The XY Model in a Transverse Magnetic Field

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Abstract

We show that the ground state of the XY model (ferromagnetic or antiferromagnetic) in a transverse magnetic field h (transverse meaning here perpendicular to the plane where the interactions occur)— for any spin value, in any dimension — is the state with all spins aligned parallel to the field, when h is bigger in magnitude than some critical value h_c. We also compute the spatial and time decay of correlation functions for the spin-1/2 linear chain and study the behavior of these quantities as a function of the external magnetic field.

1 Introduction

The Hamiltonian for the quantum XY model in the presence of a constant transverse field is given by $\mathrm{H}_\Lambda = \frac{J}{2}\sum_{\langle x,y\rangle\in\Lambda}[S_1(x)S_1(y)+S_2(x)S_2(y)] + h\sum_{x\in\Lambda}S_3(x).Asusual,\Lambda$ is a finite domain in the hypercubic lattice $\mathbf{Z}^d$ and by $\langle x,y\rangle$ we mean nearest neighbor sites on the lattice. At each site x we have a spin angular momentum $\mathbf{S}(x)$, such that $\mathbf{S}^2 = S\,(S+1)$ is the same for all spins, the system being conceived with open boundary conditions. The sign of J defines the ferromagnetic ($J<0$) or antiferromagnetic ($J>0$) nature of the system. Actually, it is a known fact that the XY model can always be equivalently described as ferromagnetic or antiferromagnetic, since there exists an unitary transformation (a rotation of spins in a sublattice around the z-axis through an angle of π) that maps $J \to -J$. This property leads to the conclusion that the sign of J is not relevant in our analysis.

In 1961, Lieb, Schultz and Mattis [1] solved the one-dimensional antiferromagnetic model by means of the Jordan-Wigner [2] transformation which maps the system into a free Fermi gas in the presence of an external potential, thereby giving the complete set of states and excitation energies through the diagonalization of the Hamiltonian in terms of fermion operators. Since this transformation is not available in more than one dimension, in this paper we use a variational principle to determine the ground state of the model in d dimensions and show that, when $|h| > 2Sd\,|J|$, all spins align antiparallel to the field (section 2 and the Appendix). For the antiferromagnetic system, our result is a proof of the so-called "Jacobsohn theorem" [3] in a special case and is a simple application of classical domination to quantum spin chains.

Next we consider the spin-1/2 one-dimensional model with open boundary conditions (section 3). The problem of determining the eigenstates and

eigenvalues for the system with periodic boundary conditions was solved by Niemeiyer [6]. In section 4, we investigate the asymptotic behavior of spatial correlations for large chains ($\Lambda \to \infty$) as functions of the external field. We show that the polynomial decay obtained in [1] for zero field survives up to the critical value of the field $h = |J|$. For $h > |J|$ the spins become independent variables, so that the correlation degenerates into delta functions of x and y. This result should be compared to the exponential decay obtained by Klein e Perez [5] for random fields.
Finally, in section 5 we study the time decay of correlation functions and find out that their behavior is also controled by the external magnetic field in the following way

$$\begin{cases} \delta_{xy} e^{-t(|h|-J)} & \text{if } h > |J| \\ \delta_{xy} \frac{1}{\sqrt{t}} & \text{if } h = |J| \\ \frac{1}{t} & \text{if } h < |J| \end{cases}$$

An important remark is that, in this kind of system, the time decay of correlation functions is a measure of the energy gap of the excited states against the ground state as a function of the external field.

2 The Ground State of the XY Model in a Strong Transverse Magnetic Field

We shall make use of the raising and lowering operators

$$S_{\pm}(x) = S_1(x) \pm i S_2(x), \tag{1}$$

$$[S_+(x), S_-(y)] = 2S_3(x)\,\delta_{x,y},$$

which obey the commutation relations $Intermsoftheseoperatorsthe Hamiltonianiswrittenas : \mathcal{H}_\Lambda =$

$$[S_3(x), S_+(y)] = S_+(x)\,\delta_{x,y}.$$

$\frac{J}{2} \sum_{x\in\Lambda} \sum_{k=1}^{d} [S_+(x)S_-(x+\mathbf{e}_k) + S_+(x)S_-(x+\mathbf{e}_k)] + h \sum_{x\in\Lambda} S_3(x)(2)$with $\mathbf{e}_k$ being an unit vector in $\mathbf{Z}^d$. In our perturbative method, we break up the Hamiltonian operator into two parts:

$$\mathcal{H}_\Lambda = \mathcal{H}^z_\Lambda + \mathcal{H}^{xy}_\Lambda,$$

where

$$\mathcal{H}_\Lambda^z = h \sum_{x\in\Lambda} S_3(x), \tag{3}$$

and

$$\mathcal{H}_\Lambda^{xy} = \frac{J}{2} \sum_{x\in\Lambda} \sum_{k=1}^{d} [S_+(x) S_-(x+\mathbf{e}_k) + S_+(x) S_-(x+\mathbf{e}_k)]. \tag{4}$$

Of these, the external potential $\mathcal{H}_\Lambda^z$ is considered the dominant term, which is perturbed by the $x-y$ planar interaction $\mathcal{H}_\Lambda^{xy}$.

The partition function of the unperturbed system, defined by $\mathcal{H}_\Lambda^z$, factorizes completely, leading to

$$\mathcal{Z}_\Lambda^{(z)} = \left[2 \sum_{j=1}^{S} \cosh\left(j\beta h\right) \right]^{|\Lambda|}.$$

The expectation values of the energy and the magnetization, in the limit $\beta \to \infty$, correspond to the state where all the spins lie antiparallel to the external field, even when $|\Lambda| \to \infty$, so that the thermodynamical limit is well defined.

We can solve immediately the unperturbed eigenvalue problem by setting up the basis that diagonalizes $\mathcal{H}_\Lambda^z$, namely

$$\otimes_\Lambda |\sigma(x)\rangle$$

where

$$S_3(x)\,|\sigma(x)\rangle = \sigma(x)\,|\sigma(x)\rangle\,, \text{ with } -S \le \sigma(x) \le S,\ \forall x \in \Lambda,$$

As a consequence of the above discussion, $\mathcal{H}_\Lambda^z$ has a unique ground state $|\Omega_\Lambda\rangle$, which is also an eigenstate of the perturbation $\mathcal{H}_\Lambda^{xy}$, with zero eigenvalue. This state is such that

$$S_3\,(x)\,|\Omega_\Lambda\rangle = -S|\Omega_\Lambda\rangle,$$

for any $x \in \Lambda$. Moreover, since $\mathcal{H}_\Lambda^{xy}$ commute with $\sum_x S_3(x)$, the total number of particles is conserved. Thus, the only effect of the perturbation $\mathcal{H}_\Lambda^{xy}$ is the mixing of the degenerate energy levels of $\mathcal{H}_\Lambda^z$.

The general $n-$angular momentum state, defined by

$$|\Psi_n\rangle = \sum_{x_1,\ldots,x_n} \Phi(x_1,\ldots,x_n) S_+(x_1)\ldots S_+(x_n)|\Omega_\Lambda\rangle$$

satisfies

$$\mathcal{H}_\Lambda^z|\Psi_n\rangle = -h(S|\Lambda| - n)|\Psi_n\rangle$$

where $n = 0, 1, 2, ..., 2S|\Lambda|$ is the change in the value of the system total angular momentum. Therefore, the spectrum of $\mathcal{H}_\Lambda^z$ is the discrete set of values from $-|\Lambda|Sh$ to $+|\Lambda|Sh$ with width h between adjacent levels. To verify that the ground state of of the complete Hamiltonian is also $|\Omega_\Lambda\rangle$, it is sufficient to show that the expectation value of the perturbation in any unperturbed eingenstate is never greater than the difference between the energies of the considered level and of the fundamental one. This situation will be achieved, as we shall see, with a sufficiently strong magnetic field. The difference from the case treated in [4] is that, now, the $|\Psi_n\rangle$ state may be generated by the application of n raising operators ($n \leq 2S$) on the same point ($x_1 = x_2 = ... = x_n = x$).

The action of the $\mathcal{H}_\Lambda^{xy}$ operator on the vector $|\Psi_n\rangle$ produces

$$\mathcal{H}_\Lambda^{xy}|\Psi_n\rangle = \left|\Psi_n^{(1)}\right\rangle + \left|\Psi_n^{(2)}\right\rangle + \left|\Psi_n^{(3)}\right\rangle,$$

where

$$\begin{aligned}
\left|\Psi_n^{(1)}\right\rangle &= J\sum_{x\in\Lambda}\sum_{k=1}^{d}\sum_{x_1\ldots x_n}\Phi(x_1,...,x_n)\times \\
&\quad\times S_+(x)S_+(x_1)...S_+(x_n)S_-(x+\mathbf{e}_k)|\Omega_\Lambda\rangle = 0, \\
\left|\Psi_n^{(2)}\right\rangle &= 4S\frac{J}{2}\sum_{k=1}^{d}\sum_{i=1}^{n}\sum_{x_1\ldots x_n}\Phi(x_1,...,x_n)\times \\
&\quad\times S_+(x_i+\mathbf{e}_k)S_+(x_1)...S_+(x_{i-1})S_+(x_{i+1})...S_+(x_n)|\Omega_\Lambda\rangle \\
\left|\Psi_n^{(3)}\right\rangle &= -4\frac{J}{2}\sum_{k=1}^{d}\sum_{i\neq j}\sum_{x_1\ldots,x_i=x_j,\ldots,x_n}\Phi(x_1,...,x_n)\times \\
&\quad\times S_+\left(x_i+\mathbf{e}_k\right)S_+\left(x_1\right)...S_+(x_{i-1})S_+(x_{i+1})...S_+\left(x_{j-1}\right)\times \\
&\qquad\times S_+\left(x_i\right)S_+\left(x_{j+1}\right)...S_+(x_n)|\Omega_\Lambda\rangle.
\end{aligned}$$

The fact that the coefficients of $\left|\Psi_n^{(3)}\right\rangle$ are less or equal than zero implies that

$$\|\ \mathcal{H}_\Lambda^{xy}|\Psi_n\rangle\ \| \leq 2nSd|J|\ \|\ |\Psi_n\rangle\ \|.$$

To obtain the above results we have used the commutation relations (2) together with the characterization of the ground state of the unperturbed model as the state where all the spins have a S_3 eigenvalue $-S$.

Thus, the expectation value of the energy in any subspace defined by $|\Psi_n\rangle$ is

$$-|\Lambda|Sh + n(h - 2Sd\,|J|)$$

which, for $|h| > 2d\,|J|\,S$, is greater than the energy of the vacuum, thus proving that the vacuum $|\Omega_\Lambda\rangle$ is the ground state of the complete system.

3 The Spin-1/2 Linear Chain with Open Boundary Conditions

We now consider a linear chain of $2L+1$ (L even) spins 1/2 in an external transverse magnetic field h, with free boundary conditions. The system, still defined by (1), is restricted to the box $\Lambda = \mathbf{Z} \cap [-L, L]$ and $\mathbf{S}^2 = \frac{3}{4}$. The problem can be exactly solved by first introducing the raising and lowering operators given by (1) and the fermion creation and annihilation operators defined by the Jordan-Wigner transformation [2]. For $-L < x \leq L$, let $a^\dagger(x) = S_+(x) \exp[i\pi \sum_{-L\leq z<x} S_+(z)S_-(z)]$

$$TheHamiltonian(1)isthenwrittenasaquadraticform\mathcal{H}_L =$$

$a(x) = \exp[-i\pi \sum_{-L\leq z<x} S_+(z)S_-(z)]S_-(x)$

$-(J/2)\sum_{-L\leq x\leq L-1}[a^\dagger(x)a(x+1)+a^\dagger(x+1)a(x)]++h\sum_{-L\leq x\leq L} a^\dagger(x)a(x).Thecanonicaltransformationdefinedby$ $\begin{array}{ll} a_k^\dagger & = \sum_{-L\leq x\leq L} \varphi_k(x)a^\dagger(x) \\ a_k & = \sum_{-L\leq x\leq L} \varphi_k(x)a(x) \end{array}$ $sends\mathrm{H}_L$

into the form

$$\mathcal{H}_L = \sum_k \Gamma_k a_k^\dagger a_k.$$

In (3), the coefficients $\varphi_k(x)$ $(x = -L, ..., L)$ are the $2L+1$ components of the vector Φ_k in the matrix equation

$$A\Phi_k = \Gamma_k^2 \Phi_k \tag{5}$$

where

$$A = \frac{1}{2}\begin{bmatrix} 2h & -J & 0 & ... & 0 & 0 \\ -J & 2h & -J & ... & 0 & 0 \\ 0 & -J & 2h & ... & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & -J & 2h & -J \\ 0 & 0 & ... & 0 & -J & 2h \end{bmatrix}$$

The set of solutions of (5) is then $\varphi_k(x) = \begin{cases} A_+ \cos kx, \quad k \in I_1 = \{\frac{(2l+1)\pi}{2L+1}, \qquad l = 0, 1, \dots, L\} \\ \\ A_- \sin kx, \quad k \in I_2 = \{\frac{l\pi}{L+1}, \qquad l = 1, 2 \dots, L\}, \end{cases}$ $whichbelongtothesetofeigenvalues\Gamma_k =$ $h - J\cos k$. It is also usefull to define $\Lambda^* = I_1 \cup I_2$. The normalization constants are given by $A_\pm = \frac{1}{\sqrt{\frac{2L+1}{2} \pm \frac{\sin k(2L+1)}{2\sin k}}} The eigenvectors of the Hamiltonian operator, when indexed by the set of energy levels, can be written as |\Psi_I\rangle = \prod_{k\in I} a_k^\dagger |\Omega\rangle with the eigenvalues E_I = \sum_{k\in I} \Gamma_k, \qquad I \subset \Lambda^*, where —\Omega\rangle$ is the Fock vaccum

$$a(x)|\Omega\rangle = 0, \quad \forall\, x \in \Lambda, \quad \text{or} \quad a_k|\Omega\rangle = 0, \quad \forall\, k \in \Lambda^*.$$

The ground state, in particular, is given by

$$|\Psi_0\rangle = |\Psi_{I_0}\rangle = \prod_{k\in I_0} a_k^\dagger |\Omega\rangle \tag{6}$$

where

$$I_0 = \{k \subset \Lambda^* : \Gamma_k < 0\}.$$

There are three regions of localization of the ground state as a function of the external field:

$$\begin{cases} \text{if} \quad |h| < J \quad \text{then} \quad I_0 \neq \emptyset, \quad I_0 \neq \Lambda^* \\ \text{if} \quad h > J \quad \Rightarrow E_I > 0 \quad \forall \quad I \subset \Lambda^* \quad \text{then} \quad I_0 = \emptyset \\ \text{if} \quad h < -J \quad \Rightarrow E_I < 0 \quad \forall \quad I \subset \Lambda^* \quad \text{then} \quad I_0 = \Lambda^* \end{cases}$$

In the first case we will prove that the correlation function between the first and last spins of the chain exhibits a polynomial decay. On the other hand, whether $h > J$ (so that $I_0 = \emptyset$), or $h < -J$ (so that $I_0 = \Lambda^*$), the correlation length will be shown to be zero.

4 Correlation Functions in One Dimension

We define the correlation function between spins at sites x and y by

$$\langle\Psi|[S_+(x)S_-(y) + S_-(x)S_+(y)]|\Psi\rangle$$

where $|\Psi\rangle$ is an eigenvector of the total particle number operator

$$\sum n(x)|\Psi\rangle = N|\Psi\rangle$$

with $n(x) = S_+(x)S_-(x)$, for some integer $N \geq 0$.
To prove that for $h < |J|$, the spin-spin correlation function between the first and last spins decays polynomially we are going, first of all, to write the correlation function in terms of Fermi operators, by inverting the system (3)

$$S_+(x)S_-(y) = a^\dagger(x) \prod_{x<z<y} e^{i\pi n(z)} a(y)$$

$$S_-(x)S_+(y) = -a(x) \prod_{x<z<y} e^{i\pi n(z)} a^\dagger(y).$$

for $x < y$.
We then obtain (again for $x < y$)

$$\langle \Psi | S_+(x)S_-(y) | \Psi \rangle = -e^{i\pi(L-1)} \langle \Psi | \prod_{-L \leq z < x} e^{i\pi n(z)} a^\dagger(x) a(y) \prod_{y < z' \leq L} e^{i\pi n(z')} | \Psi \rangle$$

$$\langle \Psi | S_-(x)S_+(y) | \Psi \rangle = e^{i\pi(L-1)} \langle \Psi | \prod_{-L \leq z < x} e^{i\pi n(z)} a(x) a^\dagger(y) \prod_{y < z' \leq L} e^{i\pi n(z')} | \Psi \rangle$$

For the end points these non-local expressions reduce to the Fermi two-point function:

$$\langle \Psi | S_+(1)S_-(L) | \Psi \rangle = -e^{i\pi(L-1)} \langle \Psi | a^\dagger(1) a(L) | \Psi \rangle$$
$$\langle \Psi | S_-(1)S_+(L) | \Psi \rangle = e^{i\pi(L-1)} \langle \Psi | a(1) a^\dagger(L) | \Psi \rangle.$$

If we now take $|\Psi\rangle$ to be the ground state $|\Psi_0\rangle$ given by (6), the two-point function of the Fermi operators can be computed to give (for $x < y$),

$$\begin{aligned} \langle \Psi_0 | a^+(x) a(y) | \Psi_0 \rangle &= \sum_{l,m} \varphi_l(x) \varphi_m(y) \langle \Psi_0 | a_l^\dagger a_m | \Psi_0 \rangle \\ &= \sum_{k_1 \in I_1} A_+^2 \cos k_1 x \cos k_1 y + \sum_{k_2 \in I_2} A_-^2 \sin k_2 x \sin k_2 y \end{aligned}$$

where $k_{1,2} = \{k : \Gamma_k < 0\}$.
In the limit $L \to \infty$ we obtain the integral form

$$\langle \Psi_0 | a^+(x) a(y) | \Psi_0 \rangle = \frac{1}{\pi} \int_{-k_c}^{k_c} \cos k(x-y)\, dk$$

where k_c is the solution of the equation

$$h - J \cos k_c = 0.$$

We have then

$$\langle\Psi_0|[S_+(1)S_-(L)+S_-(1)S_+(L)]|\Psi_0\rangle = -\frac{4}{\pi}e^{i\pi(L-1)}\frac{\sin k_c(L-1)}{L-1},$$

thus proving our assertion at the beginning of this section. To obtain the spin-spin correlation function for $h > |J|$, just note that since $S_+(x)|\Psi\rangle$ creates a particle at x, and $S_+(y)|\Psi\rangle$ creates, independently of x, another particle at y, we have simply

$$\langle\Psi|S_+(y)\sigma_+(x)|\Psi\rangle = \delta_{xy}$$

which proves that the spin variables become independent when the field is sufficiently strong.

5 Time Decay of Correlation Functions

The time decay of correlation functions, for $h < |J|$, is given by

$$\begin{aligned}
\langle\Psi_I|a^\dagger(x)e^{-t(\mathcal{H}-E_I)}a(y)|\Psi_I\rangle &= \sum_{l,m}\varphi_l(x)\varphi_m(y)\times \\
&\times \langle\Psi_I|a_l^\dagger e^{-t(\sum_k\Gamma_k a_k^\dagger a_k - E_I)}a_m|\Psi_I\rangle = \\
&= \sum_{l,m}\varphi_l(x)\varphi_m(y)\langle\Psi_I|a_l^\dagger e^{t\Gamma_m}a_m|\Psi_I\rangle = \\
&= \sum_{l\in I}\varphi_l(x)\varphi_l(y)e^{t\Gamma_l}.
\end{aligned}$$

Substituting for φ_k the expressions given in (8), we get

$$\int_{-k_c}^{k_c} dk\ \cos k(x-y)e^{t\Gamma_k}.$$

The major contribution to the integral comes from the behavior of Γ_k in the neighborhood of the extremes of integration. So for an arbitrary $0 < \epsilon < k_c$ we divide the integral into three parts:

$$\int_{-k_c}^{-k_c+\epsilon} dk\cos k(x-y)e^{t\Gamma_k} + \int_{|k|<k_c-\epsilon} dk\cos k(x-y)e^{t\Gamma_k} +$$

$$+\int_{k_c-\epsilon}^{k_c} dk \cos k(x-y) e^{t\Gamma_k}$$

For $|k| < k_c - \epsilon$ the following estimate holds

$$\int_{|k_c|<k_c-\epsilon} dk\ \cos k(x-y) e^{t\Gamma_k} \leq \int_{|k_c|<k_c-\epsilon} dk\ \cos k(x-y) e^{t\Gamma_\epsilon}$$

The expression on the r.h.s. goes to zero in the limit $\epsilon \to k_c$.
For the other two regions we expand Γ_k around the limits of integration

$$\Gamma_k = \Gamma^{'}(-k_c)(k+k_c) + \Gamma^{''}(-k_c)(k+k_c)^2 + ..., \ -k_c < k < -\epsilon$$

and

$$\Gamma_k = \Gamma^{'}(k_c)(k-k_c) + \Gamma^{''}(k_c)(k-k_c)^2 + ..., \ \epsilon < k < k_c.$$

We then perform the following change of variables

$$u = \begin{cases} t|\Gamma^{'}(-k_c)|(k+k_c) & \text{for } -k_c < k < -\epsilon \\ \\ t\Gamma^{'}(k_c)(k-k_c) & \text{for } \epsilon < k < k_c \end{cases}$$

This yields, respectively,

$$\int_0^{t|\Gamma^{'}(-k_c)|\epsilon} \frac{du}{t|\Gamma^{'}(-k_c)|} \cos\left(\frac{u}{t|\Gamma^{'}(-k_c)|} - k_c\right)(x-y) e^{-u+o(1/t)+...}$$

and

$$\int_{-t\Gamma^{'}(k_c)\epsilon}^{0} \frac{du}{t\Gamma^{'}(k_c)} \cos\left(\frac{u}{t\Gamma^{'}(k_c)} + k_c\right)(x-y) e^{u+o(1/t)+...}$$

which, for $t \gg 1$, can be approximated by

$$\frac{1}{t|\Gamma^{'}(-k_c)|} \cos k_c(x-y)$$

and

$$\frac{1}{t\Gamma^{'}(k_c)} \cos k_c(x-y).$$

So we get

$$\langle \Psi_I | a^\dagger(x) e^{-t(\mathcal{H}-E_I)} a(y) | \Psi_I \rangle \leq \frac{2}{t\Gamma^{'}(k_c)} \cos k_c(x-y) +$$
$$+ \int_{-k_c+\epsilon}^{k_c-\epsilon} dk\ \cos k(x-y) e^{t\Gamma_\epsilon}$$

Taking the limit $\epsilon \to k_c$, the integral goes to zero, so that the term on the left exhibits the $1/t$ dependence.
For $h > |J|$, the time decay of the correlation function is given by

$$\begin{aligned}
\langle \Psi_I | a(x) e^{-t(\mathcal{H}-E_I)} a^\dagger(y) | \Psi_I \rangle &= \sum_{l,m} \varphi_l(x) \varphi_m(y) \times \\
&\times \ \langle \Psi_I | a_l e^{-t(\sum_k \Gamma_k a_k^\dagger a_k - E_I)} a_m^\dagger | \Psi_I \rangle = \\
&= \sum_{l,m} \varphi_l(x) \varphi_m(y) \langle \Psi_I | a_l e^{-t\Gamma_k} a_m^\dagger | \Psi_I \rangle = \\
&= \sum_l \varphi_l(x) \varphi_l(y) e^{-t\Gamma_l}.
\end{aligned}$$

The behavior of the correlation is determined by the energy gap between the ground state and the excited ones. This gap is just $\Gamma_0 = |h| - J$. Since $\Gamma_0 \le \Gamma_k$, $\forall\, k \in \Lambda^*$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} dk\, \varphi_k(x) \varphi_k(y) e^{-t\Gamma_k} &\le \int_{-\pi}^{\pi} dk\, |\varphi_k(x)|\, |\varphi_k(y)| e^{-t(|h|-J)} \\
&= C \delta_{xy} e^{-t(|h|-J)}.
\end{aligned}$$

For $h = |J|$, we expand Γ_k around $k = 0$,

$$\int_{-\pi}^{\pi} dk\, \varphi_k(x) \varphi_k(y) e^{-t\Gamma_k} = \int_{-\pi}^{\pi} dk\, \varphi_k(x)\, \varphi_k(y) e^{-th_c(\frac{k^2}{2!} - \frac{k^4}{4!} + \ldots)} \tag{7}$$

and make the change of variable $u = k\sqrt{\frac{th_c}{2}}$, to obtain

$$\frac{1}{\sqrt{t}} \int_{-\infty}^{+\infty} du\, \varphi_u(x) \varphi_u(y) e^{-u^2 + o(1/t)}$$

6 Appendix

It is also possible to obtain the result of section 2 by formally identiffying the effect of the Hamiltonian over the state $|\Psi_n\rangle$ with the difference Laplacean in d dimensions, $-\Delta_d$, acting on the function $\Phi(x_1, ..., x_n)$, which we define in the standard way, as

$$\begin{aligned}
-\Delta_d \Phi(x_1, ..., x_n) = \sum_{i=1}^{n} \sum_{k=1}^{d} [2\Phi(x_1, ..., x_i, ..., x_n) - \\
-\Phi(x_1, ..., x_i + \mathbf{e}_k, ..., x_n) - \Phi(x_1, ..., x_i - \mathbf{e}_k, ..., x_n)].
\end{aligned}$$

The argument of the function $\Phi(x_1, x_2, \ldots, x_n)$ has now the following characteristics:

$$\begin{aligned} x_1 = x_2 = \ldots = x_{m_1} &= y_1, \\ x_{m_1+1} = x_{m_1+2} = \ldots = x_{m_1+m_2} &= y_2, \\ &\vdots \\ x_{(m_{i-1})+1} = x_{(m_{i-1})+2} = \ldots = x_{(m_i)} &= y_i, \end{aligned}$$

where we have introduced the notation $x_{(m_p)} = x_{m_1+\ldots+m_p}$. We also notice that

$$i)\; i \leq n,\;\; m_p \leq 2S \Rightarrow \sum_{p=1}^{i} m_p \leq 2nS$$

$$\begin{aligned} ii)\; &\Phi(x_1, \ldots, x_{(m_p)+1} + a, x_{(m_p+2}, \ldots, x_{(m_p)+m_{p+1}}, \ldots, x_n) = \\ &= \Phi(x_1, \ldots, x_{(m_p)+1}, x_{(m_p)+2} + a, x_{(m_p)+3}, \ldots, x_{(m_p)+m_{p+1}}, \ldots, x_n) = \\ &\vdots \\ &= \Phi(x_1, \ldots, x_{(m_p)+m_{p+1}-1}, x_{(m_p)+m_{p+1}} + a, \ldots, x_n), \end{aligned}$$

for any $a \in \mathbf{Z}^d$, $1 \leq p \leq i$. So that the action of the lattice Laplacean over the function $\Phi(x_1, \ldots, x_n)$ produces

$$\begin{aligned} -\Delta_d \Phi(x_1, \ldots, x_n) = \sum_{p=1}^{i} m_p \sum_{j=1}^{d} [2\Phi(x_1, \ldots, x_n) - \\ -\Phi(x_1, \ldots, x_{(m_p)+1} + \mathbf{e}_j, \ldots, x_n) - \Phi(x_1, \ldots, x_{(m_p)+1} - \mathbf{e}_j, \ldots, x_n)]. \end{aligned}$$

For a n-angular momentum state we are able to estimate the expectation value of the perturbation to be always less than the norm of the Laplacean operator. Using the Schwartz inequality we have

$$|\langle \Psi_n | \mathcal{H}_\Lambda^{xy} | \Psi_n \rangle| \leq \frac{|J|}{2} \| \langle \Psi | \| \, \| \, \Delta_d | \Psi \rangle \| \leq 2nSd|J|$$

Note that the proof requires the total number of particles to be conserved, and the ground state of $\mathcal{H}_z$ to be the same as that of $\mathcal{H}_{xy}$.

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