An obstruction for q-deformation of the convolution product

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Abstract

We consider two independent q-Gaussian random variables X_0 and X_1 and a function γ chosen in such a way that $\gamma(X_0)$ and X_0 have the same distribution. For $q \in (0,1)$ we find that at least the fourth moments of X_0+X_1 and $\gamma(X_0)+X_1$ are different. We conclude that no q-deformed convolution product can exist for functions of independent q-Gaussian random variables.

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1 Introduction and Notation

In 1991 Bożejko and Speicher introduced a deformation of Brownian motion by a parameter $q \in [-1,1]$ (cf. [1, 2]). Their construction is based on a q-deformation, $\mathcal{F}_q(\mathcal{H})$, of the full Fock space over a separable Hilbert space $\mathcal{H}$. Their random variables are given by self-adjoint operators of the form

$$X(f) := a(f) + a(f)^*, \qquad f \in \mathcal{H},$$

where $a(f)$ and $a(f)^*$ are the annihilation and creation operators associated to f satisfying the q-deformed commutation relation,

$$a(f)a(g)^* - qa(g)^*a(f) = \langle f, g\rangle \mathbb{1}. \tag{1}$$

This commutation relation was first introduced by Frisch and Bourret in [4] and various aspects of it are studied in a.o. [3, 5, 6, 10].

The above construction was for some time considered a good candidate for a q-deformed notion of the concept of independence itself. Indeed for $q = 1$ the random variables $X(f)$ and $X(g)$ with $f \perp g$ are independent Gaussian random variables in the classical sense, for $q = 0$ they are freely independent in the sense of Voiculescu [11]. In both cases a convolution law holds for sums of functions of $X(f)$ and $X(g)$. For $q = 1$ the convolution is ordinary convolution whereas for $q = 0$ the convolution is found to be an interesting operation involving Gauchy transforms and inverted functions [8, 11]. In this paper we show, by a simple example, that for $q \in (0, 1)$ no such convolution law holds since the distributions of functions of $X(f)$ and $X(g)$ do not determine the distribution of their sum.

The construction of the Fock representation for (1) is described in [1, 3] but for completeness we give the necessary details here. Operators $a(f)$ and $a(f)^*$ are, for all $f \in \mathcal{H}$, defined on the full Fock space $\mathcal{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ by:

$$a(f)^* h_1 \otimes \cdots \otimes h_n := f \otimes h_1 \otimes \cdots \otimes h_n, \qquad n \in \mathbb{N}, h_1, \ldots, h_n \in \mathcal{H}$$

and

$$\begin{aligned} a(f)\Omega &:= 0, \\ a(f) h_1 \otimes \cdots \otimes h_n &:= \sum_{k=1}^{n} q^{k-1} \langle f, h_k \rangle h_1 \otimes \cdots \check{h}_k \cdots \otimes h_n, \qquad n \geq 1 \end{aligned} \tag{2}$$

where the notation $h_1 \otimes \cdots \check{h}_k \cdots \otimes h_n$ stands for the tensor product $h_1 \otimes \cdots \otimes h_{k-1} \otimes h_{k+1} \otimes \cdots \otimes h_n$ and $\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots$. In order to ensure that $a(f)^*$ is the adjoint of $a(f)$ for all $f \in \mathcal{H}$, Bożejko and Speicher recursively define an inner product $\langle \cdot, \cdot \rangle_q$ on $\mathcal{F}$ as:

$$\begin{aligned} \langle g_1 \otimes \cdots \otimes g_m, h_1 \otimes \cdots \otimes h_n \rangle_q &:= \delta_{n,m} \langle g_2 \otimes \cdots \otimes g_m, a(g_1) h_1 \otimes \cdots \otimes h_n \rangle_q \\ &= \delta_{n,m} \sum_{k=1}^{n} q^{k-1} \langle g_1, h_k \rangle \langle g_2 \otimes \cdots \otimes g_m, h_1 \otimes \cdots \check{h}_k \cdots \otimes h_n \rangle_q . \end{aligned}$$

We denote the full Fock space $\mathcal{F}$ equipped with this inner product by $\mathcal{F}_q(\mathcal{H})$. By the GNS construction there exists, up to unitary equivalence, only one cyclic representation of the relations (1) and (2). For $\mathcal{H} = \mathbb{C}$ the above construction reduces to $\mathcal{F}_q(\mathbb{C}) \cong l^2(\mathbb{N}, [n]_q!)$, where $[n]_q = (1-q^n)/(1-q)$ and $[n]_q! = \prod_{j=1}^n [j]_q$ with $[0]_q! = 1$.

In [2, 10] the density of the q-Gaussian distribution, $\nu_q(dx)$, of the random variable $X_0 = a(f_0) + a(f_0)^*$ with $f_0 \in \mathcal{H}$ and $\|f_0\| = 1$ is calculated. This density is a measure on $\mathbb{R}$, where it is supported on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$. If we denote the n-fold product $\prod_{k=0}^{n-1}(1-aq^k)$ by $(a;q)_n$ and agree on $(a_1, \ldots, a_m; q)_n = (a_1;q)_n \cdots (a_m;q)_n$, then $\nu_q(dx)$ can be written as:

$$\nu_q(dx) = \nu_q'(x)dx = \frac{1}{\pi}\sqrt{1-q}\sin\theta(q, qv^2, qv^{-2}; q)_\infty \, dx,$$

where $2\cos\theta = x\sqrt{1-q}$ and $v = \exp(i\theta)$.

To state the main theorem of this paper we define X_1 to be the random variable $a(f_1) + a(f_1)^*$ for some $f_1 \in \mathcal{H}$ with $\|f_1\| = 1$ and $\langle f_0, f_1 \rangle = 0$. Then X_0 and X_1 are q-Gaussian random variables, independent in the sense of quantum probability ([1, 7]).

THEOREM 1 *There exists a function $\gamma\colon \mathbb{R} \to \mathbb{R}$ such that X_0 and $\gamma(X_0)$ are identically distributed but $X_0 + X_1$ and $\gamma(X_0) + X_1$ are not.*

The consequence of this theorem is that the distribution of the sum of two or more random variables depends on the choice of random variables and not solely on the respective distributions of these random variables. This means that a q-convolution paralleling the known convolution for probability measures for the cases $q = 0$ (cf. [8, 11]) and $q = 1$ cannot exist.

In contrast to the above, Nica, in [9], constructs a convolution law for probability distributions that interpolates between the known cases $q = 0$ and $q = 1$. Theorem 1 implies that this interpolation is incompatible with relation (1). In fact this can also be seen by explicit calculation of the moments of the distribution of $X_0^n + X_1^m$, $n, m \in \mathbb{N}$ using the convolution law Nica suggests and using the structure inherently present in $\mathcal{F}_q(\mathcal{H})$. From the fourth moment onwards the moments differ for $n, m \geq 1$, although they are the same for the cases $q = 0$ and $q = \pm 1$, as they should be.

In the next section we will prove theorem 1 by constructing the function γ and showing that the fourth moment of the distribution of $\gamma(X_0) + X_1$ is strictly smaller then the fourth moment of the distribution of $X_0 + X_1$ for $q \in (0, 1)$.

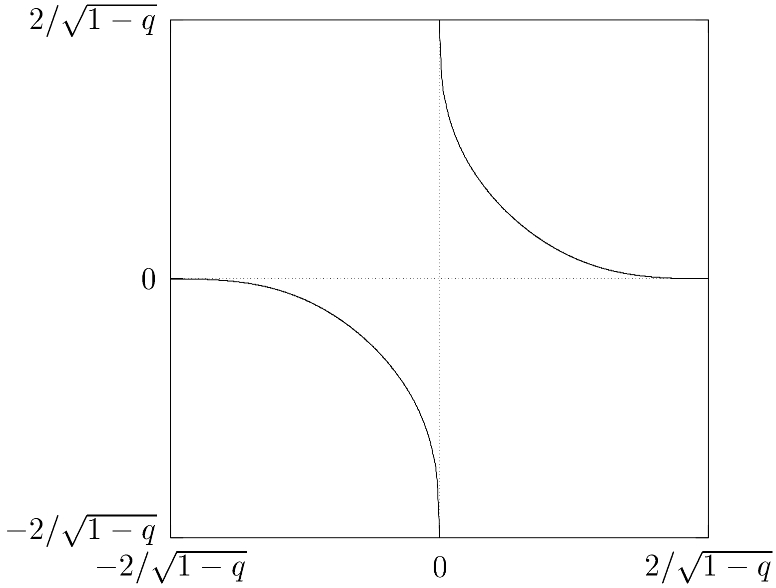

Figure 1: The function γ.

2 Construction of γ and proof of theorem

In [10] we construct the unitary operator $U\colon \mathcal{F}_q(\mathbb{C}) \to L^2(\mathbb{R}, \nu_q)$ that diagonalizes the operator $X = a + a^*$ with $a = a(1)$, such that $UX = TU$ with T the operator of pointwise multiplication on $L^2(\mathbb{R}, \nu_q)$ given by $(Tf)(x) = xf(x)$ for $f \in L^2(\mathbb{R}, \nu_q)$.

Let γ be the transformation on $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ that changes the orientations of $[-2/\sqrt{1-q}, 0]$ and $[0, 2/\sqrt{1-q}]$ in such a way that the distribution ν_q is preserved. For this γ has to satisfy the differential equation:

$$\nu_q'(x)dx + \nu_q'(\gamma(x))d\gamma(x) = 0, \tag{3}$$

with $\gamma(-2/\sqrt{1-q}) = \gamma(2/\sqrt{1-q}) = 0$. Indeed, this leads to:

$$\mathbb{P}(0 \leq T \leq x) = \mathbb{P}(0 \leq \gamma(T) \leq x) = \mathbb{P}(\gamma^{-1}(x) \leq T \leq 2/\sqrt{1-q})$$

as can be seen by differentiating both sides with respect to x. Note that the function γ is its own inverse. Figure 1 shows a typical picture of the shape of the function γ.

Let $\widehat{W}$ be the unitary operator on $L^2(\mathbb{R}, \nu_q)$ that implements γ:

$$(\widehat{W}f)(x) = f(\gamma(x)) \qquad \text{for } f \in L^2(\mathbb{R}, \nu_q).$$

This immediately implies that $\widehat{W}^2 = \mathbb{1}$ since $\gamma \circ \gamma = \mathrm{id}$, so $\widehat{W}$ is self-adjoint. If we define $\widetilde{W} := U^* \widehat{W} U$ it follows that:

$$\gamma(X) = \gamma(U^*TU) = U^*\gamma(T)U = U^*\widehat{W}T\widehat{W}U = \widetilde{W}X\widetilde{W},$$

so $\widetilde{W}$ is a unitary and self-adjoint operator on $\mathcal{F}_q(\mathbb{C})$ that implements γ on X. Note that $\widetilde{W}\Omega = \Omega$ because $\widehat{W}1 = 1$. On the canonical basis $(e_j)_{j\in\mathbb{N}}$ of $\mathcal{F}_q(\mathbb{C})$ the operator $\widetilde{W}$ can be written as:

$$\widetilde{W}e_n = \sum_{k=0}^{\infty} w_{kn} e_k \qquad \text{with } w_{00} = 1.$$

Now let us choose $\mathcal{H} = \mathbb{C}^2$, $f_0 = (1,0)$ and $f_1 = (0,1)$ and let us denote $a(f_0)$ by a_0 and $a(f_1)$ by a_1. Recall from the introduction that $X_0 = a_0 + a_0^*$ and $X_1 = a_1 + a_1^*$. In this setting we need a unitary operator W on $\mathcal{F}_q(\mathbb{C}^2)$ that satisfies: $\gamma(X) = WXW$. To this end we denote by $\mathcal{K} \subset \mathcal{F}_q(\mathbb{C}^2)$ the kernel of the operator a_0 on $\mathcal{F}_q(\mathbb{C}^2)$, then by constructing an isomorphism $V\colon \mathcal{F}_q(\mathbb{C}) \otimes \mathcal{K} \to \mathcal{F}_q(\mathbb{C}^2)$, the operator $\widetilde{W}$ can be extended to $W = V(\widetilde{W} \otimes \mathbb{1})V^*$.

Proposition 2 *The space $\mathcal{F}_q(\mathbb{C}^2)$ is canonically isomorphic to $\mathcal{F}_q(\mathbb{C}) \otimes \mathcal{K}$.*

Proof: ¿From the commutation relation (1) we find that $a_0^n (a_0^*)^n = P_n(a_0^* a_0)$, where P_n is a polynomial of degree n with constant coefficient $[n]_q!$. In fact, P_n is given by

$$P_n(x) = \prod_{j=1}^{n} (q^j x + [j]_q).$$

For $n \in \mathbb{N}$, let $\mathcal{K}_n$ denote the Hilbert subspace $(a_0^*)^n \mathcal{K}$. Note that $\mathcal{K}_n$ is indeed closed, since $(a_0^*)^n$ acts on $\mathcal{K}$ as a multiple of an isometry, for every $\varphi \in \mathcal{K}$,

$$\|(a_0^*)^n \varphi\|_q^2 = \langle \varphi, P_n(a_0^* a_0)\varphi\rangle_q = [n]_q! \|\varphi\|_q^2.$$

Furthermore, $\mathcal{K}_n \perp \mathcal{K}_m$ for $n > m$, since for $\varphi, \psi \in \mathcal{K}$,

$$\begin{aligned}
\langle (a_0^*)^n \varphi, (a_0^*)^m \psi \rangle_q &= \langle (a_0^*)^{n-m} \varphi, a_0^m (a_0^*)^m \psi \rangle_q \\
&= \langle (a_0^*)^{n-m} \varphi, P_m(a_0^* a_0) \psi \rangle_q \\
&= [m]_q! \langle \varphi, (a_0)^{n-m} \psi \rangle_q = 0.
\end{aligned}$$

Now suppose that some $\psi \in \mathcal{F}_q(\mathbb{C}^2)$ is orthogonal to all the $\mathcal{K}_n$. We claim that for all $n \in \mathbb{N}$

$$\psi \perp \ker a_0^n, \tag{4}$$

from which it follows that $\psi = 0$, since $(\mathbb{C}^2)^{\otimes n} = \mathcal{F}_q^{(n)}(\mathbb{C}^2) \subset \ker a_0^{n+1}$.
We proceed to prove (4) by induction. For $n = 1$ we already have (4) since $\ker a_0 = \mathcal{K}$. Suppose that (4) holds for some n. Then $\psi \in \overline{\mathrm{Ran}(a_0^*)^n}$, say $\psi = \lim_{k\to\infty}(a_0^*)^n \varphi_k$ with $\varphi_k \in \mathcal{F}_q(\mathbb{C}^2)$, $k \in \mathbb{N}$. Take $\theta \in \ker a_0^{n+1}$ and define $\xi := a_0^n \theta \in \mathcal{K}$, then

$$\begin{aligned}
\langle \psi, \theta \rangle_q &= \lim_{k\to\infty} \langle (a_0^*)^n \varphi_k, \theta \rangle_q = \lim_{k\to\infty} \langle \varphi_k, \xi \rangle_q \\
&= \frac{1}{[n]_q!} \lim_{k\to\infty} \langle \varphi_k, P_n(a_0^* a_0)\xi \rangle_q \\
&= \frac{1}{[n]_q!} \lim_{k\to\infty} \langle (a_0^*)^n \varphi_k, (a_0^*)^n \xi \rangle_q \\
&= \frac{1}{[n]_q!} \langle \psi, (a_0^*)^n \xi \rangle_q = 0,
\end{aligned}$$

because $(a_0^*)^n \xi \in \mathcal{K}_n \perp \psi$. The claim, (4), follows by induction.
We define an operator $V\colon \mathcal{F}_q(\mathbb{C}) \otimes \mathcal{K} \to \mathcal{F}_q(\mathbb{C}^2)$ by:

$$V(e_n \otimes \varphi) := (a_0^*)^n \varphi.$$

The operator V is an isomorphism since its range is dense by the above, and, for all $\varphi, \xi \in \mathcal{K}$,

$$\begin{aligned}
\langle V(e_n \otimes \varphi), V(e_m \otimes \xi) \rangle_q &= \langle (a_0^*)^n \varphi, (a_0^*)^m \xi \rangle_q \\
&= \delta_{n,m} \langle \varphi, P_n(a_0^* a_0)\xi \rangle_q \\
&= \delta_{n,m} [n]_q! \langle \varphi, \xi \rangle_q \\
&= \langle e_n \otimes \varphi, e_m \otimes \xi \rangle_q .
\end{aligned}$$

□

Lemma 3 *The operator W has the following properties:*

1. *W is unitary and self adjoint,*

2. *$\gamma(X_0) = W X_0 W$,*

3. $W\varphi = \varphi$ *for all* $\varphi \in \mathcal{K}$, *in particular* $W\Omega = \Omega$,

4. $W(X_0\varphi) = \sum_{k=1}^{\infty} w_{k1}(a_0^*)^k\varphi$ *for all* $\varphi \in \mathcal{K}$.

PROOF: Property 1 is clear from the definition of W since $\widetilde{W}$ is unitary and self adjoint.
To prove property 2, note that, for $\varphi \in \mathcal{K}$ and $n \in \mathbb{N}$, $V(a^* \otimes \mathbb{1})(e_n \otimes \varphi) = V(e_{n+1} \otimes \varphi) = a_0^* V(e_n \otimes \varphi)$, so $V(a^* \otimes \mathbb{1})V^* = a_0^*$ and $V(X \otimes \mathbb{1})V^* = X_0$. It follows that

$$\begin{aligned} WX_0W &= WV(X \otimes \mathbb{1})V^*W \\ &= V(\widetilde{W} \otimes \mathbb{1})(X \otimes \mathbb{1})(\widetilde{W} \otimes \mathbb{1})V^* \\ &= V(\gamma(X) \otimes \mathbb{1})V^* \\ &= V\gamma(X \otimes \mathbb{1})V^* = \gamma(X_0). \end{aligned}$$

Property 3 is immediate from definitions:

$$W\varphi = V(\widetilde{W} \otimes \mathbb{1})(e_0 \otimes \varphi) = V(e_0 \otimes \varphi) = \varphi,$$

for all $\varphi \in \mathcal{K}$.
The proof of property 4 is also immediate from definitions:

$$\begin{aligned} W(X_0\varphi) &= W(a_0^*\varphi) = V(\widetilde{W} \otimes \mathbb{1})(e_1 \otimes \varphi) = V(\widetilde{W}e_1 \otimes \varphi) \\ &= \sum_{k=1}^{\infty} w_{k1} V(e_k \otimes \varphi) = \sum_{k=1}^{\infty} w_{k1}(a_0^*)^k \varphi. \end{aligned}$$

□

We now turn to the proof of theorem 1.

PROOF: First we calculate the fourth moment of $X_0 + X_1$. Since $X(f_0 + f_1)$ is q-Gaussian with variance 2 we have:

$$\begin{aligned} \langle \Omega, (X_0 + X_1)^4 \Omega \rangle_q &= (\sqrt{2})^4 \langle \Omega, X_0^4 \Omega \rangle_q = 4\|X_0^2\Omega\|_q^2 \\ &= 4(\|\Omega\|_q^2 + \|f^{\otimes 2}\|_q^2) = 4(1 + [2]_q) \\ &= 8 + 4q, \end{aligned}$$

a linear interpolation between 8 and 12 for q varying between 0 and 1. We now turn to the calculation of the fourth moment of $\gamma(X_0) + X_1$:

$$\langle \Omega, (\gamma(X_0) + X_1)^4 \Omega \rangle_q = \|(\gamma(X_0) + X_1)^2 \Omega\|_q^2.$$

For this we need the following:

$$\begin{aligned}
\gamma(X_0)^2\Omega &= WX_0^2W\Omega = \Omega + Wf_0^{\otimes 2} \\
X_1^2\Omega &= \Omega + f_1^{\otimes 2} \\
\gamma(X_0)X_1\Omega &= WX_0WX_1\Omega = WX_0X_1\Omega \\
&= \sum_{k=1}^{\infty} w_{k1}(a_0^*)^k X_1\Omega = \sum_{k=1}^{\infty} w_{k1} f_0^{\otimes k} \otimes f_1 \\
X_1\gamma(X_0)\Omega &= X_1WX_0\Omega = \sum_{k=1}^{\infty} w_{k1} f_1 \otimes f_0^{\otimes k},
\end{aligned}$$

from which it is easy to deduce that:

$$\|(\gamma(X_0)+X_1)^2\Omega\|_q^2 = \|(\gamma(X_0)^2+X_1^2)\Omega\|_q^2 + \|(\gamma(X_0)X_1 + X_1\gamma(X_0))\Omega\|_q^2. \tag{5}$$

The first term on the right hand side of (5) is found to be:

$$\|(\gamma(X_0)^2+X_1^2)\Omega\|_q^2 = 4\|\Omega\|_q^2 + \|f_0^{\otimes 2}\|_q^2 + \|f_1^{\otimes 2}\|_q^2 = 4 + 2[2]_q = 6 + 2q.$$

The second term on the right hand side of (5) yields:

$$\begin{aligned}
\|(\gamma(X_0)X_1 + X_1\gamma(X_0))\Omega\|_q^2 &= \sum_{k=1}^{\infty} w_{k1}^2 \|(f_0^{\otimes k} \otimes f_1 + f_1 \otimes f_0^{\otimes k})\|_q^2 \\
&= \sum_{k=1}^{\infty} w_{k1}^2 (2\|f_0^{\otimes k} \otimes f_1\|_q^2 + 2\langle f_0^{\otimes k} \otimes f_1, f_1 \otimes f_0^{\otimes k}\rangle_q) \\
&= 2\sum_{k=1}^{\infty} w_{k1}^2 (1+q^k)[k]_q! \\
&= 2 + 2\sum_{k=1}^{\infty} w_{k1}^2 q^k [k]_q!.
\end{aligned}$$

To prove the theorem it remains to show that

$$\sum_{k=1}^{\infty} w_{k1}^2 q^k [k]_q! < q \qquad \text{for } q \in (0,1).$$

To this end note that $q^k < q$ for $k \geq 2$ and $q \in (0,1)$, so:

$$\sum_{k=1}^{\infty} w_{k1}^2 (q^k - q)[k]_q! < 0$$

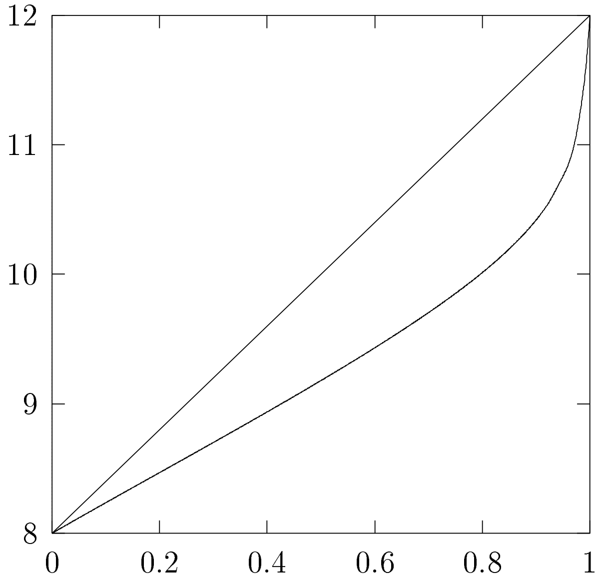

Figure 2: The fourth moment of $X + Y$ and $\gamma(X) + Y$ for $q \in (0, 1)$.

from which it follows that:

$$\sum_{k=1}^{\infty} w_{k1}^2 q^k [k]_q! < q \sum_{k=1}^{\infty} w_{k1}^2 [k]_q! = q \|\widetilde{W} e_1\|_q^2 = q.$$

We conclude that $\langle \Omega, (\gamma(X_0) + X_1)^4 \Omega \rangle_q < \langle \Omega, (X_0 + X_1)^4 \Omega \rangle_q$ for $q \in (0, 1)$. $\square$

The content of theorem 1 is shown graphically in figure 2 where the fourth moment of $X_0 + X_1$ and a numerical approximation of the fourth moment of $\gamma(X_0) + X_1$ are plotted.

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