Some explicit formulas for components of the Mandelbrot set

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Abstract

Let f be a rational function, which has k n-cycles under iteration. By using the symmetry of the underlying equation of degree $k \cdot n$, it is reduced to equations of degree k and n. This is explained in terms of Galois theory. The 3- and 4-cycles of $f_c = z^2 + c$ are obtained explicitly. This yields the corresponding multiplier, which maps hyperbolic components of the Mandelbrot set conformally onto the unit disk.

1 Introduction

For a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, denote the n-th iterate by f^n. z_0 is in the Julia set of f, if the sequence $(f^n(z))$ is not normal in any neighborhood of z_0. We consider the family of quadratic polynomials $f_c(z) = z^2 + c$. The Mandelbrot set $\mathcal{M}$ contains those $c \in \mathbb{C}$, such that the Julia set of f_c is connected. Since 0 is the only critical point of the polynomial f_c, $c \in \mathcal{M}$ iff $(f^n(0))$ is bounded [1].

A n-cycle of f_c consists of distinct points $z_1 \ldots z_n$ with $f_c(z_1) = z_2, \ldots,$ $f_c(z_n) = z_1$. The corresponding multiplier is $\lambda = {f_c^n}'(z_1) = 2^n z_1 \cdot z_2 \cdots z_n$. The cycle is attracting, if $|\lambda| < 1$. The set of those c, such that f_c has an attracting n-cycle, is a union of components of $\mathcal{M}$ (called hyperbolic). These are mapped conformaly onto the unit disk by λ. It is well known that λ is an algebraic function, with $(\lambda/2)^2 - \lambda/2 + c = 0$ for $n = 1$ and $\lambda = 4(c+1)$ for $n = 2$ [3].

We describe an algorithm to obtain these functions for every n, and give the results for period 3 and 4. Define the polynomials $g_n(z, c)$ recursively by $f_c^n(z) - z = \prod_{d|n} g_d(z, c)$, then the zeros of g_n are the $n-$periodic points of f_c. For $n \geq 3$, the degree of g_n is at least 6. In general, only polynomial equations of degree 4 or less can be solved explicitly, but g_n satisfies the symmetry relation $g_n(z, c) = 0 \Rightarrow g_n(f_c(z), c) = 0$, which is used to reduce the equation. The resulting algorithm is best understood in terms of Galois theory.

2 The cycles and multipliers of f_c

Except for some values of c, at which a bifurcation occurs, g_n has $k \cdot n$ simple zeros. These form k $n-$cycles $z_1^{(j)} \ldots z_n^{(j)}$ with $f_c(z_i^{(j)}) = z_l^{(j)}$, $l = i + 1 \pmod{n}$. This suggests the following algorithm: Define $s_n(z, c, a) = z + f_c(z) + \ldots + f_c^{n-1}(z) - a$. Then $s_n(z_i^{(j)}, c, a) = s_n(z_l^{(j)}, c, a)$, thus a can be chosen such that the greatest common divisor of g_n and s_n is of degree n. We perform Euklid's algorithm with g_n and s_n. The remainder with degree $< n$ must vanish. This yields an equation $h_n(a, c) = 0$, where h_n is the g.c.d. of the coefficients of this remainder. Denote the remainder of degree n by $j_n(z, c, a)$. It is the g.c.d. of g_n and s_n, if a satisfies $h_n(a, c) = 0$.

We have applied this algorithm to g_3, g_4 and g_5 and give the results for $n = 3, 4$ in Theorem 1. For $n \geq 6$, the algorithm might require some modifications for two reasons: If the leading coefficient of a remainder vanishes (for all c and appropriate a), this case must be treated separately. If, say, $z_1^{(1)} + \ldots + z_n^{(1)} = z_1^{(2)} + \ldots + z_n^{(2)}$ for all c, the g.c.d. will be of degree al least $2n$. Then s_n must be replaced by some higher-degree symmetric polynomial.

Theorem 1 *Determine h_n and j_n from the algorithm described above. The $n-$cycles of f_c are obtained by solving $h_n(a, c) = 0$ for a and $j_n(z, c, a) = 0$ for z. This can be done explicitly for $n = 3, 4$. We have*

$$
\begin{aligned}
h_3(a, c) &= a^2 + a + c + 2 \\
j_3(z, c, a) &= z^3 - a\,z^2 + (-a + c - 1)\,z - (ac + c + 1) \\
h_4(a, c) &= a^3 + (4c + 3)\,a + 4 \\
j_4(z, c, a) &= \left(4z^2 - 2az - \frac{a^2 + a + 4}{a}\right)^2 - (a^2 + 4)\Big(2z - a - 1\Big)^2 .
\end{aligned}
$$

We will see in the following section that the Galois group of j_n is cyclic. Thus the discriminant of j_3 is a square, namely $27^2\,(4a^2 + 6a + 9)^2$, and the resolvent of j_4 is reducible [2, p. 126], which yields the simplified form of j_4 given above. All of these formulas can be translated to the logistic map $x \mapsto A\,x\,(1-x)$.

From j_n, the multiplier $\lambda = 2^n z_1 \cdot z_2 \cdots z_n$ is obtained by using Vieta's theorem. This yields

Theorem 2 *For a hyperbolic component $\mathcal{H}$ of $\mathcal{M}$ corresponding to attractive $n-$cycles of f_c, the multiplier $\lambda: \mathcal{H} \to D_1(0)$ is a suitable branch of $k_n(c,\,\lambda) = 0$, where the polynomial k_n is obtained by eliminating a from the equations*

$$h_n(a,\,c) = 0 \qquad \text{and} \qquad \lambda = (-2)^n\,\frac{trailing\ coefficient(j_n)}{leading\ coefficient(j_n)}\ .$$

For $n = 3$ and $n = 4$ this yields

$$\begin{aligned}
k_3(c,\,\lambda) &= c^3 + 2\,c^2 + (1-\lambda/8)\,c + (1-\lambda/8)^2\\
k_4(c,\,\lambda) &= c^6 + 3\,c^5 + (\lambda/16+3)\,c^4 + (\lambda/16+3)\,c^3\\
&+ (2-\lambda/16-(\lambda/16)^2)\,c^2 + (1-\lambda/16)^3\ .
\end{aligned}$$

(The elimination can be done by applying Euklid's algorithm once more.) k_n has degree k with respect to λ and degree $nk/2$ w.r.t. c. For $n = 3$ or 4, λ is obtained explicitly from c. For $n = 3$, c is obtained from λ, and the boundary of the 3 coresponding hyperbolic components of $\mathcal{M}$ is determined by $|c + 2 \pm c\sqrt{-4c-7}| = 1/4$. In $\mathbb{R}^2$, this is a curve of order 12.

3 The Galois group of g_n

Denote the field $\mathbb{Q}(c)$ by K and the splitting field of $g_n \in K[z]$ by L. The algorithm of Theorem 1 can be understood in terms of Galois theory as follows: The Galois group G consists of those automorphisms of L, which are leaving K fixed. It is represented by permutations of the zeros of g_n. For $\alpha \in G$, we have $f_c(z \cdot \alpha) = f_c(z) \cdot \alpha$. Since f_c acts on the zeros as the permutation $\alpha_0 = (z_1^{(1)} \ldots z_n^{(1)}) \ldots (z_1^{(k)} \ldots z_n^{(k)})$, G must be contained in the centralizer of α_0 in S_{kn}, which is a wreath-product $S_k \wr C_n$. We have $S_k \wr C_n \cong S_k \times (C_n \otimes \ldots \otimes C_n)$, where S_k is permuting the different cycles of g_n, while each C_n is acting on the elements of a cycle. Now the algorithm of the previous section corresponds to the normal series $S_k \wr C_n \trianglerighteq (C_n \otimes \ldots \otimes C_n) \trianglerighteq \ldots \trianglerighteq C_n \otimes C_n \trianglerighteq C_n \trianglerighteq 1$. The Galois group of $h_n \in K[z]$ is contained in S_k, and for $h_n(a,\,c) = 0$ the Galois group of $j_n \in K(a)[z]$ is cyclic of order $\leq n$. Thus $j_n = 0$ can always be solved explicitly, but $h_n = 0$ is in general not solvable for $k > 4$. In the case of $n = 3$ or $n = 4$, the formulas of Theorem 1 show that $G \cong S_k \wr C_n$ in general, i.e., if c is transcendental, or equivalently, if $\mathbb{Q}(c)$ is understood as the field of rational functions in one variable.

4 Summary

If f is rational function with k n-cycles, the underlying equation of degree $k \cdot n$ is reduced to one equation of degree k and k equations of degree n. The first is solvable explicitly at least if $k \leq 4$, while the latter equations are always solvable, as Galois theory shows.

For $f_c = z^2 + c$, our algorithm yields the conformal mappings of hyperbolic components of the Mandelbrot set onto the unit disk. It also yields explicit results for other low-degree rational functions, e.g., the 3 2-cycles of a third-degree polynomial and the 8 3-cycles of $z - \frac{z^3-1}{3z^2}$ can be obtained.

References

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