Explicit Constants for Rellich Inequalities in $L_p(\Omega)$

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1. Introduction

Let Ω be a bounded region in a complete Riemannian manifold with smooth boundary $\partial\Omega$. Let $C^\infty(\overline{\Omega})$, $C_0^\infty(\overline{\Omega})$ and $C_c^\infty(\Omega)$ denote respectively the space of smooth functions on $\overline{\Omega}$, the subspace consisting of such functions which vanish on $\partial\Omega$, and the subspace of such functions which vanish in a neighbourhood of $\partial\Omega$. For those whose main interest is in spectral theory in Euclidean space we mention that our main results are also new in that context.

We investigate the existence and explicit determination of constants c and weights X and Y on Ω such that the *Rellich inequality*

$$\int_\Omega X|u|^p \le c\int_\Omega Y|\Delta u|^p$$

holds for some $1 \le p < \infty$ and all $u \in C_0^\infty(\overline{\Omega})$, or all $u \in C_c^\infty(\Omega)$. Although we do not claim that the constants c which we obtain are sharp for all choices of X and Y, there are important cases in which this can be shown. The power p is allowed to take any value in the range $1 \le p < \infty$.

A large literature exists for the generalized *Hardy inequality*

$$\int_\Omega X|u|^p \le c\int_\Omega Y|\nabla u|^p,$$

particularly in the one–dimensional case ([14],[4],[7],[13],[15]; see also [11; Section 2] with historical remarks), the best known being Hardy's original inequality

$$\int_a^b \frac{|u(x)|^p}{x^p}\,dx \le \left(\frac{p}{p-1}\right)^p \int_a^b |u'(x)|^p\,dx,$$

the case $p = 2$ of which appeared in [8; p. 110], the general case for $1 < p < \infty$ in [9; formula (4)]. The Rellich inequality, however, was published for the first time in the form

$$\frac{d^2(d-4)^2}{16}\int \frac{|u(x)|^2}{|x|^4}\,dx \le \int |\Delta u(x)|^2\,dx \qquad (*)$$

in the proceedings of the Amsterdam ICM in 1954 [17; pp. 247–249], but proved earlier in lectures at New York University in 1953 published posthumously in [18; Chapter II, §7, Theorem 1]. It is much less well studied, and sharp constants have previously only been obtained for particular weights in the case $p = 2$ ([2],[3]). Among the many nice proofs for $(*)$ we mention that our method of approach owes a lot to Schmincke [19; Lemma 2]. However, we treat the general value of p and make no use of special properties in Euclidean space R^d. We also obtain the sharp constant c for the higher order Rellich inequality

$$\int_{\mathrm{R}^d} \frac{|u(x)|^p}{|x|^\beta}\,dx \le c\int_{\mathrm{R}^d} \frac{|\Delta^m u(x)|^p}{|x|^\alpha}\,dx$$

for suitable α, β, m, d, p and all $u \in C_c^\infty(\mathrm{R}^d\backslash\{0\})$.

Our results have two obvious applications. The first involves putting $p = 2$ and finding explicit constants c for which a potential X satisfies the quadratic form inequality

$$X \le c(-\Delta)^m,$$

where the operator on the right–hand side satisfies Dirichlet boundary conditions in $L_2(\Omega)$. This bound enables one to define $(-\Delta)^m - \lambda X$ as a quadratic form sum for $0 \le \lambda < c^{-1}$, and ensures that its spectrum is non–negative.

For the second application we determine conditions under which a potential V has operator relative bound less than 1 with respect to $-\Delta$ in $L_p(\Omega)$. This ensures that the operator domain of $-\Delta + V$ is the same as that of $-\Delta$ in $L_p(\Omega)$; once again we impose Dirichlet boundary conditions.

2. The basic inequalities

We assume that Ω is a bounded open set with smooth boundary $\partial\Omega$ in a complete Riemannian manifold of dimension $d > 1$. We assume further that $1 \leq p < \infty$ and that $V \in L_{1,loc}(\Omega)$ has weak partial derivatives of order up to two in $L_{1,loc}(\Omega)$. When we write "'$X > 0$"', we mean that $X(x) > 0$ for almost every x in Ω. In the various integral inequalities below, we allow the values of the integrals on the right–hand sides to be $+\infty$. We start with a particular Hardy type inequality (for the case $p = 2$ cf. [13; Lemma 2]).

Theorem 1. *If $\Delta V < 0$ then*

$$\int |\Delta V||u|^p \leq p^p \int \frac{|\nabla V|^p}{|\Delta V|^{p-1}} |\nabla u|^p$$

for all $u \in C_c^\infty(\Omega)$.

Proof. Let $\epsilon > 0$. Then $0 \leq u_\epsilon := (|u|^2 + \epsilon^2)^{p/2} - \epsilon^p \in C_c^\infty(\Omega)$ (u_ϵ has the same support as u) and

$$\begin{aligned} \int |\Delta V| u_\epsilon = -\int \Delta V u_\epsilon &= \int \nabla V \cdot \nabla \left\{ (|u|^2 + \epsilon^2)^{p/2} \right\} \\ &\leq p \int |u| (|u|^2 + \epsilon^2)^{(p-2)/2} |\nabla u||\nabla V|. \end{aligned}$$

By dominated convergence and H"older's inequality:

$$\begin{aligned} \int |\Delta V||u|^p &\leq p \int |u|^{p-1} |\Delta V|^{(p-1)/p} |\nabla u||\Delta V|^{-(p-1)/p} |\nabla V| \\ &\leq p \left(\int |\Delta V||u|^p \right)^{(p-1)/p} \left(\int \frac{|\nabla V|^p}{|\Delta V|^{p-1}} |\nabla u|^p \right)^{1/p}. \end{aligned}$$

We complete the proof by cancelling and raising both sides to the power p. $\square$

The idea of this proof provides a clue to our basic tool for obtaining a Rellich inequality.

Lemma 2. *If $V \geq 0$, $\Delta V < 0$, and there is a $c \geq 0$ such that*

$$c \int |\Delta V||u|^p \leq p(p-1) \int\limits_{\{x\in\Omega;\, u(x)\neq 0\}} V|u|^{p-2}|\nabla u|^2 < \infty$$

for all $u \in C_c^\infty(\Omega)$, then

$$(1+c)^p \int |\Delta V||u|^p \leq p^p \int \frac{V^p}{|\Delta V|^{p-1}}|\Delta u|^p$$

for all $u \in C_c^\infty(\Omega)$.

Remark. Assuming V to be non– negative is not a strong restriction: if $V(x) < 0$ for all x in a neighbourhood of some $x_0 \in \Omega$, then the extra integral condition cannot be fulfilled with a non- -negative c for all u concentrated near x_0. □

Proof of Lemma 2. We may assume u to be real–valued by using the identity (cf. [5; p. 176])

$$\forall z \in \mathrm{C} : |z|^p = \left(\int_{-\pi}^{\pi} |\cos\vartheta|^p \, d\vartheta \right)^{-1} \int_{-\pi}^{\pi} |re(z)\cos\vartheta + im(z)\sin\vartheta|^p \, d\vartheta,$$

proved by writing $z = r(\cos\varphi + i\sin\varphi)$ and simplifying.

As in the proof of Theorem 1, we have for $\epsilon > 0$:

$$\begin{aligned} \int |\Delta V| u_\epsilon &= -\int \Delta V u_\epsilon = -\int V\Delta(u_\epsilon) \\ &= -\int V\{p(p-2)u^2(u^2+\epsilon^2)^{(p-4)/2} + p(u^2+\epsilon^2)^{(p-2)/2}\}|\nabla u|^2 \\ &\qquad -p\int Vu(u^2+\epsilon^2)^{(p-2)/2}\Delta u. \end{aligned}$$

Hence

$$\begin{aligned} \int [|\Delta V|u_\epsilon + V\{p(p-2)u^2(u^2+\epsilon^2)^{(p-4)/2} + p(u^2+\epsilon^2)^{(p-2)/2}\}|\nabla u|^2] \\ \leq p\int V|u|(u^2+\epsilon^2)^{(p-2)/2}|\Delta u|. \end{aligned}$$

The right–hand side goes to $p\int V|u|^{p-1}|\Delta u|$ by dominated convergence, the integrands being bounded by $V(\max|u|^2+1)^{(p-1)/2}\max|\Delta u|$, which is in L_1 since u has compact support. As the integrands on the left–hand side are non–negative and tend to $|\Delta V||u|^p+p(p-1)V|u|^{p-2}|\nabla u|^2$ pointwise if $u(x)\neq 0$, it follows from Fatou's lemma that

$$\begin{aligned}(1+c)\int|\Delta V||u|^p &\leq p\int|\Delta V|^{(p-1)/p}|u|^{p-1}V|\Delta V|^{-(p-1)/p}|\Delta u|\\ &\leq p\left(\int|\Delta V||u|^p\right)^{(p-1)/p}\left(\int\frac{V^p}{|\Delta V|^{p-1}}|\Delta u|\right)^{1/p},\end{aligned}$$

using H"older's inequality in the last step. Again the proof is completed by taking the p–th power. □

We now need an ingredient to fulfil the assumption of Lemma 2.

Lemma 3. *Let $1<p<\infty$. If $0<V\in C(\Omega)$ with $\Delta V<0$ and $\Delta(V^\delta)\leq 0$ for some $\delta>1$, then*

$$(\delta-1)\int|\Delta V||u|^p\leq p^2\int\limits_{\{x\in\Omega;u(x)\neq 0\}}V|u|^{p-2}|\nabla u|^2<\infty$$

for all $u\in C_c^\infty(\Omega)$.

Remarks. 1. If $\Delta(V^\delta)<0$, then $\Delta V<0$ is a consequence of the rest of the assumptions; see (1) in the proof below.
2. The integral on the right-hand side can be infinite for $p=1$: consider a function u which behaves like x_1 near 0; then $|u|^{-1}|\nabla u|^2\sim\frac{1}{|x_1|}$. □

Proof of Lemma 3. Note that by an interpolation argument V has square integrable first derivatives; cf. [10; Lemma 3]. We then have by regularization:

$$0\geq\Delta(V^\delta)=\delta V^{\delta-2}\left\{(\delta-1)|\nabla V|^2+V\Delta V\right\}. \qquad (1)$$

Hence $(\delta-1)|\nabla V|^2\leq V|\Delta V|$. From Theorem 1 we obtain for $u\in C_c^\infty(\Omega)$:

$$\begin{aligned}(\delta-1)\int|\Delta V||u|^2 &\leq 4(\delta-1)\int\frac{|\nabla V|^2}{|\Delta V|}|\nabla u|^2\\ &\leq 4\int V|\nabla u|^2=4\int\limits_{\{x\in\Omega;u(x)\neq 0\}}V|\nabla u|^2, \qquad (2)\end{aligned}$$

the latter since $|\{x \in \Omega; u(x) = 0 \neq |\nabla u(x)|\}| = 0$. This is the case $p = 2$.

For any other p put $w = (u^2+\epsilon^2)^{p/4} - \epsilon^{p/2}$. Because $0 \leq w \leq |u|$, the left-hand side of (2) with u replaced by w tends to $(\delta - 1) \int |\Delta V||u|^p$ by dominated convergence. Similarly, on the right-hand side since $\nabla w = \frac{p}{2} u(u^2+\epsilon^2)^{(p-4)/4} \nabla u$, we have

$$|\nabla w|^2 \leq \frac{p^2}{4} |\nabla u|^2 \cdot \begin{cases} (u^2+\epsilon^2)^{(p-2)/2}, & \text{if} \quad p > 2, \\ |u|^{p-2}, & \text{if} \quad p < 2, \end{cases}$$

and thus in both cases

$$|\nabla w(x)|^2 V(x) \to \frac{p^2}{4} |u(x)|^{p-2} |\nabla u(x)|^2 V(x), \text{ if } u(x) \neq 0.$$

As the proof of Lemma 2 shows that

$$\int\limits_{\{x \in \Omega; u(x) \neq 0\}} V|u|^{p-2} |\nabla u|^2 < \infty,$$

again dominated convergence completes the proof. $\square$

It is now easy to combine the above into our first main theorem (for the case $p = 2$ cf. [3; Theorem 5]).

Theorem 4 *If $0 < V \in C(\Omega)$ with $\Delta V < 0$ and $\Delta(V^\delta) \leq 0$ for some $\delta > 1$, then*

$$\int |\Delta V||u|^p \leq \frac{p^{2p}}{\{(p-1)\delta + 1\}^p} \int \frac{V^p}{|\Delta V|^{p-1}} |\Delta u|^p$$

for all $u \in C_c^\infty(\Omega)$.

Proof. By Lemma 3 we may put $c = \frac{p-1}{p}(\delta - 1)$ in Lemma 2. $\square$

The application of the above theorems depends upon making a choice of the potential V on Ω. Among the many possibilities those associated with the Green function of the region Ω are of particular interest. Let $H = -\Delta$ in $L_2(\Omega)$ subject to Dirichlet boundary conditions. The operator H^{-1} is bounded on $L_2(\Omega)$ and has a positive Green function $G(x, y)$ which vanishes on the boundary $\partial\Omega$ (see [6; Theorem 4.6.11]). Since H^{-1} is also a bounded operator on $L_p(\Omega)$ for any $p \geq 1$, there exists a constant $a > 0$ such that

$$0 < \int_\Omega G(x, y) dx \leq a$$

for all $y \in \Omega$. We can use the Green function to construct suitable weights in two ways.

Example 5. Let $0 < X \in L_1(\Omega)$ and put $V = (H^{-1}X)^\vartheta$ where $0 < \vartheta < 1$. Then $0 < V \in C(\Omega)$ and for $\delta = \vartheta^{-1}(> 1)$ we have $\Delta(V^\delta) = \Delta(H^{-1}X) = -X < 0$, such that the conditions of Theorem 4 are all satisfied (cf. Remark 1 after Lemma 3). $\square$

A singular case of the above is obtained in the following way.

Example 6. Choose $y \in \Omega$ and put

$$V(x) = G(x, y)^\vartheta$$

for all $x \in \Omega_y := \Omega \setminus \{y\}$, where $0 < \vartheta < 1$. Then V is positive, and as a consequence of elliptic regularity theorems it is smooth on Ω_y with

$$\begin{aligned} \Delta V &= \nabla \cdot (\vartheta G^{\vartheta-1} \nabla_x G) \\ &= \vartheta G^{\vartheta-1} \Delta_x G - \vartheta(1-\vartheta) G^{\vartheta-2} |\nabla_x G|^2 = -\vartheta(1-\vartheta) G^{\vartheta-2} |\nabla_x G|^2. \end{aligned}$$

Hence $\Delta V < 0$ if $\nabla_x G(x, y) \neq 0$ for every $x \in \Omega_y$. Furthermore, if $\delta = \vartheta^{-1}(> 1)$, then

$$\Delta(V^\delta) = \Delta_x G = 0,$$

so Theorem 4 is valid for all $u \in C_c^\infty(\Omega_y)$. $\square$

3. Relative boundedness in $L_p(\Omega)$

If $H = -\Delta$ in $L_2(\Omega)$ subject ot Dirichlet boundary conditions, then H has compact resolvent ([6; Theorem 1.6.8]). There exists a complete orthonormal set of eigenfunctions $\{\varphi_n\}_{n \in \mathrm{N}}$ such that $H\varphi_n = \lambda_n \varphi_n$, where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$. It is standard that $\lambda_n \sim n^{2/d}$ as $n \to \infty$, precise asymptotics being given by the Weyl formula. The operators e^{-Ht} for $t \geq 0$ are contractions on $L_p(\Omega)$ for all $1 \leq p < \infty$ and induce strongly continuous one-parameter contraction semigroups on each such space, with generators which we denote by H_p (cf. [6; Theorem 1.3.5]). The eigenfunctions φ_n lie in $C_0^\infty(\overline{\Omega})$ by elliptic regularity theorems ([1; Lemma 9.1, Theorem 9.9]). Furthermore we have:

Lemma 7. *The space $C_0^\infty(\overline{\Omega})$ is an operator core for H_p.*

Proof. If $f \in C_0^\infty(\overline{\Omega})$ then f lies in the domain of H_2 and $g := (H_2+1)f \in C^\infty(\overline{\Omega})$. But $(H_2+1)^{-1}$ and $(H_p+1)^{-1}$ are consistent bacause the resolvents are given by integration of the semigroups, so $f = (H_2+1)^{-1}g = (H_p+1)^{-1}g$ lies in the domain of H_p as well.

The density of $C_0^\infty(\overline{\Omega})$ in $L_p(\Omega)$ is well-known, and the inclusion

$$e^{-H_pt}C_0^\infty(\overline{\Omega}) \subset C_0^\infty(\overline{\Omega})$$

is a consequence of elliptic regularity theorems as before. We may now refer to [5; Theorem 1.9] to see that $C_0^\infty(\overline{\Omega})$ is a core for H_p. □

A potential W on Ω is relatively bounded with respect to H_p, if there are numbers $a, b > 0$ such that

$$||Wf||_p^p \leq a^p||H_pf||_p^p + b^p||f||_p^p$$

for all $f \in C_0^\infty(\overline{\Omega})$. The infimum of the numbers a is then called the relative bound of W.

It is known, that if $W \in L_q(\Omega)$ for $q \geq \frac{d}{2}$, then W has relative bound 0 with respect to H_p, if $1 < p < \frac{d}{2}$; for $\Omega = \mathrm{R}^d$ this is proved in [16; pp. 171f] if $p = 2$ (in which case it also applies to $q > 2$ if $d = 4$ and $q \geq \frac{3}{2}$ if $d = 3$) and in [12; Theorem 2.5] for other p, and the general case follows from the fact that the relevant integral operator for $L_2(\Omega)$ is pointwise dominated by the operator for $L_2(\mathrm{R}^d)$. For potentials in certain weak L_q spaces one has relative boundedness as well, but the actual relative bound is hard to compute in general (cf. [20; Theorem 3.6]). Our contribution is to clarify the issues involved in determining the relative bound and to provide some new inequalities.

The following modifications of arguments in [7] establish that the relative bound may be computed from purely local quantities. We set out from a special partition of unity.

Lemma 8. *Let $1 \leq p < \infty$ and let $\{U_1, ..., U_n\}$ be an open cover of the set $\overline{\Omega}$. Then there exist non– negative smooth functions ψ_i with supp$(\psi_i) \subset U_i$ and $\sum_{i=1}^{n} \psi_i(x)^p = 1$ for all $x \in \overline{\Omega}$.*

Proof. A standard argument shows, that there exist non–negative smooth functions u_i with $\operatorname{supp}(u_i) \subset U_i$ and $\sum_{i=1}^{n} u_i(x) > 0$ for all $x \in \overline{\Omega}$. We now define

$$\varphi_i(x) = \begin{cases} \exp\left(-\frac{1}{u_i(x)}\right), & \text{if } u_i(x) > 0, \\ 0, & \text{if } u_i(x) = 0, \end{cases}$$

to obtain smooth functions $\varphi_i \geq 0$ with $\operatorname{supp}(\varphi_i) \subset U_i$ and such that φ_i^α are smooth for any $\alpha > 0$. It is immediate that

$$\xi(x) = \left\{ \sum_{i=1}^{n} \varphi_i(x)^p \right\}^{1/p}$$

is positive and smooth in a neighbourhood of $\overline{\Omega}$, where we may define the functions ψ_i, which have the desired properties, by $\psi_i(x) = \dfrac{\varphi_i(x)}{\xi(x)}$. □

With this, relative boundedness on subsets can now be combined to yield relative boundedness on the whole region, together with an upper limit for the relative bound.

Lemma 9. *Let $1 < p < \infty$ and let $\{U_1, ..., U_n\}$ be an open cover of the set $\overline{\Omega}$. Assume that*

$$\int_{U_i} |Wf|^p \leq c_i^p \int_{U_i} |\Delta f|^p + d_i^p \int_{U_i} |f|^p$$

for all $f \in C_0^\infty\overline{(U_i \bigcap \Omega)}$. Then for any $c > \max\limits_{1 \leq i \leq n} \{c_i\}$, there exists $d > 0$ such that

$$\int_{\Omega} |Wf|^p \leq c^p \int_{\Omega} |\Delta f|^p + d^p \int_{\Omega} |f|^p$$

for all $f \in C_0^\infty(\overline{\Omega})$.

Proof. Let ψ_i be as in Lemma 8, and let $f \in C_0^\infty(\overline{\Omega})$. Then

$$\begin{aligned} \int_{\Omega} |Wf|^p &= \sum_{i=1}^{n} \int_{U_i} |W\psi_i f|^p \leq \sum_{i=1}^{n} c_i^p \int_{U_i} |\Delta(\psi_i f)|^p + \sum_{i=1}^{n} d_i^p \int_{U_i} |\psi_i f|^p \\ &\leq \left(\max_{1 \leq i \leq n} \{c_i\}\right)^p \sum_{i=1}^{n} \int_{U_i} |\Delta(\psi_i f)|^p + \left(\max_{1 \leq i \leq n} \{d_i\}\right)^p \sum_{i=1}^{n} \int_{U_i} |\psi_i f|^p. \end{aligned}$$

For $\epsilon > 0$ there exists $b_\epsilon > 0$ such that

$$\begin{aligned}\int_{U_i} |\Delta(\psi_i f)|^p &= \int_{U_i} |(\Delta\psi_i) f + 2\nabla\psi_i \cdot \nabla f + \psi_i \Delta f|^p \\ &\leq \int_{U_i} \{(1+\epsilon)\psi_i^p |\Delta f|^p + b_\epsilon |\nabla\psi_i|^p |\nabla f|^p + b_\epsilon |\Delta\psi_i|^p |f|^p\},\end{aligned}$$

and since $1 < p < \infty$ a Sobolev imbedding theorem allows us to bound the right–hand side by

$$(1+\epsilon)\int_{U_i} \psi_i^p |\Delta f|^p + \tilde{b}_\epsilon \int_{U_i} |f|^p + \epsilon \int_{U_i} |\Delta f|^p,$$

where $\tilde{b}_\epsilon$ depends upon ϵ, p and ψ_i. Summing over i completes the proof of the lemma. $\square$ For $U \subset \Omega$ let $c(U)$ denote the relative bound of W with respect to $-\Delta$ in the subspace $L_p(U)$, this being a monotonic function of the subregion U. If $x \in \overline{\Omega}$ we then define $c(x)$ to be the infimum of $c(U \bigcap \Omega)$ where U is any open subset of the manifold M such that $x \in U$. The monotonicity property establishes that we may take the infimum over a sequence of decreasing neighbourhoods of x, so $c(x)$ depends only upon the local structure of the potential W around x.

Theorem 10. *The function $c(x)$ is upper semi–continuous on $\overline{\Omega}$, and the relative bound of W in $L_p(\Omega)$ equals* $\max\left\{c(x); x \in \overline{\Omega}\right\}$.

Proof. The upper semicontinuity of $c(x)$ follows from the limiting procedure used in its definition and implies the existence of a maximum value. If $c > \max\{c(x); x \in \overline{\Omega}\}$, then for every $x \in \overline{\Omega}$ there exists a neigbourhood U_x of x such that $c(U_x \bigcap \Omega) < c$. These neighbourhoods cover $\overline{\Omega}$, and the theorem follows by applying Lemma 9 to a finite subcovering. $\square$

The preceding theorem reduces the computation of the relative bound of a potential to a local problem. There are two issues involved in the application of Theorem 4 in this context. The first is the identification of potentials V for which the weight $\dfrac{V^p}{|\Delta V|^{p-1}}$ is bounded. This is discussed in the next section. The second is the problem that Theorem 4 is proved for $u \in C_c^\infty(\Omega)$, and must be extended to $u \in C_0^\infty(\overline{\Omega})$ in view of Lemma 7.

Theorem 11. *Let V and all its derivatives of order up to two lie in $L_1(\Omega)$. If also $0 < V \in C(\Omega)$, $\Delta V < 0$ and $\Delta(V^\delta) \le 0$ for some $\delta > 1$, then*

$$\int |\Delta V||u|^p \le \frac{p^{2p}}{\{(p-1)\delta+1\}^p} \int \frac{V^p}{|\Delta V|^{p-1}} |\Delta u|^p$$

for all $u \in C_0^\infty(\overline{\Omega})$.

Proof. The boundary behaviour of u is of importance only when we perform the integrations by parts in the proofs of Theorem 1 and Lemma 2. In both cases the starting point is $\int (\Delta V) u_\epsilon$, where $u_\epsilon = (|u|^2 + \epsilon^2)^{p/2} - \epsilon^p$ as before. The formula

$$\nabla u_\epsilon = p(|u|^2+\epsilon^2)^{(p-2)/2} u \nabla u$$

establishes that if $u \in C_0^\infty(\overline{\Omega})$, then u_ϵ and ∇u_ϵ vanish on $\partial\Omega$. Thus the integration by parts procedure is valid under the stated assumptions on V. $\square$

There is a higher order analogue of Theorems 4-11 which we only investigate in the case $\Omega = \mathrm{R}^d$ in the next section. Since the Schwartz space $\mathcal{S}$ is invariant under the semigroup $e^{-H_2 t}$ by Fourier transform, $\mathcal{S}$ is an operator core for H_p in $L_p(\mathrm{R}^d)$ by [5; Theorem 1.9]. By density in $W_p^k(\mathrm{R}^d), C_c^\infty(\mathrm{R}^d)$ is a core as well, so that we may restrict ourselves to test functions again.

4. Analysis of local singularities

In this section we complete our analysis by giving a detailed treatment of local singularities in Euclidean space. In order to make the proofs transparent, we put $\Omega = \mathrm{R}^d \backslash \{0\}$, the fact that this is unbounded causing no extra problems. The extension to functions which do not vanish at 0 and which lie in the appropriate Sobolev spaces will be given as a corollary at the end. Our main achievment is to obtain sharp constants for bounds of the type

$$\int_{\mathrm{R}^d} \frac{|u(x)|^p}{|x|^\beta} dx \le c \int_{\mathrm{R}^d} \frac{|\Delta u(x)|^p}{|x|^\alpha} dx.$$

These bounds yield by iteration similar bounds in which we replace Δu on the right-hand side by $\Delta^m u$ or by $\nabla(\Delta^m u)$. It is remarkable that we are able

to show that these iterated bounds also have sharp constants. While some of our results are known for $p = 2$, we believe that they are entirely new for other values of p.
If $V(x) = |x|^{-\sigma}$ for some $\sigma > 0$, then

$$\Delta(V^\delta)(x) = \sigma\delta\,(\sigma\delta - (d-2))\,|x|^{-(2+\sigma\delta)}$$

for any $\delta \geq 1$, so we may apply Theorem 4 whenever $\sigma < d-2$ and, to get the best constant, for $\delta = \dfrac{d-2}{\sigma}$, yielding

$$\int \frac{|u(x)|^p}{|x|^{2+\sigma}} dx \leq c^p \int \frac{|\Delta u(x)|^p}{|x|^{2+\sigma-2p}} dx, \qquad (3)$$

for all $u \in C_c^\infty(\mathrm{R}^d \backslash \{0\})$ and all $0 < \sigma < d-2$, where

$$c = \frac{p^2}{\{d-2-\sigma\}\{(p-1)(d-2)+\sigma\}}.$$

This is just the case of Example 6 with $G(x,0) = |x|^{-(d-2)}$, and for $p = 2 = \sigma$ it includes Rellich's original inequality for $d > 4$; for $p = 2, \sigma > 2$, cf. [2; Lemma 2].

Repeating this process in a suitable manner we arrive at:

Theorem 12. *Let $m \in \mathrm{N}$ and $2(1+(m-1)p) < \beta < d$. Then*

$$\int \frac{|u(x)|^p}{|x|^\beta} dx \leq c(d,m,p,\beta)^p \int \frac{|\Delta^m u(x)|^p}{|x|^{\beta-2mp}} dx,$$

for all $u \in C_c^\infty(\mathrm{R}^d \backslash \{0\})$, where

$$c(d,m,p,\beta) = \prod_{k=0}^{m-1} \gamma\left(d, \beta - 2(1+kp), \frac{d-2}{\beta - 2(1+kp)}, p\right)$$

and

$$\gamma(d,\beta,\delta,p) = \frac{p^2}{\beta\{d-2-\beta\}\{(p-1)\delta+1\}}.$$

Proof. Since the inequality is obviously true for $m = 0$, we may concentrate on the induction step from $m-1$ to m, which can be carried out by applying (3) with u and σ replaced by $\Delta^{m-1}u$ and $\beta - 2(1+(m-1)p)$, respectively. $\square$

The same procedure which led to (3) can be used in Theorem 1 with the result

$$\int \frac{|u(x)|^p}{|x|^{2+\sigma}}dx \le \tilde{c}^p \int \frac{|\nabla u(x)|^p}{|x|^{2+\sigma-p}}dx, \qquad (4)$$

for all $u \in C_c^\infty(\mathrm{R}^d\backslash\{0\})$ and all $0 < \sigma < d-2$, where

$$\tilde{c} = \frac{p}{d-2-\sigma}.$$

Since there is no restriction on the sign of V in Theorem 1, however, we may put $V(x) = \mathrm{sign}(\sigma)|x|^{-\sigma}$ to prove the validity of (4) for all $-\infty < \sigma < d-2$ (in fact any $\sigma \neq d-2$ replacing $\tilde{c}$ by $|\tilde{c}|$), the case $\sigma = 0$ following from a limiting process.

Here $p = 2$ and $\sigma = 0$ leads to the Hardy inequality for $d > 2$:

$$\int \frac{|u(x)|^2}{|x|^2}dx \le \frac{4}{(d-2)^2}\int |\nabla u(x)|^2 dx.$$

For $p = 2$ and any σ, (4) has been given in [15; Theorem 1]; cf. also [2; Lemma 3], [13; Corollary to Lemma 2], and [11; Lemma 1].

Again this can be extended to a generalized *Hardy- Rellich inequality*:

Theorem 13. *Let $m \in \mathrm{N}_0$ and $2(1+(m-1)p) < \beta < d$. Then*

$$\int \frac{|u(x)|^p}{|x|^\beta}dx \le \tilde{c}(d,m,p,\beta)^p \int \frac{|\nabla(\Delta^m u)(x)|^p}{|x|^{\beta-(2m+1)p}}dx,$$

for all $u \in C_c^\infty(\mathrm{R}^d\backslash\{0\})$, where

$$\tilde{c}(d,m,p,\beta) = \frac{p}{d+2mp-\beta}c(d,m,p,\beta).$$

Proof. In (4) replace u and σ by $\Delta^m u$ and $\beta - 2(1+mp)$, respectively, and combine with Theorem 12. $\square$

The case $p = 2$ of Theorems 12 and 13 can be found in [2; Corollary 4]; cf. also [3; Corollary 6].

Remark. The constants c and $\tilde{c}$ in Theorems 12 and 13, respectively, are sharp. To see this, consider $u(x) = |x|^{-\sigma}$ (near the origin) with a $\sigma > 0$. Then the inequalities in Theorems 12 and 13 should hold as long as $\sigma < \dfrac{d-\beta}{p}$. By induction we have

$$\Delta^m u = (-1)^m \left\{ \prod_{k=0}^{m-1} (\sigma + 2k)\,(d - 2 - (\sigma + 2k)) \right\} |x|^{-(\sigma+2m)}$$

for every $m \in \mathrm{N}_0$, whence

$$|\Delta^m u| = \left\{ \prod_{k=0}^{m-1} (\sigma + 2k)\,(d - 2 - (\sigma + 2k)) \right\} |x|^{-(\sigma+2m)}$$

and

$$|\nabla(\Delta^m u)| = \left\{ (\sigma + 2m) \prod_{k=0}^{m-1} (\sigma + 2k)\,(d - 2 - (\sigma + 2k)) \right\} |x|^{-(\sigma+2m)}.$$

It is easy to see that the products in curly brackets are *equal* to $c(d, m, p, \beta)$ and $\tilde{c}(d, m, p, \beta)$, respectively, for $\sigma = \dfrac{d-\beta}{p}$. $\square$

In view of Section 3 it is finally worth noting the special case when the weights in the right-hand integrals of Theorems 12 and 13 are bounded.

Corollary 14. *Let $m \in \mathrm{N}_0$, $d \in \mathrm{N}$.*

i) If $1 < p < \infty$ and $d > 2mp$, then

$$\int \frac{|u(x)|^p}{|x|^{2mp}} dx \leq K(m,p,d)^p \int |\Delta^m u(x)|^p dx$$

for all $u \in W_p^{2m}(\mathrm{R}^d)$, where

$$K(m,p,d) = p^{2m} \left(\prod_{k=1}^{m} (d - 2kp)(2(k-1)p + (p-1)d) \right)^{-1};$$

ii) if $1 \le p < \infty$ *and* $d > (2m+1)p$, *then*

$$\int \frac{|u(x)|^p}{|x|^{(2m+1)p}} dx \le \tilde{K}(m,p,d)^p \int |\nabla(\Delta^m u)(x)|^p dx$$

for all $u \in W_p^{2m+1}(\mathrm{R}^d)$, *where*

$$\tilde{K}(m,p,d) = \frac{p^{2m+1}}{d-p} \left(\prod_{k=1}^{m} (d-(2k+1)p)((2k-1)p+(p-1)d) \right)^{-1}.$$

Proof. Put $\beta = 2mp$ in Theorem 12 and $\beta = (2m+1)p$ in Theorem 13, respectively, and note that $C_c^\infty(\mathrm{R}^d \backslash \{0\})$ is dense in $W_p^k(\mathrm{R}^d)$ as long as $d > kp$, since one can cut out a ball of radius ϵ by a smooth function whose k-th derivative behaves like ϵ^{-k}. □

As an example we recover the classical case $p = 2$:

Example 15. *Let* $m \in \mathrm{N}_0$, $n \in \mathrm{N}$. *Then*

$$K(m,2,4m+n) = \frac{\Gamma(\frac{n}{4})}{4^m \Gamma\left(\frac{n}{4} + 2m\right)},$$

$$\tilde{K}(m,2,4m+2+n) = \frac{\Gamma(\frac{n}{4})}{2 \cdot 4^m \Gamma\left(\frac{n}{4} + 2m + 1\right)},$$

in Corollary 14.

This follows from the formula $\forall x > 0 \, \forall M \in \mathrm{N}_0 : \Gamma(x+M) = \Gamma(x) \prod_{k=0}^{M-1} (x+k)$.

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