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TECHNIQUES OF DISTRIBUTIONS IN PERTURBATIVE QUANTUM FIELD THEORY.

(II) Applications to Theory of Multiloop Diagrams.

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ABSTRACT

The results of the mathematical theory of asymptotic operation developed in [?] are applied to problems of immediate physical interest. First, the problem of UV renormalization is analyzed from the viewpoint of asymptotic behaviour of integrands in momentum representation. A new prescription for UV renormalization in momentum space representation is presented (generalized minimal subtraction scheme); it ensures UV convergence of renormalized diagrams by construction, makes no use of special (e.g. dimensional) regularizations, and comprizes massless renormalization schemes (including the MS scheme). Then we present formal regularization-independent proofs of general formulae for Euclidean asymptotic expansions of renormalized Feynman diagrams (inlcuding short-distance OPE, heavy mass expansions and mixed asymptotic regimes etc.) derived earlier in the context of dimensional regularization [?], [?], [?]. This result, together with the new variant of UV renormalization, demonstrates the power of the new techniques based on a systematic use of the theory of distributions and establishes the method of As-operation as a comprehensive full-fledged—and inherently more powerful—alternative to the BPHZ approach.

1. Introduction

In the preceding paper [?] we undertook a regularization-independent formalization of the heuristic reasoning behind a series of publications [?], [?], [?], [?], [?], [?], [?] in which efficient methods of perturbative calculations were found (for references to various 2-, 3-, 4- and 5-loop calculations performed using that techniques see [?]). The new techniques for studying multiloop Feynman diagrams is based on a systematic use of the ideas of the distribution theory, and the key notion is that of asymptotic expansion in the sense of distributions [?], [?], [?]. A very general context in which to construct such expansions is established by the *extension principle* [?], [?]—an abstract functional-analytic proposition analogous to the classic Hahn-Banach theorem. A specific realization of the recipe implied by the extension principle—and the key instrument of our techniques—is the so-called asymptotic operation (As-operation) [?], [?]. The Euclidean version of As-operation constructed in [?], [?] is defined on a class of products of singular functions comprising integrands of Euclidean multiloop Feynman diagrams. The As-operation returns their expansions in powers and logarithms of a small parameter (e.g., a mass) in the sense of distributions.

In the first paper [?] an analytic technique was developed for describing singularities of distributions, as well as a combinatorial formalism (*universum of graphs*) to work with hierarchies of graphs and their subgraphs. Such a formalism makes it easy to utilize inherent recursive structures in problems involving multiloop diagrams. As a warm-up exercise, a very compact proof of a (localized) version of the familiar Bogoliubov-Parasiuk theorem in coordinate representation was presented with a purpose of illustrating in detail the typical ways of reasoning within the new techniques.

It was constructively proved that the asymptotic expansions in powers and logarithms of the small parameter exist for a large class of products of singular functions that includes integrands of Euclidean Feynman diagrams in momentum representation. The simplest example of such an expansion is with respect to a mass in a product of propagators, while the As-operation applied to such a product yields asymptotic expansions in powers and logs of the small mass, with coefficients given by compact explicit expressions suitable for studying expansion problems within the framework of applied Quantum Field Theory.

In the present paper we apply that general techniques to studying expansions of integrated Euclidean Feynman diagrams in momentum representation.

Similarly to the first part of the review [?], our purpose here is not to construct a complete perturbation theory or prove a short-distance expansion for a particular model in full detail but rather to demonstrate the general techniques of our formalism. Thus we do not attempt to write out, say, criteria for UV convergence of non-renormalized diagrams; such results are well-known and present little interest *per se*, and can be obtained in a straightforward manner from our general formulae whenever needed in a specific situation. Instead, we focus on how the techniques of the As-operation allows one to deal with the essential analytical aspects of such problems and to avoid combinatorial complexities of the usual approaches.

First of all, when performing integrations over infinite momentum space one encounters the well-known phenomenon of UV divergences that are due—in one of the possible interpretations—to slow decrease of Feynman integrands at infinite integration momenta. However, using the results of [?] one can explicitly extract those and only those terms in the asymptotic expansion of the integrand at infinite integration momenta, that are responsible for UV divergences. It turns out that such terms follow (with the opposite sign) the pattern of the terms to be introduced by the standard R-operation. Therefore, a direct subtraction of such terms from the integrand (with some natural precautions) results in a correct UV renormalization.

The UV finiteness of our construction is ensured *by definition* and what has to be proved is its equivalence to the standard formulation of the R-operation.[1] The class of subtraction schemes that naturally corresponds to the new definition of UV renormalization (the so-called *generalized minimal subtraction schemes*, or GMS schemes) comprises all massless schemes (including the MS scheme [?]) characterized by an extremely important property of polynomiality of the renormalization group functions in masses [?].

[1] Which, strictly speaking, is superfluous because one could prove correctness of the new representation directly, without reducing it to the standard construction. Such a proof would require results like existence of short-distance OPE (see e.g. [?]); cf. also [?]) which is also considered in the present paper.

A definition of the ultraviolet R-operation as a procedure of subtraction of asymptotics from the momentum representation integrands was first given by D. Slavnov et al. [?]. However, in [?] such subtractions are performed recursively with respect to loops, i.e. with respect to each integration momentum in turn, while in our approach asymptotics with respect to the entire collection of loop momenta are subtracted. This and the use of the As-operation for products of singular functions [?] in our approach allow one to easily exhibit the pattern of subtractions that is characteristic of the standard definition of the R-operation.

Our definition of the R-operation was first published in a complete form in [?]. It develops an idea from [?] and emerged from a study of diagrammatic interpretation of the non-trivial terms generated by the As-operation, not without an influence of [?].

An extremely important observation is that the coefficients of the As-operation constructed in [?] turn out to be exactly renormalized Feynman diagrams corresponding to subgraphs of the initial diagram. This fact has a dramatic technical impact on the problem of asymptotic expansions of Feynman diagrams, because it allows one to restore global OPE from expansions of individual diagrams in a very simple and straightforward fashion without any of the complexities of the combinatorial techniques of the BPHZ method.[2]

The second application of the theory of As-operation that we consider addresses the problem of Euclidean asymptotic expansions of renormalized multiloop diagrams. We present a compact and straightforward derivation of general Euclidean asymptotic expansions in the form of As-operation for integrated diagrams that was first introduced in [?]. Then the combinatorial techniques developed in [?] immediately allows one to obtain expansions for perturbative Green functions in OPE-like form.

The importance and feasibility of the general problem of Euclidean asymptotic expansions was realized in [?], [?], [?]. In those papers, a compact derivation of closed general formulae for such expansions was presented. The derivation of [?], [?], [?], however, aimed at obtaining the results in a shortest way and in a form immediately useful for phenomenological applications, so that a heavy use was made of the dimensional regularization and the MS scheme [?].[3] This left an open question of to what extent the results of [?], [?], [?] are independent of regularization.

Apart from the general interest, there are also very practical reasons for developing a regularization independent formalism. First of all, there are the notorious difficulties that the dimensional regularization encounters when applied to models involving γ_5 or supersymmetry. In particular, the γ_5 problem emerges if one wishes to use the chirality representation-based formalisms that are used to facilitate the enormously cumbersome gauge algebra in calculations of radiative corrections in QCD (cf. e.g. the superstring theory-inspired formalism developed in [?]).

Another reason is the breakdown of dimensional regularization in non-Euclidean asymptotic expansion problems [?], so that the regularization-independent formalism seems to be the only basis for construction of practical algorithms in the non-Euclidean case. This effect is connected with the insistence on expansions in "perfectly factorized" form, which is important for the following reasons:

It was an important realization of [?], [?], [?], [?] that a proof of any asymptotic expansion—be it Wilson's OPE or a heavy-mass expansion or asymptotics of the quark formfactor in the Sudakov regime—is phenomenologically irrelevant unless the result exhibits *perfect factorization* of large and small parameters. At the technical level of diagram-by-diagram expansions, perfect factorization means that the expansions run in pure powers and logarithms of the expansion parameter. Such expansions possess the property of uniqueness (cf. the discussion in [?], §15.4) which is tremendously useful from the technical point of view; for example, one immediately obtains that the As-operation commutes with multiplications by polynomials (see [?] and [?]). Another example is that one need not worry about properties like gauge invariance of the expansion in a given approximation: such properties are inherited by the expansion termwise from the initial amplitude, provided the expansion is

[2] It should be noted that the combinatorial constructions that naturally emerged within the framework of the theory of As-operation [?] (in particular, the inverse R-operation) caused an overhaul of how the combinatorial aspects are treated in the BPHZ theory (see [?])—even if the basic nature of the old approach (a complete resolution of all recursions) does not allow one to avoid having to deal with the rather cumbersome multiple summation formulas etc.

[3] One can ponder on the tremendous heuristic potential of the dimensional regularization and the MS scheme. Although the understanding of the analytic aspects of the problem—including existence of the representation of UV renormalization in the GMS form—was hardly lacking in [?], [?], [?], the compact presentation given there turned out feasible due to compactness of the formalism of dimensional regularization and its property to nullify certain types of scaleless integrals.

"perfect" in the above sense.[4]

For the above reasons, we consider it our major task to clarify the issue of existence of "perfect" expansions in regularization-independent way.

It is interesting that the derivation of OPE and, more generally, Euclidean asymptotic expansions presented in this paper—being more formalized than that of [?], [?], [?]—leads to a final formula which is much easier to deal with at the final stage of obtaining expansions for Green functions in a global "exponentiated" form. For example, unlike [?], we don't have to study inversion of the R-operation.[5]

However, as was stressed in [?], the derivation presented there was geared to the calculational needs of applied Quantum Field Theory (primarily, applications to perturbative Quantum Chromodynamics) and, therefore, dealt explicitly with UV counterterms etc. From practical point of view, the formalism of the present paper offers, at least in its current form, no advantages as compared with the explicit recipes of [?], [?], [?]—*provided* one can perform the calculations within dimensional regularization.

From theoretical point of view the formalization undertaken in the present series is more than just an exercise in rigour: there is the major unsolved problem of asymptotic expansions in non-Euclidean regimes, and it seems to be intrinsically intractable by the BPHZ method.[6] On the other hand, extension of the As-operation to non-Euclidean regimes—taking into account the accumulated experience [?] which only needs to be properly organized within an adequate technical framework—seems to be a matter of near future. We expect the experience gained in Euclidean problems to play a crucial role in the more complicated cases [?].

One of the main points in any proof of OPE is to study the interaction of UV renormalization and the expansion proper. As we are going to show, within our formalism the problem reduces to double asymptotic expansions in the sense of distributions. Indeed, a renormalized Feynman diagram in the GMS formulation has a form of an integral of the remainder of the As-expansion of the integrand in the regime when all the dimensional parameters of the diagram are much less than the implicit UV cutoff. When one applies the second expansion with respect to some of the diagram's masses or external momenta, there emerges, essentially, a double As-expansion. All one has to prove is that the double expansion thus obtained factorizes into a composition of two commuting As-expansions and that the remainder of such a double expansion is bounded by a factorizable function of the small parameters. This is done by a straightforward extension of the analytical techniques of [?].

The plan of the paper is as follows (as in [?], sections contain a preamble where further comments on its contents and results can be found).

In section **??** we present motivations and a definition of an operation $\mathcal{R}$ which subtracts asymptotics at large loop momenta from the integrand of a multiloop diagram, thus ensuring UV finiteness of the latter. In section **??** the explicit expressions for the As-operation from [?] are used to (partially) restrict the arbitrariness in the definition of the operation $\mathcal{R}$. For the purposes of illustration, section **??** establishes equivalence of $\mathcal{R}$ (with the arbitrary constants properly fixed) and the R-operation in the MS scheme.

In section **??** we obtain a useful representation for the operation $\mathcal{R}$ and apply it to deriving a "renormalization-group transformation" of the $\mathcal{R}$-renormalized diagrams. This transformation has exactly the form that is characteristic of the R-operation.

In section **??** the problem of the asymptotic expansion of renormalized Feynman diagrams is reviewed, its heuristic analysis from the point of view of As-operation is given, and a recursive expansion formula is obtained. Section **??** is devoted to combinatorial analysis of that formula. In section **??** a convenient expression for expanded renormalized diagrams is obtained in a combinatorial form similar to UV R-operation, and its

[4] As was pointed out to us by J.C. Collins, such a property should be even more important for the problem of asymptotic expansions in Minkowskian regimes where both gauge invariance plays a greater role for phenomenological reasons, and the expansions one has to deal with are considerably more complicated. It may be said that the "relentless pursuit of perfection" (in the above sense) is one of the characteristic differences of the philosophy of the As-operation from the old BPHZ paradigm.

[5] Although we do use it in establishing—for the purposes of illustration—a connection of our GMS prescription to the standard MS scheme.

[6] Because the pattern of recursions involved is so much more complicated than in the Euclidean case that it does not seem possible to resolve them in an explicit manner [?].