsgemeinschaft for its financial support.

References

[1] V. Enss and R. Weder, "The geometrical approach to multidimensional inverse scattering", J. Math. Phys. **36**, 3902–3921 (1995).

[2] V. Enss and R. Weder, "Inverse potential scattering: A geometrical approach", in: "Mathematical Quantum Theory II: Schrödinger Operators", p.p. 151–162, Proceedings of the Summer School in Mathematical Quantum Theory, August 1993, Vancouver, B.C., J. Feldman, R. Froese, and L.M. Rosen, editors, CRM Proceedings and Lecture Notes **8**, AMS, Providence (1995).

[3] V. Enss and R. Weder, "Uniqueness and reconstruction formulae for inverse N–particle scattering", in: "Differential Equations and Mathematical Physics", p.p. 55–66, Proceedings of the International Conference, Univ. of Alabama at Birmingham, March 1994, I. Knowles, editor, International Press Boston (1995).

[4] S. Helgason, "Groups and Geometric Analysis", Academic Press, Orlando (1984).

[5] L. Hörmander, "The existence of wave operators in scattering theory", Math. Z. **146**, 69–91 (1976).

[6] T. Kato, "Perturbation Theory for Linear Operators", 2nd edition, Springer, Berlin (1976).

[7] M. Reed and B. Simon, "Methods of Modern Mathematical Physics III: Scattering Theory", Academic Press, New York (1979).

[8] R. Weder, "Multidimensional inverse scattering in an electric field", Preprint 28, IIMAS–UNAM, 29 p.p., to appear in J. Func. Analysis.