Dimensional Hausdorff Properties of Singular Continuous Spectra

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Abstract

We present an extension of the Gilbert-Pearson theory of subordinacy, which relates dimensional Hausdorff spectral properties of one-dimensional Schrödinger operators to the behavior of solutions of the corresponding Schrödinger equation. We use this theory to analyze these properties for several examples having singular-continuous spectrum, including sparse barrier potentials, the almost Mathieu operator and the Fibonacci Hamiltonian.

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Singular continuous spectra have been extensively studied recently. Our interest here is in the classification and decomposition of such spectra with respect to dimensional Hausdorff measures. The measure-theoretical aspect of this point of view goes back to Rogers-Taylor [1], and it has been studied recently within spectral theory by Last [2] and by del-Rio et al. [3] who have shown that the singular-continuous spectrum which is produced by localized rank-one perturbations of Anderson-model Hamiltonians in the localized regime [4] must be purely zero-dimensional — in the sense that the associated spectral measures are supported on a set of zero Hausdorff dimension.

The main purpose of this paper is to report a general method for spectral analysis of one-dimensional Schrödinger operators from this point of view. It is a natural extension of the Gilbert-Pearson theory of subordinacy [5, 6], and it allows us to analyze the dimensional Hausdorff properties for a number of examples with singular-continuous spectrum. Below we describe the main ideas of our study and some of the main results. Mathematically complete proofs of these results will be given elsewhere [7].

Most of our discussion will be restricted to one-dimensional discrete (tight-binding) Schrödinger operators of the form

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)\ . \tag{1}$$

We shall consider two kinds of such operators: "line" operators acting on $\ell^2(\mathbf{Z})$ ($-\infty < n < \infty$), and "half-line" operators acting on $\ell^2(\mathbf{Z}^+)$ ($n > 0$), which are considered with a phase boundary condition of the form:

$$\psi(0)\cos\theta + \psi(1)\sin\theta \ = \ 0\ , \tag{2}$$

where $-\pi/2 < \theta < \pi/2$.

Before formulating our main result, which would require some definitions, we would like to describe some of its applications. We stress at this point that the dimensional Hausdorff properties which we study are those which are associated with the spectral measures of the corresponding operators. The spectra themselves, as sets, are closed sets, and their dimensions may be larger than those which are associated with the spectral measures. A description of the precise spectral-theoretic scheme which underlies our study is given below.

We start with a somewhat artificial example of "half-line" operators with sparse barrier potentials. More specifically, we consider potentials which vanish for all n's outside a sparse (fastly growing) sequence of points $\{L_n\}_{n=1}^{\infty}$ where $|V(L_n)| \to \infty$ as $n \to \infty$. Simon-Spencer [8] have shown that the Schrödinger operators corresponding to such potentials have no absolutely-continuous spectrum, and Gordon [9] has shown that if the $|V(L_n)|$'s grow sufficiently fast (compared to the growth of the L_n's) then for (Lebesgue) a.e. boundary phase θ the corresponding operators have pure-point spectrum with exponentially decaying eigenfunctions. It is easy to see [10], however, that if the L_n's grow sufficiently fast (compared to the growth of the $|V(L_n)|$'s) then, for every boundary phase θ, the spectrum in $(-2,2)$ is purely singular-continuous, and Simon [11] has recently shown that if the growth is even faster then the spectrum in $(-2,2)$ is purely one-dimensional, in the sense that the spectral measure does not give weight to sets of Hausdorff dimension less than 1. By applying Theorem 1 below, we have shown:

Theorem 2. *Let $\alpha \in (0,1)$. Let $L_n = 2^{(n^n)}$ and define a potential $V(k)$ for $k > 0$ by $V(L_n) = L_n^{(1-\alpha)/(2\alpha)}$; $V(k) = 0$ if $k \notin \{L_n\}_{n=1}^{\infty}$. Then:*

(i) *For every boundary phase θ, the spectrum of the corresponding "half-line" discrete Schrödinger operator consists of the interval $[-2,2]$ (which is the essential spectrum) along with some discrete point spectrum outside this interval.*

(ii) *For every θ, the Hausdorff dimensionality of the spectrum in $(-2,2)$ is bounded between dimensions α and $\beta \equiv 2\alpha/(1+\alpha)$, in the sense that the restriction of the spectral measure to $(-2,2)$ is supported on a set of Hausdorff dimension β and does not give weight to sets of Hausdorff dimension less than α.*

(iii) *For Lebesgue a.e. θ, the spectrum in $[-2,2]$ is of exact dimension α, namely, the restriction of the spectral measure to $[-2,2]$ is supported on a set of Hausdorff dimension α and does not give weight to sets of Hausdorff dimension less than α.*

Remark. The result only requires the L_n's to be sufficiently sparse (namely, to grow sufficiently fast). $L_n = 2^{(n^n)}$ is a particular choice for which the sufficient sparseness is easy to show.

Next we consider two examples of "line" operators with quasiperiodic potentials. The first is the almost Mathieu (also called Harper) operator $H_{\beta,\lambda,\theta}$, which is the operator of the form (1) on $\ell^2(\mathbf{Z})$ with potential $V(n) = V_{\beta,\lambda,\theta}(n) = \lambda\cos(2\pi\beta n + \theta)$, where λ, θ are any real numbers, and β is an irrational. Aubry and Andre [12] have conjectured that $H_{\beta,\lambda,\theta}$ has purely absolutely-continuous spectrum whenever $|\lambda| < 2$, and purely point spectrum (with exponentially localized eigenfunctions) whenever $|\lambda| > 2$. While the $|\lambda| < 2$ part of this conjecture may be correct (so far, the existence of absolutely-continuous spectrum [13] and absence of point spectrum [14] have been established rigorously), the $|\lambda| > 2$ case turned out to be more delicate: Absolutely-continuous spectrum is absent [15], but both pure-point and singular-continuous spectra occur, depending on arithmetical properties of both β and θ [16]. It turns out, though, that if we concentrate on the dimensional Hausdorff properties of the spectral measures, rather than distinguishing between pure-point and singular-continuous spectra, the situation becomes simpler:

Theorem 3. *For $|\lambda| > 2$, every irrational β, and every θ, $H_{\beta,\lambda,\theta}$ has purely zero-dimensional spectrum, in the sense that its spectral measures are all supported on a set of zero Hausdorff dimension.*

Remarks. (i) The spectrum of $H_{\beta,\lambda,\theta}$, as a set, is known in this case ($|\lambda| > 2$) to have positive Lebesgue measure [17]. (ii) The result extends to potentials of the form $V(n) = f(2\pi\beta n + \theta)$, where $f(x) \equiv \sum_{k=1}^{N} \lambda_k \cos(kx)$, in which case we prove that the spectrum is purely zero-dimensional whenever$|\lambda_N| > 2$.

Our second "line" example is the Fibonacci Hamiltonian H_λ, which is the operator of the form (1) on $\ell^2(\mathbf{Z})$ with potential $V(n) = \lambda([(n+1)\omega] - [n\omega])$, where $\omega = (\sqrt{5}-1)/2$ is the golden mean, and $[x] \equiv \max\{m \in \mathbf{Z} \,|\, m \leq x\}$. H_λ is the most studied of all one-dimensional quasicrystal models. It is known [18] that, for every $\lambda \neq 0$, it has purely singular-continuous spectrum, and, moreover, its spectrum (as a set) is a Cantor set of zero Lebesgue measure. We have shown:

Theorem 4. *For every λ there exists an $\alpha > 0$ such that H_λ has purely α-continuous spectrum, namely, its spectral measures do not give weight to sets of Hausdorff dimension less than α.*

Remark. There exists strong numerical evidence [19] that the spectrum of H_λ (as a set) has Hausdorff dimension strictly less than 1 (for every $\lambda \neq 0$), and this would imply that its spectrum must also be β-singular (see below) for some $\beta < 1$.

Let us now describe the spectral-theoretic scheme in the context of which the above results should be understood.

Consider a separable Hilbert space $\mathcal{H}$, and a self adjoint operator H. Recall [20] that for each $\psi \in \mathcal{H}$, the spectral measure μ_ψ (also known to physicists as the local spectral density) is the unique Borel measure obeying $\langle \psi,\, f(H)\psi \rangle = \int f(x)\, d\mu_\psi(x)$ for any measurable function f. By Lebesgue's decomposition theorem, every Borel measure μ decomposes uniquely as: $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$. The absolutely-continuous part, μ_{ac}, gives zero weight to sets of zero Lebesgue measure. The pure-point part, μ_{pp}, is a countable sum of atomic (Dirac) measures. The singular-continuous part, μ_{sc}, gives zero weight to countable sets and is supported on some set of zero Lebesgue measure (we say that a measure μ is supported on a set S if $\mu(\mathbf{R} \setminus S) = 0$). Letting $\mathcal{H}_{ac} \equiv \{\psi \,|\, \mu_\psi \text{ is purely absolutely-continuous}\}$, $\mathcal{H}_{sc} \equiv \{\psi \,|\, \mu_\psi \text{ is purely singular-continuous}\}$, and $\mathcal{H}_{pp} \equiv \{\psi \,|\, \mu_\psi \text{ is purely pure-point}\}$, one obtains a decomposition: $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}$. $\mathcal{H}_{ac}$, $\mathcal{H}_{sc}$, and $\mathcal{H}_{pp}$ are closed (in norm), mutually orthogonal subspaces, which are invariant under H. The absolutely-continuous spectrum (σ_{ac}), singular-continuous spectrum (σ_{sc}), and pure-point spectrum (σ_{pp}) are defined as the spectra of the restrictions of H to the corresponding subspaces, and $\mathrm{Spec}\,(H) \equiv \sigma = \sigma_{ac} \cup \sigma_{sc} \cup \sigma_{pp}$.

The above standard scheme of spectral theory can be extended, to further decompose the singular-continuous subspace, by using Hausdorff measures. Recall that for any subset S of $\mathbf{R}$ and $\alpha \in [0, 1]$, the α-dimensional Hausdorff measure, h^α, is given by

$$h^\alpha(S) \;\equiv\; \lim_{\delta \to 0}\; \inf_{\delta-covers} \sum_{\nu=1}^{\infty} |b_\nu|^\alpha \,, \tag{3}$$

where a $\delta-cover$ is a cover of S by a countable collection of intervals, $S \subset \bigcup_{\nu=1}^{\infty} b_\nu$, such that for each ν the length of b_ν is at most δ. h^1 coincides with Lebesgue measure, and h^0 is the counting measure (assigning to each set the number of points in it). Given any $\emptyset \neq S \subseteq \mathbf{R}$, there exists a unique $\alpha(S) \in [0,\, 1]$ such that $h^\alpha(S) = 0$ for any $\alpha > \alpha(S)$, and $h^\alpha(S) = \infty$ for any $\alpha < \alpha(S)$. This unique $\alpha(S)$ is called the Hausdorff dimension of S. A rich theory of decomposing measures with respect to Hausdorff measures has been developed by Rogers and Taylor [1]. Below we discuss only a small part of it. A much more detailed description has been given by Last [2].

Given $\alpha \in [0,1]$, a measure μ is called α-continuous (αc) if $\mu(S) = 0$ for every set S with $h^\alpha(S) = 0$. It is called α-singular (αs) if it is supported on some set S with $h^\alpha(S) = 0$. We say that μ is one-dimensional (od) if it is α-continuous for every $\alpha < 1$. We say that it is zero-dimensional (zd) if it is α-singular for every $\alpha > 0$. μ is said to have exact dimension α if, for every $\epsilon > 0$, it is both $(\alpha - \epsilon)$-continuous and $(\alpha + \epsilon)$-singular.

Given a (positive, finite) measure μ and $\alpha \in [0,1]$, we define

$$D_\mu^\alpha(x) \equiv \limsup_{\epsilon \to 0} \frac{\mu((x-\epsilon,\, x+\epsilon))}{(2\epsilon)^\alpha} \tag{4}$$

and $T_\infty \equiv \{x \mid D_\mu^\alpha(x) = \infty\}$. The restriction $\mu(T_\infty \cap \cdot\,) \equiv \mu_{\alpha s}$ is α-singular, and $\mu((\mathbf{R} \setminus T_\infty) \cap \cdot\,) \equiv \mu_{\alpha c}$ is α-continuous. Thus, each measure decomposes uniquely into an α-continuous part and an α-singular part: $\mu = \mu_{\alpha c} + \mu_{\alpha s}$. Moreover, an α-singular measure must have $D_\mu^\alpha(x) = \infty$ a.e. (with respect to it) and an α-continuous measure must have $D_\mu^\alpha(x) < \infty$ a.e.. It is important to note that $D_\mu^\alpha(x)$ is defined with a $\limsup$. The corresponding limit need not exist.

We let $\mathcal{H}_{\alpha c} \equiv \{\psi \mid \mu_\psi \text{ is } \alpha\text{-continuous}\}$ and $\mathcal{H}_{\alpha s} \equiv \{\psi \mid \mu_\psi \text{ is } \alpha\text{-singular}\}$. $\mathcal{H}_{\alpha c}$ and $\mathcal{H}_{\alpha s}$ are mutually orthogonal closed subspaces which are invariant under H, and $\mathcal{H}$ decomposes as $\mathcal{H} = \mathcal{H}_{\alpha c} \oplus \mathcal{H}_{\alpha s}$. The α-continuous spectrum ($\sigma_{\alpha c}$) and α-singular spectrum ($\sigma_{\alpha s}$) are defined as the spectra of the restrictions of H to the corresponding subspaces, and $\sigma = \sigma_{\alpha c} \cup \sigma_{\alpha s}$. Note, in particular, that when we classify spectra as being α-singular, zero-dimensional, of exact dimension α etc., we always relate to the corresponding properties of the spectral measures.

The above scheme for spectral classification can be related to the dynamics of the underlying quantum systems. A detailed account of such relations has been given by Last [2].

It should be pointed out that certain fractal and multifractal studies of some operators with singular-continuous spectrum (including some of the examples we discussed above) have been carried out by several authors [19, 21]. While such studies are related to the above decomposition theory, the relations are generally far from trivial, and we believe that they are only partial. One should exercise extreme care when attempting to interpret the results of such studies within the framework of the scheme discussed above.

¿From here on we shall restrict our discussion to one-dimensional tight-binding Schrödinger operators of the form (1). While we discuss discrete operators, the subordinacy results we describe are equally valid for continuous Schrödinger operators of the form $-\frac{d^2}{dx^2} + V$.

Consider first "half-line" operators, defined with a phase boundary condition of the form (2). For such operators, it is well known that the spectral measures for lattice-point vectors δ_n, where $\delta_n(m) = \delta_{nm}$, are all mutually equivalent (namely, they have the same sets of zero measure). Thus, the spectral problem reduces to analyzing a single spectral measure, which we choose to be $\mu = \mu_{\delta_1}$. The Gilbert-Pearson theory of subordinacy [5] relates the pointwise behavior of the spectral measure μ at some energy E to the behavior of solutions of the corresponding Schrödinger equation

$$\psi(n+1) + \psi(n-1) + V(n)\psi(n) \;=\; E\psi(n)\;. \tag{5}$$

Given a solution of (5), we let $\|\psi\|_L$ denote the norm of the solution ψ over length L. It is useful to consider the length L as a continuous variable (allowed to take any positive real value), and so we define:

$$\|\psi\|_L \;=\; \left[\sum_{n=1}^{[L]} |\psi(n)|^2 \;+\; (L-[L])|\psi([L]+1)|^2\right]^{1/2}\;, \tag{6}$$

where $[L]$ denotes the integer part of L. A (non-trivial) solution ψ of (5) is called a subordinate solution if for any other solution φ of (5), which is not a constant multiple of ψ, $\lim_{L\to\infty}\frac{\|\psi\|_L}{\|\varphi\|_L} = 0$. Note that a subordinate solution must be unique (up to constant multiples). The Gilbert-Pearson theory relates the decomposition of the spectral measure μ to subordinacy of solutions as follows: The absolutely-continuous part of μ is supported on the set of energies for which (5) has no subordinate solutions. (In fact, this set of energies is, up to a set of both Lebesgue and spectral measure zero, the set where μ has a finite non-vanishing derivative.) The singular part of μ is supported on the set of energies for which the solutions which obey the appropriate boundary condition (2) are subordinate.

Let us now denote by ψ_1 the solution of (5) which obeys the boundary condition (2) and has normalization $|\psi_1(0)|^2 + |\psi_1(1)|^2 = 1$. Let us denote by ψ_2 the solution of (5) which obeys the orthogonal boundary condition to (2), namely, $\psi_2(0)\sin\theta - \psi_2(1)\cos\theta = 0$, and has normalization $|\psi_2(0)|^2 + |\psi_2(1)|^2 = 1$. Our main result is the following:

Theorem 1. *For any $\alpha \in (0,1)$ and every $E \in \mathbf{R}$, $D_\mu^\alpha(E) = \infty$ if and only if*

$$\liminf_{L\to\infty} \frac{\|\psi_1\|_L}{\|\psi_2\|_L^\beta} \;=\; 0\;,$$

where $\beta = \alpha/(2-\alpha)$.

Remark. Theorem 1 is proven with the same ideas used by Gilbert-Pearson, but it requires some optimization of their analysis. As a by-product, we also get a simplified proof of their original results. A key observation is to assign to each $\epsilon > 0$ a length $L(\epsilon)$ via the equality $\|\psi_1\|_{L(\epsilon)}\|\psi_2\|_{L(\epsilon)} = 1/(2\epsilon)$, for which we prove the explicit inequality

$$\frac{5-\sqrt{24}}{|m(E+i\epsilon)|} \;<\; \frac{\|\psi_1\|_{L(\epsilon)}}{\|\psi_2\|_{L(\epsilon)}} \;<\; \frac{5+\sqrt{24}}{|m(E+i\epsilon)|}\;,$$

where $m(z)$ is the Weyl-Titchmarsh function [22].

For spectral analysis, Theorem 1 can be combined with the existence of generalized eigenfunctions [22], from which one can show that for a.e. E with respect to the spectral measure μ, the solution ψ_1 must obey $\limsup_{L\to\infty} \frac{\|\psi_1\|_L}{L^{1/2}\ln L} < \infty$ and $\liminf_{L\to\infty} \frac{\|\psi_1\|_L}{L^{1/2}} < \infty$. Another useful fact is the constancy of the Wronskian $\psi_1(n+1)\psi_2(n) - \psi_2(n+1)\psi_1(n)$, which implies $\|\psi_1\|_L\|\psi_2\|_L \geq (L-1)/2$.

We now discuss briefly "line" operators. The spectral measures of a "line" operator can be constructed from those of corresponding two "half-line" operators (a left and a right), and while the relations are not completely trivial, they do allow an extension of the subordinacy theory to this case. Gilbert [6] has shown that the absolutely-continuous part of the spectral measures of a "line" operator is supported on the set of energies for which at least one of the "half-line" problems has no subordinate solution. The singular part is supported on the set of energies for which (5) has a solution which is subordinate both to the right and to the left. The probing of dimensional Hausdorff properties is somewhat more delicate in this case since it involves a $\liminf$ rather than a limit. Nevertheless, in concrete settings, such as the ones discussed in Theorems 3 and 4, the required control can be obtained.

In conclusion we would like to remark the following: The classification of spectra with respect to dimensional Hausdorff measures extends the usual spectral classification in a natural way, and provides a useful way of distinguishing between different kinds of singular-continuous spectra. The subordinacy theory extends to this point of view in a natural way, and allows to answer the relevant spectral questions whenever the nature of the solutions of the corresponding Schrödinger equation is sufficiently well understood. We note, in particular, that singular-continuous spectrum which occurs in "close neighborhood" to Anderson localization (as in the case of the strongly coupled almost Mathieu operator or the rank-one perturbed Anderson model) tends to be purely zero-dimensional; while the singular-continuous spectrum of the Fibonacci Hamiltonian, which has been identified as having "critical states" in physics literature [19], is α-continuous for some positive α.

As we were completing this paper we learned of a preprint by Remling [23] which obtains a restricted version of Theorem 1.

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References

[1] C.A. Rogers and S.J. Taylor, Acta Math. Stock. **101**, 273–302 (1959); Acta Math. Stock. **109**, 207–240 (1963)

[2] Y. Last, *Quantum dynamics and decompositions of singular continuous spectra*, J. Funct. Anal., to appear

[3] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, Phys. Rev. Lett. **75**, 117–119 (1995); *Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization*, preprint

[4] A. Gordon, Commun. Math. Phys. **164**, 489–505 (1994); R. del Rio, N. Makarov, and B. Simon, Commun. Math. Phys. **165**, 59–67 (1994)

[5] D.J. Gilbert and D.B. Pearson, J. Math. Anal. Appl. **128**, 30–56 (1987); also see S. Kahn and D.B. Pearson, Helv. Phys. Acta **65**, 505–527 (1992)

[6] D.J. Gilbert, Proc. Roy. Soc. Edinburgh **112A**, 213–229 (1989)

[7] S. Jitomirskaya and Y. Last, in preparation

[8] B. Simon and T. Spencer, Commun. Math. Phys. **125**, 113–126 (1989)

[9] A. Gordon, Math. Notes **48**, 1197–1203 (1990); also see W. Kirsch, S. Molchanov, and L. Pastur, Funct. Anal. Appl. **24**, 176–186 (1990)

[10] B. Simon and G. Stolz, *Operators with singular continuous spectrum, V. Sparse potentials*, Proc. Amer. Math. Soc., to appear

[11] B. Simon, *Operators with singular continuous spectrum, VII. Examples with borderline time decay*, Commun. Math. Phys., to appear

[12] S. Aubry and G. Andre, Ann. Isr. Phys. Soc. **3**, 133–140 (1980)

[13] Y. Last, Commun. Math. Phys. **151**, 183–192 (1993); F. Gesztesy and B. Simon, *The Xi function*, Acta Math., to appear

[14] F. Delyon, J. Phys. **A 20**, L21–L23 (1987)

[15] For a.e. θ, this is essentially due to Aubry and Andre [12]. For every θ, it follows from a recent result of Y. Last and B. Simon (in preparation).

[16] J. Avron and B. Simon, Bull. AMS **6**, 81–85 (1982); S. Jitomirskaya, Commun. Math. Phys. **165**, 49–57 (1994); Commun. Math. Phys. **168**, 563–570 (1995); S. Jitomirskaya and B. Simon, Commun. Math. Phys. **165**, 201–205 (1994)

[17] D.J. Thouless, Phys. Rev. **B 28**, 4272–4276 (1983); J. Avron, P. van Mouche, and B. Simon, Commun. Math. Phys. **132**, 103–118 (1990); Y. Last, Commun. Math. Phys. **164**, 421–432 (1994)

[18] A. Sütő, Commun. Math. Phys. **111**, 409–415 (1987); J. Stat. Phys. **56**, 525–531 (1989); J. Bellisard, B. Iochum, E. Scoppola, and D. Testard, Commun. Math. Phys. **125**, 527–543 (1989)

[19] H. Hiramoto and M. Kohmoto, Int. J. Mod. Phys. B **6**, 281–320 (1992) and references therein.

[20] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I. Functional Analysis.* London, San Diego: Academic Press 1980

[21] T. Geisel, R. Ketzmerick, and G. Petschel, Phys. Rev. Lett. **66**, 1651–1654 (1991); R. Ketzmerick, G. Petschel, and T. Geisel, Phys. Rev. Lett. **69**, 695–698 (1992); M. Wilkinson and E. Austin, Phys. Rev. **B 50**, 1420–1429 (1994); F. Piechon, M. Benakli, and A. Jagannathan, Phys. Rev. Lett. **74**, 5248–5251 (1995)

[22] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators.* Boston: Birkhäuser 1990

[23] C. Remling, *Relationships between the m-function and subordinate solutions of second order differential operators*, preprint