# Large deviation rule for Anosov flows

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ABSTRACT. *The volume contraction in dissipative reversible transitive Anosov flows obeys a large deviation rule (fluctuation theorem).*

## 1. Introduction and formalism

In this paper the results in [G3] are extended to the case of Anosov flows. The interest and the physical motivation are explained in [CG1] and [G2]: we briefly review them.

For dissipative systems the existence and properties of a non equilibrium stationary state are not known in general. Nevertheless there are systems in which such a state exists and has been extensively studied: Anosov systems and, more generally, Axiom $A$ systems.

The content of the *chaotic hypothesis* proposed in [CG1] generalizing the Ruelle's principle for turbulence, [R5], is that, as far as only macroscopical quantities have to be computed, a many particle system in a stationary state out of equilibrium can be regarded as if it was an Axiom $A$ system.

In [CG1] a theorem of large deviations is heuristically proven for dissipative reversible transitive Anosov systems, by using the properties of the stationary state (SRB measure), and it is shown to agree with the results of the numerical experiments in [ECM]. A rigorous proof for Anosov diffeomorphisms is performed in [G3]. As the physical systems which one wants to study through mathematical models evolve in a continuous way, it can be interesting to check if the theorem still holds if one consider Anosov flows instead of diffeomorphisms. This program is achieved in the present paper: it will be shown that the study of Anosov flows can be reduced to the study of Anosov diffeomorphisms (more rigorously of maps which have all the "good" properties of Anosov diffeomorphisms, in a sense which will be explained below, after Proposition 1.9), so that the large deviation rule for dissipative reversible transitive Anosov diffeomorphisms is extended to the case of flows.

If the systems is Axiom $A$ but not Anosov, something can still be said: see comments after Theorem 3.6.

In this section we simply review the basic notions and results on Axiom $A$ flows, Markov partitions and symbolic dynamics, essentially taken from [B2] and [BR], and in §2 we introduce the SRB measures for Axiom $A$ flows. Even if in the end we will confine ourselves on Anosov flows, it can be worthwile to start with more general definitions (also in view of possible future extensions to Axiom $A$ flows of the results holding for Anosov flows), as the discussion in this introductory part can be carried out with no relevant change both for Axiom $A$ and Anosov flows.

In §3 we consider dissipative reversible Axiom $A$ flows, study conditions under which they reduce to Anosov flows, and state the fundamental result of the paper (a large deviation rule for the volume contraction in dissipative reversible transitive Anosov flows), which will be proven in §4.

Let $M$ be a differentiable ($C^\infty$) compact Riemannian manifold and $f^t\colon M \to M$ a differentiable flow.

**1.1.** Definition. *A closed $f^t$-invariant set $X \subset M$ containing no fixed points is hyperbolic if the tangent bundle restricted to $X$ can be written as the Whitney sum*[1] *of three $Tf^t$-invariant continuous subbundles*

$$T_X M = E + E^s + E^u \ , \tag{1.1}$$

*where $E$ is the one-dimensional bundle tangent to the flow, and*

$$\begin{array}{lll} (a) & \|Tf^t w\| \le c\, e^{-\lambda t}\|w\| \ , & \text{for } w \in E^s \ , \quad t \ge 0 \ , \\ (b) & \|Tf^{-t} w\| \le c\, e^{-\lambda t}\|w\| \ , & \text{for } w \in E^u \ , \quad t \ge 0 \ , \end{array} \tag{1.2}$$

*for some positive constants $c, \lambda$; $\|\cdot\|$ denotes the norm induced by the Riemann metric.*

More generally, if $Y$ is the union of a hyperbolic set as above and a finite number of hyperbolic fixed points, we also say that $Y$ is hyperbolic, [ER], §F.2.

**1.2.** Definition. *A closed $f^t$-invariant set $\Lambda$ is a basic hyperbolic set if*
*(1) $\Lambda$ contains no fixed points and is hyperbolic;*
*(2) the periodic orbits of $f^t|\Lambda$ are dense in $\Lambda$;*
*(3) $f^t|\Lambda$ is topologically transitive;*[2]
*(4) there is an open set $U \supset \Lambda$ with $\Lambda = \cap_{t\in\mathbb{R}} f^t U$.*

Definition 1.2 is taken from [BR], §1. Usually one defines a basic hyperbolic set as a set which either satisfies Definition 1.2 or is a hyperbolic fixed point. In the following we will be interested in basic hyperbolic sets which are not a single point: this motivates Definition 1.2.

**1.3.** Definition. *A basic hyperbolic set $\Lambda$ for which the set $U$ in item (4) can be chosen satisfying $f^t U \subset U$ for all $t \ge t_0$, for fixed $t_0$, is defined to be an attractor.*

A point $x \in M$ is called *nonwandering* if, for every neighbourhood $V$ of $x$ and every $t_0 \in \mathbb{R}$, there is a $t > t_0$ such that

$$f^t V \cap V \ne \emptyset \ .$$

The *nonwandering set* is defined as

$$\Omega = \{x \in M : x \text{ is nonwandering}\} \ .$$

**1.4.** Definition. *A flow $f^t\colon M \to M$ is said to satisfy Axiom $A$ if the nonwandering set $\Omega$ is the disjoint union of a set satisfying (1) and (2) of Definition 1.2 and a finite number of hyperbolic fixed points.*

Smale's *spectral decomposition theorem* ([Sm], Theorem 5.2; see also [PS], Theorem 2.1) states that, if the flow $f^t\colon M \to M$ satisfies Axiom $A$, and if we denote by $\mathcal{F}$ the set of

[1] That is for each $x \in X$, the decomposition (1.1) becomes $T_x M = E_x \oplus E^s_x \oplus E^u_x$.

[2] A flow $f^t : \Lambda \to \Lambda$ is topologically transitive if, for all $U$, $V \subset \Lambda$ open nonempty, $U \cap f^t V \ne \emptyset$ for some $t > 0$.

hyperbolic fixed points in $\Omega$, then $\Omega \setminus \mathcal{F}$ is the disjoint union of a finite number of basic hyperbolic sets.

**1.5.** DEFINITION. *A flow $f^t \colon M \to M$ is an Anosov flow if $M$ is hyperbolic.*

An Anosov flow satisfies Axiom A (Anosov's *closing lemma*, [A]; see also [B3], §3.8). By Smale's spectral decomposition theorem, given an Anosov flow $f^t \colon M \to M$, then one can decompose $\Omega = \cup_{j=1}^{N} \Lambda_j$, where $\Lambda_1, \ldots, \Lambda_N$ are basic hyperbolic sets, and one can consider the restriction $f^t|\Lambda_j$, $\forall j = 1, \ldots, N$, which is topologically transitive. If one has $\Omega = M$, then each $f^t|\Lambda_j$ is a *transitive Anosov flow*.[3]

Standard examples of Axiom $A$ flows are the suspension of an Axiom $A$ diffeomorphism, *e.g.* the solenoid, [Sm], and the geodesic flow on a compact manifold with negative curvature, [A], which is an Anosov flow, (see also [B4]).

Let $\Lambda$ be a basic hyperbolic set. For any $x \in \Lambda$, the stable and unstable manifolds are defined as

$$\begin{aligned} W_x^s &= \{ y \in M \ : \ \lim_{t \to \infty} d(f^t x, f^t y) = 0 \} \ , \\ W_x^u &= \{ y \in M \ : \ \lim_{t \to \infty} d(f^{-t} x, f^{-t} y) = 0 \} \ , \end{aligned} \tag{1.3}$$

where $d$ is the distance induced by the Riemann metric, and

$$W_\Lambda^s = \bigcup_{x \in \Lambda} W_x^s \ , \quad W_\Lambda^u = \bigcup_{x \in \Lambda} W_x^u \ . \tag{1.4}$$

If $\Lambda$ is an attractor, $W_\Lambda^s$ is its *basin*: $\lim_{t \to \infty} d(f^t x, \Lambda) = 0 \ \forall x \in W_\Lambda^s$.

For $x \in \Lambda$, we set

$$\begin{aligned} W_{x,\varepsilon}^s &= \{ y \in W_x^s \ : \ d(f^t x, f^t y) \le \varepsilon \ \forall t \ge 0 \} \ , \\ W_{x,\varepsilon}^u &= \{ y \in W_x^u \ : \ d(f^{-t} x, f^{-t} y) \le \varepsilon \ \forall t \ge 0 \} \ . \end{aligned}$$

Let $D$ be a differentiable closed disk (*i.e.* a closed element of a $C^\infty$ manifold of dimension $\dim(M) - 1$, if $\dim(M)$ is the dimension of $M$), containing a point $x \in \Lambda$ and trasverse to the flow. For any closed subset $T \subset D$ containing $x$ and for any $y \in T$ such that $d(x, y) \le \alpha_1$ for a suitable $\alpha_1$, let us define $\langle x, y \rangle$ as the intersection $W_{f^v x, \varepsilon}^s \cap W_{y,\varepsilon}^u$, for a suitable $v$, (by choosing $\alpha_1$ small enough, one can always take $|v| \le \varepsilon$): such an intersection is well defined (*i.e.* it is a single point) and lies in $\Lambda$, [Sm]. Then let us introduce the *canonical coordinate* $\langle x, y \rangle_D$ as the *projection* of $\langle x, y \rangle$ on $D$, [B2], §1: this means that there exists a constant $\xi \ge 0$ such that, for $|r| \le \xi$, $f^r \langle x, y \rangle_D = \langle x, y \rangle$. The subset $T$ is called a *rectangle* if $\langle x, y \rangle_D \in T$ for any $x, y \in T$, and in this case we can define $\langle x, y \rangle_T \equiv \langle x, y \rangle_D$.

Then, for $x \in T$, we set $W_x^s(T) = \{ \langle x, y \rangle \ : \ y \in T \}$ and $W_x^u(T) = \{ \langle y, x \rangle \ : \ y \in T \}$: we can say that $W_x^s(T)$ and $W_x^u(T)$ are the projection of the stable manifold and, respectively, of the unstable manifold of $x$ on the rectangle $T$ containing $x$ (the projection is meant along the flow), and $T$ becomes the direct product of $W_x^s(T)$ and $W_x^u(T)$.

**1.6.** DEFINITION. *Choose a basic hyperbolic set $\Lambda$. We call $\mathcal{T} = \{T_1, \ldots, T_{\mathcal{N}}\}$ a proper family of rectangles, if there are positive constants $\alpha$ and $\alpha_1$ such that*
*(1) $T_j \subset \Lambda$ is a closed rectangle;*
*(2) if $\Gamma(\mathcal{T}) = \bigcup_{j=1}^{\mathcal{N}} T_j$, then $\Lambda \subset \bigcup_{0 \le t \le \alpha} f^{-t} \Gamma(\mathcal{T})$;*
*(3) $T_j \subset \operatorname{int} D_j$, where $D_j$ is a $C^\infty$ closed disk transverse to the flow, such that: (3.1)* $\operatorname{diam}(D_j) \le \alpha_1 \ \forall j$, *(3.2)* $\dim(D_j) = \dim(M) - 1 \ \forall j$, *(3.3)* $T_j = \overline{\operatorname{int} T_j} \ \forall j = 1, \ldots, \mathcal{N}$,

[3] Note that there are Anosov flows for which $\Omega \ne M$, [FW]: such flows are obviously non transitive. In the case of maps, the identity $\Omega = M$ is conjectured to hold for all Anosov diffeomorphisms, [Sm], Problem 3.4.

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FIG.1. *Canonical coordinates.*

*(where* $\operatorname{int} T_j$ *is the interior of* $T_j$ *as a subset of* $\Lambda \cap D_j$*), and (3.4) for* $i \neq j$*, at least one of the sets* $D_i \cap \bigcup_{o \le t \le \alpha} f^t D_j$ *and* $D_j \cap \bigcup_{o \le t \le \alpha} f^t D_i$ *is empty (in particular* $D_i \cap D_j = \emptyset$*).*

The above definition is taken from [B2], Definition 2.1. [In [B2], $\alpha_1 = \alpha$ and $\mathcal{T}$ is called a proper family of rectangles "of size $\alpha$"; in general it can be useful to consider $\alpha$ and $\alpha_1$ as independent parameters, so that one can change one of them without affecting the other one.]

Note that, if the flow $f^t \colon M \to M$ is a topologically mixing Anosov flow,[4] then $E^s$ and $E^u$ are not jointly integrable (*i.e.* $E^s \oplus E^u$ is not integrable, [Pl], Proposition 1.6), [Pl], Lemma 1.4, Lemma 1.5, Theorem 1.8. This means that, if $\varepsilon'$ is so chosen that for any $x \in \Lambda$, $y \in W^u_{x,\varepsilon}$ and $\xi \in W^s_{x,\varepsilon'}$ one has $W^u_{\xi,\varepsilon} \cap \bigcup_{-\varepsilon \le t \le \varepsilon} f^t W^s_{y,\varepsilon} \neq \emptyset$, then $W^u_{\xi,\varepsilon} \cap W^s_{y,\varepsilon} = \emptyset$. Therefore, in such a case, the disks $D_j$'s can *not* be constructed so that the conditions $W^s_{x,\varepsilon}(T) = W^s_{x,\varepsilon} \cap T$ and $W^u_{x,\varepsilon}(T) = W^u_{x,\varepsilon} \cap T$ are simultaneously possible.

Given $x \in \Gamma(\mathcal{T})$, let $t'(x)$ be the first positive time required for $f^t x$ to cross $\Gamma(\mathcal{T})$. If $x \in T_i$, for some $i = 1, \ldots, \mathcal{N}$, then $f^{t'(x)} x \in T_j$, for some $j \neq i$; set $t(x) = t'(x)$ for $x \in \operatorname{int} T_i$ and extend it by continuity to the boundaries of $T_i$. Then define $\mathcal{H}_\mathcal{T} x = f^{t(x)} x$, for any $x \in \Gamma(\mathcal{T})$, [B2], §2: $t(x)$ is called *ceiling function* and $\mathcal{H}_\mathcal{T}$ *Poincaré map.* There exists a $t_0 \in (0, \alpha)$ such that $t(x) > t_0$ $\forall x \in \Gamma(\mathcal{T})$.

The function $\mathcal{H}_\mathcal{T}$ is continuous on

$$\Gamma'(\mathcal{T}) = \{x \in \Gamma(\mathcal{T}) \ : \ \mathcal{H}^k_\mathcal{T} x \in \bigcup_{j=1}^{\mathcal{N}} \operatorname{int} T_j \ \forall k \in \mathbb{Z}\} \ , \tag{1.5}$$

and $\Gamma'(\mathcal{T})$ is dense in $\Gamma(\mathcal{T})$, being a countable intersection of dense open subsets (*Baire's theorem*, [Bb], Ch. IX, §5.3).

**1.7.** DEFINITION. *A proper family of rectangles* $\mathcal{T}$ *is called a* Markov partition *(or* Markov family*, or* Markov pavement*), if*
*(1) for* $x \in T_i$, $\mathcal{H}_\mathcal{T} x \in T_j$*, one has* $\mathcal{H}_\mathcal{T} y \in T_j$ $\forall y \in W^s_x(T_i)$*;*
*(2) for* $x \in T_i$, $\mathcal{H}^{-1}_\mathcal{T} x \in T_j$*, one has* $\mathcal{H}^{-1}_\mathcal{T} y \in T_j$ $\forall y \in W^u_x(T_i)$.

The above definition is taken from [B2], Definition 2.3.

Define

$$\partial^s \mathcal{T} = \bigcup_{j=1}^{\mathcal{N}} \partial^s T_j \ , \quad \partial^u \mathcal{T} = \bigcup_{j=1}^{\mathcal{N}} \partial^u T_j \ , \quad \partial \mathcal{T} = \partial^s \mathcal{T} \cup \partial^u \mathcal{T} \ ,$$
$$\Delta^s \mathcal{T} = \bigcup_{0 \le t \le \alpha} f^t \partial^s \mathcal{T} \ , \quad \Delta^u \mathcal{T} = \bigcup_{0 \le t \le \alpha} f^{-t} \partial^u \mathcal{T} \ ;$$

[4] A transitive Anosov flow is said to be topologically mixing if the stable and unstable manifolds $W^s_x$ and $W^u_x$ are dense in $M$ for some (and then for each) $x \in M$.

where

$$\partial^s T = \{\langle y_1, y_2\rangle_T \; : \; y_1 \in \partial W^u_{x,\varepsilon}(T) \text{ and } y_2 \in W^s_{x,\varepsilon}(T)\} \; ,$$
$$\partial^u T = \{\langle y_1, y_2\rangle_T \; : \; y_1 \in W^u_{x,\varepsilon}(T) \text{ and } y_2 \in \partial W^s_{x,\varepsilon}(T)\} \; ,$$

if $\partial W^u_{x,\varepsilon}(T)$ and $\partial W^s_{x,\varepsilon}(T)$ denote the boundaries of $W^u_{x,\varepsilon}(T)$ and $W^s_{x,\varepsilon}(T)$ as subsets, respectively, of $W^u_x(T) \cap \Lambda$ and $W^s_x(T) \cap \Lambda$.

**1.8.** PROPOSITION. *If $\mathcal{T}$ is a Markov partition, one has $f^t \Delta^s \mathcal{T} \subset \Delta^s \mathcal{T}$ and $f^{-t} \Delta^u \mathcal{T} \subset \Delta^u \mathcal{T}$* $\forall t \geq 0$.

The proof is in [B2], Proposition 2.6.

Then the following fundamental result is proven in [B2], Theorem 2.5.

**1.9.** PROPOSITION. *Any basic hyperbolic set $\Lambda$ admits a proper family of rectangles which is a Markov partition.*

By construction, the discontinuity set of $\mathcal{H}_\mathcal{T}$, *i.e.* $\Gamma(\mathcal{T}) \setminus \Gamma'(\mathcal{T})$, is covered by the evolution of some stable and unstable manifolds, so that $\mathcal{H}_\mathcal{T}$: $\Gamma'(\mathcal{T}) \to \Gamma'(\mathcal{T})$ can be studied as it was an Axiom $A$ diffeomorphism, (see [B3] for a review).

**1.10.** *Symbolic dynamics.* Let us introduce a $\mathcal{N} \times \mathcal{N}$ matrix $A$ such that

$$A_{ij} = \begin{cases} 1 & \text{if there exists } x \in \operatorname{int} T_i \text{ such that } \mathcal{H}_\mathcal{T} x \in \operatorname{int} T_j, \\ 0 & \text{otherwise,} \end{cases}$$

(*transition matrix*), and let us define the *space of the compatible strings*

$$\mathcal{M} = \{\mathbf{m} \equiv \{m_i\}_{i \in \mathbb{Z}} \; : \; m_i \in \{1, \ldots, \mathcal{N}\} \, , \; A_{m_i m_{i+1}} = 1 \; \forall i \in \mathbb{Z}\}, \tag{1.6}$$

and the map $\sigma \colon \mathcal{M} \to \mathcal{M}$ by $\sigma \mathbf{m} = \{m'_i\}_{i \in \mathbb{Z}}$, where $m'_i = m_{i+1}$. If $\{1, \ldots, \mathcal{N}\}$ is given the discrete topology and $\{1, \ldots, \mathcal{N}\}^{\mathbb{Z}}$ the product topology, $\mathcal{M}$ becomes a compact metrizable space and $\sigma$ a topologically transitive homeomorphism (*subshift of finite type*); furthermore, because of the transitivity of $f^t$, $\sigma$ can be supposed to be topologically mixing,[5] [BR], Lemma 2.1. A metric on $\mathcal{M}$ can be $d(\mathbf{m}, \mathbf{n}) = d_1 e^{-d_2 N}$, with $d_1, d_2 > 0$, if $m_i = n_i$, $\forall |i| \leq N$, [B2], §1.

For $\psi \colon \mathcal{M} \to \mathbb{R}$ a positive continuous function, *i.e.* $\psi \in C(\mathcal{M})$, and

$$Y = \{(\mathbf{m}, s) \; : \; s \in [0, \psi(\mathbf{m})], \; \mathbf{m} \in \mathcal{M}\} \; , \tag{1.7}$$

identify the points $(\mathbf{m}, \psi(\mathbf{m}))$ and $(\sigma \mathbf{m}, 0)$ for all $\mathbf{m} \in \mathcal{M}$, so obtaining a new compact metric space $\Lambda(A, \psi)$, [BW]. If $q \, : \; Y \to \Lambda(A, \psi)$ is the quotient map, then the *suspension flow* (or *special flow*) $g^t : \Lambda(A, \psi) \to \Lambda(A, \psi)$ is defined as

$$g^t q(\mathbf{m}, s) = q(\sigma^k \mathbf{m}, v) \; ,$$

where $k$ is chosen so that

$$v = t + s - \sum_{j=0}^{k-1} \psi(\sigma^j \mathbf{m}) \; \in [0, \psi(\sigma^k \mathbf{m})] \; .$$

For $\psi \in C(\mathcal{M})$, let

$$\operatorname{var}_N \psi = \sup\{|\psi(\mathbf{m}) - \psi(\mathbf{n})| \; : \; \mathbf{m}, \mathbf{n} \in \mathcal{M}, \; m_i = n_i \; \forall |i| \leq N\}$$

[5] A homeomorphism $f \colon X \to X$ is topologically mixing if, for all $U$, $V \subset X$ open nonempty, $U \cap f^n V \neq \emptyset$ for all sufficiently large $n$.

**f2.ps**

FIG.2. *Suspension flow.*

and let

$$\mathcal{F} = \{\psi \in C(\mathcal{M}) \;:\; \exists\, c_1, c_2 > 0 \text{ so that } \mathrm{var}_N \psi \le c_1\, e^{-c_2 N}\ \forall N \ge 0\}. \tag{1.8}$$

A flow $g^t$ with $\psi \in \mathcal{F}$ is called a *hyperbolic symbolic flow*, [B2], Definition 1.3, and its class is the same as the class of all one-dimensional basic sets for flows, [B1].

**1.11.** PROPOSITION. *If $\Lambda$ is a basic hyperbolic set, there exists a positive $\psi \in \mathcal{F}$ and a continuous surjection $\rho$: $\Lambda(A,\psi) \to \Lambda$ (symbolic code) such that $\rho \circ g^t = f^t \circ \rho$.*

The proof is in [B1], §2. If $N = \bigcup_{t=-\infty}^{\infty} f^t \partial\mathcal{T}$, and we set $\Lambda_0 = \Lambda \setminus N$ and $\mathcal{M}_0 = \Lambda(A,\psi) \setminus \rho^{-1}(N)$, then $\rho$ is a continuous bijection between $\Lambda_0$ and $\mathcal{M}_0$. Note that, if $x \in \Gamma(\mathcal{T})$, then $x = \rho(\mathbf{m}, 0)$, $\mathbf{m} \in \mathcal{M}$, and one has $t(x) = \psi(\mathbf{m})$.

## 2. Equilibrium states and SRB measures

Given a homeomorphism $f$, $M(f)$ denotes the set of $f$-invariant Borel probability measures; if $F = \{f^t\}_{t\in\mathbb{R}}$, then $M(F) = \bigcap_{t\in\mathbb{R}} M(f^t)$. If $g^t$ is the suspension flow, we set $G = \{g^t\}$.

Let $\Lambda$ be a basic hyperbolic set. For $x \in \Lambda$, if $E^s_x$ and $E^u_x$ denote, respectively, the subbundles tangent to $W^s_x$ and $W^u_x$ in $x$, let $\lambda_{0,t}(x)$, $\lambda_{u,t}(x)$ and $\lambda_{s,t}(x)$ be the jacobians of the linear maps, respectively, $Df^t : E_x \to E_{f^t x}$, $Df^t : E^u_x \to E^u_{f^t x}$ and $Df^t : E^s_x \to E^s_{f^t x}$, and $\lambda_t(x) = \lambda_{0,t}(x)\, \lambda_{s,t}(x)\, \lambda_{u,t}(x)\, \chi^1_t(x)\, \chi^2_t(x)$, where $\chi^1_t(x) = \sin[\psi^1(f^t x)]/\sin[\psi^1(x)]$ and $\chi^2_t(x) = \sin[\psi^2(f^t x)]/\sin[\psi^2(x)]$, being $\psi^1(x)$ the angle between $E^s_x$ and $E^u_x$ in $x$, and $\psi^2(x)$ the angle between $E^s_x \oplus E^u_x$ and the flow direction in $x$.

Note that $\lambda_{u,t+t'}(x) = \lambda_{u,t}(f^{t'} x)\, \lambda_{u,t'}(x)$ and analogous relations hold for $\lambda_{s,t}$, $\lambda_{0,t}$, $\chi^1_t$ and $\chi^2_t$: so that all such quantities are *cocycles*, in the sense of [R4], Definition B2.

By the transversality properties and the absence of fixed points of the basic hyperbolic sets (see Definitions 1.1 and 1.2), there exists a positive constant $B_1$ such that $B_1^{-1} < \chi^1_t(x), \chi^2_t(x), \lambda_{0,t}(x) < B_1$, for each $t \in \mathbb{R}$, $x \in \Lambda$.

Then we define

$$\begin{aligned} \varphi^{(u)}(x) &= -\frac{d}{dt} \ln \lambda_{u,t}(x)\Big|_{t=0} , \\ \varphi^{(s)}(x) &= \frac{d}{dt} \ln \lambda_{s,t}(x)\Big|_{t=0} , \end{aligned} \tag{2.1}$$

so that, for $x = \rho(\mathbf{m}, 0)$ and $\tau_k(x) = \sum_{j=0}^{k-1} t(\rho(\sigma^j \mathbf{m}, 0)) \equiv \sum_{j=0}^{k-1} \psi(\sigma^j \mathbf{m})$, if

$$\begin{aligned} &J_{u,k}(x) = \lambda_{u,\tau_k(x)}(x) , \quad J_{s,k}(x) = \lambda_{s,\tau_k(x)}(x) , \quad J_k(x)(x) = \lambda_{\tau_k(x)}(x) , \\ &J_u(x) \equiv J_{u,1}(x) , \quad J_s(x) \equiv J_{s,1}(x) , \quad J(x) \equiv J_1(x) , \end{aligned}$$

we obtain

$$\ln J_{u,k}(x) = -\int_0^{\tau_k(x)} dt\, \varphi^{(u)}(f^t x) = \sum_{j=0}^{k-1} \ln J_u(\mathcal{H}_{\mathcal{T}}^j x) \;, \quad \ln J_u(x) = \int_0^{\tau_1(x)} dt\, \varphi^{(u)}(f^t x) \;,$$

$$\ln J_{s,k}(x) = -\int_0^{\tau_k(x)} dt\, \varphi^{(s)}(f^t x) = \sum_{j=0}^{k-1} \ln J_s(\mathcal{H}_{\mathcal{T}}^j x) \;, \quad \ln J_s(x) = \int_0^{\tau_1(x)} dt\, \varphi^{(s)}(f^t x) \;,$$

and we can define for even $k$

$$\begin{aligned} \ln \mathcal{J}_{u,k}^{-1} &\equiv \int_{\tau_{-k/2}(x)}^{\tau_{k/2}(x)} dt\, \varphi^{(u)}(f^t x) = \sum_{j=-k/2}^{k/2-1} \ln J_u^{-1}(\mathcal{H}_{\mathcal{T}}^j x) \;, \\ \ln \mathcal{J}_{s,k} &\equiv -\int_{\tau_{-k/2}(x)}^{\tau_{k/2}(x)} dt\, \varphi^{(s)}(f^t x) = \sum_{j=-k/2}^{k/2-1} \ln J_s(\mathcal{H}_{\mathcal{T}}^j x) \;. \end{aligned} \tag{2.2}$$

For $\nu \in M(\sigma)$ and $m$ Lebesgue measure, $\mu_\nu \equiv [\nu \times m\,(Y)]^{-1}\, \nu \times m|Y$ gives a probability measure on $\Lambda(A,\psi)$, as $\nu \times m$ gives measure zero to the identifications on $Y \to \Lambda(A,\psi)$, so that no ambiguity can arise. [$Y$ is defined in (1.7).]

**2.1.** PROPOSITION. *There exists a measure $\mu_+ \in M(G)$, which is the unique equilibrium state for $\varphi^{(u)}$ with respect to G, ergodic and positive on nonempty open sets, (forward SRB measure), and a measure $\mu_- \in M(G)$, which is the unique equilibrium state for $\varphi^{(s)}$ with respect to $G^{-1}$, ergodic and positive on nonempty open sets, (backward SRB measure). One has $\mu_\pm = \mu_{\nu_\pm}$, $\nu_\pm \in M(\sigma)$, where $\nu_+$ and $\nu_-$ are, respectively, the unique equilibrium states for $\Phi^{(u)}$ and $\Phi^{(s)}$, if*

$$\Phi^{(u)}(\mathbf{m}) = \int_0^{\psi(\mathbf{m})} dt\, \varphi^{(u)}(\rho(\mathbf{m},t)) \;, \qquad \Phi^{(s)}(\mathbf{m}) = \int_0^{\psi(\mathbf{m})} dt\, \varphi^{(s)}(\rho(\mathbf{m},t)) \;,$$

*with respect to $\sigma$.*

The proof for $\mu_+$ is in [BR], Proposition 3.1, (where the definition of equilibrium state can also be found), and for $\mu_-$ can be carried out in the same way, by studying $f^{-t}$ instead of $f^t$ and taking into account the fact that the unstable manifolds for $f^t$ becomes the stable manifolds for the opposite flow $f^{-t}$ and *viceversa.*

For transitive Anosov flows, we have the following result, (note that if $f^t\colon M \to M$ is a transitive Anosov flow, then $M = \Lambda$, see comments after Definition 1.5).

**2.2.** PROPOSITION. *Let $f^t\colon M \to M$ be a transitive Anosov flow . The volume measure $\mu_0$ on $\Lambda$ admits the representation $\mu_0 = \mu_{\nu_0}$, with $\nu_0$ formally proportional to* $\exp[-H(\mathbf{m})]$, *where the formal Hamiltonian $H(\mathbf{m})$ is given by*

$$H(\mathbf{m}) = \sum_{j=-\infty}^{-1} h_-(\sigma^j \mathbf{m}) + h_0(\mathbf{m}) + \sum_{j=0}^{\infty} h_+(\sigma^k \mathbf{m}) \;, \tag{2.3}$$

*with*

$$\begin{cases} h_-(\mathbf{m}) = -\ln J_s(\rho(\mathbf{m},0)) \;, \quad h_+(\mathbf{m}) = \ln J_u(\rho(\mathbf{m},0)) \;, \\ h_0(\mathbf{m}) = -\ln \chi(\rho(\mathbf{m},0)) \;, \end{cases}$$

*being $\chi(\rho(\mathbf{m},0))$ bounded between two constants, which we can take as $B_2^{-1} < B_2$. If $\nu_+$, $\nu_-$ are the Gibbs states with formal Hamiltonians*

$$H_+(\mathbf{m}) = \sum_{j=-\infty}^{\infty} h_+(\sigma^j \mathbf{m}) \;, \qquad H_-(\mathbf{m}) = \sum_{j=-\infty}^{\infty} h_-(\sigma^j \mathbf{m}) \;, \tag{2.4}$$

*then $\mu_{\nu_+}$, $\mu_{\nu_-}$ are, respectively, the forward and backward SRB measures $\mu_+$, $\mu_-$ on $\Lambda$.*

A statement similar to Proposition 2.2 holds for diffeomorphisms (see [G3]), and follows from the analysis in [Si1,Si2,Si3] and [G2,G3]. In Appendix A1, we show how to reduce the discussion of the flows to the case of diffeomorphisms, so that Proposition 2.2 follows.

**2.3.** *Remark.* Note that, unlike the SRB measures, the volume measure is not translation invariant (so that it is not really a Gibbs state, see [R3]): the non translation invariance is due not to any symmetry breaking phenomenon, but simply to the fact that the potential "to the right" is different from the potential "to the left".

If $g: M \to \mathbb{R}$ is smooth, the function $g(\rho(\mathbf{m},0))$, $\mathbf{m} \in \mathcal{M}$, can be represented in terms of suitable functions $\gamma_k(m_{-k},\ldots,m_k)$ as

$$g(\rho(\mathbf{m},0)) = \sum_{k=1}^{\infty} \gamma_k(m_{-k},\ldots,m_k) \ , \qquad |\gamma_k(m_{-k},\ldots,m_k)| \le \Gamma\, e^{-\lambda k} \ ,$$

where $\Gamma > 0, \lambda > 0$. In particular $h_\pm$ (and $h_0$) enjoy the above property (*short range*), by the properties of the Markov partition introduced in §1.

**2.4.** Proposition. *For any smooth function $g: M \to \mathbb{R}$, if $\Lambda$ is an attractor for the Axiom A flow $f^t: M \to M$, and $W^s_\Lambda$ is its basin, one has*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, g(f^t x) = \int_\Lambda \mu_+(dy)\, g(y)$$

*for $\mu_0$-almost all $x \in W^s_\Lambda$. Analogously, if $\Lambda'$ is an attractor for the opposite flow $f^{-t}: M \to M$, one has*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, g(f^{-t} x) = \int_{\Lambda'} \mu_-(dy)\, g(y)$$

*for $\mu_0$-almost all $x \in W^u_{\Lambda'}$, where $W^u_{\Lambda'}$ is the basin of $\Lambda'$: $\lim_{t\to\infty} d\,(f^{-t}x,\Lambda') = 0$ $\forall x \in W^u_{\Lambda'}$.*

The proof is in [BR], Theorem 5.1.

Consider transitive Anosov flows.

Let us construct the Markov partition $\mathcal{T}_L = \bigvee_{j=-L}^{L} \mathcal{H}_{\mathcal{T}}^{-j}\mathcal{T}$, (this means that, if $\mathbf{m}_{[-L,L]} \equiv (m_{-L},\ldots,m_L)$ and $T_{\mathbf{m}_{[-L,L]}} = \bigcap_{j=-L}^{L} \mathcal{H}_{\mathcal{T}}^{-j} T_{m_j}$, for $T_{m_j} \in \mathcal{T}$ $\forall j = -L,\ldots,L$, then $T_{\mathbf{m}_{[-L,L]}} \in \mathcal{T}_L$), and let $x_{\mathbf{m}^0_{[-L,L]}}$ be a suitable point in $T_{\mathbf{m}_{[-L,L]}}$, where $\mathbf{m}^0_{[-L,L]} \in \mathcal{M}$, (*i.e.* $\mathbf{m}^0_{[-L,L]}$ is a compatible string), with $(\mathbf{m}^0_{[-L,L]})_i = (\mathbf{m}_{[-L,L]})_i$, $\forall |i| \le L$; for instance we can choose the symbols corresponding to the sites $|j| \ge \pm(L+1)$ such that $A_{m_j m_{j+1}} = 1$, so that the dependence on $\mathbf{m}_{[-L,L]}$ is only via the symbols $m_{\pm L}$.

We can define

$$\int_\Lambda \mu_{L,k}(dx)\, g(x) = \frac{\sum_{\mathbf{m}_{[-L,L]}} \mathcal{J}^{-1}_{u,k}(x_{\mathbf{m}^0_{[-L,L]}}) \int_0^{\psi(\mathbf{m}^0_{[-L,L]})} dt\, g(f^t x_{\mathbf{m}^0_{[-L,L]}})}{\sum_{\mathbf{m}_{[-L,L]}} \mathcal{J}^{-1}_{u,k}(x_{\mathbf{m}^0_{[-L,L]}})\, \psi(\mathbf{m}^0_{[-L,L]})} \ , \tag{2.5}$$

which is called *approximating distribution for* $\mu_+$. In fact the following result holds.

**2.5.** Proposition (Approximation theorem). *Let $f^t: M \to M$ be a transitive Anosov flow. If $\mu_{L,k}$ is defined as in (2.5), then, for any smooth function $g: M \to \mathbb{R}$, one has*

$$\lim_{\substack{k\to\infty \\ L\ge k/2}} \int \mu_{L,k}(dx)\, g(x) = \int \mu_+(dx)\, g(x) \ ,$$

*where $\mu_+$ is the forward SRB measure.*

The measure $\mu_{L,k}$ can be written as $\mu_{L,k} \equiv \mu_{\nu_{L,k}}$, where $\nu_{L,k}$ is the approximating distribution for $\nu_+ \in M(\sigma)$ defined on $\mathcal{M}$. In the following we shall use the notation

$$\int \nu_+(d\mathbf{m})\,\overline{g}(\mathbf{m}) = \int_\Lambda \mu_+(dx)\,g(x)\;, \qquad \int \nu_{L,k}(d\mathbf{m})\,\overline{g}(\mathbf{m}) = \int_\Lambda \mu_{L,k}(dx)\,g(x)\;,$$

where

$$\overline{g}(\mathbf{m}) = \int_0^{\psi(\mathbf{m})} dt\, g(\rho(\mathbf{m},t))\;,$$

with $\psi(\mathbf{m}) \in (t_0, \alpha)\ \forall \mathbf{m} \in \mathcal{M}$. For any subset $A \subset \mathcal{M}$, we denote by $\nu_+(A)$ the $\nu_+$-measure of $A$: $\nu_+(A) = \int_A \nu_+(d\mathbf{m})$.

We conclude this section with a comment inherited from [CG2].

**2.6.** *Remark.* In Proposition 2.5, we could define the approximating distribution with $\mathcal{J}_{u,k}(x) \to \mathcal{J}_{u,k}(x)\,\delta_k(x)$, where $\delta_k(x) = \sin(\mathcal{H}_\mathcal{T}^{k/2}x)/\sin(\mathcal{H}_\mathcal{T}^{-k/2}x)$, which corresponds to considering a Gibbs state with a different boundary condition (with the difference becoming irrelevant in the limit as $k \to \infty$, because of the absence of phase transitions for one-dimensional Gibbs states with short range interactions, [R2,GL]). Note that the factors $\delta_k(x)$ are *cocycles*, according to [R4], Definition B2.

## 3. Reversible dissipative systems and results.

Let us consider flows $f^t\colon M \to M$ verifying the following conditions (A) and (B).

**3.1.** DEFINITION. *The flow $f^t\colon M \to M$ is (A) dissipative if*

$$\sigma_\pm = -\int_\Lambda \mu_\pm(dx)\,\ln J^{\pm 1}(x) > 0\;,$$

*and (B) reversible if there is an isometric involution $i : M \to M$, $i^2 = \mathbb{1}$, such that: $if^t = f^{-t}i$.*

If $f^t\colon M \to M$ is transitive and reversible, then, for any $x \in M$, the stable and the unstable manifolds have the same dimension, so that the the dimension of $M$ is odd.

By reversibility, one has $\sigma_+ = \sigma_-$, $J(x) = J^{-1}(ix)$, $iW_x^u = W_{ix}^s$, [G3], §2, and $\mathcal{J}_{u,k} = \mathcal{J}_{s,k}^{-1}(ix)$, [G3], §4. Moreover, if $\Lambda$ is an attractor for the flow $f^t\colon M \to M$, then $\Lambda' = i\Lambda$ is an attractor for the opposite flow $f^{-t}\colon M \to M$, so that $W_\Lambda^s = iW_{\Lambda'}^u$, with the notations in Proposition 2.4.

**3.2.** LEMMA. *Let $\Lambda$ be an attractor for the Axiom A flow $f^t\colon M \to M$. If*
*(a) the flow is transitive on $M$, or*
*(b) the flow is reversible and $i\Lambda = \Lambda$,*
*then $\Lambda$ is a connected component of $M$ and $f^t|\Lambda$ is an Anosov flow.*

**3.3.** *Proof of Lemma 3.2.* If $f^t\colon M \to M$ is transitive, $\Lambda$ is dense in $M$ (because of the $f^t$-invariance of $\Lambda$), and, as $\Lambda$ is closed, then $\Lambda = M$; this proves (a).[6]

If $\Lambda$ is an attractor, one has $m(W_\Lambda^s) > 0$, where $m$ is the measure on $M$ derived from the Riemann metric, [BR], Theorem 5.6. If one sets $\Lambda' = i\Lambda$, one has $iW_{\Lambda'}^u = W_\Lambda^s$, by reversibility. If $i\Lambda = \Lambda$, then $m(W_\Lambda^u) = m(W_\Lambda^s)$. But $\Lambda = W_\Lambda^u$, hence $m(\Lambda) > 0$, so that $\Lambda$ is

[6] It is not necessary to assume the existence of an attractor in order to deduce from transitivity that $f^t\colon M \to M$ is an Anosov flow: in fact transitivity implies trivially $\Omega = M$.

a connected component of $M$ and $f^t|\Lambda$ is an Anosov flow, [R5], [BR], Corollary 5.7. Then (b) follows. ■

**3.4.** REMARK. In hypothesis (b) of Lemma 3.2, one can assume $iW^s_\Lambda = W^s_\Lambda$ instead of $i\Lambda = \Lambda$: in fact $i\Lambda = \Lambda$ yields $\Lambda = W^s_\Lambda$, and, if $iW^s_\Lambda = W^s_\Lambda$, one has $i\Lambda \subset iW^s_\Lambda = W^s_\Lambda$, hence $i\Lambda = \Lambda$, (because $i\Omega = \Omega$).

Note that a stretched exponential bound on the correaltion functions is obtained, [Ch], for three-dimensional topologically mixing Anosov flows satisfying an extra assumption ("uniform nonintegrability" of the "foliations" $E^s$ and $E^u$, [Ch], §13, Assumption A5), while it is known that topologically mixing Axiom $A$ flows can have correlations function decaying arbitrarily slowly, [R6,Po].

**3.5.** DEFINITION. *We define the dimensionless volume contraction rate at $x \in \Gamma(\mathcal{T})$ and over a time $k$ as*

$$\varepsilon_k(x) = \frac{1}{\sigma_+ k} \sum_{j=-k/2}^{k/2-1} \ln J^{-1}(\mathcal{H}^j_{\mathcal{T}} x) = \frac{1}{\sigma_+ k} \ln \mathcal{J}^{-1}_k(x) \ ,$$

*where $\mathcal{J}^{-1}_k(x) \equiv \prod_{j=-k/2}^{k/2-1} J^{-1}(\mathcal{H}^j_{\mathcal{T}} x)$, and we set $\overline{\varepsilon}_k(\mathbf{m}) = \int_0^{\psi(\mathbf{m})} dt\, \varepsilon_k(\rho(\mathbf{m},t))$.*

Then the following result holds, which can be interpreted as a large deviation rule, (see [La,CG1]).

**3.6.** THEOREM (FLUCTUATION THEOREM). *Let $\Lambda$ be an attractor for the dissipative reversible transitive Anosov flow $f^t : M \to M$. There exists $p^* > 0$ such that the SRB distribution $\mu_+ = \mu_{\nu_+}$ on $\Lambda$ verifies*

$$p - \delta \le \lim_{k\to\infty} \frac{1}{\sigma_+ k} \ln \frac{\nu_+(\{\mathbf{m} :\ \overline{\varepsilon}_k(\mathbf{m}) \in [p-\delta, p+\delta]\})}{\nu_+(\{\mathbf{m} :\ \overline{\varepsilon}_k(\mathbf{m}) \in -[p-\delta, p+\delta]\})} \le p + \delta \ ,$$

*for all $p$ and $\delta$ such that $|p| + \delta < p^*$.*

If $f^t$: $M \to M$ is a dissipative reversible Axiom $A$ flow, and (1) the restriction of $f^t$ on the attractor $\Lambda$ is an Anosov flow, and (2) there exists an isometric involution $i^*$: $\Lambda \to \Lambda$, such that $i^* f^t|\Lambda = f^{-t} i^*|\Lambda$, then Theorem 3.6 still applies.

## 4. Proof of the fluctuation theorem

For $x \in \Gamma(\mathcal{T})$, the function $\varepsilon_k(x)$ can be regarded as a function on $\mathcal{M}$, by setting $\overline{\varepsilon}_k(\mathbf{m}) = \int_0^{\psi(\mathbf{m})} dt\, \varepsilon_k(\rho(\mathbf{m},t))$. Then the following two propositions hold.

**4.1.** PROPOSITION. *For a suitable $p^* > 0$ and for $p \in (-p^*, p^*)$, $|p| + \delta < p_*$, there exists the limit*

$$\lim_{k\to\infty} \frac{1}{k} \ln \nu_+(\{\mathbf{m} :\ \overline{\varepsilon}_k(\mathbf{m}) \in [p-\delta, p+\delta]\}) = \sup_{s\in[p-\delta,p+\delta]} \{ -\zeta(s) \} \ ,$$

*and $\zeta(s)$ is a real analytic strictly convex function on $(-p^*, p^*)$. The difference between the right and left hand sides tends to zero bounded by $D_1\, k^{-1}$, for some positive constant $D_1$.*

In Proposition 4.1, $p^*$ is defined as $p^* = \sup_{x\in\Lambda}\{\limsup_{k\to+\infty} \varepsilon_k(\mathcal{H}^{k/2}_T x)\}$.

**4.2.** PROPOSITION. *For $\beta \in (-\infty, \infty)$, define*

$$\lambda(\beta) = \sup_{s\in(-p^*,p^*)} \{\beta s - \zeta(s)\} \ ; \tag{4.1}$$

*then one has*

$$\lambda(\beta) = \lim_{k\to\infty} \frac{1}{k} \ln \int \nu_+(d\mathbf{m})\, e^{\beta k \overline{\varepsilon}_k(\mathbf{m})} ,$$

*and $\lambda(\beta)$ is a real analytic strictly convex function on $\mathbb{R}$, asymptotically linear to $\pm p^*$, for $\beta \to \pm\infty$. The difference between the right and left hand sides tends to zero bounded by $D_2\, k^{-1}$, for some positive constant $D_2$.*

The two statements are equivalent. In fact Proposition 4.1 yields Proposition 4.2 (and *viceversa*): the proof of such an assertion is standard, [R2], and is given in in Appendix A2. Therefore it is enough to prove one of the two results: the proof of Proposition 4.2 can be deduced from [CO], and it is reproduced in Appendix A3.

Hence it will be sufficient to prove the following result.

**4.3.** LEMMA. *If $I_{p,\delta} = [p-\delta, p+\delta]$, $|p|+\delta < p^*$, the distribution $\nu_+$ verifies the inequalities*

$$\frac{1}{\sigma_+ k} \ln \frac{\nu_+(\{\mathbf{m} :\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta\mp\eta(k)}\})}{\nu_+(\{\mathbf{m} :\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{-p,\delta\pm\eta(k)})\}} \begin{cases} < p+\delta+\eta'(k) \\ > p-\delta-\eta'(k) \end{cases}$$

*where $\eta(k), \eta'(k) > 0$ and $\eta(k), \eta'(k) \to 0$ for $k\to\infty$.*

For $q \in \mathbb{Z}$ and $n$ odd, set $X = \{q, q+1, \ldots, q+n-1\} \equiv [q, q+n-1]$ and define $\mathbf{m}_X = (m_q,\, m_{q+1},\, \ldots,\, m_{q+n-1})$ and $\overline{X} = q + (n-1)/2$ (the *center* of $X$). If $\mathbf{m} \in \mathcal{M}$, let $\mathbf{m}^0_X$ be an arbitrary configuration $\{(m^0_X)_i\}_{i\in\mathbb{Z}}$ such that $(m^0_X)_i = (m_X)_i$, $\forall i = q, \ldots, q+n-1$.

One can write

$$\frac{1}{\sigma_+} \ln J^{-1}(\rho(\mathbf{m}, 0)) = \sum_{\overline{X}=0} E_X(\mathbf{m}_X) , \qquad h_+(\mathbf{m}) = \sum_{\overline{X}=0} H_X(\mathbf{m}_X) , \tag{4.2}$$

where $E_X(\mathbf{m}_X)$ and $H_X(\mathbf{m}_X)$ are translation invariant and exponentially decaying functions, *i.e.* , if $\vartheta$ denotes translation to the right,

$$\begin{aligned} E_{\vartheta X}(\mathbf{m}_X) &\equiv E_X(\mathbf{m}_X) , \qquad & |E_X(\mathbf{m}_X)| &\le b_1^{(E)}\, e^{-b_2^{(E)} n} , \\ H_{\vartheta X}(\mathbf{m}_X) &\equiv H_X(\mathbf{m}_X) , \qquad & |H_X(\mathbf{m}_X)| &\le b_1^{(H)}\, e^{-b_2^{(H)} n} , \end{aligned}$$

for suitable positive constant $b_1^{(E)}$, $b_2^{(E)}$, $b_1^{(H)}$ and $b_2^{(H)}$.

Then $k\overline{\varepsilon}_k(\mathbf{m})$ can be written as

$$k\overline{\varepsilon}_k(\mathbf{m}) \ = \sum_{\overline{X}\in[-k/2, k/2-1]} E_X(\mathbf{m}_X) ,$$

and, if $k\overline{\varepsilon}^N_k(\mathbf{m}) = \sum^{(N)} E_X(\mathbf{m}_X)$, with $\sum^{(N)}$ denoting summation over the sets $X \subseteq [-k/2-N, k/2+N]$, $N \ge 0$, while $\overline{X} \in [-k/2, k/2-1]$, one has the approximation formula

$$|k\overline{\varepsilon}^N_k(\mathbf{m}) - k\overline{\varepsilon}_k(\mathbf{m})| \le b_1 e^{-b_2 N} ,$$

where $b_1 = (e+1)[(e-1)\,(1-\exp(-b_2^{(E)}))]^{-1} b_1^{(E)}$, $b_2 = b_2^{(E)}$, and $N$ can be chosen $N=0$. Then

$$\begin{aligned} \nu_+(\{\mathbf{m}\ :\ \overline{\varepsilon}^0_k(\mathbf{m}) \in I_{p,\delta-b_1/k}\}) \ &\le\ \nu_+(\{\mathbf{m}\ :\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta}\}) \\ &\le\ \nu_+(\{\mathbf{m}\ :\ \overline{\varepsilon}^0_k(\mathbf{m}) \in I_{p,\delta+b_1/k}\}) . \end{aligned}$$

From the general theory of one-dimensional Gibbs states, [R1,R3], (see Proposition 2.5 in §2), one has that the $\nu_+$-probability of a configuration $\mathbf{m}^0_{[-L/2,L/2]}$, $L \ge k/2$, is

$$\frac{e^{-\sum^* H_X(\mathbf{m}^0_X)}\, B\left(\mathbf{m}_{[-L/2,L/2]}\right)}{\left[\sum_{\mathbf{m}_{[-L/2,L/2]}} e^{-\sum^* H_X(\mathbf{m}^0_X)}\right]} ,$$

where $\sum^*$ denotes summation over all the $X \subseteq [-L/2, L/2]$, with $\overline{X} \in [-k/2, k/2-1]$, and $B(\mathbf{m}_{[-L/2,L/2]})$ depends on $\mathbf{m}_{[-L/2,L/2]}$, but verifies the bound $|\ln B(\mathbf{m}_{[-L/2,L/2]})| \le \ln B_2$, for a suitable $B_2 > 0$ and uniformly in $L$.

As $\psi(\mathbf{m}) \in (t_0, \alpha)$, we can define $\tilde{B}_2 = \max\{\alpha, t_0^{-1}\}$, so that $|\ln \psi(\mathbf{m})| \le \ln \tilde{B}_2$.

Then, for any $L \ge k/2$, one has, for $B_3 = B_2\tilde{B}_2$,

$$\begin{aligned}\nu_+(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta}\}) &\le \nu_+(\{\mathbf{m}:\ \overline{\varepsilon}^0_k(\mathbf{m}) \in I_{p,\delta+b_1/k}\}) \\ \le B_3\, \nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}^0_k(\mathbf{m}) \in I_{p,\delta+b_1/k}\}) &\le B_3\, \nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta+2b_1/k}\}) ,\end{aligned}$$

and likewise a lower bound is obtained by replacing $B_3$ by $B_3^{-1}$ and $b_1$ by $-b_1$.

Then, if $I_{p,\delta} \subset (-p^*, p^*)$ the set of the rectangles $T_{\mathbf{m}_{[-L,L]}} \in \mathcal{T}_L$ with center $x$ such that $\varepsilon_k(x) \in I_{p,\delta}$ is not empty, and we have obtained the following rewriting of Lemma 4.3.

**4.4.** LEMMA. *The distributions $\nu_+$ and $\nu_{L,k}$, $L \ge k/2$, verify the inequalities*

$$\frac{1}{k\sigma_+} \ln \frac{\nu_+(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta\mp 2b_3/k}\})}{\nu_+(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in -I_{p,\delta\pm 2b_3/k}\})} \begin{cases} < \frac{1}{k\sigma_+} \ln B_3^2 + \frac{1}{k\sigma_+} \ln \frac{\nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta}\})}{\nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in -I_{p,\delta}\})} \\ > -\frac{1}{k\sigma_+} \ln B_3^2 + \frac{1}{k\sigma_+} \ln \frac{\nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta}\})}{\nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in -I_{p,\delta}\})} \end{cases}$$

*for $I_{p,\delta} \subset (-p^*, p^*)$ and for $k$ so large that $p + \delta + 2b_3/k < p^*$.*

Hence Lemma 4.3 follows if the following result can be proven.

**4.5.** LEMMA. *There is a constant $\overline{b}$ such that the approximate distribution $\mu_{L,k}$ verifies the inequalities*

$$\frac{1}{\sigma_+ k} \ln \frac{\nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta}\})}{\nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in -I_{p,\delta}\})} \begin{cases} \le p + \delta + \overline{b}/k \\ \ge p - \delta - \overline{b}/k \end{cases}$$

*for $k$ large enough (so that $|p| + \delta + \overline{b}/k < p^*$) and for all $L \ge k/2$.*

If $\mathcal{T}$ is a Markov partition also $i\mathcal{T}$ is such (because $iS = S^{-1}i$ and $iW^u_x = W^s_{ix}$); furthermore if $\mathcal{T}_1$ and $\mathcal{T}_2$ are Markov partitions also $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$ is such. Therefore there exists a time reversal invariant Markov partition $\mathcal{T}$, *i.e.* a Markov partition such that $\mathcal{T} = i\mathcal{T}$: it is enough to take any Markov partition $\mathcal{T}_0$, hence to set $\mathcal{T} = \mathcal{T}_0 \vee i\mathcal{T}_0$.

Since the center of a rectangle $T_{\mathbf{m}_{-[L,L]}} \in \mathcal{T}_L$ can be taken to be any point $x_{\mathbf{m}_{[-L,L]}}$ in the rectangle $T_{\mathbf{m}_{[-L,L]}}$ (provided $x_{\mathbf{m}_{-[L,L]}} = \rho(\mathbf{m}', 0)$, $\mathbf{m}' \in \mathcal{M}$), we can and shall suppose that the centers of the rectangles in $\mathcal{T}_{\mathbf{m}_{[-L,L]}}$ have been so chosen that the center of $iT_{\mathbf{m}_{[-L,L]}}$ is $ix_{\mathbf{m}_{[-L,L]}}$, *i.e.* the time reversal of the center $x_{\mathbf{m}_{[-L,L]}}$ of $T_{\mathbf{m}_{[-L,L]}}$.

For $k$ large enough the set of configurations $\mathbf{m}_{[-L,L]}$ such that $\varepsilon_k(x) \in I_{p,\delta}$ for all (possible) $x \in T_{\mathbf{m}_{[-L,L]}}$ is not empty and the ratio in Lemma 4.5 can be written, if $x_{\mathbf{m}_{[-L,L]}}$ is the center of $T_{\mathbf{m}_{[-L,L]}}$, as

$$\begin{aligned}\frac{\nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in I_{p,\delta}\})}{\nu_{L,k}(\{\mathbf{m}:\ \overline{\varepsilon}_k(\mathbf{m}) \in -I_{p,\delta}\})} &= \frac{\sum_{\varepsilon_k(x_{\mathbf{m}_{[-L,L]}}) \in I_{p,\delta}} \mathcal{J}^{-1}_{u,k}\left(x_{\mathbf{m}_{[-L,L]}}\right)}{\sum_{\varepsilon_k(x_{\mathbf{m}_{[-L,L]}}) \in -I_{p,\delta}} \mathcal{J}^{-1}_{u,k}\left(x_{\mathbf{m}_{[-L,L]}}\right)} \\ &= \frac{\sum_{\varepsilon_k(x_{\mathbf{m}_{[-L,L]}}) \in I_{p,\delta}} \mathcal{J}^{-1}_{u,k}\left(x_{\mathbf{m}_{[-L,L]}}\right)}{\sum_{\varepsilon_k(x_{\mathbf{m}_{[-L,L]}}) \in I_{p,\delta}} \mathcal{J}^{-1}_{u,k}\left(ix_{\mathbf{m}_{[-L,L]}}\right)} .\end{aligned}$$

But the time reversal symmetry implies that $\mathcal{J}_{u,k}(x) = \mathcal{J}^{-1}_{s,k}(ix)$, so that the above ratio becomes

$$\frac{\sum_{\varepsilon_k(x_{\mathbf{m}_{[-L,L]}}) \in I_{p,\delta}} \mathcal{J}^{-1}_{u,k}\left(x_{\mathbf{m}_{[-L,L]}}\right)}{\sum_{\varepsilon_k(x_{\mathbf{m}_{[-L,L]}}) \in I_{p,\delta}} \mathcal{J}_{s,k}\left(x_{\mathbf{m}_{[-L,L]}}\right)} \begin{cases} < \max_{\mathbf{m}_{[-L,L]}} \mathcal{J}^{-1}_{u,k}\left(x_{\mathbf{m}_{[-L,L]}}\right) \mathcal{J}^{-1}_{s,k}\left(x_{\mathbf{m}_{[-L,L]}}\right) \\ > \min_{\mathbf{m}_{[-L,L]}} \mathcal{J}^{-1}_{u,k}\left(x_{\mathbf{m}_{[-L,L]}}\right) \mathcal{J}^{-1}_{s,k}\left(x_{\mathbf{m}_{[-L,L]}}\right) \end{cases}$$

where the maxima are evaluated as $\mathbf{m}_{[-L,L]}$ varies with $\varepsilon_k(x_{\mathbf{m}_{-L,L}}) \in I_{p,\delta}$.

We can replace $\mathcal{J}_{u,k}^{-1}(x)\,\mathcal{J}_{s,k}^{-1}(x)$ with $\mathcal{J}_k^{-1}(x)B_4^{\pm 1}$, $B_4 = B_1^3$, and $B_1$ is defined at the beginning of §2.

By definition of the set of $\mathbf{m}_{[-L,L]}$'s in the maximum operation in the last inequalities one has $[\sigma_+ k]^{-1}\ln \mathcal{J}_k^{-1}(x_{\mathbf{m}_{[-L,L]}}) \in I_{p,\delta}$: then Lemma 4.5 follows with $\overline{b} = \sigma_+^{-1}\ln B_4$.

From the chain of implications $4.5 \to 4.4 \to 4.3 \to 3.6$, Theorem 3.6 follows and a bound $O(k^{-1})$ is found on the speed at which the limits are approached: in fact the limit in Lemma 4.1 is reached at speed $O(k^{-1})$, and the regularity of $\zeta(s)$, the size of $\eta(k)$ and $\eta'(k)$ and the error term in Lemma 4.5 have all order $O(k^{-1})$.

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## Appendix A1. Proof of Proposition 2.2

The proof of the statements in Proposition 2.2 can be adapted from [G3]. In fact we can study the map $S = \mathcal{H}_T\colon \Gamma'(\mathcal{T}) \to \Gamma'(\mathcal{T})$ as it was an Anosov diffeomorphism. For any $x \in \Gamma(\mathcal{T})$, $W_x^s(T)$ and $W_x^u(T)$ are the stable and the unstable manifolds of $x$, if $T$ is the rectangle in $\Gamma(\mathcal{T})$ containing $x$; the angle $\alpha(x)$ between them is bounded by two constants $K_0^{-1} < K_0$, by the transversality implied by the Whitney sum decomposition (see Definition 1.1). We can denote by $|DS_u(x)|$ and $|DS_s(x)|$ the jacobians of the map $S$ restricted to the unstable manifold $W_x^u(T)$ and, respectively, to the stable manifold $W_x^s(T)$.

Then we can apply the discussion of [G1,G2], and obtain that the measure $\nu$ of a cylinder set

$$\mathcal{C}\begin{smallmatrix} -k & \cdots & k \\ m_{-k} & \cdots & m_k \end{smallmatrix} = \bigcap_{j=-k}^{k-1} S^{-j} T_{m_j} \subset T_{m_0}$$

is "essentially" given by

$$\nu\left(\mathcal{C}\begin{smallmatrix} -k & \cdots & k \\ m_{-k} & \cdots & m_k \end{smallmatrix}\right) = \beta^s_{m_{-k}}\Big[\prod_{j=-k}^{0} |DS_s(S^j x)|\Big]\alpha(x)\Big[\prod_{j=1}^{k-1} |DS_u^{-1}(S^j x)|\Big]\beta^u_{m_k}\ ,$$

where $x$ is a point in $\mathcal{C}\begin{smallmatrix} -k & \cdots & k \\ m_{-k} & \cdots & m_k \end{smallmatrix}$, $\beta^u_{m_k}$ and $\beta^s_{m_{-k}}$ denote the surfaces of the unstable boundary of $T_{m_k}$ and, respectively, of the stable boundary of $T_{m_{-k}}$. Here "essentially" has the same meaning as in [G2], arising from the approximation involved by the arbitrarity of the choise of the point $x$, and it is solved as in [G2].

When the limit as $k \to \infty$ is taken, we find that $\nu$ is formally proportional to the exponential of

$$-\Big[-\sum_{j=-\infty}^{-1} \ln|DS_s(\rho(\sigma^j\mathbf{m},0))| - \ln\alpha(\rho(\mathbf{m},0)) + \sum_{j=0}^{\infty} \ln|DS_u(\rho(\sigma^j\mathbf{m},0))|\Big]\ .$$

By using the fact that $|DS_u|$ and $|DS_s|$ are cocycles, we can write

$$\prod_{j=-k}^{0} |DS_s(S^j x)| = |DS_s^k(S^{-k}x)|\ , \qquad \prod_{j=0}^{k-1} |DS_u(S^j x)| = |DS_u^k(x)|\ ,$$

and replace $|DS_s^k(S^{-k}x)|$ with $\mathcal{J}_{s,\tau_{-k}(x)}(\mathcal{H}_{\mathcal{T}}^{-k}x)$, and $|DS_u^k(x)|$ with $\mathcal{J}_{u,\tau_k(x)}(x)$: the errors caused by such substitutions are bounded by two other pairs of constants $K_1^{-1} < K_1$ and

$K_2^{-1} < K_2$, by [B2], Lemma 7.1, (this simply means that the boundary conditions for the corresponding Gibbs state can be different for finite $k$, but the difference becomes irrelevant as $k \to \infty$; see also Remark 2.6).

Then we have $\mu = \nu \times m$, where $m$ is the Lebesgue measure and $\nu$ is a measure on $\mathcal{M}$, so that Proposition 2.2 follows with $\nu_0 = \nu$ and $B_2 = K_0 K_1 K_2$. ■

If the volume measure $\mu_0$ admits the representation $\mu_{\nu_0} = \nu_0 \times m$, then the measure $\nu_+$ describing the Gibbs state with formal Hamiltonian $H_+(\mathbf{m})$ denotes the SRB measure for the map $H_{\mathcal{T}}$: $\Gamma'(\mathcal{T}) \to \Gamma'(\mathcal{T})$, so that $\mu_{\nu_+} = \nu_+ \times m$ is the SRB measure for the $G$.

## Appendix A2. Equivalence of ensembles

In this appendix we prove the equivalence between Proposition 4.1 and Proposition 4.2. Let assume that Proposition 4.1 holds. Define

$$Q(\beta, k) = \int \nu_+(d\mathbf{m})\, e^{\beta k \overline{\varepsilon}_k(\mathbf{m})} \; .$$

Given $\delta > 0$, let $p_0$ be such that $\beta p - \zeta(p) > \lambda(\beta) - c_1\delta$ for any $p \in [p_0 - \delta, p_0 + \delta]$, for a suitable constant $c_1$. Then for $k$ large enough (so that $D_1 < k\delta$)

$$\begin{aligned} Q(\beta, k) &\geq \int \nu_+(d\mathbf{m})\, e^{\beta k \overline{\varepsilon}_k(\mathbf{m})}\, \chi(\overline{\varepsilon}_k(\mathbf{m}) \in [p_0 - \delta, p_0 + \delta]) \\ &\geq \exp\Big[\beta k (p_0 - \delta) + k\big(\sup_{s \in [p_0 - \delta, p_0 + \delta]} \big\{ -\zeta(s)\big\} - k^{-1} D_1\big)\Big] \geq e^{k(\lambda(\beta) - \delta(2\beta + 1 + c_1))} \; , \end{aligned}$$

(here $\chi$ denotes the characteristic function), so that

$$\liminf_{k \to \infty} \frac{1}{k} \ln Q(\beta, k) \geq \lambda(\beta) \; .$$

Given $\delta > 0$, one can write, for $k$ large enough (so that $D_1 < k\delta$),

$$\begin{aligned} Q(\beta, k) &= \sum_{i=1}^{k_0 - 1} \int \nu_+(d\mathbf{m})\, e^{\beta k \overline{\varepsilon}_k(\mathbf{m})}\, \chi(\varepsilon_i \leq \overline{\varepsilon}_k(\mathbf{m}) \leq \varepsilon_{i+1}) \\ &+ \sum_{i=k_0}^{k-1} \int \nu_+(d\mathbf{m})\, e^{\beta k \overline{\varepsilon}_k(\mathbf{m})}\, \chi(\varepsilon_i \leq \overline{\varepsilon}_k(\mathbf{m}) \leq \varepsilon_{i+1}) \; , \end{aligned}$$

where the strictly increasing sequence $\{\varepsilon_i\}_{i=1}^k$ is so taken that: (a) $-p_* = \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_k = p^*$; (b) $\forall i = 1, \ldots, k$, $\varepsilon_{i+1} - \varepsilon_i \leq 2\delta$; and (c) $\varepsilon_{k_0} = s_0$, where $\sup_{s \in (-p_*, p_*)}\{-\zeta(s)\} = -\zeta(s_0)$. Then

$$\begin{aligned} Q(\beta, k) &\leq \sum_{i=0}^{k_0 - 1} e^{\beta k \varepsilon_{i+1} - k\zeta(\varepsilon_{i+1}) + D_1} + \sum_{i=k_0}^{k-1} e^{\beta k \varepsilon_{i+1} - k\zeta(\varepsilon_i) + D_1} \\ &\leq (k-1)\, e^{k[\lambda(\beta) + \delta(2\beta + 1)]} \; , \end{aligned}$$

and one deduces

$$\limsup_{\kappa \to \infty} \frac{1}{k} \ln Q(\beta, k) \leq \lambda(\beta) \; ,$$

so that the first statement of Proposition 4.2 is proven. The other properties of $\lambda(\beta)$ follow from the properties of $\zeta(s)$, by taking into account that $\lambda(\beta)$ is the Legendre transform of $\zeta(s)$.

## Appendix A3. Canonical ensemble

We prove Proposition 4.2. In order to apply the methods of [CO], let us reformulate it under more general conditions as follows (recall that real analyticity means analyticity in an arbitrarily small strip around the real axis).

**A3.1.** LEMMA. *Let us denote by $X$ the subsets of $\mathbb{Z}$, by $|X|$ the number of elements in $X$, and define* $\mathrm{diam}(X) = \max_{i,j\in X} |[i,j]|$. *Let the class of the* potentials *be defined as $\Phi = \{\Phi_X(\mathbf{m}_X)\}_{X\subset\mathbb{Z}}$, where $\Phi_X(\mathbf{m}_X)$ is a function depending on the values of the symbols $\{m_p\}_{p\in X}$, and set*

$$U^{(\Phi)}(\mathbf{m}) = \sum_{X\in\mathbb{Z}} \Phi_X(\mathbf{m}_X) \ .$$

*If $\|f_X\|_\infty$ denotes the supremum norm for the continuous function $f_X \in C(\{1,\ldots,\mathcal{N}\}^{|X|})$, i.e. $\|f_X\|_\infty = \sup_{\mathbf{m}_X} |f_X(\mathbf{m}_X)|$, and $\mathcal{B}$ is the Banach space of the* potentials $\Phi$ *with the norm*

$$\|\Phi\|_{\mathcal{B}} \equiv \sum_{X\ni 0} e^{r\,\mathrm{diam}(X)} \|\Phi_X(\mathbf{m}_X)\|_\infty < \infty \ ,$$

*for some $r > 0$, then there exists a domain $\omega$ of the complex plain centered in the origin, such that, for $\Phi \in \mathcal{B}$, the limit*

$$q(\beta) = \lim_{k\to\infty} \frac{1}{k} \ln \int \nu_{k/2,k}(d\mathbf{m})\, e^{\beta U^{(\Phi)}(\mathbf{m})}$$

*exists and is analytic in $\beta \in \omega$.*

Once the existence of the limit $q(\beta)$ is proven, the convexity can be proven with standard methods, [R1], and the linearity in $\beta$ for $\beta \to \infty$ follows from the definition of $p^*$ in §4.

**A3.2.** *Decimation and first cluster expansion.* Let $\Lambda_p$ be the interval centered at the origin of lenght $|\Lambda_p| = 2pM + (2p+1)L$, and decompose $\Lambda_p$ into consecutive blocks $A_{-p}$,$B_{-p}$,$A_{-p+1}$,…,$A_{p-1}$,$B_{p-1}$,$A_p$, such that $|B_i| = M$ $\forall i = -p,\ldots,p-1$, and $|A_i| = L$ $\forall i = -p,\ldots,p$, with $L \le M$. Set $\mathbb{Z} = \lim_{p\to\infty} \Lambda_p$, and define $\Gamma_p^A = \{A_i\}_{i=-p}^{p}$ and $\Gamma_p^B = \{B_i\}_{i=-p}^{p-1}$.

Consider the function

$$Z_p(H+\beta E) = \sum_{\mathbf{m}_{\Lambda_p}} e^{-U^{(H+\beta E)}(\mathbf{m}_{\Lambda_p})} \ ,$$

where the potentials $H = \{H_X(\mathbf{m}_X)\}_{X\subset\mathbb{Z}}$ and $E = \{E_X(\mathbf{m}_X)\}_{X\subset\mathbb{Z}}$ are in $\mathcal{B}$. Note that $U^{(H+\beta E)} = U^{(H)} + U^{(\beta E)}$, so that the real analyticity in $\beta$ of

$$\lim_{p\to\infty} |\Lambda_p|^{-1} Z_p(H+\beta E)$$

yields Lemma A3.1 (the sign of $\beta$ being irrelevant).

If one chooses $H$ and $B$ as defined in §4, after Lemma 4.3, then from Lemma A3.1 Proposition 4.2 follows, for $A_{ij} \equiv 1$ ($A$ is the matrix introduced in §1.8). The extension to the general case is trivial.

Call $\mathbf{a}_i = \mathbf{m}_{A_i}$ and $\mathbf{b}_i = \mathbf{m}_{B_i}$ the configurations in the blocks $A_i$ and $B_i$, and $\mathbf{a}_S$ [$\mathbf{b}_S$] the configuration in $S$, if $S$ is the union of sets in $\Gamma_p^A$ [$\Gamma_p^B$]. We have

$$\begin{aligned} U^{(H+\beta E)}(\mathbf{m}_{\Lambda_p}) = & \sum_{i=-p}^{p} \alpha^{(H+\beta E)}(\mathbf{a}_i) + \sum_{i=-p}^{p-1} J_i^{(H)}(\mathbf{a}_i,\mathbf{b}_i,\mathbf{a}_{i+1}) \\ & + \sum_{D\in\Gamma_p^D} W_D^{(H+\beta E)}(\mathbf{a}_D) + \sum_{i=-p}^{p-1} J_i^{(\beta E)}(\mathbf{a}_i,\mathbf{b}_i,\mathbf{a}_{i+1}) + \sum_{C\in\bar{\Gamma}_p} W_C^{(H+\beta H)}(\mathbf{m}_C) \ , \end{aligned}$$

where the sets $\Gamma_p^D$ and $\bar{\Gamma}_p$ are defined as follows:

$$\Gamma_p^D = \{D = A_{i_1} \cup \ldots \cup A_{i_k} \ : \ 2 \leq k \leq p\, , \ \text{and} \ D \neq A_i \cup A_{i+1}\, , \quad \forall i \in \mathbb{Z}\}\ ,$$

$$\begin{aligned}\bar{\Gamma}_p = \{C = A_{i_1} \cup \ldots \cup A_{i_k} \cup B_{i'_1} \cup \ldots \cup B_{i'_{k'}} \ : \ 0 \leq k \leq p\, , \ 1 \leq k' \leq p+1\, ,\\ \text{and} \ C \neq A_i \cup B_i \cup A_{i+1}\, , \ C \neq A_i B_i\, , \ C \neq B_i A_{i+1}\, , \ C \neq B_i\, , \forall i \in \mathbb{Z}\}\ ,\end{aligned}$$

and

$$\begin{aligned}
\alpha^{(\Phi)}(\mathbf{a}_i) &= \sum_{X \subset A_i} \Phi_X(\mathbf{a}_X)\ ,\\
J_i^{(\Phi)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1}) &= \sum_{\substack{X \subset A_i \cup B_i \cup A_{i+1} \\ X \cap B_i \neq \emptyset}} \Phi_X(\mathbf{m}_X) + \sum_{\substack{X \subset A_i \cup A_{i+1} \\ X \cap A_i \neq \emptyset\, ,\ X \cap A_{i+1} \neq \emptyset}} \Phi_X(\mathbf{a}_X)\ ,\\
W_D^{(\Phi)}(\mathbf{a}_D) &= \sum_{\substack{X \subset D \\ X \cap A_{i_h} \neq \emptyset\ \forall A_{i_h} \subset D}} \Phi_X(\mathbf{a}_D)\ ,\\
W_C^{(\Phi)}(\mathbf{m}_C) &= \sum_{\substack{X \subset C \\ X \cap A_{i_h} \neq \emptyset\ \forall A_{i_h} \subset C\, ,\ X \cap B_{i'_h} \neq \emptyset\ \forall B_{i'_h} \subset C}} \Phi_X(\mathbf{m}_C)\ .
\end{aligned}$$

Note that $\Gamma_p^D = \emptyset$ if only connected subsets $X$ are allowed for interaction $\Phi$, as it is the case when $H$ and $E$ are given as in §4.

If we define

$$Z_{B_i}^{(\Phi)}(\mathbf{a}_i, \mathbf{a}_{i+1}) = \sum_{\mathbf{b}_i} e^{J_i^{(\Phi)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})}\ , \qquad (A3.1)$$

and

$$\begin{aligned}\exp[-\tilde{U}^{(H+\beta E)}(\mathbf{a}_{\Gamma_p^A})] = \sum_{\mathbf{b}_{\Gamma_p^B}} \Big[ \prod_{i=-p}^{p-1} \frac{e^{J_i^{(H)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})}}{\sum_{\mathbf{b}_i} e^{J_i^{(H)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})}} \Big] \cdot \\ \cdot \Big[ \prod_{C \in \bar{\Gamma}_p} e^{W_C^{(H+\beta E)}(\mathbf{m}_C)} \Big] \cdot \Big[ \prod_{i=-p}^{p-1} e^{J_i^{(\beta E)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})} \Big]\ ,\end{aligned}$$

then we can average over the variables associated to $\Gamma_p^B$ (*decimation procedure*, see [KH1, KH2])

$$\begin{aligned}\sum_{\mathbf{m}_{\Lambda_p}} U^{(H+\beta E)}(\mathbf{m}_{\Lambda_p}) = \sum_{\mathbf{a}_{\Gamma_p^A}} \Big\{ \sum_{i=-p}^{p} \alpha^{(H+\beta E)}(\mathbf{a}_i) + \sum_{i=-p}^{p-1} \ln Z_{B_i}^{(\Phi)}(\mathbf{a}_i, \mathbf{a}_{i+1}) \\ + \sum_{D \in \Gamma_p^D} W_D^{(H+\beta E)}(\mathbf{a}_D) + \tilde{U}^{(H+\beta E)}(\mathbf{a}_{\Gamma_p^A}) \Big\}\ .\end{aligned}$$

To each $C \in \bar{\Gamma}_p$ we associate a bond $\pi(C)$, and to each $B \in \Gamma_p^B$ a bond $\pi(B)$, such that the lenght of a bond $\pi(B)$ is $|\pi(B)| = 1$ and the lenght of a bond $\pi(C)$, denoted as $|\pi(C)|$, is given by the number of blocks $A$'s and $B$'s contained in $C$. Consider

$$\mathcal{R} = \{C_1, \ldots, C_k, B_1, \ldots, B_h\Big\}\ , \qquad C_s \subset \bar{\Gamma}_p\ \forall s = 1, \ldots, k\ ,$$

and set $\tilde{C} = C \cap \Gamma_p^B$ and $\tilde{\mathcal{R}} = \{\tilde{C}_1, \ldots, \tilde{C}_k, B_1, \ldots, B_h\Big\}$: $|\pi(\tilde{C})|$ is given by the number of sets $B$'s contained in $C$. We set $\tilde{B} = B$.

The set of bonds corresponding to $\mathcal{R}$, *i.e.*

$$R = \Big\{\pi(C_1), \ldots, \pi(C_k), \pi(B_1), \ldots, \pi(B_h)\Big\}\ , \qquad C_s \subset \bar{\Gamma}\ \forall s = 1, \ldots, k\ , \qquad (A3.2)$$

is a *polymer* (see [GMM]) if, for any choise of bonds $\pi(X_l)$ and $\pi(X_j)$, with $X_l, X_j \in \mathcal{R}$, there exist $X_{i_1}, \ldots, X_{i_r} \in \mathcal{R}$, $r \le k+h$, such that $X_{i_1} = X_l$, $X_{i_r} = X_j$ and $\tilde{X}_{i_h} \cap \tilde{X}_{i_h+1} \neq \emptyset$ $\forall h = 1, \ldots, r-1$. Then one has

$$\exp[-\tilde{U}^{(H+\beta E)}(\mathbf{a}_{\Gamma_p^A})] = 1 + \sum_{n=1}^{\infty} \sum_{\substack{R_1,\ldots,R_n \\ \tilde{R}_i \cap \tilde{R}_j = \emptyset}} \prod_{i=1}^{n} \zeta(R_i) ,$$

where $\tilde{R}$ is defined as $R$ in (A3.2), but with $C_s$ replaced with $\tilde{C}_s$ $\forall s = 1, \ldots, k$, and

$$\zeta(R) = \sum_{\mathbf{b}_{\tilde{\mathcal{R}}}} \Big[ \prod_{B_i \subset \tilde{\mathcal{R}}} \frac{e^{J_i^{(H)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})}}{\sum_{\mathbf{b}_i} e^{J_i^{(H)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})}} \Big] \cdot \\ \cdot \Big[ \prod_{s=1}^{k} \Big( e^{W_{C_s}^{(H+\beta E)}(\mathbf{m}_{C_s})} - 1 \Big) \Big] \cdot \Big[ \prod_{s'=1}^{h} \Big( e^{J_{i_{s'}}^{\beta E}(\mathbf{a}_{i_{s'}}, \mathbf{b}_{i_{s'}}, \mathbf{a}_{i_{s'}+1})} - 1 \Big) \Big] ,$$

is the *activity* of the polymer $R$. Here $B_i \subset \tilde{\mathcal{R}}$ means $B_i \in \mathcal{R}$ or $B_i \subset \tilde{C}_j \in \tilde{\mathcal{R}}$ for some $j = 1, \ldots, k$. We set $|\tilde{R}| = \sum_{s=1}^{k} |\pi(\tilde{C})| + h$ (as $|\pi(B)| = 1$).

**A3.3.** LEMMA. *Given a potential $\Phi \in \mathcal{B}$, and considered a polymer $R$, the activity $\zeta(R)$ satisfies the inequality*

$$|\zeta(R)| \le \rho^{|\tilde{R}|} \prod_{s=1}^{k} w_{C_s} \prod_{s'=1}^{h} j_{B_{s'}} ,$$

*where $\rho = e^{-r}$, $w_C = 2\, e^{r|\pi(\tilde{C})|} \|W_C^{(H+\beta E)}\|_\infty$ and $j_{B_i} = 2\, e^r \; \|J_i^{(\beta E)}\|_\infty$, being $\|f_X\|_\infty$ the supremum norm for the continuous function $f_X$, and $w_C$ and $j_B$ are positive constants such that $\max\{w_C, j_B\} \le \ln[\sqrt{\rho}(2-\sqrt{\rho})]^{-1}$ for $\beta$ small enough and $L$ sufficiently large.*

**A3.4.** *Proof of Lemma A3.3.* For complex $z$ such that $|z| < 1/2$, one has $|e^z - 1| \le 2|z|$. Since

$$\lim_{L\to\infty} e^{r' L} \, \|W_C^{(H+\beta E)}\|_\infty = 0 , \qquad \forall r' < r ,$$

and

$$\lim_{\beta\to 0} \beta^{1-\varepsilon} \, \|J_i^{(\beta E)}\|_\infty = 0 , \qquad \forall \varepsilon > 0 ,$$

we can apply the above inequality, and obtain

$$|\zeta(R)| \le \sum_{\mathbf{b}_{\tilde{R}}} \Big[ \prod_{B_i \subset \tilde{R}} \frac{e^{J_i^{(H)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})}}{\sum_{\mathbf{b}_i} e^{J_i^{(H)}(\mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{i+1})}} \Big] \cdot \Big( \prod_{s=1}^{k} 2\|W_C^{(H+\beta E)}\| \Big) \cdot \Big( \prod_{s'=1}^{h} 2\|J_{j_{s'}}^{(\beta E)}\| \Big) ;$$

then we can
(1) extract a factor $e^{-r|\pi(\tilde{C})|}$ from $\|W_C^{(H+\beta E)}\|_\infty$ and a factor $e^{-r}$ from $\|J_{j_{s'}}^{(\beta E)}\|_\infty$, because of the exponential decay of the interaction, and
(2) make $w_C$ and $j_B$ arbitrarily small by taking $L$ sufficiently large, and

$$L > \frac{1}{r} \ln \Big[ 4(1-\rho)^{-1} \|H + \beta E\|_{\mathcal{B}} \max\{1, \ln[\sqrt{\rho}(2-\sqrt{\rho})]\} \Big] , \\ |\beta| < \frac{\rho}{4M\|E\|_{\mathcal{B}}} \min\{1, \ln[\sqrt{\rho}(2-\sqrt{\rho})]^{-1}\} ,$$

so obtaining Lemma A3.3. ■

**A3.5.** REMARK. Note that, if we had considered an interaction with exponential decay $e^{-r|X|}$ instead of $e^{-r\,\mathrm{diam}(X)}$, the same bound as in Lemma A2.3 would have followed. In fact the same cluster expansion can be still performed, and the only difference is that now $W_C^{(H+\beta E)}$ decays as $e^{-r|\tilde C|}$: but this is sufficient in order to prove Lemma A3.3.

**A3.6.** *Second cluster expansion.* By Lemma A3.3, for suitable $\beta$ and $L$, one has

$$|\zeta(R)| \le \rho^{|\tilde R|} \prod_{C\in\mathcal{R}} \mathcal{C}_C \ , \qquad 0<\mathcal{C}_C \le K \ , \qquad K \le \ln[\sqrt{\rho}(2-\sqrt{\rho})]^{-1} \ ;$$

then the conditions of [CO], Lemma 1, are satisfied, so that we can deduce (analogously to [CO], Lemma 2)

$$\exp[-\tilde U^{(H+\beta E)}(\mathbf{a}_{\Gamma_p^A})] = \sum_{D\in\tilde\Gamma_p^D} \tilde W_D^{(H+\beta E)}(\mathbf{a}_D) \ ,$$

where $\tilde\Gamma_p^D$ is defined as $\Gamma_p^D$, but with no restriction, and $\tilde W_D^{(H+\beta E)}(\mathbf{a}_D)$ is analytic in $\beta\in\omega_M$, if $\omega_M$ is a circle around the origin of the complex plane whose radius tend to zero as $M\to\infty$. One can write

$$\tilde W_D^{(H+\beta E)}(\mathbf{a}_D) = \sum_{\substack{R_1,\dots,R_n \\ \cup_{i=1}^n \mathcal{R}_i\setminus\tilde{\mathcal{R}}_i\subseteq D}} \varphi_T(R_1,\dots,R_n)\prod_{i=1}^n \zeta(R_i) \ ,$$

where the sum is over all the polymers $R_1,\dots,R_n$ such that the product $\zeta(R_1)\ \dots\ \zeta(R_n)$ depends only on the variables $\mathbf{a}_D$, and $\varphi_T\ (R_1,\ \dots,\ R_n)$ is a suitable coefficient, (see [CO], Lemma 1; see also [GMM]).

Then we can define (recall (A3.1) and define $\mathbf{1}$ as the configuration of a block $A$ with each element set equal to 1)

$$\bar\alpha^{(H)}(\mathbf{a}_i) = \alpha^{(H)}(\mathbf{a}_i) + \ln\Big[\frac{Z_{B_i}^{(H)}(\mathbf{a}_i,\mathbf{1})\cdot Z_{B_{i-1}}^{(H)}(\mathbf{1},\mathbf{a}_i)}{Z_{B_i}^{(H)}(\mathbf{1},\mathbf{1})\cdot Z_{B_{i-1}}^{(H)}(\mathbf{1},\mathbf{1})}\Big] \ ,$$

and

$$\mathcal{V}_D(\mathbf{a}_D) = \begin{cases} \alpha_i^{(\beta E)}(\mathbf{a}_i) \ , & \text{if } D=A_i \ , \\ \tilde W_{A_i\cup A_{i+1}}^{(H+\beta E)}(\mathbf{a}_i,\mathbf{a}_{i+1}) + \ln\Big[\frac{Z_{B_i}^{(H)}(\mathbf{a}_i,\mathbf{a}_{i+1})\cdot Z_{B_i}^{(H)}(\mathbf{1},\mathbf{1})}{Z_{B_i}^{(H)}(\mathbf{a}_i,\mathbf{1})\cdot Z_{B_i}^{(H)}(\mathbf{1},\mathbf{a}_{i+1})}\Big] \ , & \text{if } D=A_i\cup A_{i+1} \ , \\ W_D^{(H+\beta E)}(\mathbf{a}_D) + \tilde W_D^{(H+\beta E)}(\mathbf{a}_D) \ , & \text{if } D\ne A_i, A_i\cup A_{i+1} \ , \end{cases}$$

and write

$$\sum_{\mathbf{m}_{\Lambda_p}} \exp[-U^{(H+\beta E)}(\mathbf{m}_{\Lambda_p})] = \text{const.}\Big[\prod_{i=-p}^{p}\sum_{\mathbf{a}_i} e^{\bar\alpha^{(H)}(\mathbf{a}_i)}\Big]\cdot$$
$$\cdot\sum_{\mathbf{a}_{\Gamma_p^A}}\Big[\prod_{i=-p}^{p}\frac{e^{\bar\alpha^{(H)}(\mathbf{a}_i)}}{\sum_{\mathbf{a}_i} e^{\bar\alpha^{(H)}(\mathbf{a}_i)}}\Big]\cdot\Big[\prod_{D\in\tilde\Gamma_p^D} e^{\mathcal{V}_D(\mathbf{a}_D)}\Big] \ .$$

We introduce a new cluster expansion by associating to each $D\in\tilde\Gamma_p^D$ a bond $\pi(D)$, and defining a polymer $S$ as

$$S = \{\pi(D_1),\dots,\pi(D_k)\} \ , \tag{A3.3}$$

and $\mathcal{S}=\{D_1,\dots,D_k\}$. Then

$$\sum_{\mathbf{m}_{\Lambda_p}} \exp[-U^{(H+\beta E)}(\mathbf{m}_{\Lambda_p})] = \Big[\prod_{i=-p}^{p}\sum_{\mathbf{a}_i} e^{\bar\alpha^{(H)}(\mathbf{a}_i)}\Big]\cdot\Big(1+\sum_{n=1}^{\infty}\sum_{\substack{S_1,\dots,S_n\\ S_i\cap S_j=\emptyset}}\prod_{i=1}^{n}\Theta(S_i)\Big) \ ,$$

where

$$\Theta(S)=\sum_{\mathbf{a}_S}\prod_{A_i\subset\mathcal{S}}\frac{e^{\bar{\alpha}^{(H)}(\mathbf{a}_i)}}{\sum_{\mathbf{a}_i}e^{\bar{\alpha}^{(H)}(\mathbf{a}_i)}}\prod_{D\in\mathcal{S}}\left(e^{\mathcal{V}_D(\mathbf{a}_D)}-1\right)$$

is the activity of the polymer $S$. Here $A_i\subset\mathcal{S}$ means $A_i\subset D_j\in\mathcal{S}$ for some $j=1,\dots,k$.

**A3.7.** LEMMA. *If $Z^{(H)}_{B_i}(\mathbf{a}_i,\mathbf{a}_{i+1})$ is defined as $Z^{(H)}_{B_i}(\mathbf{a}_i,\mathbf{a}_{i+1})=\sum_{\mathbf{b}_i}\exp J^{(H)}_i(\mathbf{a}_i,\mathbf{b}_i,\mathbf{a}_{i+1})$, then*

$$\lim_{M\to\infty}\sup_{\mathbf{a}_i,\mathbf{a}_{i+1}}\left\{\ln\left[\frac{Z^{(H)}_{B_i}(\mathbf{a}_i,\mathbf{a}_{i+1})\cdot Z^{(H)}_{B_i}(\mathbf{1},\mathbf{1})}{Z^{(H)}_{B_i}(\mathbf{a}_i,\mathbf{1})\cdot Z^{(H)}_{B_i}(\mathbf{1},\mathbf{a}_{i+1})}\right]\right\}=0\ ,$$

$\forall L<\infty$.

**A3.8.** *Proof of Lemma A3.7.* Let $\mathbb{Z}^+=\{i\in\mathbb{Z}\ :\ i\geq 0\}$ and $K_+=\{1,\dots,\mathcal{N}\}^{\mathbb{Z}^+}$. We denote by $C(K_+)$ the Banach space of the real continuous functions on $K_+$, and by $M^*(K_+)$ its dual, *i.e.* the space of real measures on $K_+$. Given a configuration $\mathbf{m}_N\in\{1,\dots,\mathcal{N}\}^N$, $N\geq 1$, and a configuration $\mathbf{m}_+\in K_+$, one can define the configuration $(\mathbf{m}_N,\mathbf{m}_+)\in K_+$ as

$$(\mathbf{m}_N,\mathbf{m}_+)_i=\begin{cases}(\mathbf{m}_N)_i\ , & \text{for } i=0,\dots,N\ ,\\ (\mathbf{m}_+)_{i-N}\ , & \text{for } i>N.\end{cases}$$

Given $\Phi\in\mathcal{B}$, an operator $\mathcal{L}_\Phi$: $C(K_+)\to C(K_+)$ is defined by

$$\mathcal{L}_\Phi f\,(\mathbf{m}_+)=\sum_{m_0=1}^{\mathcal{N}}\exp\Big[\sum_{\substack{X\subset\mathbb{Z}^+\\ X\ni 0}}\Phi_X(\mathbf{m}_X)\Big]f(m_0,\mathbf{m}_+)\ ;$$

then there exist $\lambda_\Phi>0$, $h_\Phi\in C(K_+)$ and $\nu_\Phi\in M^*(K_+)$ such that
(a) $\mathcal{L}_\Phi h_\Phi=\lambda_\Phi h_\Phi$, and
(b) if $f\in C(K_+)$, $\lim_{k\to\infty}\|\lambda_\Phi^{-k}\mathcal{L}^k_\Phi f-\nu_\Phi(f)\,h_\Phi\|_\infty=0$, uniformly for $\Phi$ in a bounded subset of a finite dimensional subspace of $\mathcal{B}$.

The proof of such a statement follow from [R1] and can be found in [G1], Ch. 18, Proposition XXXV and exercises, (it is essentially an adaptation from [R1], see also [GL]).

Define the function in $C(K_+)$

$$f_{\mathbf{a}_i}(\mathbf{m}_+)=\exp\Big[\sum_{X\subset A_i\cup A_{i+1}}\Phi_X(\mathbf{m}_X)\Big]\ ,$$

where $\mathbf{a}_i\in\{1,\dots,\mathcal{N}\}^L$. Note that $f_{\mathbf{a}_i}(\mathbf{m}_+)$ depends only on the first $L$ symbols of $\mathbf{m}_+$, (so that the successive ones can be set equal to an arbitrary value, say 1). Then one has

$$\frac{Z^{(H)}_{B_i}(\mathbf{a}_i,\mathbf{a}_{i+1})\cdot Z^{(H)}_{B_i}(\mathbf{1},\mathbf{1})}{Z^{(H)}_{B_i}(\mathbf{a}_i,\mathbf{1})\cdot Z^{(H)}_{B_i}(\mathbf{1},\mathbf{a}_{i+1})}=\frac{\mathcal{L}^M_\Phi f_{\mathbf{a}_i}(\mathbf{a}_{i+1},\mathbf{1})\mathcal{L}^M_\Phi f_{\mathbf{1}}(\mathbf{1})}{\mathcal{L}^M_\Phi f_{\mathbf{a}_i}(\mathbf{1})\mathcal{L}^M_\Phi f_{\mathbf{1}}(\mathbf{a}_{i+1},\mathbf{1})}\ ,$$

where $\mathbf{1}$ appearing in $(\mathbf{a}_{i+1},\mathbf{1})$ is an element in $K_+$, while the subscript $\mathbf{1}$ in $f_{\mathbf{1}}$ is an element in $\{1,\dots,\mathcal{N}\}^L$ (*i.e.* $\mathbf{a}_i=\mathbf{1}$).

Then from the property (b) above, Lemma A3.7 follows. ■

**A3.9.** *Proof of Lemma A3.1.* We consider the cluster expansion envisaged in §A3.5. From the interaction $\{\mathcal{V}_D\}_{D\in\tilde{\Gamma}^D_p}$, terms of the following form arise:
(a) $\sum_{R_1,\dots,R_n}\varphi_T(R_1,\dots,R_n)\,\zeta(R_1)\dots\zeta(R_n)$, where the dependence on a configuration $\mathbf{a}_i$ is only through the factors

$$U_i=\frac{e^{J^{(H)}_i(\mathbf{a}_i,\mathbf{b}_i,\mathbf{a}_{i+1})}}{\sum_{\mathbf{b}_i}e^{J^{(H)}_i(\mathbf{a}_i,\mathbf{b}_i,\mathbf{a}_{i+1})}}$$

appearing in $\zeta(R)$;
(b) for $D = A_i$, $\alpha_i^{(\beta E)}(\mathbf{a}_i)$;
(c) $\sum_{R_1,\dots,R_n} \varphi_T(R_1,\dots,R_n)\zeta(R_1)\dots\zeta(R_n)$, where the dependence on the configurations $\mathbf{a}_i$ is (also) through terms $W_C^{(H+\beta E)}(\mathbf{m}_C)$ and $J_i^{(\beta E)}(\mathbf{a}_i,\mathbf{b}_i,\mathbf{a}_{i+1})$ in $\zeta(R)$;
(d) for $D = A_i \cup A_{i+1}$, also

$$\ln\Big[\frac{Z_{B_i}^{(H)}(\mathbf{a}_i,\mathbf{a}_{i+1})\cdot Z_{B_i}^{(H)}(\mathbf{1},\mathbf{1})}{Z_{B_i}^{(H)}(\mathbf{a}_i,\mathbf{1})\cdot Z_{B_i}^{(H)}(\mathbf{1},\mathbf{a}_{i+1})}\Big] .$$

As a polymer $R$ contains at most $2|\tilde{R}|$ $A$-blocks through the factors $U_i$, we can extract also a factor $\rho^{1/4}$ from each $A$-block appearing in terms of the form (a), by simply replacing $\zeta(R)$ with a new activity $\hat{\zeta}(R) = \rho^{-|\tilde{R}|/2}\rho(R)$.

In (b), we can write $\alpha_i^{(\beta E)}(\mathbf{a}_i) = \rho^{1/2}[\rho^{-1/2}\alpha_i^{(\beta E)}(\mathbf{a}_i)]$, where $\rho^{-1/2}\|\alpha_i^{(\beta E)}\|_\infty$ can be made arbitrarily small by taking $\beta$ small enough.

As far as the terms in (c) are concerned, we can extract a factor $e^{-r} = \rho$ from each $A$-block, thanks to the exponential decay of the interaction, and the remaining factor can be made arbitrarily small by taking $L$ large and $\beta$ small, (see the proof of Lemma A3.4 and the definition of the set $\Gamma_p^D$ in §A3.2).

By Lemma A3.7, we can extract a factor $\rho$ from each term in (d) by taking $M$ sufficiently large.

Therefore we have a factor $\rho^{1/4}$ $\forall A$ arising in (a) and a factor $\rho^{1/2}$ $\forall A$ arising in (b), (c) and (d). Then we can bound

$$|\Theta(S)| \le \tilde{\rho}^{|S|} \prod_{D\in\mathcal{S}} \mathcal{C}_D ,$$

where $\tilde{\rho} = \rho^{1/4}$, and

$$\mathcal{C}_D = \begin{cases} 2\rho^{-1/2}\|\alpha^{(\beta E)}\|_\infty , & \text{if } D = A_i , \\ 2\rho^{-1}\|\mathcal{V}_D\|_\infty , & \text{if } D = A_i \cup A_{i+1} , \\ 2\big[\|W_D\| + \|\hat{W}_D\|_\infty\big] , & \text{if } D \ne A_i , \; A_i\cup A_{i+1} \end{cases} ,$$

being $\hat{W}_D$ defined as $\tilde{W}_D$, but with $\hat{\zeta}(R)$ replacing $\zeta(R)$.

If $\beta \to 0$ and $M \to \infty$, then

$$\sum_X |\mathcal{V}_X(\mathbf{a}_X)|\, e^{|X|r'}$$

can me made arbitrarily small, for some $r' < r$, in order to apply Israel's analyticity theorem, [Is], Theorem II, 4, (see also [CO]). Equivalently we can reason as before, and we can apply again [CO], Lemma 1, and deduce that, for any constant $\bar{\kappa} > 0$, we have
(1) for any $L$, $\exists M_1(L)$ such that for $\forall M \ge M_1(L)$

$$\frac{2}{\rho}\sup_{\mathbf{a}_i,\mathbf{a}_{i+1}} \Big\{\ln\Big[\frac{Z_{B_i}^{(H)}(\mathbf{a}_i,\mathbf{a}_{i+1})\cdot Z_{B_i}^{(H)}(\mathbf{1},\mathbf{1})}{Z_{B_i}^{(H)}(\mathbf{a}_i,\mathbf{1})\cdot Z_{B_i}^{(H)}(\mathbf{1},\mathbf{a}_{i+1})}\Big]\Big\} \le \frac{\bar{\kappa}}{3} ;$$

(2) $\exists L_0$ such that $\forall L \ge L_0$, $\forall M$ and $\forall\beta \in \omega_M$ (being $\omega_M$ defined in §A3.6)

$$\sup_{A\in\Gamma_p^A} \sum_{D\supset A} 2\|\hat{W}_D(\mathbf{a}_D)\|_\infty \le \frac{\bar{\kappa}}{3} ;$$

(3) for $L = L_0$, $\exists M_2(L_0)$ such that $\forall\beta \in \omega_{M_2(L_0)}$ and $\forall M \ge M_2(L_0)$

$$\sup_i \frac{2}{\sqrt{\rho}}\|\alpha_i^{(\beta E)}\|_\infty + \sup_{A\in\Gamma_p^A}\sum_{D\supset A} 2\|W_D(\mathbf{a}_D)\|_\infty \le \frac{\bar{\kappa}}{3} .$$

Therefore, if $\beta \in \omega_{M_0}$, with $M_0 = \max\{M_1(L_0), M_2(L_0)\}$, there exist $\kappa \equiv \kappa(\beta, L_0, M_0) \le \bar\kappa$ such that $\mathcal{C}_D \le \kappa$. If $\bar\kappa$ is so chosen that $\kappa \le \ln[\sqrt{\tilde\rho}\,(2-\sqrt{\tilde\rho})]^{-1}$, one can apply again [CO], Lemma 1: then there is a constant $G(\tilde\rho, \kappa)$ such that

$$\lim_{p\to\infty} \frac{1}{|\Lambda_p|} \ln Z_p(H+\beta E) \le G(\tilde\rho,\kappa) + \frac{1}{L} \ln \Big| \sum_{\mathbf{a}_i} e^{\bar\alpha^{(H)}(\mathbf{a}_i)} \Big| \,.$$

The uniformity of the bound and the existence of the second limit uniformly in $L$ (for real $H$, the proof of such a result is standard, [R2]) allow us to apply Vitali's convergence theorem, [T], §5.21, and complete the proof of Lemma A3.1. ∎

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