Localized Classical Waves Created by Defects

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Abstract

We study acoustic and electromagnetic waves in a periodic medium (or any other background medium with a spectral gap) disturbed by a single defect, i.e., a local disturbance analogous to a well potential in solid state physics. We show that defects do not change the essential spectrum of the associated nonnegative operators and can only create isolated eigenvalues of finite multiplicity in a gap of the periodic medium, with the eigenmodes decaying exponentially. We give a constructive and simple description of defects in acoustic and dielectric media, including a simple condition on the parameters of the medium and of the defect, which ensures the rise of a localized eigenmode with

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the corresponding eigenvalue in a specified subinterval of the given gap of the periodic medium.

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1 Introduction

Localization of classical waves, acoustic and electromagnetic, have received much attention in recent years (e.g., [An, J1, J2, DE, Sc, VP, M] and references therein). This phenomenon arises from coherent multiple scattering and interference and occurs when the scale of the coherent multiple scattering reduces to the wavelength itself. Numerous potential applications (e.g., [DE, J2, VP]), for instance, the optical transistor, and the fundamental significance of localization of classical waves motivate the interest in this phenomenon.

In this article we study localization phenomena due to a single defect on a periodic medium (or any other background medium with a spectral gap). We have previously given rigorous proofs of Anderson localization due to a random array of defects, both in the lattice case [FK1, FK2], and in the true continuum case [FK3, FK4].

It is a well known fact in solid state physics that a well potential in three dimensional space of depth U and of radius a generates an exponentially localized state if $a^2U > \frac{\pi^2\hbar^2}{8m}$, where m is the mass of the quantum particle (e.g., [S]). In this paper we give a similar condition on a single defect in a periodic medium with a gap in the spectrum (more generally, in any background medium with a gap in the spectrum), which ensures the rise of exponentially localized eigenmodes for classical acoustic and electromagnetic waves, with the eigenvalue in any specified closed subinterval of a given gap of the background medium.

We study acoustic and electromagnetic waves which are described by the formally self-adjoint operators

$$A = A(\varepsilon) = -\nabla \cdot \frac{1}{\varepsilon(x)}\nabla \ \text{ on } \ L^2(\mathbb{R}^d), \tag{1}$$

and

$$\mathbf{M} = \mathbf{M}(\varepsilon) = \nabla^{\times} \frac{1}{\varepsilon(x)} \nabla^{\times} \ \text{ on } \ \mathbb{S}, \tag{2}$$

where $\mathbb{S}$, the space of solenoidal fields, is the closure in $L^2(\mathbb{R}^3; \mathbb{C}^3)$ of the linear subset $\{\Psi \in C_0^1(\mathbb{R}^3; \mathbb{C}^3); \ \ \nabla \cdot \Psi = 0\}$. We use the notation

$$\nabla^{\times}\Psi = \nabla \times \Psi = \operatorname{curl}\Psi; \ \ \nabla \cdot \Psi = \operatorname{div}\Psi.$$

We also set

$$M = M(\varepsilon) = \nabla^{\times} \frac{1}{\varepsilon(x)} \nabla^{\times} \ \text{ on } \ L^2(\mathbb{R}^3; \mathbb{C}^3). \tag{3}$$

The function $\varepsilon(x)$ describes the medium; it is the position dependent mass density for acoustic waves, and the position dependent dielectric constant for electromagnetic waves. We deliberately pick the same notation $\varepsilon(x)$ for the coefficients of the above operators, even so they have different physical meaning, in order to emphasize their similarity and describe uniformly their common spectral properties.

We always assume that $\varepsilon(x)$ is a measurable real valued function satisfying

$$0 < \varepsilon_- \leq \varepsilon(x) \leq \varepsilon_+ < \infty \quad \textit{a.e. for some constants} \quad \varepsilon_- \quad \textit{and} \quad \varepsilon_+. \tag{4}$$

Localized acoustic or electromagnetic waves are finite energy solutions of the acoustic or Maxwell equations with the property that almost all of the wave's energy remains in a fixed bounded region of space at all times. They can be constructed from exponentially localized eigenmodes of the acoustic operator A or the Maxwell operator $\mathbf{M}$. (See the discussion in [FK3, FK4].)

A *defect* is a perturbation of a given medium in a finite domain. Defects in the medium are expected to generate localized waves by creating localized eigenmodes of A or $\mathbf{M}$. This phenomenon is analogous to the rise of the localized eigenmodes for electrons described by Schrödinger operators, due to defects such as a well potential satisfying some simple conditions on its width and depth (e.g., [S]).

In spite of the fundamental similarity between the creation of localized eigenmodes for classical and electron waves, there are some important differences. First of all, for the electron it suffices to perturb a homogeneous medium (i.e., a constant potential) locally in order to generate a localized eigenmode. For classical waves a local perturbation of a homogenous media (i.e., $\varepsilon_0(x)$ is constant) cannot generate a localized eigenmode. This can be easily seen from the consideration of a one dimensional model. Indeed, in that case we consider the eigenvalue problem $-\left(\frac{1}{\varepsilon(x)}u'(x)\right)' = \lambda u'(x)$, $x \in \mathbb{R}$, where $\varepsilon(x) = const$ if $|x| > R$ for some R and λ is a positive number. It is clear that this equation cannot have square integrable solutions. Since, in general, the one dimensional case is the most favorable for localization, we should not expect localization under this circumstances in the multidimensional case.

The reason for this difference between classical waves and electrons can be explained as follows. The motion of the electron in a homogenous medium is described by the Schrödinger operator $H_0 = -\Delta + V_0$ with a constant potential $V_0(x) \equiv v_0$. Clearly the spectrum $\sigma(H_0) = [v_0, \infty)$, so we may consider the infinite interval $(-\infty, v_0)$ as a gap in the spectrum of the operator H_0. Notice that the edge v_0 of the gap depends on the homogeneous medium. Hence, if we perturb this homogeneous medium by a defect, say by a potential well, the spectrum could expand in the interior of the gap $(-\infty, v_0)$, and if this happens the corresponding eigenmodes will be exponentially localized. For classical waves in a homogeneous medium , described by an acoustic operator A or Maxwell operator $\mathbf{M}$ with constant $\varepsilon(x)$, we always have $\sigma(A) = \sigma(\mathbf{M}) = [0, \infty)$, so, as for Schrödinger, we may consider the infinite interval $(-\infty, 0)$ as a gap in the spectrum. But for classical waves the bottom 0 of the spectrum does not depend on the $\varepsilon(x)$ of the medium at all. This is why any local perturbation of any medium by a defect does not expand the spectrum into the gap $(-\infty, 0)$, as we saw in the one-dimensional model.

Thus, in order to employ a mechanism for localization of classical waves similar to the one for electronic localization, we start with a medium described by a coefficient $\varepsilon_0(x)$, such that the corresponding acoustic or Maxwell operator has a gap inside its spectrum, and the edges of the gaps must depend on the medium, i.e., on the coefficient $\varepsilon_0(x)$. Such media with media dependent gaps can be perturbed locally by a defect and generate exponentially localized eigenmodes with corresponding eigenvalues in the interior of the gaps.

The most natural way to obtain media with gaps in the spectrum is to consider periodic media, i.e., media described by periodic $\varepsilon_0(x)$. In this case, the spectra of the operators A_0 and $\mathbf{M_0}$, according to Floquet-Bloch theory, have band-gap structure and can have gaps. The existence of gaps for some periodic dielectric and acoustic media has been proved in [FKu1, FKu2].

In this article we show that defects satisfying rather simple conditions do generate localized eigenmodes with corresponding eigenvalues in the gaps, and that the interior of the gaps contains no points of accumulation of those eigenvalues. We give a constructive description of defects in acoustic and dielectric media, including a simple condition on the parameters of the medium and of the defect, that deposit localized eigenvalues (i.e., with exponentially localized eigenmodes) in any specified closed subinterval of a gap.

2 Statement of results

A and M are rigorously defined as as the nonnegative self-adjoint operators on $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^3;\, \mathbb{C}^3)$, respectively, uniquely defined by the quadratic forms given by the closure of the nonnegative densely defined quadratic forms

$$\mathcal{A}(\psi) = \langle \nabla\psi, \frac{\infty}{\varepsilon(\S)}\nabla\psi\rangle \equiv \sum_{|=\infty}^{\lceil} \langle \partial_|\psi, \frac{\infty}{\varepsilon(\S)}\partial_|\psi\rangle, \text{ with } \psi \in \mathcal{C}_\prime^\infty(\mathbb{R}^\lceil), \tag{5}$$

and

$$\mathcal{M}(\ominus) = \langle \nabla\times\ominus, \frac{\infty}{\varepsilon(\S)}\nabla\times\ominus\rangle, \text{ with } \ominus \in \mathcal{C}_\prime^\infty(\mathbb{R}^\ni;\, \mathbb{C}^\ni). \tag{6}$$

We recall Weyl's decomposition (see [BS]), $L^2(\mathbb{R}^3;\, \mathbb{C}^3) = \mathbb{S} \oplus \mathbb{G}$, where $\mathbb{G}$, the space of potential fields, is the closure in $L^2(\mathbb{R}^3;\, \mathbb{C}^3)$ of the linear subset $\{\Psi \in C_0^1(\mathbb{R}^3;\, \mathbb{C}^3);\;\; \Psi = \nabla\varphi \text{ with } \varphi \in C_0^1(\mathbb{R}^3)\}$. The spaces $\mathbb{S}$ and $\mathbb{G}$ are left invariant by M, with $\mathbb{G} \subset \mathcal{D}(\mathcal{M})$ and $M|_{\mathbb{G}} = 0$. We define $\mathbf{M}$ as the restriction of M to $\mathbb{S}$, i.e., $\mathcal{D}(\mathbf{M}) = \mathcal{D}(\mathcal{M}) \cap \mathbb{S}$ and $\mathbf{M} = \mathbf{M}|_{\mathcal{D}(\mathcal{M})\cap\mathbb{S}}$. Thus

$$\mathbf{M} = \mathbf{P}_{\mathbb{S}}\mathbf{M}\mathbf{I}_{\mathbb{S}} = \mathbf{M}\mathbf{I}_{\mathbb{S}}, \tag{7}$$

with $P_{\mathbb{S}}$ the orthogonal projection onto $\mathbb{S}$ and $I_{\mathbb{S}} : \mathbb{S} \to L^2(\mathbb{R}^3;\, \mathbb{C}^3)$ the restriction of the identity map. Notice $M = \mathbf{M} \oplus \mathbf{0}_{\mathbb{G}}$, so we can work with M to answer questions about the spectrum of $\mathbf{M}$.

In this article we discuss results common to both acoustic and Maxwell operators. Since most of the discussion will apply to both cases, where it simplifies the discussion we will use W to denote either A or $\mathbf{M}$, and $\widetilde{W}$ to denote either A or M.

In this article the *background medium* will be described by $\varepsilon_0(x)$ (as in (4)), the corresponding operators will be denoted by A_0, $\mathbf{M_0}$, M_0, W_0, $\widetilde{W}_0$.

We will say that the medium described by $\varepsilon(x)$ (as in (4)) was *obtained from the background medium by the insertion of a defect*, if $\varepsilon(x)$ and $\varepsilon_0(x)$ differ only in a bounded domain, i.e., $\varepsilon(x) - \varepsilon_0(x)$ has compact support. In this case we will say that $\varepsilon(x)$ and $\varepsilon_0(x)$ *differ by a defect.*

The operator W_0 will be said to have a *gap in the spectrum*, if there exist numbers $0 \le \hat{a} < a < b < \hat{b}$ such that

$$\sigma(W_0) \bigcap [\hat{a}, \hat{b}] = [\hat{a}, a] \bigcup [b, \hat{b}].$$

The interval (a, b) is then called a *gap* in $\sigma(W_0)$.

Our first result starts by saying that the insertion of a defect does not change the essential spectrum of our operators; this fact is a corollary to Weyl's Theorem on the stability of the essential spectrum [RS4, Section XIII.4]. If the background operator W_0 has a gap in the spectrum, we show that, in the medium obtained by the insertion of a defect, eigenmodes corresponding to eigenvalues created inside the gap must decay exponentially fast. In this article we say that a function φ decays exponentially fast if it has has exponentially decaying local L^2-norms, i.e., $\|\chi_x \varphi\|_2$ decays exponentially as $|x| \to \infty$, where χ_x is the characteristic function of a cube of unit side centered at x. (Notice that if ψ is an eigenmode for an acoustic operator A, then if ψ decays exponentially fast it also decays exponentially fast pointwise, i.e.,$|\psi(x)|$ decays exponentially as $|x| \to \infty$, with at least the same rate of decay as the local L^2-norms [Ag, Theorem 5.1].) If ψ is an eigenmode for an acoustic operator A or a Maxwell operator $\mathbf{M}$, then if ψ decays exponentially fast we also have that its gradient, in the acoustic case, or its curl, in the Maxwell case, also decays exponentially fast, so the energy density of the associated solution of the acoustic equation or Maxwell's equations also decays exponentially fast [FK3, FK4].

Theorem 1 (Stability of essential spectrum) *Suppose $\varepsilon(x)$ and $\varepsilon_0(x)$ differ by a defect. Then*

$$\sigma_{\text{ess}}(W) = \sigma_{\text{ess}}(W_0) . \tag{8}$$

If (a, b) is a gap in the spectrum of W_0, the spectrum of W in (a, b) consists at most of isolated eigenvalues with finite multiplicity, with the corresponding eigenmodes decaying exponentially fast, with a rate depending on the distance from the eigenvalue to the edges of the gap.

At this point we should ask if there is a way to ensure the rise of at least one eigenvalue in a gap of W_0 by introducing a defect. By Theorem 1 such an eigenvalue will be localized. (An eigenvalue will be said to be *localized* if it is isolated with finite multiplicity, with the corresponding eigenmodes decaying exponentially fast.) The next theorem shows that one can introduce simply defined defects which generate localized eigenvalues in any gap of W_0.

A simple way to tailor these defects is as follows. Let Ω be a bounded subset of $\mathbb{R}^d$ with nonempty interior Ω°; without loss of generality we assume $0 \in \Omega^\circ$. Typically, we take Ω to be the cube Λ of side 1 centered at the origin, or the ball B of radius 1 centered at the origin. We set $\Omega_\ell = \ell\Omega$ for $\ell > 0$, so Λ_ℓ is the cube of side ℓ centered at the origin, etc. We insert a defect by changing the value of $\varepsilon_0(x)$ inside Ω_ℓ to a given constant $\epsilon > 0$, i.e., we set

$$\varepsilon\left(x\right) = \varepsilon_{\epsilon,\ell}\left(x\right) = \begin{cases} \epsilon & \text{if} \quad x \in \Omega_\ell \\ \varepsilon_0\left(x\right) & \quad \text{otherwise.} \end{cases} \tag{9}$$

If (a, b) is a gap in the spectrum of W_0, we will show that we can deposit a localized eigenvalue of W inside any specified closed subinterval of (a, b), by inserting such a defect with $\ell^2\epsilon$ large enough, how large depending only on the geometry of Ω and on the specified closed subinterval. We write $C_0^2(\Omega, \mathbb{R}) = \{\xi \in C_0^2(\mathbb{R}^d, \mathbb{R});\ \operatorname{supp}\xi \subset \Omega^\circ\}$ and use S_d to denote the unit sphere in $\mathbb{R}^d$.

Theorem 2 (Creation of localized eigenvalues) *Let (a, b) be a gap in the spectrum of W_0, select $\mu \in (a, b)$, and pick $0 < \gamma < 1$ such that the interval $[\mu(1-\gamma), \mu(1+\gamma)]$ is contained in the gap, i.e., $[\mu(1-\gamma), \mu(1+\gamma)] \subset (a, b)$. If $\varepsilon\left(x\right) = \varepsilon_{\epsilon,\ell}\left(x\right)$ is as in (9), with*

$$\ell^2\epsilon > \frac{2}{\mu\gamma^2} \inf_{\substack{\xi \in C_0^2(\Omega,\mathbb{R}),\ \|\xi\|=1 \\ \hat{\kappa}\in S_d}} \left\{ \|\hat{\kappa}\cdot\nabla\xi\|^2 \left(1 + \sqrt{1 + \frac{\|\Delta\xi\|^2}{4\|\hat{\kappa}\cdot\nabla\xi\|^4}\gamma^2}\right)\right\}, \tag{10}$$

the corresponding operator W has at least one localized eigenvalue inside the interval $[\mu(1-\gamma), \mu(1+\gamma)]$. The eigenmodes corresponding to such eigenvalues decay exponentially fast, with a rate of exponential decay bigger or equal to

$$m_{a,b,\mu,\gamma,\varepsilon_{0,-},\epsilon} = \frac{\eta_{a,b,\mu,\gamma}}{4\left[(\min\{\varepsilon_{0,-}, \epsilon\})^{-1} + \mu(1+\gamma) + \eta_{a,b,\mu,\gamma}\right]}, \tag{11}$$

where $\eta_{a,b,\mu,\gamma} = \min\{\mu(1-\gamma) - a, b - \mu(1+\gamma)\}$.

Remark 3 *The lower bound for $\ell^2\epsilon$ given in (10) to guarantee the existence of a localized eigenvalue depends only on the geometry of the support of the defect, the location μ and the relative half-width γ of of the specified interval. To guarantee the creation of at least one localized eigenvalue in the gap (a,b), it suffices to take $\mu = \frac{a+b}{2}$ and $\gamma = \frac{b-a}{a+b}$ in (10).*

Remark 4 *If Ω is the unit cube Λ in $\mathbb{R}^d$, we can take $\xi(x) = \frac{\Xi(x)}{\|\Xi(x)\|}$, where $\Xi(x) = \prod_{i=1}^d \zeta(x_i)$ with*

$$\zeta(t) = \begin{cases} \left(t-\frac{1}{2}\right)^2\left(t+\frac{1}{2}\right)^2 & \textit{if } \; -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \textit{otherwise,} \end{cases} \tag{12}$$

and $\hat{\kappa}$ to be the unit vector in the direction of a coordinate axis, so (10) is guaranteed by

$$\ell^2\epsilon > \frac{24}{\mu\gamma^2}\left(1+\sqrt{1+\frac{d\,(5+2d)}{8}\gamma^2}\right). \tag{13}$$

For Maxwell operators we always have $d=3$, in which case (13) is just

$$\ell^2\epsilon > \frac{24}{\mu\gamma^2}\left(1+\sqrt{1+\frac{33}{8}\gamma^2}\right). \tag{14}$$

3 Stability of the essential spectrum

In this section we prove Theorem 1. We start by proving (8). Let $\eta(x) = \frac{1}{\varepsilon(x)} - \frac{1}{\varepsilon_0(x)}$, by our hypotheses it is a bounded measurable function with compact support. We write $\eta(x) = \eta_+(x) - \eta_-(x)$, with $\eta_\pm(x) = \max\{\pm\eta(x), 0\}$, and define nonnegative self-adjoint operators $A_{\eta_\pm} = -\nabla\cdot\eta_\pm(x)\nabla$ and $M_{\eta_\pm} = \nabla^\times\eta_\pm(x)\nabla^\times$ by the respective quadratic forms, as in (5) and (6) (with $\frac{1}{\varepsilon(x)} = \eta_\pm(x)$). We also set $\mathbf{M}_{\eta_\pm} = P_{\mathbb{S}} M_{\eta_\pm} I_{\mathbb{S}} = M_{\eta_\pm} I_{\mathbb{S}}$ as in (7). Using $W_{\eta_\pm}$ to denote either $A_{\eta_\pm}$ or $\mathbf{M}_{\eta_\pm}$, we clearly have $W = (W_0 + W_{\eta_+}) + (-W_{\eta_-})$ as quadratic forms. Since $W_0 + W_{\eta_+} = W(\varepsilon_1)$, with $\varepsilon_1 = \frac{\varepsilon_0}{1+\eta_+\varepsilon_0}$ satisfying (4), it suffices to prove (8) when either $\eta(x) \ge 0$ or $\eta(x) \le 0$ for all x. Thus (8) follows from [RS4, Corollary 4 to Theorem XIII.14], if we can show that, for $\eta(x) \ge 0$, we have $(W_0 + I)^{-n} W_\eta (W_0 + I)^{-n}$ compact for some positive integer n. This is proven in the following lemma.

Lemma 5 *For any $\varepsilon(x)$ as in (4) and any bounded measurable function $\eta(x) \geq 0$ with compact support, $(W+I)^{-n}W_\eta(W+I)^{-n}$ is a Hilbert-Schmidt operator for either $n > \frac{d}{2}$ if $W = A$, or $n \geq 1$ if $W = \mathbf{M}$.*

Proof. Notice that if $0 \leq H_1 \leq H_2$, where H_1, H_2 are two self-adjoint operators with H_2 Hilbert-Schmidt, then H_1 is also Hilbert-Schmidt. Thus, since we can always find a continuously differentiable function $\tilde{\eta}$ with compact support such that $\eta(x) \leq \tilde{\eta}(x)$ for all x, it suffices to consider the case when $\eta \geq 0$ is a continuously differentiable function with compact support; χ_η will denote the characteristic function of its support. Notice that we then have

$$\begin{aligned} A_\eta &= -(\eta\nabla\cdot\nabla + (\nabla\eta)\cdot\nabla) = -\chi_\eta(\eta\nabla\cdot\nabla + (\nabla\eta)\cdot\nabla), && (15)\\ M_\eta &= \eta\nabla^\times\nabla^\times + (\nabla\eta)\times\nabla^\times = \chi_\eta(\eta\nabla^\times\nabla^\times + (\nabla\eta)\times\nabla^\times). && (16)\end{aligned}$$

In addition, we have

$$\|\nabla\cdot\nabla(A+I)^{-1}\| \leq \varepsilon_+, \quad \|\nabla^\times\nabla^\times(M+I)^{-1}\| \leq \varepsilon_+, \tag{17}$$

and

$$\|\nabla(A+I)^{-1}\| \leq \sqrt{\varepsilon_+}, \quad \|\nabla^\times(M+I)^{-1}\| \leq \sqrt{\varepsilon_+}. \tag{18}$$

The estimates (17) are obvious, we prove (18) for M, the other case being similar. To do so we just notice that, for any $\Psi \in L^2(\mathbb{R}^3;\mathbb{C}^3)$, we have

$$\begin{aligned} \|\nabla^\times(M+I)^{-1}\Psi\|^2 &\leq \varepsilon_+\langle(M+I)^{-1}\Psi, M(M+I)^{-1}\Psi\rangle \\ &\leq \varepsilon_+\langle\Psi,(M+I)^{-1}\Psi\rangle \leq \varepsilon_+\|\Psi\|^2. && (19)\end{aligned}$$

Since $(A+I)^{-n}\chi_\eta$ is a Hilbert-Schmidt operator for $n > \frac{d}{2}$ [FK3, Proposition 42], the lemma follows from from (15),(17), (18) for $W = A$. If $W = \mathbf{M}$, we have

$$(\mathbf{M}+I)^{-1}\mathbf{M}_\eta(\mathbf{M}+I)^{-1} = P_{\mathbb{S}}(M+I)^{-1}M_\eta(M+I)^{-1}I_{\mathbb{S}}, \tag{20}$$

so the lemma follows from (16)-(18), since $P_{\mathbb{S}}(M+I)^{-1}\chi_\eta$ is Hilbert-Schmidt [FK4]. $\square$

Let (a,b) be a gap in the spectrum of W_0 , and suppose $\lambda \in \sigma(W) \cap (a,b)$. It follows from (8) that λ must be an isolated eigenvalues with finite multiplicity; let ψ be a corresponding eigenmode: $\psi \in \mathcal{D}(W)$ with $W\psi = \lambda\psi$.

In the case of Maxwell operators, (a,b) is also a gap in the spectrum of M_0, $\psi \in \mathcal{D}(M)$, and $M\psi = \lambda\psi$. Let us use $\widetilde{W}$ to denote either A or M. For any $\varphi \in \mathcal{D}(\widetilde{W}_0)$ we have

$$\langle \varphi, (\widetilde{W}_0 - \lambda I)\psi \rangle = -\langle \varphi, \widetilde{W}_\eta \psi \rangle, \tag{21}$$

so, taking $\varphi = (\widetilde{W}_0 - \lambda I)^{-1}\zeta$ we get

$$\langle \zeta, \psi \rangle = -\langle (\widetilde{W}_0 - \lambda I)^{-1}\zeta, \widetilde{W}_\eta \psi \rangle = -\epsilon \langle \chi_\eta \nabla^\sharp (\widetilde{W}_0 - \lambda I)^{-1}\zeta, \eta \nabla^\sharp \psi \rangle, \tag{22}$$

where either $\nabla^\sharp = \nabla$ and $\epsilon = -1$ if $\widetilde{W} = A$, or $\nabla^\sharp = \nabla^\times$ and $\epsilon = 1$ if $\widetilde{W} = M$. Choosing $\zeta = \chi_x \psi$ we get

$$\begin{aligned} \|\chi_x \psi\|^2 &\leq \left\| \chi_\eta \nabla^\sharp (\widetilde{W}_0 - \lambda I)^{-1} \chi_x \right\| \|\psi\| \|\eta \nabla^\sharp \psi\| \\ &\leq \sqrt{\lambda \varepsilon_{0,+}} \|\eta\|_\infty \left\| \chi_\eta \nabla^\sharp (\widetilde{W}_0 - \lambda I)^{-1} \chi_x \right\| \|\psi\|^2. \end{aligned} \tag{23}$$

Since we have [FK3, Lemma 13], [FK4]

$$\left\| \chi_y \nabla^\sharp (\widetilde{W}_0 - \lambda I)^{-1} \chi_x \right\| \leq C \frac{1+\lambda}{\lambda_{a,b}} \mathrm{e}^{-m_\lambda |x-y|}, \tag{24}$$

where $C < \infty$ is some constant (depending only on $\varepsilon_{0,\pm}$ and the dimension d), $\lambda_{a,b} = \min\{\lambda - a, b - \lambda\}$, and

$$m_\lambda = \frac{\lambda_{a,b}}{4\left[\varepsilon_-^{-1} + \lambda + \lambda_{a,b}\right]}, \tag{25}$$

it follows from (23) that ψ decays exponentially fast, with a rate depending on the distance $\lambda_{a,b}$ from the eigenvalue to the edges of the gap.

Theorem 1 is proven.

4 Creation of localized eigenvalues

We now prove Theorem 2. In view of Theorem 1, if we can show that W has spectrum in the gap (a,b) in the spectrum of W_0 (i.e., $\sigma(W) \cap (a,b) \neq \emptyset$), this spectrum must consist of localized eigenvalues only.

So let us select $\mu \in (a,b)$, and pick $0 < \gamma < 1$ such that the interval $[\mu(1-\gamma), \mu(1+\gamma)]$ is contained in the gap, i.e., $[\mu(1-\gamma), \mu(1+\gamma)] \subset (a,b)$.

Since $\sigma(\mathbf{M}) = \sigma(M)$, to show $\sigma(W) \cap [\mu(1-\gamma), \mu(1+\gamma)] \neq \emptyset$ it suffices to show that

$$\sigma(\widetilde{W}) \cap [\mu(1-\gamma), \mu(1+\gamma)] \neq \emptyset. \tag{26}$$

Recall

$$\operatorname{dist}\left(\mu, \sigma\left(\widetilde{W}\right)\right) = \min_{\varphi \in \mathcal{D}(\widetilde{W})} \frac{\left\| \left(\widetilde{W} - \mu I\right) \varphi \right\|}{\|\varphi\|}. \tag{27}$$

In particular, if we can find $\varphi \in \mathcal{D}(\widetilde{W})$ such that

$$\left\| \left(\widetilde{W} - \mu I\right) \varphi \right\| \leq \gamma\mu \|\varphi\|, \tag{28}$$

then (27) and (28) will imply (26), and we can conclude that the operator W has at least one localized eigenvalue in the interval $[\mu(1-\gamma), \mu(1+\gamma)]$.

So let $\varepsilon(x) = \varepsilon_{\epsilon,\ell}(x)$ be as in (9). We will construct a function $\varphi \in \mathcal{D}(\widetilde{W})$, with $\|\varphi\| = 1$ and support in Ω_ℓ°, such that (28) holds. In this case the inequality (28) takes the following simple form:

$$\left\| \left(\epsilon^{-1}\Gamma - \mu I\right) \varphi \right\| \leq \gamma\mu, \tag{29}$$

where $\Gamma = -\nabla \cdot \nabla = -\Delta$ in the acoustic case, and $\Gamma = \nabla^\times \nabla^\times$ in the Maxwell case. Notice that (29) is the same as

$$\|(\Gamma - \mu' I)\varphi\| \leq \delta', \tag{30}$$

with $\mu' = \mu\epsilon$ and $\delta' = \gamma\mu\epsilon$.

We start with generalized eigenfunctions for the operator Γ. If $\Gamma = -\Delta$, we pick $\kappa \in \mathbb{R}^d$ such that $|\kappa|^2 = \mu'$, and set $f_A(x) = \mathrm{e}^{i\kappa\cdot x} \in C^\infty(\mathbb{R}^d)$. If $\Gamma = \nabla^\times\nabla^\times$, we pick $\kappa, \sigma \in \mathbb{R}^3$ such that $|\kappa|^2 = \mu'$, $|\sigma| = 1$, $\kappa \cdot \sigma = 0$, and set $f_M(x) = \mathrm{e}^{i\kappa\cdot x}\sigma = f_A(x)\sigma \in C^\infty(\mathbb{R}^3; \mathbb{C}^3)$. We will write $f_\Gamma(x)$ for either $f_A(x)$ or $f_M(x)$; notice that, pointwise, we have

$$(\Gamma f_\Gamma)(x) = \mu' f_\Gamma(x). \tag{31}$$

To produce the desired φ satisfying (28), we will restrict f_Γ to Ω_ℓ° in suitable manner, and prove (30). We now take

$$\varphi_\Gamma(x) = \xi_\ell(x) f_\Gamma(x), \tag{32}$$

where ξ_ℓ is a real valued C^2 function with support in Ω_ℓ°, with $\|\xi_\ell\| = 1$. Notice $\varphi_M(x) = \varphi_A(x)\sigma$. It is clear that $\varphi_\Gamma \in \mathcal{D}(\widetilde{W}) \cap \mathcal{D}(\Gamma)$, with support in Ω_ℓ°, and

$$\|\varphi_\Gamma\| = \|\xi_\ell\| = 1. \tag{33}$$

We consider first the acoustic case. We have

$$\begin{aligned} -\Delta\varphi_A &= \xi_\ell(-\Delta f_A) + (-\Delta\xi_\ell)f_A - 2(\nabla\xi_\ell)\cdot(\nabla f_A) && (34)\\ &= \epsilon\mu'\varphi_A + (-\Delta\xi_\ell)f_A - 2if_A\kappa\cdot(\nabla\xi_\ell), && (35)\end{aligned}$$

so

$$\begin{aligned} (-\Delta - \mu' I)\varphi_A &= (-\Delta\xi_\ell)f_A - 2(\nabla\xi_\ell)\cdot(\nabla f_A) && (36)\\ &= (-\Delta\xi_\ell - 2i\kappa\cdot\nabla\xi_\ell)f_A. && (37)\end{aligned}$$

Thus

$$\|(-\Delta - \mu' I)\varphi_A\|^2 = \|\Delta\xi_\ell\|^2 + 4\mu'\|\hat\kappa\cdot\nabla\xi_\ell\|^2, \tag{38}$$

where $\hat\kappa = |\kappa|^{-1}\kappa$.

We now use a scaling argument (i.e., write $\xi_\ell(x) = \xi(\ell^{-1}x)$) to conclude that to obtain (30), it suffices to find $\xi \in C_0^2(\Omega, \mathbb{R})$ with $\|\xi\| = 1$, and a unit vector $\hat\kappa$, such that

$$\ell^{-4}\|\Delta\xi\|^2 + 4\ell^{-2}\mu'\|\hat\kappa\cdot\nabla\xi_\ell\|^2 \le \delta'^2, \tag{39}$$

so we have (26) if

$$\ell^2\epsilon > \inf_{\xi,\hat\kappa}\left\{\frac{2}{\gamma^2\mu}\|\hat\kappa\cdot\nabla\xi\|^2\left(1 + \sqrt{1 + \frac{\|\Delta\xi\|^2}{4\|\hat\kappa\cdot\nabla\xi\|^4}\gamma^2}\right)\right\}. \tag{40}$$

Thus (10) implies (26).

We now turn to the electromagnetic case. Now $d = 3$ and we have

$$\nabla^\times\nabla^\times\varphi_M = (-\nabla\cdot\nabla\varphi_A)\sigma. \tag{41}$$

Thus

$$\left\|\left(\epsilon^{-1}\nabla^\times\nabla^\times - \mu I\right)\varphi_M\right\| = \left\|\left(-\epsilon^{-1}\nabla\cdot\nabla - \mu I\right)\varphi_A\right\|, \tag{42}$$

so in the electromagnetic case we also have (26) if (10) holds with $d = 3$.

Since (11) follows from (25), Theorem 2 is proven.

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