# Semi-classical analysis for the transfer operator: formal WKB constructions in large dimension

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November 8 , 1996

**Abstract**

This paper is devoted to problems coming from statistical mechanics. The transfer matrix (or transfer operator) approach consists in reducing the analysis of asymptotic properties of statistical systems to the analysis of the spectral properties of their transfer operator. Sometimes the new problem appears to have a semi-classical nature. Although the problem is similar to the semiclassical study of the Kac's operator presented in our paper with M.Brunaud [BruHe] which was devoted to the study of $\exp -\frac{V}{2} \cdot \exp h^2\Delta \cdot \exp -\frac{V}{2}$ for $h$ small, new features appear for the model $\exp -\frac{V}{2h} \cdot \exp \frac{h}{a}\Delta \cdot \exp -\frac{V}{2h}$. Our first results concern semi-classical analysis of the ground state for this second operator. We then analyze the two models in the large dimension situation. One basic technique is Sjöstrand's formalism of the 0-standard functions as introduced in [Sj5]. The one-dimensional case was presented in [He11].

# 1 Introduction

We consider a model generalizing the Kac model with nearest neighbors interaction:

$$\Phi(x) = \sum_{j\in \mathbb{Z}/m\cdot\mathbb{Z}} V(x_j) + \sum_{j\in \mathbb{Z}/m\cdot\mathbb{Z}} W(x_j, x_{j+1}) \;, \tag{1.1}$$

for $x \in \mathbb{R}^{mp}$, $x = (x_1, \cdots, x_m)$, $x_j = (x_{j1}, \cdots, x_{jp}) \in \mathbb{R}^p$, and we assume that

$$\Phi(-x) = \Phi(x) \;. \tag{1.2}$$

$V$ is now a potential defined on $\mathbb{R}^p$ (which is usually invariant by circular permutation of the variables) and we shall try to follow most of the arguments with respect to $p$ in order to take the limit $p \to \infty$ after having first taken the limit $m \to \infty$. We are interested in the asymptotic properties of the measure

$$(\pi h)^{-\frac{mp}{2}} \exp -\frac{\Phi}{h} dx^{(mp)}$$

as $m$ and $1/h$ tend to $+\infty$. More precisely we are mainly interested in the asymptotic behavior of the total measure

$$\mathrm{Vol}(\mathbb{R}^{mp}) := \int_{\mathbb{R}^{mp}} \exp -\frac{\Phi}{h}\; dx^{(mp)} \tag{1.3}$$

as $m$ and $p$ tend to $\infty$. As often in statistical mechanics, the order in which we take the limit could be quite important.
Following the general reduction called the "transfer matrix method", we have seen that we have to analyze the following transfer operator $K^{tf} = K_V$ whose kernel is given by

$$K_V(x,y) = (\pi h)^{-\frac{p}{2}} \exp -\left(\frac{1}{2h}\cdot (V(x) + V(y))\right) \cdot \exp -h^{-1}W(x,y) \;.$$

We recall indeed the following relation (see for example (12.19) in [Pa] or [He8], [He9]) between the transfer operator and the total volume:

$$\lim_{m\to+\infty} \frac{\ln \mathrm{Vol}(\mathbb{R}^{mp})}{m} = \ln \mu_1 \;, \tag{1.4}$$

where $\mu_1$ is the largest eigenvalue of $K^{tf}$.
The second step could be to analyze $\lim_{p\to+\infty} \frac{\ln \mu_1}{p}$ but this will not be done

here and we shall only speak about uniform control with respect to $p$ of $\frac{\ln \mu_1}{p}$.
Although our methods are potentially more general, we shall concentrate our study on the case when

$$W(x,y) = \frac{a}{4}|x-y|^2\ ,$$

with $a > 0$.
As a typical $V$ we take

$$V^{(p)}(y) = \sum_{i=1}^{p} v(y_i) + b\sum_{i=1}^{p} |y_i - y_{i+1}|^2\ ,$$

and we assume that $b > 0$ and $v$ is a single or double well one dimensional potential with the symmetry

$$v(-t) = v(t)\ .$$

We recognize indeed a "Kac" like operator (up to a multiplicative constant) that we already studied previously ([BruHe], [He3])

$$K^{tf} = \exp -\frac{V}{2h} \cdot \exp \frac{h}{a}\Delta \cdot \exp -\frac{V}{2h}\ ,$$

with $a > 0$ and $\Delta = -\sum_j \partial_{x_j}^2$, but the semiclassical parameter is not inserted as in the standard semiclassical analysis for which we had

$$K^{kac} = \exp -\frac{V}{2} \cdot \exp h^2\Delta \cdot \exp -\frac{V}{2}\ .$$

We note that the potential $V^{(m,p)}$ can in the case $a = b$ be written in the form

$$V^{(m,p)} = \sum_{(j,i)\in(\mathbb{Z}/m\mathbb{Z})\times(\mathbb{Z}/p\mathbb{Z})} v(x_{ji}) + a\sum_{(i,j)\sim(k,\ell)} |x_{ji} - x_{k\ell}|^2\ ,$$

where $(i,j) \sim (k,\ell)$ means here that $(i,j)$ and $(k,\ell)$ are nearest neighbors in $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$.
Our main study here will be to **analyze the existence of formal WKB approximations of groundstate eigenfunctions for these operators.**

Simultaneously, this will give a semiclassical expansion for the first (= largest) eigenvalue and the other important point is that **we shall analyze the behavior with respect to the dimension of the coefficients of this expansion**. This is one step in the direction of understanding the semi-classical analysis of the first eigenvalue with control with respect to the dimension. This step was done in the large dimension context for the Schrödinger operator by J. Sjöstrand [Sj1], [Sj2] and [Sj5] ( see also [HeSj2] ) and for the transfer or Kac operator without control with respect to the dimension in [BruHe], [He3] and [He11].

# 2 Lower bound of the first eigenvalue

We first give a lower bound for the largest eigenvalue $\mu_1$ of $K$ which is here defined as

$$K^{tf} = \exp -\frac{V}{2h} \cdot \exp \frac{h}{a}\Delta \cdot \exp -\frac{V}{2h} \; . \tag{2.1}$$

The distribution kernel of this operator is given by

$$K^{tf}(x,y) = (4\pi \frac{h}{a})^{-\frac{p}{2}} \cdot \exp -\frac{V(x)}{2h} \cdot \exp -\frac{a}{4h}|x-y|^2 \cdot \exp -\frac{V(y)}{2h} \; . \tag{2.2}$$

We recall that the Perron-Frobenius theorem (more precisely, its extension to compact integral operators with strictly positive kernel) gives that the largest eigenvalue is simple and equal to the norm of the operator with kernel $K$. Moreover, the first eigenvector can be chosen strictly positive.
Let us assume that the potential $V$ satisfies

- $$V \geq 0 \; , \tag{2.3}$$

- $$V(x) \to +\infty \text{ as } x \to \infty \; , \tag{2.4}$$

- $$|D_x^\alpha V(x)| \leq C_\alpha (1 + V(x))^{(1-\rho|\alpha|)_+} \; , \tag{2.5}$$

  for some $\rho > 0$ and suitable constants $C_\alpha$,

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  the minima of $V$ are isolated, non-degenerate and in $V^{-1}(0)$ . (2.6)

One can use Segal's Lemma (cf [ReSi], p. 333) giving a direct comparison between the largest eigenvalue of the Kac operator and the smallest eigenvalue $\lambda_1$ of the corresponding Schrödinger operator

$$-\frac{h}{a}\Delta + \frac{V}{h} \,,$$

with the following inequality

$$\mu_1 \geq \exp -\lambda_1 \,. \tag{2.7}$$

We then get that $\mu_1$ is bounded from below independently of $h$ for $0 < h \leq h_0$, by using the standard semi-classical results on the Schrödinger operator $\frac{1}{ha}(-h^2\Delta + aV(x))$ presented for example in [HeSj1] or [Si2].

**Remark 2.1** *:*
*It is possible to verify, by direct comparison with harmonic potentials, that*

$$\lim_{h\to 0} \mu_j(h) = \mu^\infty_{\ell(j),k(j)} \,,$$

*where $\mu^\infty_{\ell,k}$ is for some $k$ the eigenvalue of the operator whose kernel is*

$$(\frac{4}{a}\pi)^{-\frac{p}{2}} \exp -\frac{1}{4}\left[\langle\ \mathit{Hess}\, V(x_k)x|x\rangle + \langle\ \mathit{Hess}\, V(x_k)y|y\rangle + a|x-y|^2\right] \,,$$

*where $x_k$ is one of the minima. These eigenvalues are actually explicitly computable by the Mehler's Formula (see for example [He7]).*

# 3 Decay properties of the first eigenfunction.

## 3.1 A rough control

The first point is to obtain an a priori control on the decay of the groundstate eigenfunction.
This was already explained in [He11] in the one dimensional case and the

proof is essentially unchanged in the general case. So we shall be very sketchy. We keep the assumptions of the preceding section. We now write the property that $u_1$ is the first normalized eigenfunction in the form

$$u_1(x;h) = \frac{1}{\mu_1(h)} \int_{I\!R^p} K_V(x,y;h) u_1(y;h)\, dy \; . \tag{3.1}$$

Using Cauchy-Schwarz, we get immediately, for some $h$-independent $\alpha > 0$,

$$0 < u_1(x;h) \leq \alpha^{-1} (\int K_V(x,y)^2 dy)^{\frac{1}{2}} \; . \tag{3.2}$$

We then deduce the existence of a constant $C$ such that the following estimate is satisfied, for all $h \in ]0,1]$,

$$0 < u_1(x;h) \leq C \cdot h^{-\frac{p}{4}} \cdot \exp -\frac{V(x)}{2h} \; . \tag{3.3}$$

This means that the eigenfunction is strongly localized near the minima of $V$. This will permit us to use cut-off functions in order to construct quasimodes attached to each well. We observe here that we are working with operators which are not local, but pseudolocal in a weak sense. The pseudolocality is obtained through the interaction term $\frac{a}{h} \cdot |x-y|^2$. This is a weaker pseudolocality than the pseudolocality observed in the semi-classical study of the Kac operator where the pseudolocality was obtained through an interaction in $\frac{1}{h^2} \cdot |x-y|^2$. This last case was analyzed in detail in [BruHe] and in [He3] (but without a control with respect to the dimension). This will change the properties and the way of thinking for this problem. The case when the interaction is equal to $\frac{a}{h} \cdot |x-y|^2$ with $a$ large will be more reminiscent of the semiclassical case as observed in [He11].

## 3.2 Toward improvements on the decay

As presented in [He11], these estimates can be improved by recursion. We recall some of the arguments. Our starting point is (3.2). We immediately observe that we can implement this upper bound in (3.1) and we get, for any compact $K$, the upper bound

$$0 < u_1(x;h) \leq C_K \cdot h^{-\frac{3p}{4}} \exp -\varphi_1(x)/h \; , \tag{3.4}$$

for some constant $C_K$ and with

$$\varphi_1(x) = \frac{1}{2}V(x) + \inf_y[\frac{a}{4}(x-y)^2 + V(y)] \; .$$

We see that it is strictly better outside the minima of $V$ and we observe the inequalities

$$\frac{1}{2}V(x) \leq \varphi_1(x) \leq \frac{1}{2}V(x) + \frac{a}{4} \text{ dist } (x, \{V^{-1}(0)\})^2 \; .$$

If we try to iterate, we finally are interested in the largest function $\varphi_\infty(x)$ which satisfies

$$|\varphi_\infty(x) - \varphi_\infty(y)| \leq \frac{1}{2}V(x) + \frac{a}{4}|x-y|^2 + \frac{1}{2}V(y) \; . \tag{3.5}$$

This will be indeed the natural candidate for measuring the decay.

One has actually a natural construction in order to get this function. We take $\varphi_0 = \frac{V}{2}$ and $\varphi_1$ as before and introduce by recursion

$$\varphi_n(x) = \frac{V(x)}{2} + \inf_y[\frac{V(y)}{2} + \frac{a}{4}|x-y|^2 + \varphi_{n-1}(y)] \; . \tag{3.6}$$

It is easy to see that this sequence is increasing with respect to $n$ and that

$$\frac{1}{2}V(x) \leq \varphi_n(x) \leq \frac{1}{2}V(x) + \frac{a}{4} \text{ dist } (x, \{V^{-1}(0)\})^2 \; . \tag{3.7}$$

The limit $\varphi_\infty(x)$ of the sequence $\varphi_n(x)$ is well-defined and satisfies

$$\varphi_\infty(x) = \frac{V(x)}{2} + \inf_y[\frac{V(y)}{2} + \frac{a}{4}|x-y|^2 + \varphi_\infty(y)] \; . \tag{3.8}$$

This limit can be seen as locally Lipschitzian (cf [He11]). As observed in [He11], one can recover in the case $a$ large, the Agmon distance in the sense that $d_\infty$ is asymptotically given by $a^{\frac{1}{2}} d_V$ where $d_V$ is the Agmon distance to the set of minimas.

The explicit computation when $V$ is quadratic is also enlightening. After diagonalization, we can reduce the study to the one-dimensional case when $V(x) = c\frac{x^2}{2}$ and we then find

$$\varphi_\infty(x) = b\frac{x^2}{2} \tag{3.9}$$

with

$$b = \frac{1}{2}\sqrt{c(2a+c)}\ . \tag{3.10}$$

These computations are of course immediately related to the explicit standard computations concerning the harmonic Kac operator (Mehler's Formula) which can be represented as $\exp -S$ where $S$ is an explicit harmonic'type oscillator[1] (see [He7]).

### 3.3 A short discussion in the convex case

We briefly explain how to follow the convexity in the construction given above. We consequently assume

$$\mathrm{Hess}\, V(x) \geq \rho > 0\ ,\ \forall x \in I\!R^p\ , \tag{3.11}$$

and we assume that $V$ has a unique minimum at 0 with

$$V(0) = 0\ . \tag{3.12}$$

It is clear that we have a family of convex functions $\varphi_n$ defined by (3.6) and that there exists, for each $n$, an unique $C^\infty$ function $x \mapsto y_n(x)$ such that

$$\varphi_n(x) = \varphi_{n-1}(y_n(x)) + \Theta(x, y_n(x))\ , \tag{3.13}$$

with

$$\Theta(x,y) = \frac{V(x)}{2} + \frac{V(y)}{2} + \frac{a}{4}|x-y|^2\ . \tag{3.14}$$

Fore simplicity, we discuss only the case when the dimension is one. We have for $\varphi_n''(x)$ the following formula

$$\varphi_n''(x) = \frac{1}{2}[V''(x) + a] - \frac{a^2}{[2v''(y_n(x)) + 2a + 4\varphi_{n-1}''(y_n(x))]} \tag{3.15}$$

[1] More precisely, a Schrödinger operator with a polynomial potential of degree 2.

Given a global lower bound of $\varphi''_{n-1}$ by $\sigma_{n-1}$, we are looking for a lower bound for $\varphi''_n(x)$. But we get from (3.14) and (3.11) the following lower bound

$$\varphi''_n(x) \geq \frac{1}{2}(\rho + a) - \frac{a^2}{2\rho + 2a + 4\sigma_{n-1}} \ . \tag{3.16}$$

This suggests to define the following sequence $(\sigma_n)_{n\in \mathbb{N}}$ by

$$\begin{aligned} \sigma_0 &= \tfrac{\rho}{2} \ , \\ \sigma_n &= \tfrac{1}{2}(\rho + a) - \tfrac{a^2}{2\rho+2a+4\sigma_{n-1}} \ . \end{aligned} \tag{3.17}$$

It is clear that we have

$$0 \leq \sigma_n \leq \frac{1}{2}(\rho + a)$$

and the only possible limit for $\sigma_n$ is

$$\sigma_\infty = \frac{1}{2}\sqrt{(\rho + a)^2 - a^2} \ .$$

The associated sequence

$$w_n = \sigma_\infty - \sigma_n$$

satisfies

$$w_n = \frac{[\frac{1}{2}(\rho + a) - \sigma_\infty]w_{n-1}}{[\frac{1}{2}(\rho + a) + \sigma_\infty] - w_{n-1}} \ .$$

Observing that $0 < w_0 < 2\sigma_\infty < \rho + a$, it is easy to get the convergence of $w_n$ to 0. Modulo some limit argument, which we have not verified in detail we surely obtain that the limit function $\varphi_\infty$ satisfies

$$\varphi''_\infty(x) \geq \frac{1}{2}\sqrt{(\rho + a)^2 - a^2} \ . \tag{3.18}$$

Coming back to the general case, we can also obtain

$$\mathrm{Hess}\, \varphi_\infty(x) \geq \frac{1}{2}\sqrt{(\rho + a)^2 - a^2} \ . \tag{3.19}$$

This gives a universal lower bound measuring the strict convexity of $\mathrm{Hess}\, \varphi_\infty(x)$. The inequality is actually an equality in the quadratic case (see (3.10)). This is of course related to the logconcavity of the first eigenfunction of the Kac operator and we refer to Helffer [He4] and Brascamp-Lieb [BraLi] for more details on this point.

# 4 WKB constructions for $K^{tf}$

## 4.1 Introduction

As a standard step for the construction of an approximate solution, we shall look for the construction of a formal WKB solution in the form
$u^{WKB} = a(x;h)\exp -\frac{\phi_0(x)}{h}$ with $a(x;h) \sim \sum_j a_j(x)h^j$,
or in the form
$u^{WKB} = \exp -\frac{\phi(x;h)}{h}$ with $\phi(x;h) \sim \sum_j \phi_j(x)h^j$.
We then try to find a formal $F(h) \sim \sum_j h^j F_j$ such that

$$Ku^{WKB} \sim \exp -F(h)\, u^{WKB}\ .$$

By formal we mean here that we just consider formal expansions[2] in powers of $h$ and that the expansion of $Ku^{WKB}$ is obtained by applying the formal Laplace integral method.

## 4.2 The eikonal equation

We want to control the $C^\infty$ character of the solution in the preceding construction when we are near the minimum of the potential. We have already defined a Lipschitzian function $\varphi_\infty$ which plays in some sense the role of the Agmon distance to the minima of the potential $V$.
It was observed in [HeSj1] that this Agmon's distance $d$ becomes $C^\infty$ in the neighborhood of the minima if they are assumed to be non-degenerate. The distance $d$ was in this case a solution of the eikonal equation

$$|\nabla d|^2 = V - \inf V\ ,\ d \geq 0\ , \tag{4.1}$$

in the neighborhood of a minimum.
We shall start from the following identities that we obtain first formally by assuming that everything is differentiable. The starting point is

$$\phi(x) = \inf_y [\Theta(x,y) + \phi(y)]\ , \tag{4.2}$$

[2]More precisely, we compare the expansions in powers of $h$ of $\exp \frac{\phi_0(x)}{h}\ K\ u^{WKB}$ and of $\exp \frac{\phi_0(x)}{h}\ \exp -F(h)\ u^{WKB}$ for $x$ in the neighborhood of one minimum of $V$

with

$$\Theta(x,y) = W(x,y) + \frac{V(x)}{2} + \frac{V(y)}{2}\ . \tag{4.3}$$

If we assume that the minimum is obtained for a unique regular $y(x)$ (and this will be proved for $x$ small enough), we get

$$\phi(x) = \Theta(x, y(x)) + \phi(y(x))\ , \tag{4.4}$$

and

$$\partial_y \Theta(x, y(x)) + \nabla\phi(y(x)) = 0\ . \tag{4.5}$$

Differentiating with respect to $x$, we first obtain

$$\nabla\phi(x) = \partial_x \Theta(x, y(x))\ . \tag{4.6}$$

Following an enlightening discussion with J.Sjöstrand, we are actually looking for a Lagrangian manifold

$$\Lambda_\phi = \{(x,\xi) \in I\!R^{2p} | \xi = \nabla\phi(x)\}$$

which is invariant under the canonical relation generated by $\Theta$:

$$C_\Theta = \{(x,\xi,y,\eta) \in I\!R^{4p} | \xi = \partial_x\Theta(x,y)\ ,\ \eta = -\partial_y\Theta(x,y)\} \tag{4.7}$$

We recall that, in the case of the Schrödinger operator, we found a Lagrangian manifold inside $p(x,\xi) =: \xi^2 - V(x) = 0$ by looking for an invariant outgoing Lagrangian manifold for the flow $H_p$. Here we find the same phenomenon in a discrete version.
Let us express the diffeomorphism $\kappa$ whose graph is $C_\Theta$ in the case when $W(x,y) = \frac{a}{4}|x-y|^2$.
We find the equations :

$$\begin{aligned} \xi &= (\nabla V)(x) + \tfrac{a}{2}(x-y) \\ \eta &= -(\nabla V)(y) + \tfrac{a}{2}(x-y)\ . \end{aligned} \tag{4.8}$$

and this gives the expression for $(y,\eta) = \kappa(x,\xi)$

$$\begin{aligned} y &= x + \tfrac{2}{a}(\nabla V)(x) - \tfrac{2}{a}\xi && := y(x,\xi)\ , \\ \eta &= -(\nabla V)(y(x,\xi)) + \xi - (\nabla V)(x) && := \eta(x,\xi)\ . \end{aligned} \tag{4.9}$$

We recall that $\kappa$ is a canonical transformation, that is a diffeomorphism which preserves the canonical symplectic 2-form on $I\!R^{2p}$: $\sum_j d\xi_j \wedge dx_j$.
One gets also a similar expression for $\kappa^{-1}$:

$$\begin{array}{lll} x & = y + \frac{2}{a}(\nabla V)(y) + \frac{2}{a}\eta & := x(y,\eta) \ , \\ \xi & = (\nabla V)(x(y,\eta)) + \eta + (\nabla V)(x) & := \xi(y,\eta) \ . \end{array} \tag{4.10}$$

Let us look at the fixed points of this diffeomorphism. We are looking for the points such that $y(x,\xi) = x$, $\eta(x,\xi) = \xi$ and we get $\xi = 0$, $\nabla V(x) = 0$. The fixed points of the diffeomorphism $\kappa$ are the points $(x,0)$ where $x$ is a critical value of $V$. In particular, if we assume that 0 is a non-degenerate minimum of $V$, $(0,0)$ is a fixed point. We now want to apply the general theory of Smale [Sm1]. We have to verify that $(0,0)$ is an hyperbolic point for $\kappa$. This will be verified in a next subsection by looking at the quadratic case.
The regularity of the "local" stable manifold is an old result due to Smale [Sm1] (See also [Sm2] (p.751) or [Irw]). The property that these stable manifolds are Lagrangian is obtained as in the continuous case (see for example [HeSj1]) once we have observed that the number of eigenvalues of modulus $< 1$ is exactly the dimension $p$.

**Remark 4.1** *:*
*One easily deduces from the symmetry of* $\Theta$

$$\Theta(x,y) = \Theta(y,x) \ ,$$

*the property that*

$$(x,\xi,y,\eta) \in C_\Theta \quad \textit{iff} \quad (y,-\eta,x,-\xi) \in C_\Theta \ .$$

The function $\phi$ appears simply as the generating function of the Lagrangian outgoing manifold normalized by the condition $\phi(0) = 0$ and using the preceding remark one get also that $-\phi$ is the generating function for the ingoing manifold.
We have found consequently, for a given non-degenerate minimum of the potential, a ball $B$ centered at this minimum, in which $\phi$ is defined and satisfies

$$\phi(x) = \inf_{y \in B} [\Theta(x,y) + \phi(y)] \ , \ \forall x \in B \ .$$

## 4.3 The first transport equation

Once we have found the phase $\phi$, we arrive naturally to the study of the WKB problem: find $a(x;h)$ and $E(h)$ such that

$$(K_V a \exp -\phi(\cdot)/h)(x) \sim E(h)a(\cdot;h) \exp -\phi(\cdot)/h \ . \tag{4.11}$$

We want the following formal identity to be satisfied:

$$E(h)a(x;h) \sim h^{-\frac{p}{2}} \int a(y;h) \cdot \exp -\frac{1}{h}[\Theta(x,y) + \phi(y) - \phi(x)]\, dy \ . \tag{4.12}$$

Applying the Laplace integral method, we get

$$E_0\, a_0(x) = C^p \cdot a_0(y(x))(\det[\nabla^2_{yy}\Theta + \nabla^2\phi])^{-\frac{1}{2}}(y(x)) \ , \tag{4.13}$$

where $C$ is a universal constant.
A necessary condition, in order to solve (4.13) is

$$E_0 = C^p \cdot (\det[\nabla^2_{yy}\Theta + \nabla^2\phi])^{-\frac{1}{2}}(0) \ . \tag{4.14}$$

Then we have for $a_0$ the formula

$$a_0(x) = \prod_{n=1}^{\infty} \theta(y^{(n)}(x)) \ , \tag{4.15}$$

with by definition

$$y^{(n)}(x) = y(y^{(n-1)}(x)) \ , \tag{4.16}$$

and

$$\theta(y) = \left(\frac{\det[\nabla^2_{yy}\Theta + \nabla^2\phi](0)}{\det[\nabla^2_{yy}\Theta + \nabla^2\phi](y)}\right)^{\frac{1}{2}} \ . \tag{4.17}$$

We observe here that, for any $x$ sufficiently near the bottom 0 of $V$, we have

$$y^{(n)}(x) \to 0 \tag{4.18}$$

and the convergence is exponentially rapid. Let us observe also that, for our particular function $\Theta$, the Hessian $\nabla^2_{yy}\Theta$ is independent of $x$:

$$(\nabla^2_{yy}\Theta)(y,x) = \frac{a}{2}\, I + \frac{1}{2}\, \mathrm{Hess}\, V(y) \ . \tag{4.19}$$

In order to see what is going on, let us come back to the one dimensional quadratic case. We recall that, with $V(x) = \frac{c}{2}x^2$, we have

$$\phi(y) = \frac{b}{2}y^2 \tag{4.20}$$

with

$$b = \frac{1}{2}\sqrt{c(2a+c)} \tag{4.21}$$

and

$$y(x) = \frac{a}{c+a+2b}\, x\ . \tag{4.22}$$

Of course we do not need the discussion in this case, but what will be relevant for our discussion is the property that

$$||y(x)|| \leq k||x|| \tag{4.23}$$

for $0 < k < 1$ for a suitable norm (as can be seen in the proof of the stable manifold theorem). This is what will make the procedure convergent and actually exponentially convergent.

## 4.4 The analysis of the diffeomorphism in the quadratic case

In order to verify this property of hyperbolicity, let us make explicit the computation of the diffeomorphism $\kappa_0$, $(y, \eta) = \kappa_0(x, \xi)$, attached to the quadratic form $V_0(x) = \frac{1}{2}\langle\, \mathrm{Hess}\, V(0)x, x\rangle$ instead of $V$. After diagonalization of this potential attached to $\mathrm{Hess}\, V$ at the minimum, we can reduce our considerations to the two-dimensional case. The map $\kappa$ is in this case linear and given by the $2\times 2$ matrix

$$M_\kappa = \begin{pmatrix} \frac{(a+c)}{a} & -\frac{2}{a} \\ -\frac{1}{2a}c(c+2a) & \frac{(a+c)}{a} \end{pmatrix}$$

The two eigenvalues are given by

$$\lambda_\pm = \frac{(a+c)}{a} \pm \frac{1}{a}\sqrt{c(c+2a)}$$

and one verifies that

$$0 < \lambda_- < 1$$

and

$$\lambda_-^{-1} = \lambda_+ > 1 \; .$$

We have consequently an hyperbolic point (saddle point situation) and a one dimensional subspace $\Lambda_-$ on which $\kappa$ is contracting, that is the eigenspace attached to $\lambda_-$. One finds the equation

$$\xi = \frac{1}{2}\sqrt{c(c+2a)}x$$

and, as in the case of the Schrödinger operator, one has, with $\phi$ introduced in (4.20),

$$\xi = \phi'(x)$$

that is $\phi$ is the generating function of $\Lambda_-$.

## 4.5 Remarks in the convex case

We have already mentioned that in the strictly convex case a global strictly convex function $\phi(x)$ can be defined. Moreover, we can obtain a global control of the function $x \mapsto y(x)$. Differentiating (4.5) with respect to $x$, we obtain

$$(\text{ Hess}\,\phi(x) + \text{ Hess}_{yy}\Theta(x,y(x)))\,\nabla y(x) + \partial_x\partial_y\Theta(x,y(x)) = 0 \; . \tag{4.24}$$

This gives

$$\nabla y(x) = \frac{a}{2}\left(\text{ Hess}\,\phi(x) + \text{ Hess}_{yy}\Theta(x,y(x))\right)^{-1} \; . \tag{4.25}$$

This gives

$$||\nabla y(x)|| \leq \frac{a}{a+\rho} < 1 \; , \; \forall x \; . \tag{4.26}$$

This gives that the map $I\!R^p \ni x \mapsto y(x)$ is a contracting map

$$||y(x)|| \leq k||x|| \; , \tag{4.27}$$

with $k = \frac{a}{a+\rho}$.

**Remark 4.2** *:*
*We observe also that we know (through Brascamp-Lieb [BraLi]) from for example [He4] that the first eigenfunction is logconcave and that it is possible to measure its logconcavity.*

# 5 Transport equations

## 5.1 First approach

We recall that the first equation was

$$E_0 a_0(x) = C^p \cdot a_0(y(x))(\det[\nabla^2_{yy}\Theta + \nabla^2\phi])^{-\frac{1}{2}}(y(x)) \ , \tag{5.1}$$

where $C$ was a universal constant. We have also obtained

$$E_0 = C^p(\det[\nabla^2_{yy}\Theta + \nabla^2\phi])^{-\frac{1}{2}}(0) \ , \tag{5.2}$$

By continuing to apply the Laplace integral method in order to cancel the different powers of $h$, we obtain as next equation

$$E_1 a_0(x) + E_0 a_1(x) = C^p \cdot a_1(y(x))(\det[\nabla^2_{yy}\Theta + \nabla^2\phi])^{-\frac{1}{2}}(y(x)) + c_1(x) \ , \tag{5.3}$$

where $c_1$ depends only on $a_0$ and on its derivatives of order less or equal to 2.
It seems better to look for $a_1$ in the form

$$a_1(x) = a_0(x) \cdot b_1(x) \ , \tag{5.4}$$

and write $c_1$ in the form

$$c_1(x) = E_0\, a_0(x) \cdot d_1(x) \ .$$

We then obtain

$$\begin{aligned} &E_1\, a_0(x) + E_0\, a_0(x) \cdot b_1(x) = \\ &\qquad C^p \cdot a_0(y(x)) \cdot b_1(y(x))(\det[\nabla^2_{yy}\Theta + \nabla^2\phi])^{-\frac{1}{2}}(y(x)) \ + \ E_0\, a_0(x) \cdot d_1(x) \ , \end{aligned}$$

or using the equation satisfied by $a_0$

$$b_1(x) = b_1(y(x)) + (d_1(x) - \frac{E_1}{E_0}) \ .$$

We meet the necessary condition which determines actually $E_1$

$$d_1(0) - \frac{E_1}{E_0} = 0 \ , \tag{5.5}$$

and the equation takes the structure

$$b_1(x) = b_1(y(x)) + \theta_1(x) \ , \tag{5.6}$$

with $\theta_1(x) = d_1(x) - d_1(O)$ , satsfying in particular

$$\theta_1(0) = 0 \ . \tag{5.7}$$

We then obtain the following formula for $b_1$ by iterating the functional equation

$$b_1(x) = \sum_{n=0}^{\infty} \theta_1(y^{(n)}(x)) \ . \tag{5.8}$$

The successive transport equations are treated in the same way.

## 5.2 Second approach

As was observed by J.Sjöstrand in the case of the Schrödinger operator, it can be easier, particularly if our idea is to study later the thermodynamic limit, to look for a WKB solution in the form

$$u^{wkb}(x) = \exp -\frac{\phi(x;h)}{h} \ , \tag{5.9}$$

with

$$\phi(x;h) \sim \sum_{j=0}^{\infty} h^j \ \phi_j(x) \ . \tag{5.10}$$

Here $\phi_0$ is the solution we have determined before in our study of the eikonal equation. We emphasize that this approach is only adapted for the first eigenfunction which can be chosen to be strictly positive.

We shall consequently look (after a renormalization by a multiplicative factor) for a solution satisfying formally (in the sense of formal series in powers of $h$, the Laplace integral being considered only formally in a neighborhood of a minimum)

$$(\pi h)^{-\frac{p}{2}} \int_{\mathbb{R}^p} \exp -\frac{1}{h} \left(-\phi(x;h) + \Theta(x,y) + \phi(y;h)\right) \ dy \ \sim \ \exp -F(h) \ , \tag{5.11}$$

for all $x$ in a neighborhood of the origin, with

$$F(h) \sim \sum_{j=0}^{\infty} h^j \ F_j \ . \tag{5.12}$$

We shall follow the treatment which was used by J. Sjöstrand [Sj1] (and later by the author [He5]) for Laplace integrals by reproving a Laplace integral method with parameters (the parameter will be $x$). Following J. Sjöstrand, the key point is to reprove also the Morse's Lemma which is used in the standard proof of the Laplace integral method. The idea is actually to look for a change of variables in the Laplace integral taking account also of the Jacobian and in the form

$$y \mapsto z \sim \nabla_y f(x, y; h) \ , \tag{5.13}$$

with

$$f(x, y; h) \sim \sum_{j=0}^{\infty} h^j \ f_j(x, y) \ . \tag{5.14}$$

We find the following equation

$$\begin{aligned} &-\phi(x; h) + \Theta(x, y) + \phi(y; h) \\ &\quad \sim |\nabla_y f(x, y; h)|^2 - h \ln \det(\nabla^2_{yy} f(x, y; h)) + hF(h) \ , \end{aligned} \tag{5.15}$$

that we have to solve formally in powers of $h$.

Let us look first at the first equation. We recall that we have already chosen $\phi_0$ such that the map $y \mapsto \Psi_x(y) = -\phi_0(x) + \Theta(x, y) + \phi_0(y)$ vanishes to order 2 at $y = y(x)$.
We find an "eikonal" equation for $f_0$,

$$-\phi_0(x) + \Theta(x, y) + \phi_0(y) = |\nabla_y f_0(x, y)|^2 \ , \tag{5.16}$$

and we are looking for a function $f_0$ satisfying

$$f_0(x, y) \geq 0 \ , \tag{5.17}$$

and vanishing at the critical point $y = y(x)$ of the function $\Psi_x(y)$, that is,

$$f_0(x, y(x)) = 0 \ . \tag{5.18}$$

We get also from the equation satisfied by $f_0$

$$(\nabla_y f_0)(x, y(x)) = 0 \; . \tag{5.19}$$

If the Hessian of $\Psi_x$ satisfies the condition

$$(\nabla^2_{yy}\Psi_x)(y(x)) > 0 \tag{5.20}$$

then $f_0$ exists with a good control if there is a corresponding good control of $\Psi_x$.
In particular if $\operatorname{Hess} V(0)$ is a sufficiently small perturbation of a diagonal matrix $D_0$:

$$\operatorname{Hess} V(0) = D_0 + W_0 \; ,$$

then the same will be true for $\operatorname{Hess}\phi_0(0)$ which will be indeed close to $\frac{1}{2}\sqrt{D_0(D_0+2aI)}$. (This remark is for the future when we shall have to control all the construction with respect to the dimension).
Let us now look at the $\mathcal{O}(h)$ term in (5.15). We find (think of $x$ as a parameter)

$$-\phi_1(x) + \phi_1(y) = 2\nabla_y f_0(x,y)\cdot\nabla_y f_1(x,y) - \ln\det(\nabla^2_{yy} f_0(x,y)) + F_0 \; . \tag{5.21}$$

Let us explain now how we can solve this equation. We first consider necessary conditions and note that

$$-\phi_1(x) + \phi_1(y(x)) = -\ln\det(\nabla^2_{yy} f_0(x, y(x))) + F_0 \; . \tag{5.22}$$

Here the unknowns are $\phi_1$ and $F_0$. By taking $x = 0$, we obtain the new equation

$$F_0 = \ln\det(\nabla^2_{yy} f_0(0,0)) \; , \tag{5.23}$$

which actually defines $F_0$.
Once this condition is satisfied, we shall get the "transport" equation

$$\phi_1(x) = \phi_1(y(x)) + \theta_1(x) \; , \tag{5.24}$$

with

$$\theta_1(x) = \ln\det\Big(\, \nabla^2_{yy} f_0\,(x, y(x))\,\Big) - F_0 \; , \tag{5.25}$$

and we recall that, by our choice of $F_0$,

$$\theta_1(0) = 0 \; . \tag{5.26}$$

Imposing that

$$\phi_1(0) = 0 \; , \tag{5.27}$$

we find

$$\phi_1(x) = \sum_{j=0}^{\infty} \theta_1(y^{(j)}(x)) \; . \tag{5.28}$$

Once we have solved (5.28), then we can solve (5.21) as was done for example in [HeSj1]. The convergence is not too difficult to obtain once we have observed the property that $||y(x)|| \leq k||x||$ with $k < 1$ for a suitable norm $|| \cdot ||$.
Now the general compatibility equation is

$$-\phi_j(x) + \phi_j(y(x)) = -[\ln\det(\nabla^2_{yy} f(x, y(x); h))]_{j-1} + F_{j-1} \; . \tag{5.29}$$

$[\ln\det(\nabla^2_{yy} f(x, y(x); h))]_{j-1}$ is the coefficient of $h^{j-1}$ in the expansion of $\ln\det(\nabla^2_{yy} f(x, y(x); h))$ and we observe that it depends only on $f_\ell$ and $F_\ell$ for $\ell \in \{0, \cdots, j-1\}$. This can be solved in the same way as for $\phi_1$.
Once (5.29) is satisfied, we can solve the equation corresponding to the vanishing of the $j$-th coefficient of (5.15)

$$-\phi_j(x)+\phi_j(y) = \sum_{\ell=0}^{j} \nabla_y f_\ell(x,y)\cdot\nabla_y f_{j-\ell}(x,y) - [\ln\det(\nabla^2_{yy} f(x, y; h))]_{j-1} + F_{j-1} \; . \tag{5.30}$$

What we see here is that, once we have found that $\phi_0$ has good properties[3], all the other operations appearing in the construction have essentially been considered in [Sj1], [He5] and [Sj5]. We shall indeed see later how this can be used in order to control the construction with respect to the dimension.

# 6 WKB for the Kac model

Before considering the large dimension problem, we think it is worth while to analyze in the same spirit the other transfer matrix problem that we have

[3]This means in particular that $\phi_0(x)$ is a $C^\infty$ strictly positive function with strictly positive Hessian at 0 but also various controls with respect to the dimension which will be detailed in Section 7.

first considered in [BruHe] and [He3]. Trying to answer a problem posed by M. Kac, we have considered another regime of parameters and constructed the WKB ground state solution for $\exp -\frac{V}{2} \exp h^2 \Delta \exp -\frac{V}{2}$ whose kernel is

$$K_V(x,y;h) = (4\pi h^2)^{-\frac{p}{2}} \exp -\frac{V(x)}{2} \ \exp -\frac{|x-y|^2}{4h^2} \ \exp -\frac{V(y)}{2} \ . \tag{6.1}$$

We shall look, in a suitable neighborhood of a non-degenerate minimum of $V$, for a WKB solution in the form

$$u^{wkb}(x) \sim \exp -\frac{\phi(x;h)}{h} \ ,$$

with

$$\phi(x,h) \sim \sum_{j=0}^{\infty} h^j \ \phi_j(x) \ . \tag{6.2}$$

Our previous proof was based on the observation that this operator can be considered as a $h$-pseudodifferential operator and that we can adapt in this context the standard WKB constructions introduced for the Schrödinger operator, but which were actually true for more general Hamiltonians, if some analyticity condition in the impulsion variable is satisfied. But it seems difficult to control this technique as the dimension tends to $\infty$. The reason is that a theory of $h$-pseudodifferential operators with control with respect to the dimension is still at a primitive stage and we know only about one paper by J. Sjöstrand [Sj5] which does not seem sufficient for our purpose. We propose here a more direct approach which will use the specific structure of the integral operator. This seems however more difficult than in the previous case because the parameter $h$ does not appear in an homogeneous way. We have to deal consequently with a kind of Laplace integral method with a degenerate critical set of minima for the phase.

We are consequently looking for a solution satisfying formally

$$\int_{\mathbb{R}^p} K_V(x,y;h) \exp -\frac{1}{h}\left(-\phi(x;h) + \phi(y;h)\right) \ dy \ \sim \exp -F(h) \tag{6.3}$$

with

$$F(h) \sim \sum_{j=0}^{\infty} h^j \ F_j \ . \tag{6.4}$$

As in the preceding section, we shall look for a change of variables in the Laplace integral taking account also of the Jacobian and in the form

$$z \sim \nabla_y f(x,y;h) \;, \tag{6.5}$$

with

$$f(x,y;h) \sim \sum_{j=0}^{+\infty} h^j \; f_j(x,y) \;. \tag{6.6}$$

The equation we find is the following

$$\begin{array}{r} -h^{-1}\phi(x;h) + \frac{1}{4}h^{-2}|x-y|^2 + h^{-1}\phi(y;h) + \frac{V(x)}{2} + \frac{V(y)}{2} \\ \sim h^{-2}|\nabla_y f(x,y;h)|^2 - \ln\det(\nabla^2_{yy} f(x,y;h)) + F(h) - p\ln 2 \;, \end{array} \tag{6.7}$$

that we have to solve formally by identifying the coefficients of the formal expansion in powers of $h$.

**The coefficient of $h^{-2}$** is eliminated by taking

$$f_0(x,y) = \frac{1}{4}|x-y|^2 \;. \tag{6.8}$$

This means that the first change of variable is just the map $y \mapsto \frac{1}{2}(y-x)$.

**The coefficient of $h^{-1}$.**
We get

$$\begin{array}{rl} \phi_0(y) - \phi_0(x) & = 2(\nabla_y f_0)(x,y)\cdot(\nabla_y f_1)(x,y) \\ & = (y-x)\cdot(\nabla_y f_1)(x,y) \;, \end{array} \tag{6.9}$$

and we choose the initial condition

$$f_1(x,x) = 0 \;. \tag{6.10}$$

Although, for the moment, neither $f_1$ nor $\phi_0$ are determined, we observe as a consequence of (6.9) that,

$$(\nabla\phi_0)(x) = (\nabla_y f_1)(x,x) \;. \tag{6.11}$$

**The coefficient of $h^0$.**
We get

$$\begin{array}{l} \phi_1(y) - \phi_1(x) + \frac{V(x)}{2} + \frac{V(y)}{2} \\ \quad = (y-x)\cdot(\nabla_y f_2)(x,y) + (|\nabla_y f_1|^2)(x,y) \\ \quad\quad - \ln\det(\nabla^2_{yy} f_0(x,y)) + F_0 + p\ln\frac{1}{2} \;, \end{array} \tag{6.12}$$

and we take as initial condition

$$f_2(x,x) = 0 \; . \tag{6.13}$$

According to (6.8), we then obtain

$$\phi_1(y) - \phi_1(x) + \frac{V(x)}{2} + \frac{V(y)}{2} = (y-x)\cdot(\nabla_y f_2)(x,y) + (|\nabla_y f_1|^2)(x,y) + F_0 \; . \tag{6.14}$$

A necessary and sufficient condition in order to solve (6.12)-(6.13) is to have

$$V(x) = (|\nabla_y f_1|^2)(x,x) + F_0 \; , \tag{6.15}$$

which gives according to (6.11)

$$V(x) = |\nabla \phi_0(x)|^2 + F_0 \; . \tag{6.16}$$

This is actually the standard eikonal equation that we can solve locally as in [HeSj1] under the condition

$$F_0 = 0 \; , \tag{6.17}$$

adding the conditions

$$\phi_0(x) \geq 0 \; , \; \phi_0(0) = 0 \; .$$

There is no surprise in the sense that we have already proved in [BruHe], [He3] that the WKB solution exists and has this form. But the study which is presented here will have the specific advantage of permitting an easier approach for the large dimension problem. This study of the phase $\phi_0$ was already done in [Sj1].

Once $\phi_0$ is determined, we can determine $f_1$ as a solution of (6.9)-(6.10).

If we solve (6.12), we get as a consequence the following relation between $\phi_1$ and $f_2$ by differentiating with respect to $x$ and taking $x = y$

$$-(\nabla\phi_1(x)) + \frac{1}{2}(\nabla V)(x) = -(\nabla_y f_2)(x,x) + (\nabla_x(|\nabla_y f_1|^2))(x,x) \tag{6.18}$$

This gives us a relation between $\nabla\phi_1$ and $\nabla_y f_2$ restricted to the diagonal $y = x$ of the same nature as in (6.11).

**The coefficient of $h$.**
The new equation is

$$\begin{array}{l}\phi_2(y)-\phi_2(x)\\ =(y-x)\cdot(\nabla_y f_3)(x,y)+2(\nabla_y f_1\cdot\nabla_y f_2)(x,y)\\ -\Big(\ln\det(\nabla^2_{yy}f(x,y;h)\Big)_1+F_1\ ,\end{array}\tag{6.19}$$

with the initial condition

$$f_3(x,x)=0\ .\tag{6.20}$$

This can be rewritten as

$$\begin{array}{l}\phi_2(y)-\phi_2(x)\\ =(y-x)\cdot(\nabla_y f_3)(x,y)+2(\nabla_y f_1\cdot\nabla_y f_2)(x,y)\\ -(\text{ Tr }\nabla^2_{yy}f_1)(x,y)+F_1\quad .\end{array}\tag{6.21}$$

The necessary and sufficient condition in order to solve is

$$2(\nabla_y f_1)\cdot(\nabla_y f_2)(x,x)-(\text{ Tr }\nabla^2_{yy}f_1)(x,x)+F_1\ =\ 0\ .\tag{6.22}$$

This can be written in the form

$$2(\nabla\phi_0)(x)\cdot(\nabla_y f_2)(x,x)-(\text{ Tr }\nabla^2_{yy}f_1)(x,x)+F_1=0\ .\tag{6.23}$$

But now $(\nabla_y f_2)(x,x)$ can be expressed, according to (6.18), in terms of $\nabla\phi_1$ and this gives the transport equation for $\phi_1$, which takes the form

$$\begin{array}{l}2(\nabla\phi_0)(x)\cdot(\nabla\phi_1)(x)-(\text{ Tr }\nabla^2_{yy}f_1)(x,x)+F_1=\\ (\nabla\phi_0)(x)\cdot(\nabla V)(x)-2(\nabla\phi_0)(x)\cdot(\nabla_x(|\nabla_y f_1|^2))(x,x)\ .\end{array}\tag{6.24}$$

We meet here, in order to solve with the initial condition $\phi_1(0)=0$, a condition determining $F_1$ (take $x=0$)

$$F_1=(\text{ Tr }\nabla^2_{yy}f_1)(0,0)\ ,\tag{6.25}$$

(which is not astonishing if we compare with the Schrödinger operator) and we can then determine $\phi_1$. Coming back to (6.12), we have also determined $f_2$.
But we now observe from (6.19) the following relation between $\nabla\phi_2$ and $(\nabla_y f_3)(x,x)$,

$$-\nabla_x\phi_2(x)=-(\nabla_y f_3)(x,x)+2\nabla_x((\nabla_y f_1\cdot\nabla_y f_2))(x,x)-(\text{ Tr }\nabla_x\nabla^2_{yy}f_1)(x,x)\ .\tag{6.26}$$

To summing up at this stage, we have determined $\phi_0$, $\phi_1$, $f_0$, $f_1$, $f_2$ and $F_0$, $F_1$. At this stage we can find for any $\phi_2$, $f_3$ and $f_3$ satisfies in particular (6.26).

**The coefficient of $h^2$.**
After this step, the general way for solving these equations becomes clear. The next equation is

$$\begin{aligned}\phi_3(y)-\phi_3(x) \; &= (y-x)\cdot(\nabla_y f_4)(x,y)+2(\nabla_y f_1\cdot\nabla_y f_3)(x,y) \\ &\quad +|\nabla_y f_2(x,y)|^2-[\ln\det(\nabla^2_{yy}f(x,y;h))]_2+F_2\;,\end{aligned} \tag{6.27}$$

with the initial condition

$$f_4(x,x)=0\;. \tag{6.28}$$

The necessary condition at $y=x$ takes the form

$$2(\nabla_y f_1\cdot\nabla_y f_3)(x,x)+|\nabla_y f_2(x,x)|^2-[\ln\det((\nabla^2_{yy}f)(x,x;h))]_2+F_2\;, \tag{6.29}$$

and using (6.26) we find a transport equation for $\phi_2$ which can be solved under a condition determining $F_2$. This permits to find $f_3$. (6.27) gives us again a relation between $\nabla\phi_3$ and $(\nabla_y f_4)(x,x)$ which will be used in the next step.

It is now time to establish the general transport equation corresponding to the coefficient of $h^k$ with $k\geq 2$. We assume that we have already found $f_k, F_{k-1}, \phi_{k-2}$ and we have seen that we can determine $f_{k+1}$ once we found some $\phi_k$ satisfying some condition relating $(\nabla_y f_{k+1})(x,x)$ and $\nabla\phi_k(x)$.
The equation corresponding to the cancellation of the coefficients of $h^k$ in (6.7) is

$$\begin{aligned}&-\phi_{k+1}(x)+\phi_{k+1}(y)\\ &\qquad=\nabla_y f_0(x,y)\cdot\nabla_y f_{k+2}(x,y)+\nabla_y f_1(x,y)\cdot\nabla_y f_{k+1}(x,y)\\ &\qquad\quad+\textstyle\sum_{j=2}^{k+2}\nabla_y f_j(x,y)\cdot\nabla_y f_{k+2-j}(x,y)-(\ln\det(\nabla^2_{yy}f(x,y;h)))_k+F_k\;,\end{aligned} \tag{6.30}$$

We have only here to know that $(\ln\det(\nabla^2_{yy}f(x,y;h)))_k$ depends only on the $f_\ell$ for $\ell\leq k$.
In order to solve the equation, we meet the following compatibility condition for $x=y$,

$$\begin{aligned}&\nabla_y f_1(x,x)\cdot(\nabla_y f_{k+1}(x,x))+\\ &+\textstyle\sum_{j=2}^{k+2}\nabla_y f_j(x,x)\cdot\nabla_y f_{k+2-j}(x,x)-(\ln\det(\nabla^2_{yy}f(x,x;h)))_k+F_k=0\;.\end{aligned} \tag{6.31}$$

But the relation obtained by recursion and relating $(\nabla_y f_{k+1})(x,x)$ and $\nabla\phi_k(x)$ gives the existence of $x \mapsto v_k(x) \in I\!R^p$ such that

$$\begin{array}{l}\nabla\phi_0(x)\cdot\nabla\phi_k(x)\\ +\sum_{j=2}^{k+2}\nabla_y f_j(x,x)\cdot\nabla_y f_{k+2-j}(x,x)\\ -(\ln\det(\nabla^2_{yy}f(x,x;h)))_k + F_k + \nabla\phi_0(x)\cdot v_k(x) = 0\ .\end{array} \tag{6.32}$$

This gives a transport equation in the form

$$\nabla\phi_0(x)\cdot\nabla\phi_k(x) = e_k(x) - F_k\ , \tag{6.33}$$

where $x\mapsto e_k(x)$ is a known function and this equation can be solved under the condition that

$$F_k = e_k(0)\ , \tag{6.34}$$

which determines $F_k$.
Once we have determined $F_k$, we solve the equation for $\phi_k$ by imposing the initial condition $\phi_k(0) = 0$. From the preceding step (that is (6.30) for $k-1$), we can now get $f_{k+1}$. Coming back to (6.30) we see that we can find $f_{k+2}$ if we know $\phi_{k+1}$ and, by differentiating with respect to $x$ (and taking $y = x$), we get the following necessary condition relating $\nabla\phi_{k+1}$ and $\nabla_y f_{k+2}(x,x)$

$$\begin{array}{l}\nabla\phi_{k+1}(x)\\ \quad = (\nabla_y f_{k+2})(x,x) - \nabla_x(\nabla_y f_1\cdot\nabla_y f_{k+1})(x,x)\\ \quad\quad -\nabla_x(\sum_{j=2}^{k+2}\nabla_y f_j\cdot\nabla_y f_{k+2-j})(x,x) + \nabla_x(\ln\det(\nabla^2_{yy}f(\cdot,\cdot;h)))_k)(x,x)\ .\end{array} \tag{6.35}$$

The recursion is satisfied in the general case.

Everything is consequently determined for fixed dimension and it remains only to follow all the construction with respect to the dimension in order to achieve what was done already for the Schrödinger operator and the Laplace integral. This will be rather long but we have now a general scheme to follow which is quite analogous to what was done already in the previous papers mainly of J. Sjöstrand. This will be sketched in Section 10.

**Applications**:

Once we have constructed the formal WKB construction, it is easy to construct an approximate solution by introducing[4] for example the function

$$u_M^{app}(x;h) = h^{-\frac{p}{4}}\, \chi_0(x)(\sum_{j=0}^{M} a_j(x)\, h^j) \exp -\frac{\phi_0(x)}{h}$$

where $\chi_0$ is a cutoff function with compact support in the neighborhood of the minimum where we were able to perform the construction, and which is equal to 1 in another smaller neighborhood of the minimum.
The pseudo-local character of these transfer operators permits us to verify that the cutoff function introduces only exponentially small errors.
The cutoff of the sum at $M$ leads to an error of order $\mathcal{O}(h^{M+1})$.

# 7 Brief presentation of Sjöstrand's formalism of the 0-standard functions

We have until now described, in two situations presented respectively in Sections 4 and 6, a WKB construction for the largest eigenvalue of the transfer operator. This gives the existence of a formal expansion in powers of $h$ of this WKB-eigenvalue. The next step is then to control the behavior with respect to the dimension $p$. Our main goal is here to obtain that for $F(h;p) \sim \sum_{j=0}^{\infty} h^j F_j(p)$ and for all $j \in I\!N$, there exists $C_j$ such that

$$|F_j(p)| \leq C_j \cdot (p+1)\ . \tag{7.1}$$

This is the type of results which was obtained by J. Sjöstrand in [Sj1] for the Schrödinger operator. We also hope that it is a first step in order to get, under additional conditions on the potential (see [HeSj2]), semi-classical expansions on the thermodynamic limit

$$\varphi_j = \lim_{p\rightarrow+\infty} \frac{1}{p} F_j(p)\ .$$

[4]or

$$u_M^{app}(x;h) = h^{-\frac{p}{4}}\, \chi_0(x)\, \exp -\frac{\sum_{j=0}^{M} \phi_j(x)\, h^j}{h}$$

We just mention that this is just the beginning of the program and that the next step would be to prove that the largest eigenvalue $E(h;p)$ has effectively this behavior, that is to have a semiclassical expansion of

$$-\lim_{p\to+\infty}\frac{1}{p}\ln(E(h;p)) \sim \sum_j \varphi_j h^j \ .$$

¿From now on, in order to facilitate the reference to the papers by J. Sjöstrand, **we change $p$ into $m$ for the dimension and the character $p$ is used for the norms $\ell^p$ on $I\!R^m$.** We refer the interested reader to what has been done in the case of the Schrödinger operator [HeSj2] and to the case of the Laplace integrals [He5] for the description of what are the next questions and statements.
We recall here the formalism introduced by J.Sjöstrand [Sj5] and some of the basic results of this theory.

**Definition 7.1** *:*
*A scalar function*[5] *$a(x)$ on $\Omega \subset I\!R^m$ is 0-standard if, for $k = 1, 2, \cdots$,*

$$|\langle \nabla^k a(x), t_1 \otimes \cdots \otimes t_k\rangle| \leq C_k \ |t_1|_{p_1} \cdots |t_k|_{p_k} \ , \tag{7.2}$$

*for all $p_j \in [1,\infty]$ with $1 = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$ and all $t_j$ in $I\!R^m$.*

We recall that, for $u \in I\!R^m$, $|u|_p$ is the norm in $\ell^p$ of $u$, $|u|_p := \left(\sum_{j=1}^m |u_j|^p\right)^{\frac{1}{p}}$.
**Although it is not clear for the moment, it would be better to say that we are actually looking for a family of scalar functions which are indexed by the dimension $m$ and that the constant $C_k$ are $m$-independent.**
We shall also meet different extensions of this definition to

- the case of a function depending also on a finite[6] number $\ell$ of variables[7], denoted by $s = (s_1, s_2, \cdots, s_\ell)$,

[5] We shall all the time take as $\Omega$ a suitable $\ell^\infty$- ball with $m$-independent radius.
[6] Here we mean by finite that $\ell$ is fixed and independent of our play with the dimension.
[7] In this case we assume that

$$|\partial_s^\alpha \langle \nabla_x^k a(x,s), t_1 \otimes \cdots \otimes t_k\rangle| \leq C_{k,\alpha} \ |t_1|_{p_1} \cdots |t_k|_{p_k} \ . \tag{7.3}$$

- the case when we have to consider a function $(x,y) \mapsto a(x,y)$ depending on two variables $(x,y) \in \Omega \times \Omega' \subset I\!R^m \times I\!R^m$,
- the case when we are considering a sequence $(a^{(n)})_{n\in I\!N}$ of 0-standard functions.

J.Sjöstrand considers also the case with weights which is also quite important in different contexts ([HeSj4]) but we will not consider this case in a first step.
We shall meet also naturally the notion of $k_0$-standard function that we explain now.
For a given integer $k_0 \geq 1$, we say that $u(x)$ is $k_0$-standard for $k_0 \geq 1$ if $u$ is a smooth function of $x \in \Omega \subset I\!R^m$ with values in the dual of $I\!R^{J_1} \otimes \cdots \otimes I\!R^{J_{k_0}}$, where $J_j$ are some subsets of $\{1, \cdots, m\}$, such that for every $k = 0, 1, 2, \cdots$

$$|\langle\langle \nabla^k u, t_1 \otimes \cdots \otimes t_k \rangle, r_1 \otimes \cdots r_{k_0} \rangle| \leq C_k \left( \prod_1^k |t_j|_{p_j} \right) \cdot \left( \prod_1^{k_0} |r_\ell|_{\tilde{p}_\ell} \right) , \qquad (7.4)$$

with

$$1 = \frac{1}{p_1} + \cdots + \frac{1}{p_k} + \frac{1}{\tilde{p}_1} + \cdots + \frac{1}{\tilde{p}_{k_0}} .$$

Of course, in order to be completely rigorous, we would have to be precise in the order of the operations, but we shall work most of the time with symmetric tensors and so a notation like $\langle \nabla^k u, r_1 \otimes \cdots \otimes r_{k_0} \otimes t_1 \otimes \cdots \otimes t_k \rangle$ will sometimes be used instead of $\langle\langle \nabla^k u, t_1 \otimes \cdots \otimes t_k \rangle, r_1 \otimes \cdots \otimes r_{k_0} \rangle$.
The introduction of these $k_0$-standard functions appears more natural if we observe the

**Lemma 7.2** *:*
*Let $u(x)$ be $k_0$-standard (possibly depending on parameters in the sense of (7.3)). Then $\nabla_x^k u$ viewed as a $(k_0+k)$-dual tensor is $(k_0+k)$-standard.*

The other important lemma in our study of the $WKB$ construction is

**Lemma 7.3** *:*
*Let $u(x)$ be $k_0$-standard (possibly in the parameter dependent sense). Then $\Delta_x u$ is $k_0$-standard.*

This lemma is based on the following result on matrices

$$| \text{ Tr } A\,| \leq ||A||_{\mathcal{L}(\ell^\infty,\ell^1)} \ . \tag{7.5}$$

**Lemma 7.4** *:*
*Let $u$ be $(k_0+\ell_0)$-standard and let $u_1,\cdots,u_{\ell_0}$ be 1-standard with values in appropriate spaces, so that $\langle u(x), u_1\otimes\cdots\otimes u_{\ell_0}\rangle$ is a well defined $k_0$-tensor by*

$$\langle\langle u(x), u_1\otimes\cdots\otimes u_{\ell_0}\rangle, r_1\otimes\cdots\otimes r_{k_0}\rangle \ .$$

*Then $\langle u(x), u_1\otimes\cdots\otimes u_{\ell_0}\rangle$ is $k_0$-standard.*

As particular case we shall meet the property that

- If $u$ and $v$ are 1-standard then $\langle u,v\rangle$ is 0- standard.
- If $a$ and $b$ are 0-standard then $\langle\nabla a,\nabla b\rangle$ is 0-standard.

**Lemma 7.5** *:*
*If $x\mapsto y=v(x)$ is 1-standard and $y\mapsto z=u(y)$ is $k_0$-standard, then $x\mapsto z=u(v(x))$ is $k_0$-standard.*

We now consider the implicit function theorem.

**Lemma 7.6** *:*
*Let $x\mapsto y=f(x)$ be a 1-standard map from $\Omega\subset I\!\!R^m$ into $I\!\!R^m$. Let us assume that $(\nabla f)(x)$ is uniformly invertible in $\mathcal{L}(\ell^p)$ for $p\in[1,+\infty]$, then the local[8] inverse $g=f^{-1}$ is 1-standard.*

This lemma admits the following useful extension. Let us consider the case when we have three groups of variables $(x,y,z)$ where $y$ and $z$ contain the same number of variables and we assume that the map $(x,y)\mapsto z=f(x,y)$ is 1-standard with uniformly invertible differential $\partial_y f$. Then the local solution $y=g(x,z)$ of $f(x,y)=z$ is 1-standard. In particular, the solution

[8] The mention "local" means here that if for some $x_0(m)$ such that $B_\infty(x_0(m);\delta_0)\subset\Omega(m)$ ($\delta_0$ independent of the dimension) and $(\nabla f)(x_0(m))$ is invertible and satisfies $||(\nabla f)(x_0(m))^{-1})||_{\mathcal{L}(\ell^p)}\leq C$, then there exists $\delta_1$ (independent of the dimension) such that, for $y\in B_\infty(f(x_0(m));\delta_1)$, there exists a unique solution $x\in B_\infty(x_0(m);\delta_0)$ of $f(x)=y$.

$y = g(x,0)$ of $f(x,y) = 0$ is 1-standard.

We now recall what was obtained by J.Sjöstrand concerning the flow associated to 1-standard vector fields $v(x, \partial_x)$.
The main result is the following lemma

**Lemma 7.7** *:*
*Let us denote by* $t \mapsto \langle u(x), u_1 \otimes \cdots \otimes u_{\ell_0}\rangle$ *the solution of*

$$\partial_t \Phi(t,x) = v(\Phi(t,x)) \ , \ \Phi(0,x) = x \ . \tag{7.6}$$

*Then the function* $x \mapsto \Phi(t,x)$ *is* 1*-standard (in the* $t$*-dependent sense for* $|t| \leq T$*).*

The last results we shall recall are more deep in nature and only true under stronger assumptions.

**Proposition 7.8** *:*
*Let* $\phi \in C^\infty(B_\infty(0,b); I\!R)$ *be a* 0*-standard function with* $\phi(0) = \nabla\phi(0) = 0$*,* $Hess\,\phi(0) = D + A$*, where* $D$ *is diagonal* $D \geq r_0$*,* $||A||_{\mathcal{L}(\ell^p)} \leq r_1$*, where* $0 < r_1 < r_0$ *are fixed numbers. Then the integral curve* $t \mapsto x(t,y), -\infty < t \leq 0$*, of the vector field* $\nabla\phi$ *such that* $x(0,y) = y$ *is* 1*-standard and the corresponding constants tend to* 0 *exponentially fast as* $t \to -\infty$*:*

$$|\langle \nabla_y^k x(t,y), t_1 \otimes \cdots t_k\rangle|_p \leq C_k \cdot \exp\frac{t}{\lambda_0} \ \cdot \ (\prod_1^k |t_j|_{p_j}) \ , \tag{7.7}$$

*for* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$*,* $t \leq 0$ *and some* $\lambda_0 > 0$*.*

Finally we recall that the existence of the solution of the eikonal equation was also proven.

**Proposition 7.9** *:*
*Let* $V \in C^\infty(B_\infty(0,b); I\!R)$ *be a* 0*-standard function such that* $V(0) = \nabla V(0) = 0$*,* $Hess\,V(0) = D + A$*, where* $D$ *is diagonal* $D \geq r_0$*,* $||A||_{\mathcal{L}(\ell^p)} \leq r_1$*, where* $0 < r_1 < r_0$ *are fixed numbers. Then, in a possibly smaller* $\ell^\infty$*-ball, one can find a* 0*-standard solution* $\phi$ *of*

$$|\nabla\phi|^2 = V(x) \ , \ \phi(0) = 0 \ , \ \phi(x) \geq 0 \ . \tag{7.8}$$

*Moreover* $\phi$ *satisfies in this ball similar properties as* $V$*, for suitable new* $\tilde{r}_0$ *and* $\tilde{r}_1$*:*

$$\tilde{r}_0 = \sqrt{r_0} \ , \ \tilde{r}_1 = \sqrt{r_0} - \sqrt{r_0 - r_1} \ . \tag{7.9}$$

# 8 The solution of the eikonal equation in large dimension

This section is mainly concerned with the control with respect to the dimension of the construction of the phase solution of the eikonal equation

$$\phi_0(x) = \inf_y [\Theta(x,y) + \phi_0(y)] \ ,$$

in the neighborhood of the origin (cf subsection 4.2). As we already said in this subsection, the existence of the phase is just a consequence of the stable manifold theorem applied to the diffeomorphism $\kappa$ defined by

$$\begin{array}{llll} y & = x + \frac{2}{a}(\nabla V)(x) - \frac{2}{a}\xi & := y(x,\xi)\ , \\ \eta & = -(\nabla V)(y(x,\xi)) + \xi - (\nabla V)(x) & := \eta(x,\xi)\ . \end{array} \tag{8.1}$$

What we have just to do is to follow the general construction of the stable manifold theorem in order to maintain control with respect to the dimension. When considering the more standard eikonal equation

$$|\nabla\phi_0|^2 = V \ ,$$

J. Sjöstrand was doing the same in controlling with respect to the dimension the proof of the standard stable manifold theorem for flows. We have consequently to prove the analogue for the stable manifold theorem attached to an hyperbolic diffeomorphism. The usual relation between the two theorems is that it is rather easy to deduce the stable theorem for flows once one has proved the stable manifold theorem for diffeomorphisms. We are in the opposite situation so it seems useful to explain how to treat the case of our diffeomorphism $\kappa$.

Instead of adapting what was done by J. Sjöstrand in [Sj1] or [Sj2] for the theorem for the flows, we choose to follow here the nice exposition of Irwin [Irw] for a proof of the stable manifold theorem for diffeomorphisms. His proof is indeed described essentially in the context of Banach spaces and we have just to verify some specific points where the finite dimension character was used through equivalence of norms.

We start from a diffeomorphism $\kappa$: $X \mapsto X$, and let $p$ be a fixed point of $\kappa$. The stable set of $\kappa$ at $p$ is the set

$$\{x \in X \; : \; \kappa^{(n)}(x) \to p \;\text{ as }\; n \to \infty\}. \tag{8.2}$$

We notice that this set is always non-empty, since it contains $p$. The unstable set of $\kappa$ at $p$ is the stable set of $\kappa^{-1}$ at $p$.
Given any open neighborhood $U$ of $p$, we define the local stable set of $\kappa_U$ at $p$ to be the set of all $x \in U$ such that $(\kappa^{(n)}(x) : n \geq 0)$ is a sequence in $U$ converging to $p$. We shall say that the point $p$ is hyperbolic if the tangent map of $\kappa$ at $p$, $T$, is a hyperbolic linear automorphism, that is, if it has no eigenvalue of module 1. We can then decompose the tangent space $T_pX$ as the sum

$$E := T_pX = E_s \oplus E_u$$

where $E_s$ is the stable subspace corresponding to the subspace attached to the eigenvalues of module $< 1$ (the complex space attached to these eigenvalues being stable by complex conjugation, it is the complexification of a real space that we call $E_s$) and $E_u$ is the unstable space. We shall denote by $T_s$ the restriction of $T$ to $E_s$ and by $T_u$ the restriction of $T$ to $E_u$. Because we shall apply a fixed point theorem it is better to introduce the following suitable norm

$$|| \cdot ||_E = \max(| \cdot |_s \, , \; | \cdot |_u)$$

where $| \cdot |_s$ (resp. $| \cdot |_u$ ) is chosen such that

$$||T_s||_{\mathcal{L}(E_s)} < a < 1$$

and

$$||T_u^{-1}||_{\mathcal{L}(E_u)} < a < 1 \, .$$

This is actually the only point, we shall have to verify in our case:

**Can we find such norms $| \cdot |_s$ and $| \cdot |_u$ on $E_s$ and $E_u$ such that the corresponding norm $|| \cdot ||_E$ is equivalent to the preceding one with constants which are uniform with respect to the dimension**?

We are indeed actually working with a family of diffeomorphisms $\kappa_m$ defined on $\ell^\infty$-balls in $I\!\!R^m$ and we need to control all the constants with respect to the dimension.

The proof of the basic stable manifold theorem as given by Irwin can be formulated in the following form:

**Theorem 8.1** *(Local stable manifold theorem):*
*Let $E$ be a Banach space $(E, ||\cdot||_E)$ and let $T$ be a linear map admitting two closed stable subspaces such that*

$$E := T_p X = E_s \oplus E_u \ , \tag{8.3}$$

*and for suitable norms on $E_s$ and $E_u$*

$$||T_s||_{\mathcal{L}(E_s)} < a < 1 \ , \tag{8.4}$$

$$||T_u^{-1}||_{\mathcal{L}(E_u)} < a < 1 \ , \tag{8.5}$$

*and*

$$||\cdot||_E = \max(|\cdot|_s \ , \ |\cdot|_u) \ . \tag{8.6}$$

*Let $B$ a closed ball in $E$ with center $0$ and radius $b$ for the norm $||\cdot||_E$. Let $\eta : B \mapsto E$ be Lipschitz with Lipschitz constant $\delta$, where $\delta < 1 - a$. Suppose $\eta(0) = 0$. Then there is a unique map $h : B_s \mapsto B_u$ such that graph $h$ is the stable set of $(T+\eta)_{/B}$ at $0$. Moreover the map $h$ is $(a+\delta)$-Lipschitz, and $C^r$ when $\eta$ is $C^r$.*

Of course, we have still the theorem if $||\cdot||_E$ is changed in an equivalent norm but, if we apply the theorem as we shall do for a family of normed spaces $E^{(m)}$, the constants appearing in the equivalence have to be controlled independently of the dimension and we have in this case to take the Lipschitz constant accordingly smaller. This will be usually done by taking $b$ smaller.

So we shall now see how to verify the assumptions of the theorem. The main point is that our space $E$ is the space $I\!R^m$ with the $\ell^\infty$-norm and that we have to find a suitable equivalent norm for which the map $T$ is contractive on the corresponding $E_s$.

Our $\eta$ will be actually a sum of two terms:

- A linear term that we get from the linearized diffeomorphism after subtraction of a diagonal part,

  and

- a term corresponding to the difference between the diffeomorphism and its linearized approximation at the fixed point which will be small for $b$ small enough.

Let us first limit our study to the linearized diffeomorphism. Let us start from the quadratic approximation of $\Theta$ which is denoted by $\Theta^0$.

$$\Theta^0(x,y) = \frac{1}{4}\langle \text{ Hess}\, V(0)x, x\rangle + \frac{a}{4}|x-y|^2 + \frac{1}{4}\langle \text{ Hess}\, V(0)y, y\rangle \tag{8.7}$$

Let us recall the type of assumptions which was introduced in [Sj1]. We assume that

$$A_0 := \text{ Hess}\, V(0) = D_0 + W_0 \tag{8.8}$$

where $D_0$ was a diagonal matrix satisfying

$$0 < r_0 \leq D_0 \leq R_0 \ , \tag{8.9}$$

and $W_0$ was a "perturbation" satisfying

$$||W_0||_{\mathcal{L}(\ell^\infty)} \leq \delta_0 \ , \tag{8.10}$$

and we shall choose later our $\delta_0$ small enough (surely strictly less than $r_0$) and this choice will be independent of the dimension as are all the constants appearing in our statements or assumptions. With this in mind we write shortly

$$\delta_0 << r_0 \ . \tag{8.11}$$

This will lead to a new decomposition of $\Theta^0$

$$\Theta^0 = \Theta^{00} + \Theta^{01} \ , \tag{8.12}$$

with

$$\Theta^{00}(x,y) := \frac{1}{4}\langle D_0 x, x\rangle + \frac{a}{4}|x-y|^2 + \frac{1}{4}\langle D_0 y, y\rangle \ . \tag{8.13}$$

The corresponding map is $T^{00}$ and is given by

$$T^{00} = \begin{pmatrix} (I + \frac{1}{a}D_0) & -\frac{2}{a}\, I \\ -D_0(I + \frac{1}{2a}D_0) & (I + \frac{1}{a}D_0) \end{pmatrix} \tag{8.14}$$

The corresponding eigenvalues are given by

$$\lambda_{\pm}^{j} = \frac{(a+\lambda_j^0)}{a} \pm \frac{1}{a}\sqrt{\lambda_j^0(\lambda_j^0+2a)}\ , \tag{8.15}$$

where $\lambda_j^0$ is an eigenvalue of $D_0$. Let us also observe the relation

$$\lambda_{-}^{j} \cdot \lambda_{+}^{j} = 1 \tag{8.16}$$

and the inequalities

$$\begin{aligned} \lambda_{-}^{j} &\leq \tfrac{1}{\sqrt{(1+\frac{r_0}{a})^2}+\sqrt{(1+\frac{r_0}{a})^2-1}} \\ &\leq \tfrac{a}{r_0+a} \\ &< 1 \end{aligned} \tag{8.17}$$

Following the proof of the stable manifold theorem, we perform the following change of variables

$$(x,\xi) \mapsto (\xi_{-},\xi_{+})$$

with

$$\xi_{\pm} = \xi \pm \frac{1}{2}\sqrt{D_0(2a+D_0)}\ x \tag{8.18}$$

We now observe that the map $U$: $(x,\xi) \mapsto (\xi_{-},\xi_{+})$ is an isomorphism in $\mathcal{L}(\ell^{\infty})$ with explicit constants, independent of the dimension, for the norms of $U$ and $U^{-1}$.
We get indeed the following inequalities

$$\begin{aligned} &\left(\tfrac{1}{\max(\frac{1}{2},\frac{1}{\sqrt{r_0(r_0+2a)}})}\right) ||(x,\xi)||_{\infty} \\ &\quad \leq \max(|\xi_{+}|_{\infty}\, ,\, |\xi_{-}|_{\infty}) \\ &\quad \leq \max\left(1\, ,\, \tfrac{1}{2}\sqrt{R_0(R_0+2a)}\ \right) ||(x,\xi)||_{\infty} \end{aligned} \tag{8.19}$$

In these new cordinates the map $T^{00}$ is decomposed in two diagonal blocks $T_s$ and $T_u$ whose norms in $\mathcal{L}(\ell^{\infty})$ simply satisfy:

$$||T_s||_{\mathcal{L}(\ell^{\infty})} \leq \sup_j |\lambda_j^{-}| \leq \frac{a}{r_0+a}\ , \tag{8.20}$$

and similarly using (8.16)

$$||T_u^{-1}||_{\mathcal{L}(\ell^{\infty})} \leq \frac{a}{r_0+a}\ . \tag{8.21}$$

The last step is to control the error which is called $\eta$ in the stable manifold theorem.
The function $\eta$ is the sum of two terms:

$$\eta = T^{01} + \tau \tag{8.22}$$

where

$$T_0 = T^{00} + T^{01} \tag{8.23}$$

where $T_0$ the linear map associated with $\Theta^0$, $\tau$ is the remainder and the decomposition (8.23) corresponds to the decomposition (8.12).
More explicitly this gives for $T^{01}$ the following expression

$$T^{01} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \frac{2}{a} W_0 x \\ -W_0 x - W_0(x + \frac{2}{a} A_0 x - \frac{2}{a}\xi) - \frac{2}{a} W_0 A_0 x \end{pmatrix} ,$$

that is

$$T^{01} = \begin{pmatrix} \frac{2}{a} W_0 & 0 \\ -2W_0[I + \frac{1}{a} A_0] & \frac{2}{a} W_0 \end{pmatrix} \tag{8.24}$$

It is then clear that the norm $||T^{01}||_{\mathcal{L}(\ell^\infty)}$ is in $\mathcal{O}(\delta_0)$ for each of the two norms we were considering.
Similarly, if one assumes that $V$ is 0-standard, one get also that $\tau$ satisfies

$$||\tau(x) - \tau(y)||_{\ell^\infty} \leq Cb\, ||x - y||_{\ell^\infty} \ , \ \forall\, x, \forall\, y \ \in B_\infty(0, b) \ , \tag{8.25}$$

We can then consequently have the assumptions of the theorem satisfied for $\delta_0$ and $b$ small enough.
In order to come back to our parametrization of the ingoing and of the outgoing manifold, we observe that the parametrization of these Lagrangians by $\xi_+$ for $\Lambda_-$ or $\xi_-$ for $\Lambda_+$ can be easily replaced by a parametrization by $x$.
The second point is to be more precise when we say that $h$ is $C^r$ if $\eta$ is $C^r$.
We need indeed to express everything independently of the dimension. But once the norm on the vector space $I\!R^m$ is fixed, this means that, if there exists a constant $C_r$, such that, for all $k \leq r$, $(x, \xi) \in B_E(0, b)$ and all $t_j \in E$ $(j = 1, \cdots, k)$

$$||(\nabla^k \eta)_{(x,\xi)}(t_1 \otimes t_2 \otimes \cdots \otimes t_k)|| \leq C_r\, ||t_1|| \cdots ||t_k|| \tag{8.26}$$

then there exists a constant $D_r$ depending only on $C_r$ and the different constants introduced in the theorem $r_0$, $\delta_0$ and $R_0$ such that $h$ satisfies the same property for all $k \leq r$ , $\xi_+ \in B_{E_s}(0,b)$ and all $t_j \in E_s$ $(j = 1, \cdots, k)$

$$||(\nabla^k h)_{(\xi_+)}(t_1 \otimes t_2 \otimes \cdots \otimes t_k)|| \leq D_r \, ||t_1|| \cdots ||t_k|| \, . \tag{8.27}$$

Parametrizing $\Lambda_-$ with the projection on the $x$-variable will give a new function $\tilde{h}(x)$ with the same properties for $x \in B_\infty(0, \tilde{b})$:

$$||(\nabla^k \tilde{h})_{(x)}(t_1 \otimes t_2 \otimes \cdots \otimes t_k)||_\infty \leq E_r \, ||t_1||_\infty \cdots ||t_k||_\infty \, . \tag{8.28}$$

The symplectic character of $\kappa$ leads finally to the existence of the phase $\phi_0$ such that $\tilde{h} = \nabla \phi_0(x)$ and we finally get for $\phi_0$ that, if $V$ is 0-standard in the sense that, for all $k \geq 1$, for all $t_j \in I\!R^m$, there exists $C_k$ such that the following estimates are satisfied:

$$||(\nabla^k V)_{(x)}(t_1 \otimes t_2 \otimes \cdots \otimes t_k)|| \leq C_k \, ||t_1||_{p_1} \cdots ||t_k||_{p_k} \, , \tag{8.29}$$

for all $p_1, \cdots, p_k$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1$ and all $x$ in a sufficiently small $\ell^\infty$-ball around 0 $B_\infty(0,b)$, then the same is true for $\phi_0$.
We have already written the property for $\nabla \phi_0 = \tilde{h}$ in (8.28) and this gives (8.29) with $p_1 = 1$, and $p_j = \infty$ for $j = 2, \cdots, k$.
We then get the general result by using the symmetry of $(\nabla^k \phi_0)$ and the usual complex interpolation.
So we have obtained finally a generating function $\phi_0$ for the Lagrangian manifold defined in a suitable $\ell^\infty$-ball around 0 whose radius is independent of $m$ and which is 0-standard.

**Proposition 8.2** *:*
*Let $V \in C^\infty(B_\infty(0,b); I\!R)$ a 0-standard function such that $V(0) = \nabla V(0) = 0$, $Hess\, V(0) = D + A$, where $D$ is diagonal $D \geq r_0$, $||A||_{\mathcal{L}(\ell^p)} \leq r_1$, where $0 < r_1 < r_0$ are fixed numbers. Then, in a possibly smaller $\ell^\infty$-ball $B_\infty(O,c)$ and if $r_1$ is small enough, one can find a 0-standard solution $\phi_0$ of*

$$\phi_0(x) = \inf_{y \in B_\infty(0,c)} (\Theta(x,y) + \phi_0(y)) \, , \; \phi_0(0) = 0 \, , \; \phi_0(x) \geq 0 \, . \tag{8.30}$$

*Moreover $\phi_0$ satisfies in $B_\infty(0,c)$ similar properties as $V$, for suitable new $\tilde{r}_0$ and $\tilde{r}_1$.*

# 9 The transport equation in large dimension

We first establish some properties of the function $x \mapsto y(x)$ that we already mentioned for fixed $m$.
The stable manifold theorem gives us the existence of a constant $k < 1$ and an $\ell^\infty$-ball around 0 $B_\infty(0,b)$ such that

$$|y(x)|_\infty \leq k \; |x|_\infty \;, \; \forall x \;\; \text{s.t. } |x|_\infty \leq b \;. \tag{9.1}$$

Coming back to the "eikonal equation", we also have

$$y(x) = x - \frac{2}{a}\nabla_x \phi_0(x) + \frac{2}{a}\nabla_x V(x) \;. \tag{9.2}$$

According to the definitions of J.Sjöstrand, this means that
$x \mapsto y(x)$ is 1-standard.
We can also observe that

$$y(x) = \nabla \psi_0(x) \;\; \text{with} \;\; \psi_0(x) = \frac{1}{2}x^2 + \frac{2}{a}V(x) - \frac{2}{a}\phi_0(x) \;, \tag{9.3}$$

and that $\psi_0$ is 0-standard. Analytically this means in particular the existence of constants $C_k$ such that

$$\langle(\nabla^k \langle y(x), s\rangle), t_1 \otimes \cdots \otimes t_k\rangle \leq C_k \; |s|_1 \cdot |t_1|_\infty \cdots |t_k|_\infty \;. \tag{9.4}$$

Let us also recall from the proof of the stable manifold theorem that

$$||(\nabla y)(x)||_{\mathcal{L}(\ell^\infty)} \leq k < 1 \;, \; \forall x \;\; \text{s.t. } |x|_\infty \leq b \;. \tag{9.5}$$

The first transport equation was introduced in (5.24) and was

$$\phi_1(x) = \phi_1(y(x)) + \ln \det \left(\nabla^2_{yy} f_0(x, y(x))\right) - F_0 \tag{9.6}$$

We have now to prove that

- The map $(x,y) \mapsto f_0(x,y)$ is 0-standard.
- The map $x \mapsto \ln \det \left(\nabla^2_{yy} f_0(x, y(x))\right)$ is 0-standard.

The first statement comes from a version with parameter of the proposition given in Proposition 7.9 where $V$ is replaced by the function $y \mapsto \Theta(x,y) + \phi_0(y) - \phi_0(x)$ (cf (5.16)). We have proved already that $\phi_0$ is 0-standard. We consequently get the existence[9] of a 0-standard function $(x,y) \mapsto f_0(x,y)$ solution of the eikonal equation.

The second statement comes from the following properties.
The map $(x,y) \mapsto (\nabla^2_{yy} f_0)(x,y)$ is 2-standard (using Lemma 7.2) and consequently, using Lemma 7.5 and the 1-standardness of $x \mapsto y(x)$, the map $x \mapsto (\nabla^2_{yy} f_0)(x, y(x))$ is 2-standard (with value in $\mathcal{L}(I\!\!R^m)$). Because $(\nabla^2_{yy} f_0)(0,0)$ is near a diagonal matrix with strictly positive coefficients, we get that $x \mapsto \ln(\nabla^2_{yy} f_0)(x, y(x))$ is 2-standard. We then get (using again Lemma 7.5) that the map $x \mapsto \mathrm{Tr}\ \ln(\nabla^2_{yy} f_0)(x, y(x))$ and consequently the map $x \mapsto \ln\det\left(\nabla^2_{yy} f_0(x, y(x))\right)$ are 0-standard.
All these properties have been established in [Sj5] (or in [He5]) and have been recalled in Section 7.
Let us focus, at least as a starting point, on the solution of

$$\phi_1(x) = \phi_1(y(x)) + \theta_1(x)\ ,\ \phi_1(0) = 0 \tag{9.7}$$

where $x \mapsto \theta_1(x)$ is a 0-standard function satisfying

$$\theta_1(0) = 0\ . \tag{9.8}$$

This system is quite similar to what was met in the study of the Schrödinger operator where the following equation was obtained

$$\nabla\phi_0(x) \cdot \nabla\phi_1(x) = \theta_1(x)\ ,\ \phi_1(0) = 0. \tag{9.9}$$

This last equation was also solved in [Sj5] by integrating over the flow of $\nabla\phi_0$ and controlling the flow in the 0-standard category. We are just perform here a discrete integration:

$$\phi_1(x) = \sum_{n=0}^{\infty} \theta_1\left(y^{(n)}(x)\right)\ ,$$

[9] The version with parameters exists already in Section 3 in [Sj6] (See the second remark p.33). We cannot nevertheless apply directly what is done there because our phase is unknown at the beginning.

instead of

$$\phi_1(x) = \int_{-\infty}^{0} \theta_1(\Phi_t(x))dt \; .$$

The estimates established for $x \mapsto y(x)$ are similar to the estimates obtained for the flow for the continuous case and the proof can be adapted to this discrete situation. We need actually the following lemma which is the discrete analog of Proposition 7.8,

**Proposition 9.1** *:*
*Let $V \in C^\infty(B_\infty(0,b); I\!R)$ a 0-standard function such that $V(0) = \nabla V(0) = 0$, $Hess\, V(0) = D + A$, where $D$ is diagonal $D \geq r_0$, $||A||_{\mathcal{L}(\ell^p)} \leq r_1$, where $0 < r_1 < r_0$ are fixed numbers with $r_1 << r_0$. Then the function $x \mapsto y(x)$ is 1-standard and the iterate $y^{(n)}(x)$ satisfies for $n \in I\!N$:*

$$|\langle \nabla_x^k y^{(n)}(x), t_1 \otimes \cdots t_k\rangle|_p \leq C_k \exp - \frac{n}{\lambda_0} \, (\prod_1^k |t_j|_{p_j}) \; , \tag{9.10}$$

*for $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$, and some $\lambda_0 > 0$.*

The proof is by recursion. For $n = 1$, this is simply the fact that $x \mapsto y(x)$ is 1-standard. We now consider

$$y^{(n)}(x) = y^{(n-1)}(y(x)) \; ,$$

and differentiate this identity. In order to see the main problems we just analyze the two first derivatives and refer to [Sj6] for the general technique. It consists simply in analyzing more precisely the constants in the proof of Lemma 7.5.
We first analyze the first derivative and write for any $t \in I\!R^m$

$$\langle \nabla y^{(n)}(x), t\rangle = \langle \nabla y^{(n-1)}(y(x)), \langle \nabla y(x), t\rangle\rangle \; . \tag{9.11}$$

If we assume that we have found a constant $C_1$ up to order $n-1$, we get from (9.10) and (9.5) with $\exp - \frac{1}{\lambda_0} = k$

$$|\langle \nabla_x y^{(n)}(x), t\rangle|_p \leq C_1 k^{n-1} |\langle \nabla y(x), t\rangle|_{p_1} \leq C_1 k^n |t|_{p_1} \; .$$

Here we have observe that the matrix $\nabla y$ is symmetric and this permits to obtain first from (9.5) that

$$||\nabla y(x)||_{\mathcal{L}(\ell^1)} \leq k \; ,$$

and by interpolation

$$||\nabla y(x)||_{\mathcal{L}(\ell^p)} \leq k \ , \forall p \in [1, +\infty] \ . \tag{9.12}$$

By choosing $C_1$ in order to have the result for $n = 1$, we have consequently treated the case $k = 1$ (all $n$).
The case $k = 2$ shows the general new difficulty. When differentiating (9.11), we get the sum of two terms

$$\begin{aligned} \langle \nabla^2 y^{(n)}(x), t_1 \otimes t_2 \rangle \ &= \langle \nabla^2 y^{(n-1)}(y(x)), \langle \nabla y(x), t_1 \rangle \otimes \langle \nabla y(x), t_2 \rangle \rangle \\ &\quad + \langle \nabla y^{(n-1)}(y(x)), \langle \nabla^2 y, t_1 \otimes t_2 \rangle \rangle \ . \end{aligned} \tag{9.13}$$

If we assume the result for $k = 1$ and the case $k = 2$ till $n - 1$ for some constant $C_2$, we obtain that it will be verified to step $n$ if $C_2$ satisfies the condition

$$C_2 k^n \geq C_2 k^{n+1} + C C_1 k^{n-1} \ ,$$

where $C$ is an upperbound for the norm of $\nabla^2 y$.
We consequently get that $C_2$ satisfies

$$C_2 \geq \frac{C C_1}{k(1-k)} \ ,$$

and we see that the recursion is satisfied if we take

$$C_2 \geq \max(\frac{C C_1}{k(1-k)}, C).$$

The general case can be done in the same way.
Once we have this property, we obtain through (5.28) that $\phi_1$ is 0-standard and we can then solve the transport equation (5.21) giving $f_1$ in the class of the 0 standard functions with respect to $(x, y)$. This leads to the equation (5.30) with $j = 2$. We first get the compatibilty condition (5.29) permitting to determine $\phi_2$ as a 0-standard function and the estimate on $F_2$ and we then obtain $f_2$ as a 0-standard function. The other equations do not lead to new problems if one compares with what was done for Laplace integrals in [He5].
We have finally obtained **(coming back to our previous notation for the dimension)** the following theorem

**Theorem 9.2** *:*
*Let $V \in C^\infty(B_\infty(0,b); I\!R)$ be a 0-standard function such that $V(0) = \nabla V(0) = 0$, $Hess\,V(0) = D + A$, where $D$ is diagonal $D \geq r_0$, $||A||_{\mathcal{L}(\ell^q)} \leq r_1$, and $0 < r_1 < r_0$ are fixed numbers with $r_1 << r_0$. Then one can find in an $\ell^\infty$-ball $B_\infty(0,b')$ a formal series*

$$\phi(x;h) \sim \sum_j \phi_j(x) h^j$$

*with $\phi_j$ 0-standard and a formal eigenvalue $E(h) \sim \exp -F(h)$ with $F(h) \sim \sum_j F_j(p) h^j$ satisfing for suitable $p$-independent constants $C_j$*

$$|F_j(p)| \leq C_j\, p \ , \ \forall p \in I\!N \ ,$$

*such that, by applying the formal Laplace method, one gets formally the identity*

$$K_V^{tf} \exp -\phi(.;h)/h \sim \exp -F(h) \ \cdot \ \exp -\phi(.;h)/h \ .$$

**Remark 9.3** *:*
The next step would be to control the remainder when cutting $\phi$ and $F$. This has been done for the equation of Schrödinger as well as for the Laplace integral. So we do not think that there are fundamental problems to extend the techniques to this case (cf for example the discussion of Section 5 in [Sj1] or [He5]). We shall probably meet the condition $p \leq h^{-N_0}$ for $h \leq h_0$.

# 10 The Kac operator in large dimension

In this section, we shall analyze briefly how to control with respect to the dimension (which will be called $p$ in this section) the construction we have proposed for the Kac model.
We keep the same assumptions on $V$ as in the preceding sections. We first solve (6.16) in the 0-standard category and $\phi_0$ get the same properties as $V$. $f_0$ being obviously 0 standard with respect to $(x,y)$, we can solve (6.9) in the 0-standard category and determine $f_1$. $F_1$ is explicitly seen as equal to $\mathrm{Tr}\ \nabla_{yy} f_1(0,0)$ (see (6.25)). Everything is proved at the level 0.
We now find in (6.24) an equation for $\phi_1$ attached to the 1-standard vector field $\nabla\phi_0$ and we observe also that $(\nabla\phi_0)(x)\cdot(\nabla V)(x)$ and $(\nabla\phi_0)(x)\cdot(\nabla_x(|\nabla_y f_1|^2))(x,x)$ using the different lemmas from Section 7. Observing

that $V$, $\phi_1$ and $(x,y) \mapsto (|\nabla_y f_1|^2)(x,y)$ are 0-standard, we can then solve (6.12) and find $f_2$ in the 0-standard category. The estimate for $F_2$ is then determined through the compatibility condition for solving (6.29) and we determine $\phi_2$ in the 0- standard category.
The analysis of the general transport equation does not lead to new problems and we get the analog of Proposition 9.2 in the preceding section, that is

**Theorem 10.1** *:*
*Let $V \in C^\infty(B_\infty(0,b); I\!R)$ a 0-standard function such that*

$$V(0) = \nabla V(0) = 0 \,,$$

$$Hess\, V(0) = D + A \,,$$

*where $D$ is diagonal, $D \geq r_0$, $||A||_{\mathcal{L}(\ell^q)} \leq r_1$, and $0 < r_1 < r_0$ are fixed numbers with $r_1 << r_0$ and $q \in [1, +\infty]$.*
*Then one can find in an $\ell^\infty$-ball $B_\infty(0,b')$ a formal series*

$$\phi(x;h) \sim \sum_{j=0}^{\infty} \phi_j(x) h^j$$

*with $\phi_j$ 0-standard and a formal eigenvalue $E(h) \sim \exp -F(h)$ with $F(h) \sim \sum_{j=0}^\infty F_j(p) h^j$ satisfing, for suitable $p$-independent constants $C_j$,*

$$|F_j(p)| \leq C_j\, p$$

*such that, by applying the formal Laplace method, one gets the identity between formal series in $h$*

$$K_V^{kac} \exp -\phi(.;h)/h \sim \exp -F(h) \;\cdot\; \exp -\phi(.;h)/h \,.$$

# 11 Conclusion and open problems

We have only analyzed the formal construction for the large dimension problem. We do not think that there are essential difficulties to deduce the approximation of the real largest eigenvalue of the transfer operator under the usual condition $p \leq C \cdot h^{-N_0}$ but this has not been explicitly done.
What we have not performed also is the study of the thermodynamic limit for specific potentials coming from statistical mechanics, as presented in the

introduction. We think that it can be probably done along the same line as for the Schrödinger operator in [HeSj2] and for Laplace integrals [He5] using the improved techniques introduced by J. Sjöstrand [Sj5] and Helffer-Sjöstrand [HeSj4]. There is in particular a need of a notion of weighted standard functions depending on parameters. This theory exists (cf [Sj5], [HeSj4]) and we have probably to apply it to a family of interpolating potentials between $V^{(m+n)}(x)$ and $V^{(m)}(x')+V^{(n)}(x'')$ for $x=(x',x'')\in I\!R^m\times I\!R^n$. This could be long and technical to write down the details, but we do not see any fundamental obstruction for applying this idea in the same way as for the Schrödinger operator or the Laplace integrals, although we have not verified the details of the argument. We hope to come back to this point elsewhere.
Another interesting question will be to analyze the global structure attached to the symplectic diffeomorphism $\kappa$ introduced in (4.8) when we consider a double well potential (the convex case is probably easier to treat, see Subsection 4.5). The problem is already interesting in dimension 1 and with the potential $x\mapsto v(x)=(x^2-1)^2$. The map $\kappa$ has three fixed points $(-1,0)$, $(0,0)$ and $(1,0)$. If the situation is clear near the two hyperbolic points $(-1,0)$ and $(1,0)$, the global picture and in particular what is going on near the last point $(0,0)$ may depend on $a$. This analysis could be important in order to analyze the tunneling between the two largest eigenvalues of the transfer operator (see [He11] and [He12] for preliminary discussions).

**Acknowledgements :**

Preliminary results were announced at the conference in Lund for the Danish-Swedish meeting in March 1995. Further improvements were obtained during my stays at Caltech, the IMA in Minneapolis and at the ESI in Vienna in August 1995.
I thank J.Sjöstrand for useful remarks about preliminary steps of this work and M. Chaperon for pointing out to me the reference of M.C.Irwin for a nice proof of the stable manifold theorem.
I thank also the European Union which partially supported this research through the TMR Programme of the European Commission - Network Post-doctoral training programme in partial differential equations and application in quantum mechanics-.
Finally, I would like to thank the referee who kindly suggests many improve-

ments.

# References


[Ag] S.Agmon: *Lecture on exponential decay of solutions of second order elliptic equations.* Math. Notes, t.29, Princeton University Press (1982).

[BraLi] H.J.Brascamp, E. Lieb: On extensions of the Brunn-minkovski and prékopa-Leindler theorems, including inequalities for logconcave functions, and with an application to diffusion equation, *Journal of Functional Analysis* 22 (1976).

[BruHe] M.Brunaud and B.Helffer: Un problème de double puits provenant de la théorie statistico-mécanique des changements de phase (ou relecture d'un cours de M.Kac). *Preprint LMENS*, March 1991.

[GlJa] J.Glimm and A.Jaffe: *Quantum physics (a functional integral point of view).* Springer Verlag, Second edition (1987).

[Ha] E.M.Harrell: On the rate of asymptotic eigenvalue degeneracy. *Comm. Math. Phys.* **75** (1980), p.239-261.

[He1] B.Helffer: *Introduction to the semiclassical analysis for the Schrödinger operator and applications.* Springer lecture Notes in Math., $n^0$ 1336 (1988).

[He2] B.Helffer: Problèmes de double puits provenant de la théorie statistico-mécanique des changements de phase, II Modèles de Kac avec champ magnétique. étude de modèles près de la température critique. *Preprint LMENS*, March 1992.

[He3] B.Helffer: Décroissance exponentielle des fonctions propres pour l'opérateur de Kac: le cas de la dimension $> 1$, in *Operator calculus and spectral theory*, Symposium on operator calculus and spectral theory, Lambrecht (Germany), December 1991, edited by M.Demuth, B.Gramsch and B.-W.Schulze, Operator Theory: Advances and Applications, Vol.57; Birkhäuser Verlag Basel, (1992), p.99-116.

[He4] B.Helffer: Spectral properties of the Kac operator in large dimension. Proceedings on Mathematical Quantum Theory II: Schrödinger Operators, (August 1993), edited by J.Feldman, R.Froese, and L.M.Rosen. CRM Proceedings and Lecture Notes.

[He5] B.Helffer: Around a stationary phase theorem in large dimension. *Journal of Functional Analysis* **119** (1), January 1994.

[He6] B.Helffer: Universal estimate of the gap for the Kac operator in the convex case. *Comm. in Math. Physics* **161** (3) (1994), p.631-642.

[He7] B.Helffer: Correlation decay and gap of the transfer operator. Proceedings of the conference in the Euler Institut in St Petersbourg, Preprint April 1994, to appear in *St Petersburg Math. Journal* (1996).

[He8] B.Helffer: Around the transfer operator and the Trotter-Kato formula. Proceedings of the conference in Holzhau, Birkhäuser, Series of Operator Theory: Advances and Applications, p.161-174, M. Demuth et B.-W. Schulze (editors),

[He9] B.Helffer: Recent results and open problems on Schrödinger operators, Laplace integrals and transfer operators in large dimension, Preprint November 1994, to appear in *Advances in PDE*, Akademie Verlag (1996).

[He10] B.Helffer: *Semiclassical analysis for Schrödinger operators, Laplace integrals and transfer operators in large dimension: an introduction,* Course in Orsay 1995. Editions Paris-Sud (1995).

[He11] B.Helffer: Semiclassical analysis for the transfer operator: WKB constructions in dimension 1. Partial differential equations and mathematical physics, the danish-swedish analysis seminar (1995), Progress in nonlinear differential equations and their applications, Vol. 21, Birkhäuser (1996), p. 168-180.

[He12] B.Helffer: On Laplace integrals and transfer operators in large dimension: examples in the non convex case, *Letters in Math. Phys.***38**, p. 297-312 (1996).

[HeSj1] B.Helffer and J.Sjöstrand: Multiple wells in the semiclassical limit. I *Comm. in PDE* **9** (4) (1984), p.337-408.

[HeSj2] B.Helffer, J.Sjöstrand: Semiclassical expansions of the thermodynamic limit for a Schrödinger equation. *Astérisque* **210** (1992), p.135-181.

[HeSj3] B.Helffer and J.Sjöstrand: Semiclassical expansions of the thermodynamic limit for a Schrödinger equation II. *Helvetica Physica Acta* **65** (1992), p.748-765 and Erratum **67** (1994), p.1-3.

[HeSj4] B.Helffer, J.Sjöstrand: On the correlation for Kac like models in the convex case. *J. Statist. Physics*, Vol. 74, $n^0$1-2, p. 349-369, (1994).

[Irw] M.C.Irwin: *Smooth dynamical systems,* Academic Press (1970).

[ItDr] C.Itzykson, J.M. Drouffe: *Statistical field theory*, Vol 1, Cambridge Monographs on Mathematical Physics (1988).

[Ka1] M.Kac: Statistical mechanics of some one-dimensional systems. Studies in mathematical analysis and related topics: *Essays in honor of Georges Polya*, Stanford University Press, Stanford, California (1962), p.165-169.

[Ka2] M. Kac: Mathematical mechanisms of phase transitions. Brandeis lectures (1966), Gordon and Breach.

[HelKa] M.Kac and E.Helfand: *J. Math. Phys.* **4** (1963), p.1078-1088 .

[KaTho1] M.Kac, C.Thompson: On the mathematical mechanism of phase transition. *Proc.N.A.S.* **55** (1966), p.676-683 and Erratum **56** (1966), p.1625.

[KaTho2] M.Kac, C.Thompson: Phase transition and eigenvalue degeneracy of a one dimensional anharmonic oscillator. *Studies in Applied Mathematics* **48** (1969), p.257-264.

[Pa] G.Parisi: *Statistical field theory.* Frontiers in Physics, Addison-Wesley Publishing Company, (1984).

[ReSi] M. Reed, B. Simon: *Methods of modern mathematical Physics. II. Fourier analysis, selfadjointness.* Academic Press, (1975).

[Si1] B.Simon: Instantons, double wells and large deviations. *Bull. AMS.* **8** (1983), p.323-326.

[Si2] B.Simon: Semi-classical Analysis of low lying eigenvalues.
I *Ann. Inst. H. Poincaré* **38** (1983), p.295-307.
II *Annals of Mathematics* **120** (1984), p.89-118.

[Sj1] J.Sjöstrand: Potential wells in high dimensions I. *Ann. Inst. Poincaré*, Section Physique théorique **58** (1), (1993), p.1-41.

[Sj2] J.Sjöstrand: Potential wells in high dimensions II, more about the one well case. *Ann. Inst. H. Poincaré*, Section Physique théorique **58** (1), (1993), p.42-53.

[Sj3] J.Sjöstrand: On ferromagnetic Laplace integrals and maximum principles. *Ann. Inst. Fourier* **44**(2) (1994), p.601-628.

[Sj4] J.Sjöstrand: Estimations sur les corrélations et principes du maximum. *Séminaire Equations aux dérivées partielles de l'Ecole Polytechnique*, October 1993.

[Sj5] J.Sjöstrand: Evolution equations in a large number of variables. Mathematische Nachrichten 166 (1994), p.17-53.

[Sj6] J.Sjöstrand: Correlation asymptotics and Witten Laplacians. Preprint, June 1994 and to appear in *St Petersburg Mathematical Journal* (1996).

[Sm1] S. Smale: Stable manifolds for differential equations and diffeomorphisms. Ann. Scuola Normale Superiore Pisa 18 (1963), p. 97-116.

[Sm2] S.Smale: Differentiable dynamical systems. Bull. Amer. Math. Soc. 73 (1967), p. 747-817.

[Thi] W.Thirring: Volume 4, *Quantum Mechanics of large systems.* (translated by E.M.Harrell), Springer Verlag, (1983).