Electrons in a lattice with an incommensurate potential

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ABSTRACT. *A system of fermions on a one-dimensional lattice, subject to a periodic potential whose period is incommensurate with the lattice spacing and verifies a diophantine condition, is studied. The Schwinger functions are obtained, and their asymptotic decay for large distances is exhibited for values of the Fermi momentum which are multiple of the potential period.*

1. Introduction.

1.1. The *Static Holstein model* [P, H] describes a system of fermions (electrons) in a linear lattice interacting with a classical phonon field. It is obtained from a tight-binding Hamiltonian with neglect of the vibrational kinetic energy of the lattice (an approximation which can be justified in physical models as the atom mass is much larger than the electron mass).

The Hamiltonian of the model, if we neglect all internal degrees of freedom (the spin, for example), which play no role, is given by

$$H = \sum_{x,y\in\Lambda} t_{xy}\, \psi_x^+ \psi_y^- - \mu \sum_{x\in\Lambda} \psi_x^+ \psi_x^- - \lambda \sum_{x\in\Lambda} \varphi_x \psi_x^+ \psi_x^- + \frac{1}{2} \sum_{x\in\Lambda} \varphi_x^2 \,, \tag{1.1}$$

where x, y are points on the one-dimensional lattice Λ with unit spacing, length L and periodic boundary conditions; we shall identify Λ with $\{x \in \mathbb{Z} :\ -[L/2] \le x \le [(L-1)/2]\}$. Moreover the matrix t_{xy} is defined as $t_{xy} = \delta_{x,y} - (1/2)[\delta_{x,y+1} + \delta_{x,y-1}]$, where $\delta_{x,y}$ is the Kronecker delta. The fields $\psi_x^\pm$ are creation $(+)$ and annihilation $(-)$ fermionic fields, satisfying periodic boundary conditions: $\psi_x^\pm = \psi_{x+L}^\pm$. We define also $\psi_{\mathbf{x}}^\pm = e^{tH} \psi_x^\pm e^{-Ht}$, with $\mathbf{x} = (x,t)$, $-\beta/2 \le t \le \beta/2$ for some $\beta > 0$; on t antiperiodic boundary conditions are imposed. The potential φ_x is a real function representing the classical phonon field, of a form which will be specified below (see §1.3). In (1.1) μ is the chemical potential, and λ is the interaction strenght.

The expectation value of an observable $\mathcal{O}$ in the Grand-canonical state at inverse temperature β and volume Λ is given by $\langle \mathcal{O} \rangle \equiv \mathrm{Tr}[\exp(-\beta H)\mathcal{O}]/\mathrm{Tr}[\exp(-\beta H)]$. If $\mathcal{O} = \mathbf{T}[\psi_{\mathbf{x}_1}^- \cdots \psi_{\mathbf{x}_n}^- \psi_{\mathbf{y}_1}^+ \cdots \psi_{\mathbf{y}_n}^+]$, where $\mathbf{T}$ denotes the anticommuting time ordering operator, we get the $2n$*-point Schwinger functions* of the model.

The most interesting problem about the model (1.1) is to find the minima with respect to φ_x of the ground state energy of the system $E(\varphi)$, in the thermodynamic limit. It is easy

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to show that all stationary points of $E(\varphi)$ satisfy the condition

$$\varphi_x = \lambda\rho_x \ , \qquad \rho_x \equiv \lim_{\beta\to\infty}\lim_{L\to\infty}\langle\psi^+_x\psi^-_x\rangle \ . \tag{1.2}$$

This equation has been rigorously studied, up to now, only in the case of density 1/2 [KL, LM]. However, if $\rho = \lim_{L\to\infty} L^{-1}\sum_x \rho_x$ is an irrational number, there have been recently, starting from [AAR], some numerical studies of the model, which led, through a strong numerical evidence, to the conjecture that, for small coupling, the ground state energy of the system $E(\varphi)$ has a minimum for a potential of the form $\varphi_x = \bar\varphi(2px)$, where $\bar\varphi(u)$ is a 2π-periodic real function of the real variable u and $p = \pi\rho$.

The conjecture has a physical interest to explain the properties of strongly anisotropic compounds which can be considered as one-dimensional systems; in such systems one finds a charge density wave incommensurate with the lattice, according to (1.2).

In this paper, we shall not study the minimization problem of $E(\varphi)$, but we shall analyze the properties of the two-point Schwinger function $S_2(\mathbf{x};\mathbf{y}) = \langle\mathbf{T}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}\rangle$, for a suitable set of values of μ and $\varphi_x = \bar\varphi(2px)$, with p/π irrational. [Note that all the Schwinger functions can be expressed in terms of the two-point Schwinger functions, as the interaction is quadratic in the fermionic fields]. We shall do that by constructing a convergent expansion for $S_2(\mathbf{x};\mathbf{y})$, that we hope will be useful in studying the equation (1.2).

In any case, this expansion allows to prove some properties of $S_2(\mathbf{x};\mathbf{y})$, which are interesting by themselves; these properties imply known results about the Schroedinger equation related to the model (1.1), but are not a trivial consequence of them (see discussion in §1.6 below).

1.2. As it is well known, the Schwinger functions can be written as power series in λ, convergent for $|\lambda| \le \varepsilon_\beta$, for some constant ε_β (the only trivial bound of ε_β goes to zero, as $\beta\to\infty$). This power expansion is constructed in the usual way in terms of Feynman graphs (in this case only chains, since the interaction is quadratic in the field), by using as *free propagator* the function

$$\begin{aligned} g^{L,\beta}(\mathbf{x};\mathbf{y}) \equiv g^{L,\beta}(\mathbf{x}-\mathbf{y}) &= \frac{\mathrm{Tr}\left[e^{-\beta H_0}\mathbf{T}(\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}})\right]}{\mathrm{Tr}[e^{-\beta H_0}]} = \\ &= \frac{1}{L}\sum_{k\in\mathcal{D}_L} e^{-ik(x-y)}\left\{\frac{e^{-\tau e(k)}}{1+e^{-\beta e(k)}}\mathbf{1}(\tau>0) - \frac{e^{-(\beta+\tau)e(k)}}{1+e^{-\beta e(k)}}\mathbf{1}(\tau\le 0)\right\} \ , \end{aligned} \tag{1.3}$$

where H_0 is the free Hamiltonian ($\lambda=0$), $\mathbf{x}=(x,x_0)$, $\mathbf{y}=(y,y_0)$, $\tau = x_0-y_0$, $\mathbf{1}(E)$ denotes the indicator function ($\mathbf{1}(E)=1$, if E is true, $\mathbf{1}(E)=0$ otherwise), $e(k) = 1-\cos k-\mu$ and $\mathcal{D}_L \equiv \{k = 2\pi n/L, n\in\mathbb{Z}, -[L/2]\le n\le[(L-1)/2]\}$.

It is easy to prove that, if $x_0\ne y_0$,

$$g^{L,\beta}(\mathbf{x}-\mathbf{y}) = \lim_{M\to\infty}\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{-ik_0+\cos p_F-\cos k} \ , \tag{1.4}$$

where $\mathbf{k}=(k,k_0)$, $\mathbf{k}\cdot\mathbf{x} = k_0x_0+kx$, $\mathcal{D}_{L,\beta}\equiv\mathcal{D}_L\times\mathcal{D}_\beta$, $\mathcal{D}_\beta\equiv\{k_0 = 2(n+1/2)\pi/\beta, n\in\mathbb{Z}, -M\le n\le M-1\}$ and p_F is the *Fermi momentum*, defined so that $\cos p_F = 1-\mu$ and $0<p_F<\pi$. [M is an (arbitrary) integer].

Hence, if we introduce a finite set of Grassmanian variables $\{\psi^\pm_{\mathbf{k}}\}$, one for each of the allowed $\mathbf{k}$ values, and a linear functional $P(d\psi)$ on the generated Grassmanian algebra, such that

$$\int P(d\psi)\psi^-_{\mathbf{k}_1}\psi^+_{\mathbf{k}_2} = L\beta\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat g_{\mathbf{k}_1} \ , \quad \hat g_{\mathbf{k}} = \frac{1}{-ik_0+\cos p_F-\cos k} \ , \tag{1.5}$$

we have

$$\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\hat g_{\mathbf{k}} = \int P(d\psi)\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}} \equiv g^{L,\beta}(\mathbf{x};\mathbf{y}) \ , \tag{1.6}$$

where the *Grassmanian field* $\psi_{\mathbf{x}}$ is defined by

$$\psi^{\pm}_{\mathbf{x}} = \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \psi^{\pm}_{\mathbf{k}} e^{\pm i\mathbf{k}\cdot\mathbf{x}} \; . \tag{1.7}$$

The "Gaussian measure" $P(d\psi)$ has a simple representation in terms of the "Lebesgue Grassmanian measure" $d\psi^- d\psi^+$, defined as the linear functional on the Grassmanian algebra, such that, given a monomial $Q(\psi^-,\psi^+)$ in the variables $\psi^-_{\mathbf{k}}, \psi^+_{\mathbf{k}}$,

$$\int d\psi^- d\psi^+ Q(\psi^-,\psi^+) = \begin{cases} 1 & \text{if } Q(\psi^-,\psi^+) = \prod_{\mathbf{k}} \psi^-_{\mathbf{k}}\psi^+_{\mathbf{k}} \; , \\ 0 & \text{otherwise} \; . \end{cases} \tag{1.8}$$

We have

$$P(d\psi) = \Big\{ \prod_{\mathbf{k}} (L\beta \hat g_{\mathbf{k}}) \Big\} \exp\Big\{ -\sum_{\mathbf{k}} (L\beta \hat g_{\mathbf{k}})^{-1} \psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}} \Big\} d\psi^- d\psi^+ \; . \tag{1.9}$$

Note that, since $(\psi^-_{\mathbf{k}})^2 = (\psi^+_{\mathbf{k}})^2 = 0$, $e^{-z\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}}} = 1 - z\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}}$, for any complex z.

By using standard arguments (see, for example, [NO], where a different regularization of the propagator is used), one can show that the Schwinger functions can be calculated as expectations of suitable functions of the Grassmanian field with respect to the "Gaussian measure" $P(d\psi)$. In particular, the two-point Schwinger function, which in our case determines the other Schwinger functions through the Wick rule, can be written, if $x_0 \neq y_0$, as

$$S^{L,\beta}(\mathbf{x};\mathbf{y}) = \lim_{M\to\infty} \frac{\int P(d\psi)\, e^{\mathcal{V}(\psi)}\, \psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}}{\int P(d\psi)\, e^{\mathcal{V}(\psi)}} \; , \tag{1.10}$$

where

$$\mathcal{V}(\psi) = \sum_{x\in\Lambda} \int_{-\beta/2}^{\beta/2} dx_0 \left[\lambda \varphi_x \psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} \right] . \tag{1.11}$$

If $x_0 = y_0$, $S^{L,\beta}(\mathbf{x};\mathbf{y})$ must be defined as the limit of (1.10) as $x_0 - y_0 \to 0^-$, as we shall understand always in the following.

Remark. The *ultraviolet cutoff* M on the k_0 variable was introduced in order to give a precise meaning to the Grassmanian integration (the numerator and the denominator in the r.h.s. of (1.10) are indeed finite sums), but it does not play any essential role in this paper, since all bounds will be uniform with respect to M and they easily imply the existence of the limit. Hence, we shall not stress anymore the dependence on M of the various quantities we shall study.

1.3. We now define precisely the potential φ_x. We are interested in studying potentials which, in the limit $L\to\infty$, are of the form $\varphi_x = \bar\varphi(2px)$, where $\bar\varphi$ is a real function on the real line 2π-periodic and p/π is an irrational number, so that the phonon field has a period which is incommensurate with the period of the lattice. We also impose that $\bar\varphi(u)$ is of mean zero (its mean value can be absorbed in the chemical potential), even and analytic in u, so that

$$\bar\varphi(u) = \sum_{0\neq n\in\mathbb{Z}} \hat\varphi_n\, e^{inu} \; , \qquad |\hat\varphi_n| \le F_0\, e^{-\xi|n|} \; , \qquad \hat\varphi_n = \hat\varphi_{-n} = \hat\varphi^*_n \; . \tag{1.12}$$

At finite volume we need a potential satisfying periodic boundary conditions; hence, at finite L, we approximate φ_x by

$$\varphi^{(L)}_x = \sum_{n=-[L/2]}^{[(L-1)/2]} \hat\varphi_n\, e^{2inp_L x} \; , \tag{1.13}$$

where p_L tends to p as $L\to\infty$ and is of the form $p_L = n_L\pi/L$, with n_L an integer, relatively prime with respect to L. The definition of p_L implies that $2np_L$ is an allowed momentum (modulo 2π), for any n, and that the sum in (1.13) is indeed a sum over all allowed values of k, except $k=0$.

If we insert (1.13) in the r.h.s. of (1.11), we get

$$\mathcal{V}(\psi)=\sum_{n=-[L/2]}^{[(L-1)/2]}\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\lambda\hat\varphi_n\,\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}+2n\mathbf{p}_L}\;, \tag{1.14}$$

where $\mathbf{p}_L=(p_L,0)$ and $k+2np_L$ is of course defined modulo 2π.

Let us now suppose that $p_F=mp_L$, for some integer $m\ge 1$, so that the $\hat g_{\mathbf{k}}^{-1}$ is small for $\mathbf{k}\simeq\pm m\mathbf{p}_L$. In this case (see (1.9)) there is no hope to treat perturbatively the terms with $n=\pm m$ and $\mathbf{k}$ near $\mp m\mathbf{p}_L$, but we can at most expect that the interacting measure is a perturbation of the measure

$$\bar P_\lambda(d\psi)\equiv\frac{1}{\mathcal{N}}P(d\psi)\exp\Big\{\lambda\hat\varphi_m\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\frac{1}{L}\sum_{k\in I_-}[\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}+2m\mathbf{p}_L}+\psi^+_{\mathbf{k}+2m\mathbf{p}_L}\psi^-_{\mathbf{k}}]\Big\}\;, \tag{1.15}$$

where $\mathcal{N}$ is a normalization constant and I_- is a small interval centered in $-p_F$, so small that $I_-\cap I_+=\emptyset$, if $I_+\equiv\{k=\bar k+2p_F,\bar k\in I_-\}$.

It is very easy to study the measure (1.15); in fact $\bar P_\lambda(d\psi)=\mathcal{N}'^{-1}d\psi d\bar\psi\exp[-J(\bar\psi,\psi)]$, where $J(\bar\psi,\psi)$ is a quadratic form, which can be simply diagonalized, since it only couples $\psi^\pm_{\mathbf{k}}$, $k\in I_-$, with $\psi^\pm_{\mathbf{k}+2m\mathbf{p}_L}$. One can show that there is an orthogonal transformation from the variables $\psi^\pm_{\mathbf{k}}$ to new variables $\chi^\pm_{\mathbf{k}}$, such that, if $I\equiv I_-\cup I_+$, then

$$J(\bar\psi,\psi)=\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\Big\{\frac{1}{L}\sum_{k\notin I}\chi^+_{\mathbf{k}}\chi^-_{\mathbf{k}}[-ik_0+e(k)]+\frac{1}{L}\sum_{k\in I}\chi^+_{\mathbf{k}}\chi^-_{\mathbf{k}}[-ik_0+E(k)]\Big\}\;, \tag{1.16}$$

where $e(k)=\cos p_F-\cos k$ is the free dispersion relation, while $E(k)$ is the new dispersion relation near $\pm p_F$. $E(k)$ is such that

$$E(k)\,\mathrm{sign}\,(|k|-p_F)\ge|\lambda\hat\varphi_m|\;. \tag{1.17}$$

Note that p_F is an allowed momentum, if mn_L is even. In this case, there are two eigenstates of the one-particle Hamiltonian h_{xy} corresponding to (1.16) with energy μ, for $\lambda=0$ (*i.e.* of $h_{xy}=t_{xy}$; see (1.1)); the coupling removes the degeneracy and the corresponding interacting eigenstates have energy $\mu\pm\lambda\hat\varphi_m$.

Given a one-particle Hamiltonian h_{xy} (in the model (1.1) $h_{xy}=t_{xy}-\lambda\varphi_x\delta_{xy}$) with spectrum Σ, we define as usual the *spectral gap* around the level μ in the following way:

$$\Delta=\inf\{E\in\Sigma:E>\mu\}-\sup\{E\in\Sigma:E<\mu\} \tag{1.18}$$

The bound (1.17) implies that the measure $\bar P_\lambda(d\psi)$ is associated with a one-particle Hamiltonian with a spectral gap $2|\lambda\hat\varphi_m|+O(L^{-1})$ around the level μ.

It is also easy to prove that the zero temperature density ρ^L, defined as the limit as $\beta\to\infty$ of the finite β density, given by

$$\rho^{L,\beta}=-\lim_{\tau\to0^-}\frac{1}{L}\sum_x S^{L,\beta}(x,\tau;x,0)\;, \tag{1.19}$$

is independent of λ and L, for the approximated model, and is given by $\rho^L=p_F/\pi$. This follows from the previous calculations and from the remark that ρ^L is equal to the number of eigenvalues lower than μ of the one-particle Hamiltonian plus half the number of eigenvalues equal to μ, divided by L; hence the two eigenstates that degenerate for $\lambda=0$ (if they are present) give the same contribution to ρ^L for any value of λ, as well as all the others, thanks to (1.17).

We shall prove that there is a diverging sequence of volumes L_i, such that the measure $P(d\psi)\exp[\mathcal{V}(\psi)]$ is a perturbation of $\bar{P}_\lambda(d\psi)$, for λ small enough, uniformly in i and β. In particular, we shall prove that there is a spectral gap of order $|\lambda\hat{\varphi}_m|$ around μ, independent of i, if $p_F = mp_{L_i}$.

We shall prove also that the density $\rho^{L_i,\beta}$ is a continuous function of λ near 0, uniformly in i and β, as well as its limit as $\beta\to\infty$. This result implies that $\lim_{\beta\to\infty}\rho^{L_i,\beta} = p_F/\pi$, independently of λ and L_i; in fact, at finite volume and zero temperature, the density can take only a finite set of values , hence it is constant if it is continuous.

In order to implement this program, one must face one main difficulty, related to the fact that p_{L_i} converges to an irrational number as $L_i\to\infty$, so that there are terms in the interaction (1.14), which are almost equal to those included in the definition of $\bar{P}_\lambda(d\psi)$, without being exactly equal. These terms can not be simply included in the definition of $\bar{P}_\lambda(d\psi)$ and make difficult to control the perturbation theory. This difficulty will be cared by using the decreasing property of $\hat{\varphi}_n$, see (1.12), and a diophantine condition hypothesis on p.

1.4. Denote by $\|\alpha-\beta\|_{\mathbb{T}^1}$ the distance on $\mathbb{T}^1$ of $\alpha,\beta\in\mathbb{T}^1$, and, for $\mathbf{x}=(x,x_0),\mathbf{y}=(y,y_0)\in\mathbb{R}^2$, by $|\mathbf{x}-\mathbf{y}|$ the distance $|\mathbf{x}-\mathbf{y}| = \sqrt{(x-y)^2+(x_0-y_0)^2}$.

We shall prove the following theorem.

THEOREM 1. *Let us consider a sequence L_i, $i\in\mathbb{Z}^+$, such that*

$$\lim_{i\to\infty} L_i = \infty\ , \qquad \lim_{i\to\infty} p_{L_i} = p\ ,$$

and let $S^{L_i,\beta}(\mathbf{x};\mathbf{y})$ be the Schwinger function (1.10). Suppose also that there is a positive integer m such that $p_F = mp_{L_i}\ (\mathrm{mod}\, 2\pi)$, $\hat{\varphi}_m\neq 0$ and p_{L_i} satisfies the diophantine condition

$$\|2np_{L_i}\|_{\mathbb{T}^1}\geq C_0|n|^{-\tau}\ ,\qquad \forall\, 0\neq n\in\mathbb{Z}\ ,\qquad |n|\leq\frac{L_i}{2}\ , \tag{1.20}$$

for some positive constants C_0 and τ independent of i. Then there exists $\varepsilon_0>0$, such that, if $\lambda\in\mathbb{R}$ and $|\lambda|\leq\varepsilon_0$, the following sentences are true.

(i) There exists the limit

$$\lim_{\substack{\beta\to\infty\\ i\to\infty}} S^{L_i,\beta}(\mathbf{x};\mathbf{y}) = S(\mathbf{x};\mathbf{y}) = S_1(\mathbf{x};\mathbf{y}) + \lambda S_2(\mathbf{x};\mathbf{y})\ , \tag{1.21}$$

where

$$S_1(\mathbf{x};\mathbf{y}) = g^{(1)}(\mathbf{x};\mathbf{y}) + \int\frac{d\mathbf{k}}{(2\pi)^2}\,[1-\hat{f}_1(\mathbf{k})]\,\phi(k,x,\sigma)\,\phi^*(k,y,\sigma)\,\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\varepsilon(k,\sigma)}\ , \tag{1.22}$$

with

$$\begin{aligned}
g^{(1)}(\mathbf{x};\mathbf{y}) &= \int\frac{d\mathbf{k}}{(2\pi)^2}\,\hat{f}_1(\mathbf{k})\,\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+e(k)}\ ,\\
\sigma &= \lambda\hat{\varphi}_m\ ,\\
\varepsilon(k,\sigma) &= [1-\cos(|k|-p_F)]\cos p_F\\
&\quad + \mathrm{sign}\,(|k|-p_F)\,\sqrt{[\sin(|k|-p_F)\sin p_F]^2+\sigma^2}\ ,\\
\phi(k,x,\sigma) &= e^{-ikx}u(k,x,\sigma)\ ,\\
u(k,x,\sigma) &= e^{i\mathrm{sign}\,(k)p_Fx}\left[\cos(p_Fx)\sqrt{1-\frac{\mathrm{sign}\,(|k|-p_F)\sigma}{\sqrt{(\sin(|k|-p_F)\sin p_F)^2+\sigma^2}}}\right.\\
&\quad\left. -i\,\mathrm{sign}\,(k)\sin(p_Fx)\sqrt{1+\frac{\mathrm{sign}\,(|k|-p_F)\sigma}{\sqrt{(\sin(|k|-p_F)\sin p_F)^2+\sigma^2}}}\ \right]\ .
\end{aligned} \tag{1.23}$$

Here $\hat{f}_1(\mathbf{k})$ denotes a cutoff function with support far enough from the two singular points $\mathbf{k} = (\pm p_F, 0)$, see (2.4) in §2 for a precise definition.

(ii) $S(\mathbf{x};\mathbf{y})$ is continuous as a function of λ; moreover there are three constants K_1, K_2, K_3 and, for any $N > 1$, a constant C_N, such that, if $|\mathbf{x}-\mathbf{y}| \geq K_3|\sigma|^{-1}$,

$$|S_1(\mathbf{x};\mathbf{y})|, |S_2(\mathbf{x};\mathbf{y})| \leq K_1|\sigma| \frac{C_N}{1 + (|\sigma|\,|\mathbf{x}-\mathbf{y}|)^N} , \tag{1.24}$$

while, for $|\mathbf{x}-\mathbf{y}| \leq K_3|\sigma|^{-1}$, one has

$$|S_2(\mathbf{x};\mathbf{y})| \leq K_2\,(1 + |\mathbf{x}-\mathbf{y}|)^{-1} . \tag{1.25}$$

Finally, for any $|\mathbf{x}-\mathbf{y}|$, one has

$$|S_1(\mathbf{x};\mathbf{y}) - g(\mathbf{x};\mathbf{y})| \leq K_2|\sigma| \log(|\sigma|^{-1} + 1) , \tag{1.26}$$

where $g(\mathbf{x};\mathbf{y}) \equiv \lim_{\beta\to\infty, i\to\infty} g^{L_i,\beta}(\mathbf{x};\mathbf{y})$ (it satisfies the bound (1.25) for any $|\mathbf{x}-\mathbf{y}|$).

(iii) For any i, the density $\rho^{L_i,\beta}$, given by (1.19), is a continuous function of λ, uniformly in β, as well as its limit as $\beta \to \infty$.

(iv) For any i, there is a spectral gap $\Delta \geq |\sigma|/2$ around μ.

1.5. The above theorem states that, if in the infinite volume limit the Fermi momentum is a multiple of the period of the potential ($p_F = mp$) and p/π is an irrational number verifying a diophantine condition, then the two-point Schwinger function decays as the free one ($\lambda = 0$) for $|\mathbf{x}-\mathbf{y}| \leq K_3|\sigma|^{-1}$ and faster than any power if $|\mathbf{x}-\mathbf{y}| \geq K_3|\sigma|^{-1}$, for some constant K_3. The region in which the behaviour is a power law enlarges taking larger and larger m. As the points of the form mp are dense on $\mathbb{T}^1$, very small changes in the Fermi momentum (related to changes of the density of the system) can correspond to very different values of m, and so to very different asymptotic behaviour of $S(\mathbf{x};\mathbf{y})$ (one can pass for instance with a very small difference in p_F from a situation in which the faster than any power decay is observable to a situation in which it occurs at so large distances to becomes unobservable).

1.6. The infinite volume two-point Schwinger function is obtained as the limit of $S^{L_i,\beta}(\mathbf{x};\mathbf{y})$, when $p_{L_i}/(2\pi)$ is a sequence of rational numbers verifying the generalized diophantine condition (1.20) and converging to an irrational diophantine number. A sequence with the above property is constructed in Appendix 1, for any diophantine number.

Finally, by looking at the proof of Theorem 1, one can see that, if there are sequences L_i, $p_{F,L_i} = 2\pi n_{F,L_i}/L_i$, p_{L_i} such that $\lim_{i\to\infty} L_i = \infty$ and

$$\|2np_{L_i}\|_{\mathbb{T}^1}, \ \|p_{F,L_i} + np_{L_i}\|_{\mathbb{T}^1} \geq C_0|n|^{-\tau} , \qquad \forall\, 0 \neq n \in \mathbb{Z} , \qquad |n| \leq \frac{L_i}{2} , \tag{1.27}$$

for some positive constants C_0 and τ, then the two-point Schwinger function is given by

$$S(\mathbf{x};\mathbf{y}) = g(\mathbf{x};\mathbf{y}) + \lambda S_2(\mathbf{x};\mathbf{y}) , \tag{1.28}$$

where $g(\mathbf{x};\mathbf{y})$ is defined after (1.26) and

$$|g(\mathbf{x};\mathbf{y})|, |S_2(\mathbf{x};\mathbf{y})| \leq \frac{K_4}{1 + |\mathbf{x}-\mathbf{y}|} , \tag{1.29}$$

for some constant K_4. However, since in this case the construction of a sequence of L_i, p_{F,L_i}, p_{L_i} verifying (1.27) seems to be much more involved, while the renormalization group analysis, to which mainly is devoted this paper, seems to be essentially the same, we prefer not to discuss this case here.

1.7. Systems of fermions on a lattice subject to a periodic potential *incommensurate* with the period of the lattice are widely studied, starting from [P], in which this problem was considered relevant to understand a system of electrons in a lattice and subject to a magnetic field and was faced by studying the spectrum of the finite difference Schroedinger equation

$$-\psi(x+1)-\psi(x-1)+\lambda\varphi_x\psi(x)=E\psi(x)\ , \tag{1.30}$$

where φ_x is defined as before. This problem is of course closely related to the study of the spectrum of the Schroedinger equation

$$-\frac{d^2\psi(x)}{dx^2}+\lambda\varphi_x\psi(x)=E\psi(x)\ , \tag{1.31}$$

where $\varphi_x=\bar\varphi(\boldsymbol{\omega}x)$, $\boldsymbol{\omega}\in\mathbb{R}^d$ is a vector with rationally independent components and $\bar\varphi(\mathbf{u})$ is 2π-periodic in all its d arguments.

In fact in (1.30) there are two periods, the one of the potential and the intrinsic one of the lattice, and this makes the properties of (1.30) and of (1.31) (with $d=2$) very similar to each other.

The eigenfunctions and the spectrum strongly depend on λ. For large λ there are eigenfunctions with an exponential decay for large distances; this phenomenon is called *Anderson localization* (for details, see for instance [PF] and references therein). On the other hand, for small λ and for certain values of E, there are eigenfunctions which are *quasi-Bloch waves* of the form $e^{ik(E)x}\,u(x)$ with $u(x)=\bar u(px)$ for (1.30) and $u(x)=\bar u(\boldsymbol{\omega}x)$ for (1.31), $\bar u$ being 2π-periodic in its arguments.

This was proved for (1.31) in [DS], with the condition that there exist positive constants C_0,τ such that

$$|\boldsymbol{\omega}\cdot\mathbf{n}|\ge C_0|\mathbf{n}|^{-\tau}\ ,\qquad |k(E)+\boldsymbol{\omega}\cdot\mathbf{n}|\ge C_0|\mathbf{n}|^{-\tau}\ ,\qquad \forall\mathbf{0}\neq\mathbf{n}\in\mathbb{Z}^d\ , \tag{1.32}$$

by using KAM techniques modulo some technical assumptions (like the condition of large E) which were relaxed in [E]. An analogous statement was proved for (1.30), with the condition $\|k+np\|_{\mathbb{T}^1}\ge C_0|n|^{-\tau}$ (see Theorem 1 for notations), in [BLT], by using essentially the same ideas as in [DS].

The existence of quasi-Bloch waves for (1.31) with $k(E)$ verifying $k(E)=\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{n}$ and $|\boldsymbol{\omega}\cdot\mathbf{n}|\ge C_0|\mathbf{n}|^{-\tau}$ was proved, together with the existence of gaps in the spectrum, in [JM,MP] with some additional assumption removed in [E].

Our results are in agreement with those contained in the papers referenced above, but we think that they do not follow completely from them. In particular, the properties of the Schwinger functions do not seem to us a consequence of the known properties of the one-particle Hamiltonian spectrum.

1.8. The proof of Theorem 1 is performed by using renormalization group techniques combined with the diophantine condition (1.20). The proof of the convergence of the perturbative series for the two-point Schwinger function is similar to the proof of the convergence of the Lindstedt series for the invariant tori of a mechanical system, [G,GM], in which a notion of resonance is introduced and it is shown that, thanks to the diophantine condition, if one subtracts the relevant part of the value associated to the resonances (*resonance value*, see [GM] and §3.3 below), the resulting series is convergent. In the Lindstedt series the sum of the relevant part of the resonance values is vanishing; this is not true in this case, in which the relevant part of the resonance value is a *running coupling constant*, in the renormalization group sense. However, here a different mechanism still ensures the convergence of the perturbative series.

The paper is organized as follows. In §2 we introduce the multiscale decomposition of the propagators and set up the graph formalism, which allows us to treat all contributions corresponding to graphs not belonging to a certain class (graphs without resonances, see the definition in §2.5); this will lead to Lemma 1. In §3 we show that a more refined renormalization procedure (which consists essentially in changing suitably the "Grassmanian integration" at each step of the renormalization procedure) allows us to extend the result of §2 to all graphs (Lemma 2); then the convergence of the effective potential follows. In §4 we study the two-point Schwinger functions, with the same techniques of §3, and we prove Theorem 1.

2. Multiscale decomposition

2.1. In order to simplify the notation, in this section and in the following one we shall not stress anymore the dependence on β and $L \equiv L_i$ of the various quantities; in particular p_{L_i}, $g^{L_i,\beta}(\mathbf{x};\mathbf{y})$ will be written simply as p, $g(\mathbf{x};\mathbf{y})$.

It is convenient to decompose the Grassmanian integration $P(d\psi)$ into a finite product of independent integrations:

$$P(d\psi) = \prod_{h=h_\beta}^{1} P(d\psi^{(h)}) \ , \tag{2.1}$$

where $h_\beta > -\infty$ will be defined below (before (2.9)) This can be done by setting

$$\psi^{\pm}_{\mathbf{k}} = \bigoplus_{h=h_\beta}^{1} \psi^{(h)\pm}_{\mathbf{k}} \ , \qquad \hat g_{\mathbf{k}} = \sum_{h=h_\beta}^{1} \hat g^{(h)}_{\mathbf{k}} \ , \tag{2.2}$$

where $\psi^{(h)\pm}_{\mathbf{k}}$ are families of Grassmanian fields with propagators $\hat g^{(h)}_{\mathbf{k}}$ which are defined in the following way.

We introduce a *scaling parameter* $\gamma > 1$ and a function $\chi(\mathbf{k}') \in C^\infty(\mathbb{T}^1 \times \mathbb{R})$, $\mathbf{k}' = (k', k_0)$, such that, if $|\mathbf{k}'| \equiv \sqrt{k_0^2 + ||k'||^2_{\mathbb{T}^1}}$,

$$\chi(\mathbf{k}') = \chi(-\mathbf{k}') = \begin{cases} 1 & \text{if } |\mathbf{k}'| < t_0 \equiv a_0/\gamma \ , \\ 0 & \text{if } |\mathbf{k}'| > a_0 \ , \end{cases} \tag{2.3}$$

where $a_0 = \min\{p_F/2, (\pi - p_F)/2\}$. This definition is such that the supports of $\chi(k - p_F, k_0)$ and $\chi(k + p_F, k_0)$ are disjoint and the C^∞ function on $\mathbb{T}^1 \times \mathbb{R}$

$$\hat f_1(\mathbf{k}) \equiv 1 - \chi(k - p_F, k_0) - \chi(k + p_F, k_0) \tag{2.4}$$

is equal to 0, if $\| \, |k| - p_F \|^2_{\mathbb{T}^1} + k_0^2 < t_0^2$.

We define also, for any integer $h \le 0$,

$$f_h(\mathbf{k}') = \chi(\gamma^{-h}\mathbf{k}') - \chi(\gamma^{-h+1}\mathbf{k}') \ ; \tag{2.5}$$

we have, for any $\bar h < 0$,

$$\chi(\mathbf{k}') = \sum_{h=\bar h+1}^{0} f_h(\mathbf{k}') + \chi(\gamma^{-\bar h}\mathbf{k}') \ . \tag{2.6}$$

Note that, if $h \le 0$, $f_h(\mathbf{k}') = 0$ for $|\mathbf{k}'| < t_0\gamma^{h-1}$ or $|\mathbf{k}'| > t_0\gamma^{h+1}$, and $f_h(\mathbf{k}') = 1$ for $|\mathbf{k}'| = t_0\gamma^h$.

We finally define, for any $h \le 0$,

$$\hat f_h(\mathbf{k}) = f_h(k - p_F, k_0) + f_h(k + p_F, k_0) \ , \tag{2.7}$$

$$\hat g^{(h)}_{\mathbf{k}} \equiv \frac{\hat f_h(\mathbf{k})}{-ik_0 + \cos p_F - \cos k} \ . \tag{2.8}$$

The label h is called *scale* or *frequency* label.

Note that, if $\mathbf{k} \in \mathcal{D}_{L,\beta}$, then $|k_0| \geq \pi/\beta$, implying that $\hat f_h(\mathbf{k}) = 0$ for any $h < h_\beta = \min\{h : t_0\gamma^{h+1} > \pi/\beta\}$. Hence, if $\mathbf{k} \in \mathcal{D}_{L,\beta}$, the definitions (2.4) and (2.7), together with the identity (2.6), imply that

$$1 = \sum_{h=h_\beta}^{1} \hat f_h(\mathbf{k}) \; . \tag{2.9}$$

The definition (2.7) implies also that, if $h \leq 0$, the support of $\hat f_h(\mathbf{k})$ is the union of two disjoint sets, A_h^+ and A_h^-. In A_h^+, k is strictly positive and $||k - p_F||_{\mathbb{T}^1} \leq a_0\gamma^h \leq a_0$, while, in A_h^-, k is strictly negative and $||k+p_F||_{\mathbb{T}^1} \leq a_0\gamma^h$. Therefore, if $h \leq 0$, we can write $\psi_{\mathbf{k}}^{(h)\pm}$ as the sum of two independent Grassmanian variables $\psi_{\mathbf{k},\omega}^{(h)\pm}$ with propagator

$$\int P(d\psi^{(h)})\, \psi^{(h)-}_{\mathbf{k}_1,\omega_1} \psi^{(h)+}_{\mathbf{k}_2,\omega_2} = L\beta \delta_{\mathbf{k}_1,\mathbf{k}_2}\, \delta_{\omega_1,\omega_2}\, \hat g^{(h)}_{\omega_1}(\mathbf{k}_1) \; , \tag{2.10}$$

so that

$$\psi_{\mathbf{k}}^{(h)\pm} = \bigoplus_{\omega=\pm 1} \psi_{\mathbf{k},\omega}^{(h)\pm} \; , \qquad \hat g_{\mathbf{k}}^{(h)} = \sum_{\omega=\pm 1} \hat g_\omega^{(h)}(\mathbf{k}) \; , \tag{2.11}$$

$$\hat g_\omega^{(h)}(\mathbf{k}) = \frac{\theta(\omega k)\, \hat f_h(\mathbf{k})}{-ik_0 + \cos p_F - \cos k} \; , \tag{2.12}$$

where $\theta(k)$ is the (periodic) step function. If $\omega k > 0$, we will write in the following $k = k' + \omega p_F$, where k' is the *momentum measured from the Fermi surface* and we shall define, if $h \leq 0$,

$$\tilde g_\omega^{(h)}(\mathbf{k}') \equiv \hat g_\omega^{(h)}(\mathbf{k}) = \frac{f_h(\mathbf{k}')}{-ik_0 + v_0\omega \sin k' + (1 - \cos k') \cos p_F} \; , \tag{2.13}$$

where $v_0 = \sin p_F$.

In order to simplify the notation, it will be useful in the following to denote $\hat g_{\mathbf{k}}^{(1)}$ also as $\tilde g_1^{(1)}(\mathbf{k}')$, with $k = k' + p_F$.

It is easy to prove that, for any $h \leq 1$ and any ω,

$$|\tilde g_\omega^{(h)}(\mathbf{k}')| \leq G_0 \gamma^{-h} \; , \tag{2.14}$$

for a suitable positive constant G_0, depending on p_F and diverging as $a_0 \to 0$.

In the following we shall use also the definitions

$$\begin{aligned} \psi_{\mathbf{k},\omega}^{(\leq h)\pm} &= \bigoplus_{j=h_\beta}^{h} \psi_{\mathbf{k},\omega}^{(j)\pm} \; , \qquad \tilde g_\omega^{(\leq h)}(\mathbf{k}') = \sum_{j=h_\beta}^{h} \tilde g_\omega^{(j)}(\mathbf{k}') \; , \\ \psi_{\mathbf{k}}^{(\leq h)\pm} &= \bigoplus_{j=h_\beta}^{h} \psi_{\mathbf{k}}^{(j)\pm} \; , \qquad \hat g_{\mathbf{k}}^{(\leq h)} = \sum_{j=h_\beta}^{h} \hat g_{\mathbf{k}}^{(j)} \; . \end{aligned} \tag{2.15}$$

Of course $\psi_{\mathbf{k}}^{\pm} \equiv \psi_{\mathbf{k}}^{(\leq 1)\pm}$, and $\hat g_{\mathbf{k}} \equiv \hat g^{(\leq 1)}(\mathbf{k})$.

2.2. The most naive definition for the *effective potential* "at scale" h is the following:

$$e^{\mathcal{V}^{(h)}(\psi^{(\leq h)}) + E_h} = \int P(d\psi^{(h+1)}) \ldots \int P(d\psi^{(1)}) e^{\mathcal{V}(\psi^{(\leq 1)})} \; , \tag{2.16}$$

where E_h is defined so that $\mathcal{V}^{(h)}(0) = 0$.

If we define $\mathbf{p} = (p, 0)$ and $\mathbf{p}_F = (p_F, 0)$, $\mathcal{V}(\psi^{(\leq 1)})$ can be written as

$$\mathcal{V}(\psi^{(\leq 1)}) = \sum_{n=-[L/2]}^{[(L-1)/2]} \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \lambda \hat\varphi_n\, \psi_{\mathbf{k}}^{(\leq 1)+} \psi_{\mathbf{k}+2n\mathbf{p}}^{(\leq 1)-} \; , \tag{2.17}$$

with $\hat{\varphi}_0 = 0$, see (1.12). Hence the effective potential on scale $h \le 0$ can be represented as

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{n=-[L/2]}^{[(L-1)/2]} \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \mathcal{W}_n^{(h)}(\mathbf{k})\, \psi^{(\le h)+}_{\mathbf{k},\omega} \psi^{(\le h)-}_{\mathbf{k}+2n\mathbf{p},\omega'} \,. \tag{2.18}$$

Note that here, as always in the following, the momentum k is defined modulo 2π.

The kernel $\mathcal{W}_n^{(h)}(\mathbf{k})$ admits the diagrammatic representation in terms of chain graphs described in §2.3 below. Note that a sum over the labels ω, ω' could be introduced in (2.18), but it is useless as the labels ω and ω' are uniquely determined by the signs of k and $k+2np$ respectively: $\omega = \mathrm{sign}\,(k)$ and $\omega' = \mathrm{sign}\,(k+2np)$ (see comments after (2.12)).

We shall study the convergence of the effective potential in terms of the norm

$$\|\mathcal{V}^{(h)}\| \equiv \sum_{n=-[L/2]}^{[(L-1)/2]} \sup_{\mathbf{k}\in\mathcal{D}_h} \left| \mathcal{W}_n^{(h)}(\mathbf{k}) \right| \,, \tag{2.19}$$

where $\mathcal{D}_h \equiv \{\mathbf{k} \in \mathcal{D}_{L,\beta} : \sum_{h'=h_\beta}^{h} \hat{f}_{h'}(\mathbf{k}) \neq 0\}$.

2.3. A graph ϑ of order q (see Fig.2.1 below) is a chain of $q+1$ lines $\ell_1, \ldots, \ell_{q+1}$ connecting a set of q ordered points (*vertices*) $v_1, \ldots, v_q$, so that ℓ_i enters v_i and ℓ_{i+1} exits from v_i; ℓ_1 and ℓ_{q+1} are the *external lines* of the graph and both have a free extreme, while the others are the *internal lines*; we shall denote $\mathrm{int}(\vartheta)$ the set of all internal lines. We say that $v_i < v_j$ if v_i precedes v_j and we denote v_j' the vertex immediately following v_j, if $j < q$. We denote also by ℓ_v the line entering the vertex v, so that $\ell_i \equiv \ell_{v_i}$, $1 \le i \le q$. We say that a line ℓ emerges from a vertex v if ℓ either enters v ($\ell = \ell_v$) or exits from v ($\ell = \ell_{v'}$).

We shall say that ϑ is a *labeled graph* of order q and *external scale* h, if ϑ is a graph of order q, to which the following *labels* are associated:
• a label n_v for each vertex,
• a frequency (or scale) label h for both the external lines and a frequency label $h_\ell \ge h+1$ for each internal line, $\ell \in \mathrm{int}(\vartheta)$,
• a label $\omega_\ell = \pm 1$ for each line of frequency label $h_\ell \le 0$ and a label $\omega_\ell = +1$ for each line of frequency label $h_\ell = 1$,
• a momentum $k_{\ell_1} = k = k' + \omega_1 p_F$ for the first line,
• a momentum $k_{\ell_v} = k + \sum_{\tilde{v}\le v} 2n_{\tilde{v}}p = k' + \sum_{\tilde{v}\le v} 2n_{\tilde{v}}p + (\omega_1 - \omega_{\ell_v})p_F + \omega_{\ell_v}p_F$, for each other line.
Moreover, $h_\vartheta \equiv \min_{\ell\in\mathrm{int}(\vartheta)} h_\ell$ will be called the *internal scale* or simply the *scale* of ϑ.

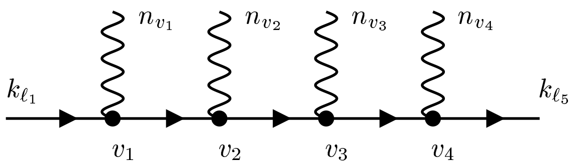

FIG.2.1. *A graph of order $q = 4$.*

A graph can be imagined to be obtained from q *graph elements* (see Fig.2.2), each of which is formed by a vertex with two emerging half-lines (representing the first one a ψ^+ field and the second one a ψ^- field), by pairing the half-lines (*contractions*) in such a way that the resulting graph turns out to be connected and only two half-lines remain non contracted (the external lines of the graph). Each line arises from the contraction of a half-line representing the ψ^- field of a vertex v with a half-line representing the ψ^+ field of a vertex w: then the line is supposed to carry an arrow pointing from v to w. We suppose also that a waving line emerges from each vertex v: it represents the component $\hat{\varphi}_{n_v}$ of the phonon field.

For each line ℓ we set $\mathbf{k}_\ell' = (k_\ell', k_0)$, and we associate to it a propagator $\tilde{g}^{(h_\ell)}_{\omega_\ell}(\mathbf{k}_\ell')$.

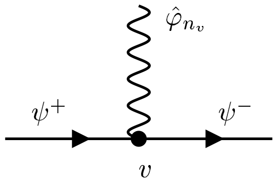

FIG.2.2 *A graph element.*

The value of the graph is given by

$$\mathrm{Val}(\vartheta) = \Big(\prod_{v\in\vartheta} \lambda\hat\varphi_{n_v}\Big)\Big(\prod_{\ell\in\mathrm{int}(\vartheta)} \tilde g^{(h_\ell)}_{\omega_\ell}(\mathbf{k}'_\ell)\Big) \ . \tag{2.20}$$

Let $\mathcal{T}^h_{n,q}$ denote the set of all labeled graphs of order q and external scale h, such that $\sum_{v\in\vartheta} 2n_v p = 2np$ and $-[L/2] \le n \le [(L-1)/2]$.

Then we have, if $\mathbf{k}\in\mathcal{D}_h$,

$$\mathcal{W}^{(h)}_n(\mathbf{k}) = \sum_{q=1}^\infty \mathcal{W}^{(h)}_{n,q}(\mathbf{k}) \ , \qquad \mathcal{W}^{(h)}_{n,q}(\mathbf{k}) = \sum_{\vartheta\in\mathcal{T}^h_{n,q}} \mathrm{Val}(\vartheta) \ . \tag{2.21}$$

Since the topological form of the graphs of order q is given (they are all chains of q vertices, see Fig.2.1), the sum in (2.21) is over all the possible assignments of the labels $\{n_v\}$ and $\{\omega_\ell, h_\ell\}$, with the constraint that

- $\sum_{v\in\vartheta} n_v = n \bmod L$,
- $h_\ell \ge h+1$ $\forall\ell\in\mathrm{int}(\vartheta)$, and
- $\omega_\ell = \mathrm{sign}\,(k_\ell)$, if $h_\ell\le 0$, and $\omega_\ell = +1$, if $h_\ell = 1$.

Note that $k_{\ell_v} = k_{\ell_{v'}} + 2n_v p$. The constraint on ω_ℓ arises from the comments after (2.18): it will disappear in §3, where new graphs will be introduced in which also "non diagonal" propagators will be allowed.

2.4. Given a labeled graph ϑ, we can consider a connected subset of its lines, T, carrying the same labels they have in ϑ. If the external lines of T (that is the lines that have only one vertex inside T) have frequency labels smaller than h_T, where h_T denotes the minimum between the frequency labels of its internal lines (*i.e.* the lines with both vertices inside T), we shall say that T is a *cluster* of scale h_T. An inclusion relation can be established between the clusters, in such a way that the innermost clusters are the clusters with the highest scale (*minimal clusters*), and so on. Note that ϑ itself is a cluster (of scale h_ϑ).

Each cluster T has an incoming line ℓ^i_T and an outgoing line ℓ^o_T; we set $2n_T p = k_{\ell^o_T} - k_{\ell^i_T}$, so that $n_T = \sum_{v\in T} n_v$. The maximum between $h_{\ell^i_T}$ and $h_{\ell^o_T}$ will be called the *external scale* of T. If a line ℓ is internal to a cluster T, we write $\ell\in T$.

Given a cluster T we introduce the following notations.

(1) We call T_0 the collection of internal lines in T which are on scale h_T (*i.e.* the lines $\ell\in T$ such that $h_\ell = h_T$), and denote by L_T the number of elements in T_0. We denote also by q_T the number of vertices inside T; of course $q_T \ge L_T + 1$.

(2) Let D_T be the *depth* of the cluster defined recursively in the following way: $D_T = 1$, if T is a minimal cluster, and $D_T = 1 + \max_{T'\subset T} D_{T'}$, otherwise. We shall denote $\mathcal{T}_D(\vartheta)$ the family of all clusters of depth D contained in ϑ.

(3) We say that a vertex v is in T_0, $v\in T_0$, if $v\in T$ and there are no other clusters inside T containing v. We define $M^{(2)}_T$ the number of vertices in T_0.

(4) We call $M^{(1)}_T$ the number of maximal clusters strictly contained in T; of course $M^{(1)}_T + M^{(2)}_T - 1 = L_T$. Define also $M_T = M^{(1)}_T + M^{(2)}_T$.

(5) We say that a line ℓ intersects a cluster T ($\ell\cap T \ne \emptyset$), if ℓ is either internal or external to T.

2.5. DEFINITION (RESONANCES). *Given a labeled graph ϑ and a cluster V contained in ϑ, we say that V is a resonant cluster of ϑ, if the momenta measured from the Fermi surface* *of the incoming and of the outgoing lines are the same, i.e.* $k'_{\ell^i_V} = k'_{\ell^o_V}$. *We define resonant vertex a vertex* v *with* $k'_{\ell_{v'}} = k'_{\ell_v}$ *and* $h_{\ell_v}, h_{\ell_{v'}} \le 0$. *We call resonances the set of resonant clusters and resonant vertices.*

Note that, if V is a resonance, $|h_{\ell^i_V} - h_{\ell^o_V}| \le 1$, as a consequence of the definition above. Moreover, if $k'_1 + \omega_1 p_F$ and $k'_2 + \omega_2 p_F$ are the momenta of the incoming and outgoing line in a cluster or a vertex, the momentum conservation says that

$$k'_1 - k'_2 + (\omega_1 - \omega_2) p_F + 2np = 0 \; , \tag{2.22}$$

where $n = n_V$ for a cluster V and $n = n_v$ for a vertex v.

In the case of a resonance $k'_1 = k'_2$ so that

$$(\omega_1 - \omega_2) p_F + 2np = 0 \; , \tag{2.23}$$

which can be verified only in two cases, if $p_F = mp$ for some integer m, as we shall suppose from now on:

$$\begin{cases} (1) \quad \omega_1 = \omega_2 \; , & n = 0 \; , \\ (2) \quad \omega_1 = -\omega_2 \; , & n = -\omega_1 m \; , \end{cases} \tag{2.24}$$

(*resonance conditions*).

We say that in the first case we have a ν-type resonance, while in the second case we have a σ-type resonance. Note also that there is no resonant vertex of ν-type, since $\hat\varphi_0 = 0$, (see (1.12)).

2.6. Lemma 1. *If* $\tilde{\mathcal{W}}^{(h)}_{n,q}(\mathbf{k})$ *is defined as the sum in the r.h.s. of (2.21), restricted to the family of graphs without resonances, then, if* $\gamma > 2^\tau$,

$$\sup_{\mathbf{k} \in \mathcal{D}_h} \left| \tilde{\mathcal{W}}^{(h)}_{n,q}(\mathbf{k}) \right| \le (|\lambda| B_1)^q \, e^{-\frac{\xi}{2}|n|} \; , \tag{2.25}$$

for some constant B_1.

The proof of Lemma 1 is in Appendix 2, §A2.2.

To obtain a bound like (2.25) also for graphs with resonances (and *so* for all graphs), a more refined procedure is required, which next section will be devoted to.

3. Renormalization

3.1. We introduce a *localization operator* $\mathcal{L}$, which acts on the effective potential in the following way:

$$\begin{aligned} & \mathcal{L} \left\{ \frac{1}{L\beta} \sum_{\mathbf{k}' \in \mathcal{D}_{L,\beta}} \psi^{(\le h)+}_{\mathbf{k}'+\omega_1 \mathbf{p}_F, \omega_1} \psi^{(\le h)-}_{\mathbf{k}'+\omega_2 \mathbf{p}_F + 2n\mathbf{p}, \omega_2} \mathcal{W}^{(h)}_n(\mathbf{k}' + \omega_1 \mathbf{p}_F) \right\} \\ & = \delta_{(\omega_1 - \omega_2) p_F + 2np, 0} \frac{1}{L\beta} \sum_{\mathbf{k}' \in \mathcal{D}_{L,\beta}} \psi^{(\le h)+}_{\mathbf{k}'+\omega_1 \mathbf{p}_F, \omega_1} \psi^{(\le h)-}_{\mathbf{k}'+\omega_2 \mathbf{p}_F, \omega_2} \mathcal{W}^{(h)}_n(\omega_1 \mathbf{p}_F) \; . \end{aligned} \tag{3.1}$$

Note that $\mathbf{k}' = 0$ is not an allowed value, but $\mathcal{W}^{(h)}_n(\mathbf{k}' + \omega_1 \mathbf{p}_F)$ is a well defined expression for any real values of $\mathbf{k}'$, so that $\mathcal{W}^{(h)}_n(\omega_1 \mathbf{p}_F)$ is well defined.

The effect of this operator is to "isolate" the problem connected to the resonances, in order to treat it separately in a way that we shall discuss below. We say that $\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)})$ is the *relevant part* (or *localized part*) of the effective potential $\mathcal{V}^{(h)}(\psi^{(\le h)})$.

We perform the integration $P(d\psi)$ in the following way. First we integrate the field with frequency $h=1$ (ultraviolet integration), which can be written, up to a constant,

$$
\begin{aligned}
P(d\psi^{(1)}) = & \prod_{\mathbf{k}} d\psi_{\mathbf{k}}^{(1)+} d\psi_{\mathbf{k}}^{(1)-} \\
& \exp\Big\{ -\frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \hat f_1^{-1}(\mathbf{k}) \Big[\big(-ik_0 + \cos p_F - \cos k \big) \psi_{\mathbf{k}}^{(1)+} \psi_{\mathbf{k}}^{(1)-} \Big] \Big\} ,
\end{aligned}
\tag{3.2}
$$

and we obtain $\mathcal{V}^{(0)}(\psi^{(\le 0)})$ as a power series in λ, convergent in the norm (2.19), for $|\lambda|$ small enough, say $|\lambda| \le \bar\varepsilon_0$, by (2.25). Then we decompose $\psi^{(\le 0)}$ as in (2.11), and we write, using also the evenness of the potential,

$$
\mathcal{L}\mathcal{V}^{(0)} = \nu_0 F_\nu^{(0)} + s_0 F_\sigma^{(0)} ,
\tag{3.3}
$$

where $F_\nu^{(0)}$ and $F_\sigma^{(0)}$ are given by

$$
\begin{aligned}
F_\nu^{(h)} &= \sum_{\omega=\pm1} \frac{1}{L\beta} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} \psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \psi^{(\le h)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} , \\
F_\sigma^{(h)} &= \sum_{\omega=\pm1} \frac{1}{L\beta} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} \psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \psi^{(\le h)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega} ,
\end{aligned}
\tag{3.4}
$$

with $h=0$. Note that, if $|\lambda|$ is small enough, by (2.25) there exists A_0 such that

$$
|s_0 - \lambda\hat\varphi_m| \le A_0 |\lambda|^2 e^{-m\xi/2} , \qquad |\nu_0| \le A_0|\lambda|^2 .
\tag{3.5}
$$

We have to study

$$
\int P(d\psi^{\le 0})\, e^{\mathcal{V}^{(0)}(\psi^{\le 0})} ,
\tag{3.6}
$$

where $P(d\psi^{(\le 0)})$ is the Grassmanian integration with propagator

$$
\begin{aligned}
& g^{(\le 0)}(\mathbf{x};\mathbf{y}) = \sum_{\omega,\omega'=\pm1} \frac{1}{L\beta} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}\, e^{-i(\omega x-\omega' y)p_F}\, \tilde g^{(\le 0)}_{\omega,\omega'}(\mathbf{k}') , \\
& \tilde g^{(\le 0)}_{\omega,\omega'}(\mathbf{k}') = \delta_{\omega,\omega'}\, \tilde g^{(\le 0)}_{\omega}(\mathbf{k}') ,
\end{aligned}
\tag{3.7}
$$

with

$$
\tilde g^{(\le 0)}_\omega(\mathbf{k}') = \frac{C_0^{-1}(\mathbf{k}')}{-ik_0 + (1-\cos k')\cos p_F + v_0\omega \sin k'} , \qquad C_h^{-1}(\mathbf{k}') = \sum_{j=h_\beta}^{h} f_j(\mathbf{k}') ,
\tag{3.8}
$$

see (2.13), (2.15).

We write

$$
\int P(d\psi^{(\le 0)})\, e^{\mathcal{V}^0(\psi^{(\le 0)})} = \frac{1}{\mathcal{N}_0} \int \tilde P(d\psi^{(\le 0)})\, e^{\tilde{\mathcal{V}}^{(0)}(\psi^{\le 0})} ,
\tag{3.9}
$$

where $\mathcal{N}_0$ is a suitable constant and, again up to a constant,

$$
\begin{aligned}
\tilde P(d\psi^{(\le 0)}) = & \prod_{\mathbf{k}} \prod_{\omega=\pm1} d\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} d\psi^{(\le 0)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \\
& \exp\Big\{ -\sum_{\omega=\pm1} \frac{1}{L\beta} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} C_0(\mathbf{k}') \Big[\Big(-ik_0 - (\cos k' - 1)\cos p_F + \omega v_0 \sin k' \Big) \\
& \psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \psi^{(\le 0)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} - \sigma_0(\mathbf{k}')\, \psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} \psi^{(\le 0)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega} \Big] \Big\} ,
\end{aligned}
\tag{3.10}
$$

with $\sigma_0(\mathbf{k}') = C_0^{-1}(\mathbf{k}')\, s_0$ and $\tilde{\mathcal{V}}^{(0)} = \mathcal{L}\tilde{\mathcal{V}}^{(0)} + (1-\mathcal{L})\mathcal{V}^{(0)}$, if

$$
\mathcal{L}\tilde{\mathcal{V}}^{(0)} = \nu_0 F_\nu^{(0)} .
\tag{3.11}
$$

The r.h.s of (3.9) can be written as

$$\frac{1}{\mathcal{N}_0}\int P(d\psi^{(\le -1)})\int \tilde P(d\psi^{(0)})\, e^{\tilde{\mathcal{V}}^{(0)}(\psi^{(\le 0)})}\ , \tag{3.12}$$

where $P(d\psi^{(\le -1)})$ and $\tilde P(d\psi^{(0)})$ are given by (3.10) with $C_0(\mathbf{k}')$ replaced with $C_{-1}(\mathbf{k}')$ and $f_0^{-1}(\mathbf{k}')$ respectively, and $\psi^{(\le 0)}$ replaced with $\psi^{(\le -1)}$ and $\psi^{(0)}$ respectively.

The Grassmanian integration $\tilde P(d\psi^{(\le 0)})$ has propagator

$$g^{(0)}(\mathbf{x};\mathbf{y}) = \sum_{\omega,\omega'=\pm 1} e^{-i(\omega x-\omega' y)p_F}\, g^{(0)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})\ , \tag{3.13}$$

if

$$g^{(0)}_{\omega,\omega'}(\mathbf{x};\mathbf{y}) \equiv \int \tilde P(d\psi^{(0)})\, \psi^{(0)-}_{\mathbf{x},\omega}\psi^{(0)+}_{\mathbf{y},\omega'} \tag{3.14}$$

is given by

$$g^{(0)}_{\omega,\omega'}(\mathbf{x};\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})} f_0(\mathbf{k}')[T_0^{-1}(\mathbf{k}')]_{\omega,\omega'}\ , \tag{3.15}$$

where the 2×2 matrix $T_0(\mathbf{k}')$ has elements

$$\begin{cases} [T_0(\mathbf{k}')]_{1,1} = (-ik_0 - (\cos k'-1)\cos p_F + v_0\sin k')\ ,\\ [T_0(\mathbf{k}')]_{1,2} = [T_0(\mathbf{k}')]_{2,1} = -\sigma_0(\mathbf{k}')\ ,\\ [T_0(\mathbf{k}')]_{2,2} = (-ik_0 - (\cos k'-1)\cos p_F - v_0\sin k')\ ,\end{cases} \tag{3.16}$$

which is well defined on the support of $f_0(\mathbf{k}')$, so that, if we set

$$A_0(\mathbf{k}') = \det T_0(\mathbf{k}') = [-ik_0 - (\cos k'-1)\cos p_F]^2 - (v_0\sin k')^2 - [\sigma_0(\mathbf{k}')]^2\ , \tag{3.17}$$

then

$$T_0^{-1}(\mathbf{k}') = \frac{1}{A_0(\mathbf{k}')}\begin{pmatrix} [\tau_0(\mathbf{k}')]_{1,1} & [\tau_0(\mathbf{k}')]_{1,2}\\ [\tau_0(\mathbf{k}')]_{2,1} & [\tau_0(\mathbf{k}')]_{2,2}\end{pmatrix}\ , \tag{3.18}$$

with

$$\begin{cases} [\tau_0(\mathbf{k}')]_{1,1} = [-ik_0 - (\cos k'-1)\cos p_F - v_0\sin k']\ ,\\ [\tau_0(\mathbf{k}')]_{1,2} = [\tau_0(\mathbf{k}')]_{2,1} = \sigma_0(\mathbf{k}')\ ,\\ [\tau_0(\mathbf{k}')]_{2,2} = [-ik_0 - (\cos k'-1)\cos p_F + v_0\sin k']\ .\end{cases} \tag{3.19}$$

We perform the integration

$$\int \tilde P(d\psi^{(0)})\, e^{\tilde{\mathcal{V}}^{(0)}(\psi^{(\le 0)})} \equiv e^{\mathcal{V}^{-1}(\psi^{(\le -1)})+\tilde E_0}\ , \tag{3.20}$$

where $\tilde E_0 = \log\int \tilde P(d\psi^{(0)})\ \exp\{\tilde{\mathcal{V}}^{(0)}(\psi^{(0)})\}$. We can write

$$\mathcal{L}\mathcal{V}^{(-1)} = \gamma^{-1}\nu_{-1}F_\nu^{(-1)} + s_{-1}F_\sigma^{(-1)}\ , \tag{3.21}$$

with suitable constants ν_{-1} and s_{-1}, and, by following the same procedure which led from (3.9) to (3.12), we have

$$\begin{aligned}\int P(d\psi^{(\le -1)})\, e^{\mathcal{V}^{(-1)}(\psi^{(\le -1)})} &= \frac{1}{\mathcal{N}_1}\int \tilde P(d\psi^{(\le -1)})\, e^{\tilde{\mathcal{V}}^{(-1)}(\psi^{(\le -1)})}\\ &= \frac{1}{\mathcal{N}_1}\int P(d\psi^{(\le -2)})\int \tilde P(d\psi^{(-1)})\, e^{\tilde{\mathcal{V}}^{(-1)}(\psi^{(\le -1)})}\ ,\end{aligned} \tag{3.22}$$

where, up to a constant,

$$\begin{aligned}\tilde P(d\psi^{(\le -1)}) = &\prod_{\mathbf{k}'}\prod_{\omega=\pm 1} d\psi^{(\le -1)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} d\psi^{(\le -1)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\\ &\exp\Big\{-\sum_{\omega=\pm 1}\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} C_{-1}(\mathbf{k}')\Big[\Big(-ik_0-(\cos k'-1)\cos p_F+\omega v_0\sin k'\Big)\\ &\psi^{(\le -1)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le -1)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} - \sigma_{-1}(\mathbf{k}')\,\psi^{(\le -1)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le -1)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega}\Big]\Big\}\ ,\end{aligned} \tag{3.23}$$

with $\sigma_{-1}(\mathbf{k}') = \sigma_0(\mathbf{k}') + C_{-1}^{-1}(\mathbf{k}')s_{-1}$ and

$$\mathcal{L}\tilde{\mathcal{V}}^{(-1)} = \gamma^{-1}\nu_{-1}F_\nu^{(-1)} \ . \tag{3.24}$$

The above procedure can be iterated, and at each step one has to perform the integration

$$\begin{aligned}&\int P(d\psi^{(\le h)})\, e^{\mathcal{V}^{(h)}(\psi^{(\le h)})} = \frac{1}{\mathcal{N}_h}\int \tilde{P}(d\psi^{(\le h)})\, e^{\tilde{\mathcal{V}}^{(h)}(\psi^{(\le h)})} = \\ &= \frac{1}{\mathcal{N}_h}\int P(d\psi^{(\le h-1)})\int \tilde{P}(d\psi^{(h)})\, e^{\tilde{\mathcal{V}}^{(h)}(\psi^{(\le h)})} = \frac{1}{\mathcal{N}_h}\int P(d\psi^{(\le h-1)}) e^{\mathcal{V}^{(h-1)}(\psi^{(\le h-1)})+\tilde{E}_h} \ ,\end{aligned} \tag{3.25}$$

which gives $\sigma_{h-1}(\mathbf{k}') = \sigma_h(\mathbf{k}') + C_h^{-1}(\mathbf{k}')s_h$, and defines the propagator

$$g^{(h)}(\mathbf{x};\mathbf{y}) = \sum_{\omega,\omega'=\pm 1} e^{-i(\omega x - \omega' y)p_F}\, g_{\omega,\omega'}^{(h)}(\mathbf{x};\mathbf{y}) \ , \tag{3.26}$$

with

$$\begin{aligned} g_{\omega,\omega'}^{(h)}(\mathbf{x};\mathbf{y}) &\equiv \int \tilde{P}(d\psi^{(h)})\, \psi_{\mathbf{x},\omega}^{(h)-}\psi_{\mathbf{y},\omega'}^{(h)+} \\ &= \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})} f_h(\mathbf{k}')[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'} \ ,\end{aligned} \tag{3.27}$$

and

$$\begin{cases} [T_h(\mathbf{k}')]_{1,1} = (-ik_0 - (\cos k' - 1)\cos p_F + v_0 \sin k') \ , \\ [T_h(\mathbf{k}')]_{1,2} = [T_h(\mathbf{k}')]_{2,1} = -\sigma_h(\mathbf{k}') \ , \\ [T_h(\mathbf{k}')]_{2,2} = (-ik_0 - (\cos k' - 1)\cos p_F - v_0 \sin k') \ , \end{cases} \tag{3.28}$$

so that

$$T_h^{-1}(\mathbf{k}') = \frac{1}{A_h(\mathbf{k}')}\begin{pmatrix} [\tau_h(\mathbf{k}')]_{1,1} & [\tau_h(\mathbf{k}')]_{1,2} \\ [\tau_h(\mathbf{k}')]_{2,1} & [\tau_h(\mathbf{k}')]_{2,2} \end{pmatrix} \ , \tag{3.29}$$

where

$$\begin{cases} [\tau_h(\mathbf{k}')]_{1,1} = [-ik_0 - (\cos k' - 1)\cos p_F - v_0 \sin k'] \ , \\ [\tau_h(\mathbf{k}')]_{1,2} = [\tau_h(\mathbf{k}')]_{2,1} = \sigma_h(\mathbf{k}') \ , \\ [\tau_h(\mathbf{k}')]_{2,2} = [-ik_0 - (\cos k' - 1)\cos p_F + v_0 \sin k'] \ , \end{cases} \tag{3.30}$$

and

$$A_h(\mathbf{k}') = [-ik_0 - (\cos k' - 1)\cos p_F]^2 - (v_0 \sin k')^2 - [\sigma_h(\mathbf{k}')]^2 \ . \tag{3.31}$$

We can define also

$$\begin{aligned} g_{\omega,\omega'}^{(\le h)}(\mathbf{x};\mathbf{y}) &\equiv \int \tilde{P}(d\psi^{(\le h)})\, \psi_{\mathbf{x},\omega}^{(\le h)-}\psi_{\mathbf{y},\omega'}^{(\le h)+} \\ &= \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})} C_h^{-1}(\mathbf{k}')[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'} \ ,\end{aligned} \tag{3.32}$$

where the last identity follows from (3.23) (with h in place of -1) and (3.25). Set

$$\tilde{g}_{\omega,\omega'}^{(h)}(\mathbf{k}') = f_h(\mathbf{k}')[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'} \ , \qquad \tilde{g}_{\omega,\omega'}^{(\le h)}(\mathbf{k}') = C_h^{-1}(\mathbf{k}')[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'} \ , \tag{3.33}$$

so that

$$\begin{aligned} g_{\omega,\omega'}^{(h)}(\mathbf{x};\mathbf{y}) &= \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}\, \tilde{g}_{\omega,\omega'}^{(h)}(\mathbf{k}') \ , \\ g_{\omega,\omega'}^{(\le h)}(\mathbf{x};\mathbf{y}) &= \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}\, \tilde{g}_{\omega,\omega'}^{(\le h)}(\mathbf{k}') \ .\end{aligned} \tag{3.34}$$

The localized part of the effective potential will be written as

$$\mathcal{L}\tilde{\mathcal{V}}^{(h)} = \gamma^h \nu_h F_\nu^{(h)} \ , \tag{3.35}$$

which defines the *running coupling constants* ν_h. Moreover

$$\sigma_h(\mathbf{k}') = \sum_{j=h}^{0} C_j^{-1}(\mathbf{k}')\, s_j \; . \tag{3.36}$$

Note that, thanks to the definition of $\chi(\mathbf{k}')$, see (2.3), if $f_h(\mathbf{k}') \neq 0$, we have

$$\sigma_h(\mathbf{k}') = C_h^{-1}(\mathbf{k}')\, s_h + \sum_{j=h+1}^{0} s_j \; . \tag{3.37}$$

Hence, by (2.5) and the second equation in (3.8), $\sigma_h(\mathbf{k}')$ is a smooth function on $\mathbb{T}^1 \times \mathbb{R}$, such that $\sigma_h(\mathbf{k}') = \sum_{j=h}^{0} s_j$ for $0 \le |\mathbf{k}'| \le t_0\gamma^h$ and $\sigma_h(\mathbf{k}') = \sum_{j=h+1}^{0} s_j$ for $|\mathbf{k}'| = a_0\gamma^h$; we define $\sigma_h \equiv \sum_{j=h}^{0} s_j$.

Note that

$$\mathrm{Re}\,[A_h(\mathbf{k}') + \sigma_h(\mathbf{k}')^2] = -k_0^2 - 4\sin^2\frac{k'}{2}\sin\left(p_F + \frac{k'}{2}\right)\sin\left(p_F - \frac{k'}{2}\right) \; , \tag{3.38}$$

and $p_F \pm k'/2 > 0$ on the support of $f_h(\mathbf{k}')$. Hence there is a constant G_3, such that, on the support of $f_h(\mathbf{k}')$

$$|A_h(\mathbf{k}') + \sigma_h(\mathbf{k}')^2| \ge G_3^2\gamma^{2h} \; . \tag{3.39}$$

Let us now define, for any complex λ with $|\lambda| \le \bar\varepsilon_0$,

$$h^* \equiv \inf\{h \ge h_\beta : G_3\gamma^h \ge 2\bar\sigma\} \; , \qquad \bar\sigma = |\lambda\hat\varphi_m| \neq 0 \; , \tag{3.40}$$

and let us suppose that there exists $\varepsilon_0 \le \bar\varepsilon_0$, such that, for $|\lambda| \le \varepsilon_0$ and $h \ge h^*$,

$$\frac{1}{2}\bar\sigma \le |\sigma_h(\mathbf{k}')| \le \frac{3}{2}\bar\sigma \; ; \tag{3.41}$$

such an assumption will be justified by Lemma 2, in §3.4.

It follows that there exists a constant G_1, such that, for any $h \ge h^*$,

$$|\tilde g^{(h)}_{\omega,\omega'}(\mathbf{k}')| \le G_1\gamma^{-h} \; , \tag{3.42}$$

and, if λ is real,

$$|\tilde g^{(\le h^*)}_{\omega,\omega'}(\mathbf{k}')| \le G_1\gamma^{-h^*} \; . \tag{3.43}$$

Note that, if λ is real, the factor 2 in front of $\bar\sigma$ in the definition of h^* could be substituted with any constant, without loosing the bounds (3.42) and (3.43)

Finally, since $|s_h| \le |\sigma_h| + |\sigma_{h+1}| \le 3\bar\sigma$, it easy to prove that, for $|\lambda| \le \varepsilon_0$, $h \ge h^*$, $0 \le t \le 1$ and any $\mathbf{q}$,

$$\left|\frac{d}{dt}\tilde g^{(h)}_{\omega,\omega'}(t\mathbf{k}' + \mathbf{q})\right| \le G_2|\mathbf{k}'|\gamma^{-2h} \; , \tag{3.44}$$

for a suitable constant G_2, a bound which will play an important role in the following.

3.2. The new effective potential $\tilde{\mathcal{V}}^{(h)}$ can be written as in (2.18), by substituting the kernel $\mathcal{W}_n^{(h)}(\mathbf{k})$ with a new kernel $\mathcal{W}_{\mathcal{R},n}^{(h)}(\mathbf{k})$, which admits a graph representation in terms of new labeled graphs $\vartheta_{\mathcal{R}}$, the *renormalized graphs*, which differ from the ones described in §2.3 in the following points:

- there is no resonant vertex with $n_v = \pm m$;
- there are new resonant vertices with $n_v = 0$, produced by the renormalization procedure, to which we associate a label $h_v \le 0$ and a factor $\gamma^{h_v}\nu_{h_v}$;
- at least one of the lines emerging from the resonant vertices with label h_v has frequency label h_v, while the other has frequency label h_v or $h_v - 1$ (it is an immediate consequence

of the renormalization procedure and momentum conservation);
• the internal lines ℓ's carry two labels ω_ℓ^i, $i=1,2$;
• given a line ℓ, the corresponding propagator is $\tilde g^{(h_\ell)}_{\omega_\ell^1,\omega_\ell^2}(\mathbf{k}'_\ell)$;
• on each resonant cluster V (including $\vartheta_{\mathcal R}$ itself, if it is a resonance) the $\mathcal{R}\equiv 1\!\!1-\mathcal{L}$ operator acts;
• the conservation of the momentum measured from the Fermi surface in each vertex gives the constraint

$$k'_{\ell_v} = k' + \sum_{\tilde v\le v}[2n_{\tilde v}p + (\omega^1_{\ell_{\tilde v'}} - \omega^2_{\ell_{\tilde v}})p_F] \; . \tag{3.45}$$

Then the second equation in (2.21) is replaced with

$$\mathcal{W}^{(h)}_{\mathcal{R},n,q}(\mathbf{k}) = \sum_{\vartheta_{\mathcal R}\in\mathcal{T}^h_{\mathcal{R},n,q}} \mathrm{Val}(\vartheta_{\mathcal R}) \; , \tag{3.46}$$

where $\mathcal{T}^h_{\mathcal{R},n,q}$ denotes the family of renormalized graphs of order q and scale h, such that $\sum_{v\in\vartheta_{\mathcal R}} 2n_v p + \sum_{\ell\in\mathrm{int}(\vartheta)}(\omega^1_\ell-\omega^2_\ell)p_F = 2np$ and $-[L/2]\le n\le[(L-1)/2]$, and $\mathrm{Val}(\vartheta_{\mathcal R})$ is computed following the rules listed above.

3.3. The renormalization procedure allows us to improve the bound of the graph values, and to extend Lemma 1 to cover also the case of graphs with resonances. As an example, let us consider a resonant cluster V, which does not contain other resonant clusters; we can associate to it a *resonance value*

$$\Xi^h_V(\mathbf{k}') = \prod_{\ell\in V} \tilde g^{(h_\ell)}_{\omega_\ell^1,\omega_\ell^2}(\mathbf{k}'_\ell) \; ; \tag{3.47}$$

the lines ℓ_V^i and ℓ_V^o have a momentum measured from the Fermi surface (by the Definition in §2.5) $k'\equiv k'_{\ell_V^i}=k'_{\ell_V^o}$, and $h=\min_{\ell\in V}\{h_\ell\}$ is the scale of V. The effect of the localization operator is to replace $\Xi^h_V(\mathbf{k}')$ with $\mathcal{L}\Xi^h_V(\mathbf{k}')\equiv\Xi^h_V(\mathbf{0})$, so that the effect of the $\mathcal R$ operator is to replace $\Xi^h_V(\mathbf{k}')$ with

$$\mathcal{R}\Xi^h_V(\mathbf{k}') \equiv \Xi^h_V(\mathbf{k}') - \Xi^h_V(\mathbf{0}) = \int_0^1 dt\Big[\frac{d}{dt}\Xi^h_V(t\mathbf{k}')\Big] \; . \tag{3.48}$$

Hence, by using (3.42) and (3.44), we can bound $\mathcal{R}\Xi^h_V(\mathbf{k}')$ in the following way:

$$|\mathcal{R}\Xi^h_V(\mathbf{k}')| \le \sum_{\ell'\in V} a_0 G_2 G_1^{L_V-1}\gamma^{h_V^e-h_{\ell'}}\prod_{\ell\in V}\gamma^{-h_\ell} \; , \tag{3.49}$$

where h_V^e is the external scale of V (see §2.4). Hence, with respect to the unrenormalized bound, $\mathcal R$ produces an extra factor of the form $\gamma^{h_V^e-h_{\ell'}}$, which can be used to compensate the lack of the small factor associated to non resonant clusters, as a consequence of the condition (1.27).

Concerning the resonant vertices, the renormalization procedure eliminated those with $n_v=\pm m$, but introduced new resonant vertices with $n_v=0$. The new vertices carry a factor γ^h, which is a real gain in the power counting, if one can prove that ν_h is uniformly bounded.

In fact the discussion can be generalized to graphs containing an arbitrary number of resonances: all these improvements will be used in Appendix 2, §A2.3, to prove the following extension of Lemma 1.

3.4. LEMMA 2. *If $\hat\varphi_m\neq 0$ and $\gamma>2^\tau$, there exists $\varepsilon_0\le\bar\varepsilon_0$, such that, for $|\lambda|\le\varepsilon_0$ and $h^*\le h\le 0$, we have*

$$\sup_{\mathbf{k}\in\mathcal{D}_h}\left|\mathcal{W}^{(h)}_{\mathcal{R},n,q}(\mathbf{k})\right| \le (|\lambda|B_2)^q\, e^{-\frac{\xi}{2}|n|} \; , \tag{3.50}$$

$$|\sigma_h-\lambda\hat\varphi_m| \le A|\lambda|^2 e^{-\frac{\xi}{2}m} \; , \qquad |\nu_h|\le B_3|\lambda| \; , \tag{3.51}$$

for some constants B_2, B_3, A.

REMARK. Lemma 2 implies that the series defining the kernel of the effective potential is convergent in the norm (2.19), uniformly in $L=L_i$ and β.

4. The two-point Schwinger function

4.1. In this section we define a perturbative expansion, similar to the one discussed for the effective potential in §3, for the two-point Schwinger function, defined by (1.10), which can be rewritten

$$S^{L,\beta}(\mathbf{x};\mathbf{y})=\lim_{M\to\infty}\frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}}\frac{1}{\mathcal{N}_1}\int P(d\psi)\,e^{\mathcal{V}(\psi)+\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}+\psi^+_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)}\Big|_{\phi^+=\phi^-=0}\,, \tag{4.1}$$

where $\int d\mathbf{x}$ is a shortcut for $\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2}dx_0$, $\mathcal{N}_1=\int P(d\psi)\,e^{\mathcal{V}(\psi)}$ and $\{\phi^\pm_{\mathbf{x}}\}$ are Grassmanian variables (the *external field*), anticommuting with $\{\psi^\pm_{\mathbf{x}}\}$. Note that all objects appearing in the r.h.s. of (4.1), as well as the other defined below, depend on L and β, but we shall not indicate explicitly this dependence, in order to simplify the notation.

Setting $\psi=\psi^{(\le 0)}+\psi^{(1)}$ and performing the integration over the field $\psi^{(1)}$ (ultraviolet integration), we find

$$\begin{aligned}S^{L,\beta}(\mathbf{x};\mathbf{y})=&\lim_{M\to\infty}\frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi_{\mathbf{y}}}e^{\int d\mathbf{x}d\mathbf{y}\,\phi^+_{\mathbf{x}}\,V^{(0)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\,\phi^-_{\mathbf{y}}}\\&\frac{1}{\mathcal{N}_0}\int P(d\psi^{(\le 0)})\,e^{\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^{(\le 0)-}_{\mathbf{x}}+\psi^{(\le 0)+}_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)}e^{\mathcal{V}^{(0)}(\psi^{(\le 0)})+W^{(0)}(\psi^{(\le 0)},\phi)}\Big|_{\phi^+=\phi^-=0}\,,\end{aligned}\tag{4.2}$$

where $\mathcal{N}_0=\int P(d\psi^{(\le 0)})\,e^{\mathcal{V}^{(0)}(\psi^{(\le 0)})}$,

$$W^{(0)}(\psi^{(\le 0)},\phi)=\int d\mathbf{x}d\mathbf{y}\left(\phi^+_{\mathbf{x}}\,K^{(0)}_{\phi,\psi}(\mathbf{x};\mathbf{y})\,\psi^{(\le 0)-}_{\mathbf{y}}+\psi^{(\le 0)+}_{\mathbf{x}}\,K^{(0)}_{\psi,\phi}(\mathbf{x};\mathbf{y})\,\phi^-_{\mathbf{y}}\right)\,, \tag{4.3}$$

$$V^{(0)}_{\phi,\phi}(\mathbf{x};\mathbf{y})=g^{(1)}(\mathbf{x};\mathbf{y})+K^{(0)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\,, \tag{4.4}$$

and

$$\begin{aligned}\int d\mathbf{x}d\mathbf{y}\,\chi^{(1)+}_{\mathbf{x}}K^{(0)}_{\chi^{(1)},\chi^{(2)}}(\mathbf{x};\mathbf{y})\,\chi^{(2)-}_{\mathbf{y}}&\\=\sum_{n=L/2}^{[(L-1)/2]}\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\hat K^{(0)}_{\chi^{(1)},\chi^{(2)},n}(\mathbf{k})\chi^{(1)+}_{\mathbf{k}}\chi^{(2)-}_{\mathbf{k}+2n\mathbf{p}}\,.&\end{aligned}\tag{4.5}$$

The kernels $\hat K^{(0)}_{\chi^{(1)},\chi^{(2)},n}(\mathbf{k})$ can be represented as sums of graphs of the same type of those appearing in the graph expansion of the effective potential $\mathcal{V}^{(0)}$; the new graphs differ only in the following respects:

• if $\chi^{(2)}=\phi$, the right external line is associated to the ϕ^- field and the graph ends with a vertex carrying no $\lambda\hat\varphi_n$ factor;

• if $\chi^{(1)}=\phi$, the left external line is associated to the ϕ^+ field and the graph begins with a vertex carrying no $\lambda\hat\varphi_n$ factor;

Note that there is at least one vertex carrying a $\lambda\hat\varphi_n$ factor. The propagators of the internal lines emerging from vertices without a $\lambda\hat\varphi_n$ factor will be called the *external propagators*.

We have, in particular,

$$K^{(0)}_{\phi,\phi}(\mathbf{x};\mathbf{y})=\sum_{q=3}^{\infty}\sum_{n=-[L/2]}^{[(L-1)/2]}\sum_{\vartheta\in\mathcal{T}^{\phi\phi,0}_{n,q}}\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})+2inpy}\,\mathrm{Val}(\vartheta)\,, \tag{4.6}$$

where $\mathcal{T}_{n,q}^{\phi\phi,0}$ is the set of all labeled graphs of order q with two external propagators, such that $\sum_{v\in\vartheta} n_v = n$ and $h_\ell = 1\ \forall\ell$; moreover, $\mathrm{Val}(\vartheta)$ is obtained from (2.20) by adding the two external propagators.

It is very easy to take the limit $M\to\infty$ in (4.2). In fact, for M large enough, the measure $P(d\psi^{(\le 0)})$ is independent of M; hence, the limit is obtained by taking the limit as $M\to\infty$ of the r.h.s. of (4.4). The limit of $K_{\phi,\phi}^{(0)}(\mathbf{x};\mathbf{y})$ is trivial; in fact each graph contributing to it behaves as $1/|k_0|^2$, as $k_0\to\infty$, so that the sum over k_0 is absolutely convergent. On the contrary, the limit of $g^{(1)}$ has to be done carefully, since it involves a sum over k_0, which is not absolutely convergent; however, by using standard techniques, one can show easily that the limit does exist and, uniformly in L and β, if $|x_0-y_0|\le\beta/2$ and $|x-y|\le L/2$,

$$\left|g^{(1)}(\mathbf{x};\mathbf{y})\right| \le \frac{C_N}{1+|\mathbf{x}-\mathbf{y}|^N}\ , \tag{4.7}$$

for any $N\ge 0$ and suitable constants C_N. Hence, from now on, we shall suppose that the limit $M\to\infty$ has been performed, but we shall still use the same notation for $g^{(1)}$ and $K_{\phi,\phi}^{(0)}(\mathbf{x};\mathbf{y})$.

Equation (4.2) can be written

$$S^{L,\beta}(\mathbf{x};\mathbf{y}) = V_{\phi,\phi}^{(0)}(\mathbf{x};\mathbf{y}) + S^{(0)}(\mathbf{x};\mathbf{y})\ , \tag{4.8}$$

where

$$\begin{aligned} S^{(0)}(\mathbf{x};\mathbf{y}) = &\frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}}\frac{1}{\mathcal{N}_0}\int P(d\psi^{(\le 0)}) \\ &e^{\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^{(\le 0)-}_{\mathbf{x}}+\psi^{(\le 0)+}_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)} e^{\mathcal{V}^{(0)}(\psi^{(\le 0)})+W^{(0)}(\psi^{(\le 0)},\phi)}\Big|_{\phi^+=\phi^-=0}\ . \end{aligned} \tag{4.9}$$

4.2. We proceed now as in §3, using the same notations. We write

$$\begin{aligned} S^{(0)}(\mathbf{x};\mathbf{y}) = &\frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}}\frac{1}{\tilde{\mathcal{N}}_0}\int \tilde{P}(d\psi^{(\le 0)}) \\ &e^{\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^{(\le 0)-}_{\mathbf{x}}+\psi^{(\le 0)+}_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)} e^{\tilde{\mathcal{V}}^{(0)}(\psi^{(\le 0)})+W^{(0)}(\psi^{(\le 0)},\phi)}\Big|_{\phi^+=\phi^-=0}\ , \end{aligned} \tag{4.10}$$

and decompose $\tilde{\mathcal{V}}^{(0)} = \mathcal{L}\tilde{\mathcal{V}}^{(0)} + \mathcal{R}\mathcal{V}^{(0)}$. On the contrary, we do not split $W^{(0)}$ into a relevant and an irrelevant part.

The integration over $\psi^{(0)}$ gives

$$\begin{aligned} S^{(0)}(\mathbf{x};\mathbf{y}) = &\frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}}\, e^{\int d\mathbf{x}d\mathbf{y}\,\phi^+_{\mathbf{x}}V^{(-1)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\phi^-_{\mathbf{y}}}\frac{1}{\tilde{\mathcal{N}}_1}\int \tilde{P}(d\psi^{(\le -1)}) \\ &e^{\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^{(\le -1)-}_{\mathbf{x}}+\psi^{(\le -1)+}_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)} e^{\mathcal{V}^{(-1)}(\psi^{(\le -1)})+W^{(-1)}(\psi^{(\le -1)},\phi)}\Big|_{\phi^+=\phi^-=0}\ , \end{aligned} \tag{4.11}$$

with

$$W^{(-1)}(\psi^{(\le -1)},\phi) = \int d\mathbf{x}d\mathbf{y}\left(\phi^+_{\mathbf{x}}\,K^{(-1)}_{\phi,\psi}(\mathbf{x},\mathbf{y})\,\psi^{(\le -1)-}_{\mathbf{y}} + \psi^{(\le -1)+}_{\mathbf{y}}\,K^{(-1)}_{\psi,\phi}(\mathbf{x};\mathbf{y})\,\phi^-_{\mathbf{x}}\right)\ , \tag{4.12}$$

$$V^{(-1)}_{\phi,\phi}(\mathbf{x};\mathbf{y}) = g^{(0)}(\mathbf{x};\mathbf{y}) + K^{(-1)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\ . \tag{4.13}$$

The kernels $\hat{K}^{(-1)}_{\chi^{(1)},\chi^{(2)},n}(\mathbf{k})$ can be represented as sums of graphs of the same type of those appearing in the graph expansion of the effective potential $\mathcal{V}^{(-1)}$; the new graphs differ only in the following respects:

• if $\chi^{(2)}=\phi$, the right external line is associated to the ϕ^- field and the graph ends with a vertex carrying no $\lambda\hat{\varphi}_n$ factor;

• if $\chi^{(1)} = \phi$, the left external line is associated to the ϕ^+ field and the graph begins with a vertex carrying no $\lambda\hat\varphi_n$ factor;
• $\mathcal{R} \equiv 1\!\!1$ on resonances containing an external propagator (defined as before).
• $h_\vartheta = 0$ for all graphs, if $\chi^{(1)} = \chi^{(2)} = \phi$.

4.3. The above construction can be iterated and we find, for any $h^* \le h \le 0$,

$$S^{L,\beta}(\mathbf{x};\mathbf{y}) = \sum_{h'=h}^{0} V^{(h')}_{\phi,\phi}(\mathbf{x};\mathbf{y}) + S^{(h)}(\mathbf{x};\mathbf{y}) \; , \tag{4.14}$$

where

$$\begin{aligned} S^{(h)}(\mathbf{x};\mathbf{y}) = {} & \frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}} \frac{1}{\mathcal{N}_h} \int P(d\psi^{(\le h)}) \\ & e^{\int d\mathbf{x}\left(\phi^+_{\mathbf{x}}\psi^{(\le h)-}_{\mathbf{x}} + \psi^{(\le h)+}_{\mathbf{x}}\phi^-_{\mathbf{x}}\right)} e^{\mathcal{V}^{(h)}(\psi^{(\le h)}) + W^{(h)}(\psi^{(\le h)},\phi)} \Big|_{\phi^+=\phi^-=0} \; , \end{aligned} \tag{4.15}$$

$$W^{(h)}(\psi^{(\le h)},\phi) = \int d\mathbf{x}d\mathbf{y} \left(\phi^+_{\mathbf{x}}\, K^{(h)}_{\phi,\psi}(\mathbf{x},\mathbf{y})\, \psi^{(\le h)-}_{\mathbf{y}} + \psi^{(\le h)+}_{\mathbf{y}}\, K^{(h)}_{\psi,\phi}(\mathbf{x};\mathbf{y})\, \phi^-_{\mathbf{x}} \right) \; , \tag{4.16}$$

$$V^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y}) = g^{(h+1)}(\mathbf{x};\mathbf{y}) + K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y}) \; . \tag{4.17}$$

The kernels $\hat K^{(h)}_{\chi^{(1)},\chi^{(2)},n}(\mathbf{k})$ can be represented as sums of graphs of the same type of those appearing in the graph expansion of the effective potential $\mathcal{V}^{(h)}$; the new graphs differ only in the following respects:

- if $\chi^{(2)} = \phi$, the right external line is associated to the ϕ^- field and the graph ends with a vertex carrying no $\lambda\hat\varphi_n$ factor;
- if $\chi^{(1)} = \phi$, the left external line is associated to the ϕ^+ field and the graph begins with a vertex carrying no $\lambda\hat\varphi_n$ factor;
- $\mathcal{R} \equiv 1\!\!1$ on resonances containing an external propagator (defined as before);
- $h_\vartheta = h+1$ for all graphs, if $\chi^{(1)} = \chi^{(2)} = \phi$.

Let us now suppose that $L = L_i$, $i \in \mathbb{Z}^+$, so that the condition (1.20) is satisfied. The integration over the field $\psi^{(\le h^*)}$ can be performed in a single step, since the covariance $g^{(\le h^*)}$ satisfies the same bound as $g^{(h^*)}$, see (3.42) and (3.43). Then the functional derivatives in (4.1) give

$$S^{L_i,\beta}(\mathbf{x};\mathbf{y}) = \sum_{h=h^*}^{0} \left(g^{(h+1)}(\mathbf{x};\mathbf{y}) + K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y}) \right) + g^{(\le h^*)}(\mathbf{x};\mathbf{y}) + K^{(<h^*)}_{\phi,\phi}(\mathbf{x};\mathbf{y}) \; , \tag{4.18}$$

where

$$K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y}) = \sum_{q=3}^{\infty} \sum_{n=-[L/2]}^{[(L-1)/2]} \sum_{\vartheta_{\mathcal{R}} \in \mathcal{T}^{\phi\phi,h}_{\mathcal{R},n,q}} \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) + 2inpy}\, \mathrm{Val}(\vartheta_{\mathcal{R}}) \; . \tag{4.19}$$

and $\mathcal{T}^{\phi\phi,h}_{\mathcal{R},n,q}$ is the set of all labeled graphs of order q with two external propagators, such that $\sum_{v\in\vartheta} n_v + \sum_{\ell\in\mathrm{int}(\vartheta)} (\omega^1_\ell - \omega^2_\ell) p_F = n$, and $\mathrm{Val}(\vartheta_{\mathcal{R}})$ is computed with the rules explained after (4.17). A similar expression is valid for $K^{(<h^*)}_{\phi,\phi}(\mathbf{x};\mathbf{y})$, with $g^{(\le h^*)}$ in place of $g^{(h+1)}$.

All the functions appearing in the r.h.s. of (4.19) have the fast decay property. In fact, as we show in Appendix 3, §A3.1 and §A3.2, if $|x_0 - y_0| \le \beta/2$, $|x-y| \le L/2$, we have, for any $N \ge 0$ and suitable constants C_N, independent of i and β,

$$\left| g^{(h)}(\mathbf{x};\mathbf{y}) \right| \le \frac{C_N \max\{\gamma^h, L^{-1}\}}{1 + \gamma^{hN}|\mathbf{x}-\mathbf{y}|^N} \; , \tag{4.20}$$

$$\left|K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\right| \le \frac{C_N|\lambda|\max\{\gamma^h, L^{-1}\}}{1+\gamma^{hN}|\mathbf{x}-\mathbf{y}|^N} \;. \tag{4.21}$$

Similar bounds are verified by $K^{(<h^*)}_{\phi,\phi}(\mathbf{x};\mathbf{y})$ and $g^{(\le h^*)}$.

4.4. We are now ready to prove Theorem 1. First of all, we note that the r.h.s. of (4.19) has a meaning also if we take the formal limit $i\to\infty$, $\beta\to\infty$, (recall that $L=L_i$), by doing the substitution

$$\frac{1}{L_i\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L_i,\beta}} \to \int_{\mathbb{T}^1\times\mathbb{R}} \frac{d\mathbf{k}}{(2\pi)^2} \;. \tag{4.22}$$

In fact the integral over $\mathbf{k}$ is well defined, since $k\in\mathbb{T}^1$ and k_0 belongs to a bounded set, except in the case $h=0$, and, for $h=0$, each graph contributing to $K^{(0)}_{\phi,\phi}(\mathbf{x};\mathbf{y})$ decays at least as $|k_0|^{-2}$ for large values of $|k_0|$, (see comments between (4.6) and (4.7)).

The substitution (4.22), applied to (3.34), allows to define $g^{(h)}(\mathbf{x};\mathbf{y})$ in the limit $i\to\infty$, $\beta\to\infty$, at least for $h\le 0$. For $h=1$, one has to be careful, since the integral over k_0 is not absolutely convergent. However it is easy to prove, by using standard well known arguments, that the limit as $i\to\infty$ and $\beta\to\infty$ of $g^{(1)}(\mathbf{x};\mathbf{y})$ is well defined for $x_0\ne y_0$ and has the same discontinuity in $x_0=y_0$ of the same limit taken on the free propagator (1.3). We shall denote this limit, as usual, by doing again the substitution (4.22) in the finite L and β expression (see again comments between (4.6) and (4.7)).

The previous considerations suggest that, for λ real and small enough, there exists the limit

$$S(\mathbf{x};\mathbf{y}) = \lim_{\substack{\beta\to\infty\\ i\to\infty}} S^{L_i,\beta}(\mathbf{x};\mathbf{y}) \tag{4.23}$$

and that this limit is obtained by doing the substitution (4.22) in all quantities appearing in the r.h.s. of (4.18). In Appendix 3, §A3.3, we shall prove that this is indeed the case.

We want now to prove that $S(\mathbf{x};\mathbf{y})$ can be decomposed as in (1.21). Therefore, we shall suppose that the substitution (4.22) has been done everywhere. Note that the bounds (4.20) and (4.21) and the similar ones for $K^{(<h^*)}_{\phi,\phi}(\mathbf{x};\mathbf{y})$ and $g^{(\le h^*)}$ are still valid, without any restriction on $\mathbf{x}$, $\mathbf{y}$ and γ^h in place of $\max\{\gamma^h, L^{-1}\}$.

If $h\le 0$, we can write

$$g^{(h)}(\mathbf{x};\mathbf{y}) = \bar g^{(h)}(\mathbf{x};\mathbf{y}) + r^{(h)}(\mathbf{x};\mathbf{y}) \;, \tag{4.24}$$

where $\bar g^{(h)}(\mathbf{x};\mathbf{y})$ is obtained by $g^{(h)}(\mathbf{x};\mathbf{y})$, by substituting in (3.29) $\sigma_h(\mathbf{k}')$ with $\sigma\equiv s_0=\lambda\hat\varphi_m$.

By proceeding as in the proof of (A3.3) and using (3.51), it is easy to prove that, if $|\lambda|\le\varepsilon_0$ and $h\ge h^*$, for any $N\ge 0$,

$$\left|r^{(h)}(\mathbf{x};\mathbf{y})\right| \le \frac{C_N|\lambda|\gamma^h}{1+\gamma^{hN}|\mathbf{x}-\mathbf{y}|^N} \;, \tag{4.25}$$

where C_N, here and everywhere from now on, denotes a generic suitable constant, only depending on N. A bound similar to (4.25) is valid for $r^{(\le h^*)}(\mathbf{x};\mathbf{y})$, if λ is real.

By diagonalizing the quadratic form $\sum_{\omega,\omega'=\pm1} e^{-i(\omega x-\omega' y)p_F}[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'}$, it is possible to see that

$$\bar g^{(h)}(\mathbf{x};\mathbf{y}) = \int \frac{d\mathbf{k}'}{(2\pi)^2} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})} f_h(\mathbf{k}') \left[\frac{F_{xy}(k',\sigma)}{A+B} + \frac{F_{xy}(-k',-\sigma)}{A-B}\right] \;, \tag{4.26}$$

where

$$\begin{aligned} F_{xy}(k',\sigma) &= \tilde\phi(k',x,\sigma)\tilde\phi(k',-y,\sigma) \;,\\ \tilde\phi(k',x,\sigma) &= \frac{1}{\sqrt{2B}}\left[\sqrt{B-C}\; e^{ip_F x} - \frac{\sigma}{\sqrt{B-C}} e^{-ip_F x}\right] \;,\\ A &= -ik_0 + (1-\cos k')\cos p_F \;,\\ C &= v_0 \sin k' \;,\\ B &= \sqrt{C^2+\sigma^2} \;. \end{aligned} \tag{4.27}$$

We can rewrite the integral in (4.26) in terms of the k variable. Recall that, if $\omega = \mathrm{sign}(k)$ and $f_h(\mathbf{k}') \neq 0$, $k = \omega p_F + k'$ and $k' = \omega(|k| - p_F)$. Hence, if we write (recall that $\theta(k)$ denotes the step function),

$$\begin{aligned}\left[\frac{F_{xy}(k',\sigma)}{A+B} + \frac{F_{xy}(-k',-\sigma)}{A-B}\right] &= \\ &= \sum_{\omega=\pm1}\left[\frac{F_{xy}(k',\sigma)\theta(\omega k')}{A+B} + \frac{F_{xy}(-k',-\sigma)\theta(-\omega k')}{A-B}\right]\ ,\end{aligned} \tag{4.28}$$

it is easy to see that

$$\begin{aligned}\bar g^{(h)}(\mathbf{x};\mathbf{y}) = \int \frac{d\mathbf{k}}{(2\pi)^2}\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{\hat f_h(\mathbf{k})}{-ik_0+\varepsilon(k,\sigma)} \sum_{\omega=\pm1}\theta(\omega k)e^{i\omega p_F(x-y)} \\ \left[F_{xy}(k',\sigma)\theta(|k|-p_F) + F_{xy}(-k',-\sigma)\theta(-|k|+p_F)\right]\ ,\end{aligned} \tag{4.29}$$

where

$$\varepsilon(k,\sigma) = [1-\cos(|k|-p_F)]\cos p_F + \mathrm{sign}(|k|-p_F)\sqrt{v_0^2\sin^2(|k|-p_F)+\sigma^2}\ . \tag{4.30}$$

By doing some other simple algebraic calculations and by using (2.3), we get

$$\begin{aligned}&\sum_{h=h^*+1}^{1} g^{(h)}(\mathbf{x};\mathbf{y}) + g^{(\le h^*)}(\mathbf{x};\mathbf{y}) = S_1(\mathbf{x};\mathbf{y}) + \bar S_2(\mathbf{x};\mathbf{y})\ , \\ &S_1(\mathbf{x};\mathbf{y}) = g^{(1)}(\mathbf{x};\mathbf{y}) + \int \frac{d\mathbf{k}}{(2\pi)^2}\,[1-\hat f_1(\mathbf{k})]\,\phi(k,x,\sigma)\,\phi(k,-y,\sigma)\,\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\varepsilon(k,\sigma)}\ ,\end{aligned} \tag{4.31}$$

where $\phi(k,x,\sigma)$ is defined as in (1.23), and, if $\lambda \in \mathbb{R}$, $|\lambda| \le \varepsilon_0$, we have

$$\left|\bar S_2(\mathbf{x};\mathbf{y})\right| \le |\lambda| \sum_{h=h^*}^{0} \frac{C_N\gamma^h}{1+\gamma^{hN}|\mathbf{x}-\mathbf{y}|^N}\ . \tag{4.32}$$

We now define

$$\lambda S_2(\mathbf{x};\mathbf{y}) = \bar S_2(\mathbf{x};\mathbf{y}) + \sum_{h=h^*}^{0} K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\ ; \tag{4.33}$$

then (4.21) and (4.32) yield

$$\left|S_2(\mathbf{x};\mathbf{y})\right| \le \sum_{h=h^*}^{0} \frac{C_N\gamma^h}{1+\gamma^{hN}|\mathbf{x}-\mathbf{y}|^N}\ , \tag{4.34}$$

and the same bound can be shown to hold also for $S_1(\mathbf{x};\mathbf{y})$.

From (4.34) the bounds (1.24) and (1.25) follow. In fact, if $1 \le |\mathbf{x}-\mathbf{y}| \le \gamma G_3(2|\sigma|)^{-1}$ (see (3.39) for the definition of G_3) and $h_x \ge h^*$ is such that $\gamma^{-h_x-1} < |\mathbf{x}-\mathbf{y}| \le \gamma^{-h_x}$, (4.34) gives, if $N > 1$,

$$\left|S_2(\mathbf{x};\mathbf{y})\right| \le C_N \sum_{h=h^*}^{h_x-1}\gamma^h + \sum_{h=h_x}^{0}\frac{C_N\gamma^h}{\gamma^{Nh}|\mathbf{x}-\mathbf{y}|^N} \le \gamma^{h_x}C_N \le \frac{C_N}{1+|\mathbf{x}-\mathbf{y}|}\ . \tag{4.35}$$

On the other hand, if $|\mathbf{x}-\mathbf{y}| \ge \gamma G_3(2|\sigma|)^{-1}$, (4.34) implies that

$$\left|S_2(\mathbf{x};\mathbf{y})\right| \le \frac{C_N}{|\mathbf{x}-\mathbf{y}|^N}\sum_{h=h^*}^{0}\gamma^{-(N-1)h} \le \frac{C_N}{|\mathbf{x}-\mathbf{y}|^N}|\sigma|^{-N+1} \le \frac{C_N|\sigma|}{1+|\sigma|^N|\mathbf{x}-\mathbf{y}|^N}\ , \tag{4.36}$$

provided that $N > 1$.

The proof of (1.26) is an easy consequence of the definition (3.26). In fact, by proceeding as in Appendix 3, one can prove that

$$\left| g^{(h)}(\mathbf{x};\mathbf{y}) - \sum_{\omega=\pm1} \int \frac{d\mathbf{k}'}{(2\pi)^2} \tilde g^{(h)}_\omega(\mathbf{k}') \right| \le \frac{C_0\gamma^{2h}}{\gamma^{2h}+\bar\sigma^2} \left[\frac{\bar\sigma^2\gamma^h}{\gamma^{2h}+\bar\sigma^2} + \bar\sigma \right] \le C_0\bar\sigma \; . \tag{4.37}$$

A similar bound is valid for $g^{(\le h^*)}(\mathbf{x};\mathbf{y})$; hence we have

$$|S_1(\mathbf{x};\mathbf{y}) - g(\mathbf{x}-\mathbf{y})| \le \sum_{h=h^*}^{0} C_0\bar\sigma \le C_0\bar\sigma \log(\bar\sigma^{-1}) \; . \tag{4.38}$$

The continuity of $S(\mathbf{x};\mathbf{y})$ as a function of $\lambda \in \mathbb{R}$, $|\lambda| \le \varepsilon_0$, is completely trivial for $\lambda \ne 0$ and, in $\lambda = 0$, immediately follows from (1.21), (1.24) and (1.26).

4.5. The proof of item (iii) of Theorem 1 is based on similar arguments, applied to the finite $L = L_i$ and β case, but one has to consider more carefully the contribution of the scales $h < h_L \equiv \min\{h \ge h_\beta : (2\pi)/L > a_0\gamma^h\}$. In fact, for $h < h_L$, one looses a γ^h factor in the bound analogous to the bound (4.37), valid for finite L and β.

Note that $S^{L,\beta}(\mathbf{x};\mathbf{y}) - g^{(1)L,\beta}(\mathbf{x};\mathbf{y})$, as well as its limit as $\beta \to \infty$, is continuous as a function of $x_0 - y_0$ and that $g^{(1)L,\beta}(\mathbf{x};\mathbf{y})$ is independent of λ. Hence we can write, by using (1.19), (4.18) and (3.27),

$$\rho^{L,\beta} = \rho_0^{L,\beta} - \sum_{\substack{h=h^*+1,\dots,0\\ \omega=\pm1}} g^{(h)}_{\omega,\omega}(\mathbf{0};\mathbf{0}) - \sum_{h=h^*,\dots,0} \hat K^{(h)}_{\phi,\phi,0} - \sum_{\omega=\pm1} g^{(\le h^*)}_{\omega,\omega}(\mathbf{0};\mathbf{0}) - \hat K^{(<h^*)}_{\phi,\phi,0} \; , \tag{4.39}$$

where $\hat K^{(h)}_{\phi,\phi,0}$ denotes the contribution with $n = 0$ to the sum in the r.h.s. of (4.19), an analogous meaning has to be given to $\hat K^{(<h^*)}_{\phi,\phi,0}$ and $\rho_0^{L,\beta} = -\lim_{\tau\to0^-} g^{(1)L,\beta}(0,\tau;0,0)$ (it is bounded uniformly in β and has a well defined limit as $\beta \to \infty$).

Note now that, if $h < h_L$, the support of $f_h(\mathbf{k}')$ is restricted to $\mathbf{k}'$ of the form $(0, k_0)$. Hence, if $p_F = mn_L\pi/L$ is not an allowed momentum, that is if mn_L is an odd integer, the support of $f_h(\mathbf{k}')$ is empty and $g^{(h)}_{\omega,\omega}(\mathbf{0};\mathbf{0}) = 0$; if, on the contrary, p_F is an allowed momentum, the support is not empty, but $g^{(h)}_{\omega,\omega}(\mathbf{0};\mathbf{0})$ can be expressed as the sum over k_0 of a function odd in k_0, hence it vanishes again. The same considerations apply to $g^{(\le h^*)}_{\omega,\omega}(\mathbf{0};\mathbf{0})$, if $h^* < h_L$. Moreover, there is a finite L and β version of the bound (4.37), for $h \ge h^*$, so that we have

$$\begin{aligned} &\sum_{h=h^*+1}^{0} \sum_{\omega=\pm1} \left| g^{(h)}_{\omega,\omega}(\mathbf{0};\mathbf{0}) - \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \tilde g^{(h)}_\omega(\mathbf{k}') \right| + \\ &+ \sum_{\omega=\pm1} \left| g^{(\le h^*)}_{\omega,\omega}(\mathbf{0};\mathbf{0}) - \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \tilde g^{(\le h^*)}_\omega(\mathbf{k}') \right| \le \sum_{h=\max\{h^*,h_L\}}^{0} C_0\bar\sigma \le C_0\bar\sigma\log(\bar\sigma^{-1}) \; . \end{aligned} \tag{4.40}$$

Finally, we have, if $h^* < h_L$,

$$\sum_{h=h^*}^{0} |\hat K^{(h)}_{\phi,\phi,0}| + |\hat K^{(<h^*)}_{\phi,\phi,0}| \le C_0|\lambda| \left[\sum_{h=h_L}^{0} \gamma^h + \sum_{h=h^*}^{h_L-1} L^{-1} \right] \le C_0 \left|\frac{\lambda}{L}\right| \log\left|\frac{\lambda}{L}\right|^{-1} \; , \tag{4.41}$$

and a similar bound, without the factor $\log|\frac{\lambda}{L}|^{-1}$, is verified, if $h^* \ge h_L$.

The bounds (4.40), (4.41) and equation (4.39) immediately imply that $\rho^{L,\beta}$ is a continuous function of λ in $\lambda = 0$, the only point where the continuity is not completely obvious.

4.6. We now prove item (iv) of Theorem 1. We first note that, at finite volume $L = L_i$ and zero temperature, the two-point Schwinger function $S^L(\mathbf{x};\mathbf{y})$ can be written in the form

$$S^L(x,t;y,0) = \frac{1}{L} \sum_k \sum_{n=-[L/2]}^{[(L-1)/2]} e^{-ik(x-y)+2inpy} \int \frac{dk_0}{2\pi} e^{-ik_0t} S_n(\mathbf{k}) \; . \tag{4.42}$$

This follows from (4.18), by taking the limit $\beta \to \infty$ as in §4.4.

For $t \neq 0$, $S^L(x,t;y,0)$ can be also expressed in terms of the spectrum $\Sigma \equiv \{E_1, \ldots, E_L\}$ of the one-particle Hamiltonian h_{xy}, by the well known equation, easily following from its definition,

$$S^L(x,t;y,0) = \frac{\text{sign}\ t}{2} \sum_{r:E_r=\mu} u_r(x)u_r^*(y) + \sum_{r:E_r\neq\mu} u_r(x)u_r^*(y) \int \frac{dk_0}{2\pi} \frac{e^{-ik_0 t}}{-ik_0 + E_r - \mu} \ , \quad (4.43)$$

where $u_r(x)$ is the normalized eigenfunction of h_{xy} with eigenvalue E_r.

The fast decay in t of $S^L(x,t;y,0)$ implies that there is no eigenvalue equal to μ; hence, by comparing (4.42) and (4.43) with $x = y$, we get:

$$\sum_r \frac{|u_r(x)|^2}{-ik_0 + E_r - \mu} = \frac{1}{L} \sum_{k\in\mathcal{D}_L} \sum_{n=-[L/2]}^{[(L-1)/2]} e^{2inpx} S_n(\mathbf{k}) \ . \quad (4.44)$$

It follows that, in order to prove that there is a gap $\Delta \geq \bar\sigma/2$, it is sufficient to prove that the r.h.s of (4.44) is analytic as a function of k_0, in the strip $\{|\text{Im}\, k_0| \leq \bar\sigma/4\}$.

For any fixed real λ small enough and k_0 real, the r.h.s. of (4.44) can be bounded by proceeding as in §4.5. The only difference is that there is no integral over k_0, so that we loose a factor γ^h in the contributions of scale h. It is easy to see that

$$\left| \frac{1}{L} \sum_{k\in\mathcal{D}_L} \sum_{n=-[L/2]}^{[(L-1)/2]} e^{2inpy} S_n(\mathbf{k}) \right| \leq C_0 \left(\log\frac{1}{|\lambda|} + \frac{1}{L|\lambda|} \right) \quad (4.45)$$

We want to show that the same bound can be obtained, also if we take k_0 complex, with $|\text{Im}\, k_0| \leq \bar\sigma/4$. Of course, the expansion discussed before is not suitable for such a task, since the cutoff functions $\hat f_h(\mathbf{k})$ are not analytic in k_0. However, we can consider a different multiscale decomposition, involving only the k' variables. It is sufficient to modify the equations (2.3)÷(2.8), by substituting everywhere $\mathbf{k}$, $\mathbf{k}'$ and $|\mathbf{k}'|$ with k, k' and $\|k'\|_{\mathbb{T}^1}$, respectively. Moreover equation (2.9) must be modified, by further substituting $\hat f_{h_\beta}(\mathbf{k})$ with $\chi(\gamma^{-h_\beta}(k - p_F)) + \chi(\gamma^{-h_\beta}(k + p_F))$.

The analysis of §2 can be repeated, without any problem, since the bound (2.14) is still valid; in particular, Lemma 1 is still true. Also the analysis of §3 can be repeated, but now one has to be careful when bounding the contribution of a resonance, since the discussion leading to the bound (3.49) is not valid anymore. The reason is that the factor $|\mathbf{k}'|$ in the r.h.s. of (3.44) is not of order $\gamma^{h_V^e}$, since k_0 is not constrained anymore to be small by the support properties of the cutoff functions. We shall now discuss how this part of the analysis of §3 has to be modified.

We first note that (2.14), (3.42) and (3.43) can be replaced by

$$\begin{aligned}
&|\tilde g^{(h)}_\omega(\mathbf{k}')| \leq G_0\, |-ik_0 + \gamma^h|^{-1} \ ,\\
&|\tilde g^{(h)}_{\omega,\omega'}(\mathbf{k}')| \leq G_1\, |-ik_0 + \gamma^h|^{-1} \ ,\\
&|\tilde g^{(\leq h^*)}_{\omega,\omega'}(\mathbf{k}')| \leq G_1\, |-ik_0 + \gamma^{h^*}|^{-1} \ ,
\end{aligned} \quad (4.46)$$

while (3.44) becomes

$$\left| \frac{d}{dt} \tilde g^{(h)}_{\omega,\omega'}(t\mathbf{k}' + \mathbf{q}) \right| \leq \frac{G_2}{2} \left[\frac{|\mathbf{k}'|}{|-itk_0 + \gamma^h|^2} + \frac{\|k'\|_{\mathbb{T}^1}}{\gamma^h|-itk_0 + \gamma^h|} \right] \ , \quad (4.47)$$

for a suitable constant G_2, and (3.49) becomes

$$\begin{aligned}
|\mathcal{R}\Xi^h_V(\mathbf{k}')| \leq &\sum_{\ell'\in V} \frac{a_1 G_2 G_0^{L_V-1}}{2} \\
&\int_0^1 dt \left[\frac{|-ik_0 + \gamma^{h_V^e}|}{|-itk_0 + \gamma^{h_{\ell'}}|} + \frac{\gamma^{h_V^e}}{\gamma^{h_{\ell'}}} \right] \prod_{\ell\in V} |-itk_0 + \gamma^{h_\ell}|^{-1} \ ,
\end{aligned} \quad (4.48)$$

where the second factor inside the square brackets can be bounded by the first one.

At first sight, the bound (4.48) is not as good as the bound (3.49), since we do not get the factor $\gamma^{h^e_V - h_\ell}$, which we claimed in §3 is necessary to compensate, in the case of a resonance, the lack of a small factor associated to non resonant clusters; even worse, the factor $|-ik_0+\gamma^{h^e_V}|/|-itk_0+\gamma^{h_{\ell'}}|$ is not bounded. However, it is easy to see, by using the first equation in (4.46), that the product of $|\mathcal{R}\Xi^h_V(\mathbf{k}')|$ by a propagator of scale h^e_V is bounded as in §3.

It follows that we can certainly bound the value of a graph $\vartheta_\mathcal{R}$ as in §3, if it contains only a resonance, not coinciding with the whole graph (so that there is at least one propagator external to the resonance). If $\vartheta_\mathcal{R}$ itself is a resonance, but it does not contain other resonances, the previous argument does not apply, but in this case the factor $\gamma^{h^e_V-h_\ell}$ was not used in the proof of Lemma 2, see (A2.29); hence it is sufficient to bound $|\mathcal{R}\Xi^h_V(\mathbf{k}')|$ by $|\Xi^h_V(\mathbf{k}')|+|\Xi^h_V(\mathbf{0})|$ and the problem associated with the use of (4.47) disappears.

In Appendix 4, we show that the previous considerations can be generalized, in order to treat a general graph, so that we get a new expansion of the effective potentials, satisfying the same bounds as before.

The analysis in Appendix 4 also shows that the bound (4.45) is still valid, for k_0 real and λ small enough. However, it is very easy to see that it is valid also if we substitute everywhere k_0 with $k_0+i\eta$, $|\eta|\le\bar\sigma/4$. In fact, the dependence on k_0 is now restricted to the factor $(-ik_0+\cos p_F-\cos k)^{-1}$ in the definition of $\hat g^{(1)}(\mathbf{k})$ and to $[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'}$, see (3.29). It is very easy to see that $\hat g^{(1)}(k,k_0+i\eta)$ can be bounded as before, if λ is small enough, and the same is true for $\tilde g^{(h)}_{\omega,\omega'}(\mathbf{k}')$, because, on the support of $f_h(k')$, thanks to (3.38) and (3.41),

$$\begin{aligned} |\mathrm{Re}\,[A_h(k',k_0+i\eta)]| &\ge |\mathrm{Re}\,[A_h(k',k_0)]| - |\eta|^2 - 4|\eta|\cos p_F \sin^2\frac{k'}{2} \\ &\ge \bar\sigma^2/8 + c_1\gamma^{2h}(1-c_2\bar\sigma)\;, \end{aligned} \tag{4.49}$$

with suitable constants c_1 and c_2. It follows that the bounds (4.46), hence the bound (4.45) too, are satisfied also for k_0 complex, if $|\mathrm{Im}\,k_0|\le\bar\sigma/4$.

Appendix 1. Uniform bound on rational approximations of diophantine numbers

A1.1. Let ω be an irrational number such that $0 < \omega < 1$ and let $\{p_i/q_i\}$, $i \geq 0$, the sequence of its *convergents*, that is the sequence of its truncated continued fractions. We have $p_0 = 1$, $q_0 = [1/\omega] \geq 1$, hence $p_0/q_0 > \omega$. We define, for $i \geq 0$,

$$\begin{aligned} h_+(x) &= p_{2i} + \frac{p_{2i+2} - p_{2i}}{q_{2i+2} - q_{2i}}(x - q_{2i}) \quad \text{if } q_{2i} \leq x \leq q_{2i+2} \\ h_-(x) &= p_{2i+1} + \frac{p_{2i+3} - p_{2i+1}}{q_{2i+3} - q_{2i+1}}(x - q_{2i+1}) \quad \text{if } q_{2i+1} \leq x \leq q_{2i+3} \end{aligned} \tag{A1.1}$$

Note that the graph of $h_+(x)$ ($h_-(x)$) is made by a sequence of segments joining the points (q_{2i}, p_{2i}) and (q_{2i+2}, p_{2i+2}) ((q_{2i+1}, p_{2i+1}) and (q_{2i+3}, p_{2i+3})).

The well known properties of the convergents (see, for example, [D]) imply that

(a) $h_+(x) > \omega x > h_-(x)$, $\forall x \geq q_1$.

(b) $\delta_+(x) \equiv h_+(x) - \omega x$ and $\delta_-(x) \equiv \omega x - h_-(x)$ are strictly decreasing functions and $\lim_{x\to\infty} \delta_\pm(x) = 0$.

(c) If $k, n \in \mathbb{N}$, $n \geq q_0$ and $\omega n - k < 0$, then $k - \omega n \geq \delta_+(n)$, the equality being satisfied iff $k = p_{2i}$, $n = q_{2i}$, $i \geq 0$; vice versa, if $k, n \in \mathbb{N}$, $n \geq q_1$ and $\omega n - k > 0$, then $\omega n - k \geq \delta_-(n)$, the equality being satisfied iff $k = p_{2i+1}$, $n = q_{2i+1}$, $i \geq 0$.

A1.2. Lemma 3. *If $k, n \in \mathbb{N}$, $i \geq 2$ and $q_1 \leq n \leq q_i/2$, then*

$$|n\frac{p_i}{q_i} - k| \geq \delta_n \equiv \frac{1}{2}\min\{\delta_+(n), \delta_-(n)\} .$$

A1.3. *Proof of Lemma 3.* Suppose that $i \geq 2$ is even; the property (a) implies that $n\frac{p_i}{q_i} - k > n\omega - k$, so that, by (c), $\forall k, n \in \mathbb{N}$, $n \geq q_1$,

$$n\omega - k > 0 \ \Rightarrow\ n\frac{p_i}{q_i} - k > \delta_-(n) .$$

If $n\omega - k < 0$ and $q_1 \leq n \leq q_i/2$, we have, by (a), (b) and (c),

$$\begin{aligned} -n\frac{p_i}{q_i} + k &= -n\omega + k - n(\frac{p_i}{q_i} - \omega) \geq \delta_+(n) - \frac{n}{q_i}(p_i - \omega q_i) = \\ &= \delta_+(n) - \frac{n}{q_i}\delta_+(q_i) \geq \delta_+(n) - \frac{1}{2}\delta_+(q_i) \geq \frac{1}{2}\delta_+(n) . \end{aligned}$$

Hence, if $i \geq 2$ is even and $q_1 \leq n \leq q_i/2$, $|n\frac{p_i}{q_i} - k| \geq \min\{\frac{1}{2}\delta_+(n), \delta_-(n)\}$. Analogously, if i is odd, one can show that, if $q_1 \leq n \leq q_i/2$, $|n\frac{p_i}{q_i} - k| \geq \min\{\frac{1}{2}\delta_-(n), \delta_+(n)\}$. The claim of Lemma 3 immediately follows from the previous remarks. ∎

A1.4. Lemma 4. *If there exist $c > 0$ and $\tau \geq 1$, such that $|n\omega - k| \geq cn^{-\tau}$, $\forall k, n \in \mathbb{N}$, $n > 0$, then*

$$\delta_n \geq \frac{1}{2}\frac{c}{n^\tau} , \qquad \forall n \geq q_1 .$$

A1.5. *Proof of Lemma 4.* The function $\delta_+(x)$ is a convex function, linear between q_{2i} and q_{2i+2}; moreover

$$\delta_+(q_{2i}) = p_{2i} - \omega q_{2i} \geq \frac{c}{q_{2i}^\tau} .$$

Since $cx^{-\tau}$ is a convex function too, we have

$$\delta_+(x) \geq \frac{c}{x^\tau} , \qquad \forall x \geq q_0 .$$

We can show analogously that $\delta_-(x) \geq cx^{-\tau}$ $\forall x \geq q_1$ and Lemma 4 immediately follows from the definition of δ_n. ■

Lemma 3 and Lemma 4 immediately imply the following result.

A1.6. PROPOSITION 1. *If there exist $c > 0$ and $\tau \geq 1$, such that $|n\omega - k| \geq cn^{-\tau}$, $\forall k, n \in \mathcal{N}$, $n > 0$, then, for any $i \geq 2$,*

$$|n\frac{p_i}{q_i} - k| \geq \frac{1}{2}\frac{c}{n^\tau} \quad , \qquad \textit{if } q_1 \leq n \leq \frac{1}{2}q_i \; .$$

By using Proposition 1, one can define the sequence of p^{L_i} verifying (1.20) with $C_0 = \pi c$, by setting $L_i = q_i$, $p^{L_i} = \frac{\pi p_i}{q_i}$.

Appendix 2. Proof of Lemmata 1 and 2

A2.1. Let us consider the quantity $\mathcal{W}^{(h)}_{n,q}(\mathbf{k})$ introduced in (2.21):

$$\mathcal{W}^{(h)}_{n,q}(\mathbf{k}) = \sum_{\vartheta \in \mathcal{T}^{q,h}_n} \mathrm{Val}(\vartheta) \; , \tag{A2.1}$$

where, if we denote by $\mathbf{T}$ the set of clusters contained in ϑ (including ϑ),

$$\sum_{\vartheta \in \mathcal{T}^h_{n,q}} \mathrm{Val}(\vartheta) = \sum_{\substack{n_{v_1},\ldots,n_{v_q} \\ n_{v_1}+\ldots+n_{v_q}=n \bmod L}} \lambda^q \, \hat\varphi_{n_{v_1}} \ldots \hat\varphi_{n_{v_q}} \sum_{\{h_\ell\}} \Big(\prod_{\ell \in \mathrm{int}(\vartheta)} \tilde g^{(h_\ell)}_{\omega_\ell}(\mathbf{k}'_\ell) \Big) \tag{A2.2}$$

can be rewritten as

$$\sum_{\vartheta \in \mathcal{T}^h_{n,q}} \mathrm{Val}(\vartheta) = \sum_{\substack{n_{v_1},\ldots,n_{v_q} \\ n_{v_1}+\ldots+n_{v_q}=n \bmod L}} \lambda^q \, \hat\varphi_{n_{v_1}} \ldots \hat\varphi_{n_{v_q}} \sum_{\{h_\ell\}} \prod_{T \in \mathbf{T}} \Big(\prod_{\ell \in T_0} \tilde g^{(h_T)}_{\omega_\ell}(\mathbf{k}'_\ell) \Big) \; , \tag{A2.3}$$

where the first product is over all the clusters contained in ϑ (which are uniquely determined by the frequency labels assignment), h_T is the scale of the cluster T and T_0 is the collection of lines inside T which are outside the clusters internal to T (so that the last product is over the lines on scale h_T contained in T, see §2.4). Finally, we shall suppose that $\mathbf{k} \in \mathcal{D}_h$.

A2.2. *Proof of Lemma 1.* The case $q = 1$ is trivial. Let us suppose that $q \geq 2$ and let us consider one of the graphs contributing to the sum in the r.h.s. of (A2.3) and suppose that it satisfies the non resonance condition assumed in the statement of Lemma 1; this means that there are neither clusters nor vertices for which the resonance conditions (2.24) can occur.

We start by considering a cluster $T \in \mathcal{T}_1(\vartheta)$, that is a minimal cluster (see §2.4). By (2.14) and (1.12), we have

$$\Big| \Big(\prod_{v \in T} \hat\varphi_{n_v} \Big) \Big(\prod_{\ell \in T} \tilde g^{(h_\ell)}_{\omega_\ell}(\mathbf{k}'_\ell) \Big) \Big| \leq F_0^{M^{(2)}_T} \Big(\prod_{v \in T} e^{-\xi |n_v|} \Big) (G_0 \gamma^{-h_T})^{L_T} \; , \tag{A2.4}$$

where $M^{(2)}_T = M_T$ as we are considering clusters with depth $D_T = 1$, (see §2.4 for notations). If $h_T = 1$, the fact that the vertices are not resonant gives no constraint on the values of n_v. However, if $h_T \leq 0$, by the support properties of $f_h(\mathbf{k}')$ and (1.20), we have, for any $v \in T$,

$$\begin{aligned} 2a_0 \gamma^{h_T} \geq |k'_{\ell_v} - k'_{\ell_{v'}}| &= |2n_v p + (\omega_{\ell_v} - \omega_{\ell_{v'}}) p_F| = \\ &= |2n_v p + (\omega_{\ell_v} - \omega_{\ell_{v'}}) mp| \geq C_0 \left(|n_v| + m \right)^{-\tau} \; , \end{aligned} \tag{A2.5}$$

since $2n_v + (\omega_{\ell_v} - \omega_{\ell_{v'}})m \neq 0$ by hypothesis. Hence, if we define $C_1 = (C_0/2a_0)^{1/\tau}$, we have

$$|n_v| \geq C_1 \gamma^{-h_T/\tau} - m \ . \tag{A2.6}$$

The inequalities (2.14), (A2.4) and (A2.6) easily imply that, for any $T \in \mathcal{T}_1(\vartheta)$,

$$\begin{aligned}\Big|\Big(\prod_{v\in T}\hat\varphi_{n_v}\Big)\Big(\prod_{\ell\in T}\tilde g^{(h_\ell)}_{\omega_\ell}(\mathbf{k}'_\ell)\Big)\Big| \leq G_0^{L_T}(F_0\bar C_2)^{M_T^{(2)}}\Big(\prod_{v\in T}e^{-\frac{3\xi}{4}|n_v|}\Big)e^{-2^{-3}\xi|n_T|}\\ \Big[\gamma^{-h_T L_T}\, e^{-2^{-3}\xi M_T^{(2)} C_1\gamma^{-h_T/\tau}}\Big] \ ,\end{aligned} \tag{A2.7}$$

where $\bar C_2 = \max\{e^{m\xi/8}, e^{C_1\gamma^{-1/\tau}\xi/8}\}$ and we used the trivial bound $\sum_{v\in T}|n_v| \geq |n_T|$.

Next we consider a cluster $T \in \mathcal{T}_2(\vartheta)$, that is a cluster of depth $D_T = 2$. Since $h_T < 1$, we can use again (A2.5) for any $v \in T_0$; moreover, given a cluster $\tilde T \subset T$, since $2n_T + (\omega_{\ell^\circ_T} - \omega_{\ell^i_T})m \neq 0$, we have an analogous bound:

$$2a_0\gamma^{h_T} \geq |k'_{\ell^i_{\tilde T}} - k'_{\ell^\circ_{\tilde T}}| \geq C_0\,(|n_T| + m)^{-\tau} \ , \tag{A2.8}$$

where $k_{\ell^i_T}$ are $k_{\ell^\circ_T}$ are defined as in §2.4.

By using (A2.5) and (A2.8), it is easy to see that

$$\begin{aligned}&\Big|\Big(\prod_{\tilde T\subset T}e^{-2^{-3}\xi|n_{\tilde T}|}\Big)\Big(\prod_{v\in T_0}\hat\varphi_{n_v}\Big)\Big(\prod_{\ell\in T_0}\tilde g^{(h_l)}_{\omega_\ell}(\mathbf{k}'_\ell)\Big)\Big|\\ &\qquad\leq G_0^{L_T}F_0^{M_T^{(2)}}\bar C_2^{M_T}\Big(\prod_{v\in T_0}e^{-\frac{3\xi}{4}|n_v|}\Big)e^{-2^{-4}\xi|n_T|}\\ &\qquad\Big[\gamma^{-h_T L_T}\, e^{-2^{-4}\xi M_T C_1\gamma^{-h_T/\tau}}\Big] \ ,\end{aligned} \tag{A2.9}$$

where we used the bound $\sum_{\tilde T\subset T}|n_{\tilde T}| + \sum_{v\in T_0}|n_v| \geq |n_T|$.

By iterating the previous procedure, and noting that

- $\sum_{T\in\mathbf{T}}\sum_{v\in T_0}|n_v| = \sum_{v\in\vartheta}|n_v| \geq \Big|\sum_{v\in\vartheta}n_v\Big| = |n|$,
- $\sum_{T\subset\mathbf{T}}M_T^{(2)} = \sum_{T\subset\mathbf{T}}L_T + 1 = q$,
- $M_T^{(1)} + M_T^{(2)} \equiv M_T = L_T + 1$,

we obtain in the end

$$\begin{aligned}\sup_{\mathbf{k}\in\mathcal{D}_h}\Big|\tilde{\mathcal{W}}^{(h)}_{n,q}(\mathbf{k})\Big| \leq |\lambda|^q e^{-\frac{\xi}{2}|n|}G_0^{q-1}F_1^q\bar C_2^{M_{\mathbf{T}}}\\ \sum_{\{h_\ell\}}\prod_{T\in\mathbf{T}}\gamma^{-h_T L_T}e^{-2^{-(D_T+2)}\xi C_1 M_T\gamma^{-h_T/\tau}} \ ,\end{aligned} \tag{A2.10}$$

where $F_1 = F_0\sum_{n\in\mathbb{Z}}e^{-\xi|n|/4}$ and $M_{\mathbf{T}} = \sum_{T\in\mathbf{T}}M_T$.

The r.h.s. of (A2.10) can be further bounded by
(1) neglecting the ordering relation between the frequency labels, and
(2) taking into account only the fact that, if a cluster T has depth D_T, then $h_T \leq -D_T + 2$.
We get

$$\begin{aligned}\sup_{\mathbf{k}\in\mathcal{D}_h}\Big|\tilde{\mathcal{W}}^{(h)}_{n,q}(\mathbf{k})\Big| \leq e^{-\frac{\xi}{2}|n|}G_0^{q-1}F_1^q\,|\lambda|^q\\ \sum_{\mathbf{T}}{}^{*}\,\bar C_2^{M_{\mathbf{T}}}\prod_{T\in\mathbf{T}}\Big[\sum_{h_T\leq -D_T+2}\gamma^{h_T}\Big(\gamma^{-h_T}e^{-2^{-(D_T+2)}\xi C_1\gamma^{-h_T/\tau}}\Big)^{M_T}\Big] \ ,\end{aligned} \tag{A2.11}$$

where $\sum_{\mathbf{T}}{}^{*}$ is the sum over all the possible choices of arrangements of the clusters over a chain of q vertices, which is bounded by 4^q. Hence we have

$$\begin{aligned}\sup_{\mathbf{k}\in\mathcal{D}_h}\Big|\tilde{\mathcal{W}}^{(h)}_{q}(\mathbf{k};\mathbf{k}+2n\mathbf{p})\Big| \leq e^{-\frac{\xi}{2}|n|}G_0^{q-1}(4F_1)^q\,|\lambda|^q\\ \max_{\mathbf{T}}\Big\{\bar C_2^{M_{\mathbf{T}}}\prod_{T\in\mathbf{T}}\Big[\sum_{r=D_T-2}^{\infty}\gamma^{-r}\Big(\gamma^{r}e^{-2^{-4}\xi C_1(\gamma^{1/\tau}/2)^r}\Big)^{M_T}\Big]\Big\} \ .\end{aligned} \tag{A2.12}$$

Suppose now γ so large that $\tilde{\gamma} \equiv \gamma^{1/\tau}/2 > 1$, and note that, $\forall N > 0$, $\exists\, C_N > 0$ such that

$$e^{-2^{-4}C_1\xi\tilde{\gamma}^r} \leq \frac{C_N}{1+(2^{-4}C_1\xi\tilde{\gamma}^r)^N} \,, \tag{A2.13}$$

(one can take $C_N = 1 + N!$). Choose N so that $\tilde{\gamma}^N \geq 2\gamma$; then, since $M_T \geq 1$,

$$\begin{aligned}\sum_{r=D_T-2}^{\infty} \gamma^{-r}\Big(\gamma^r e^{-2^{-4}\xi C_1\tilde{\gamma}^r}\Big)^{M_T} &\leq \\ \leq \gamma\Big(\sum_{r=D_T-2}^{\infty} \frac{C_N\gamma^r}{1+(2^{-4}C_1\xi)^N(2\gamma)^r}\Big)^{M_T} &\leq \gamma(C_4 2^{-D_T})^{M_T} \,,\end{aligned} \tag{A2.14}$$

where $C_4 = 8C_N/(2^{-4}C_1\xi)^N$. Since $\sum_{T\in\mathbf{T}} M_T \leq 2q$, (A2.12) and (A2.14) yield (2.25) for some constant B_1. ■

A2.3. *Proof of Lemma 2.* If there are resonances, the proof in §A2.2 does not apply, as (A2.6) and (A2.8) do not hold for resonances, and we have to carefully analyze the effect of the renormalization procedure described in §3. In particular we shall need the bound (3.44), which depend on the hypothesis (3.41). However, to check the validity of (3.41), we need a bound on the effective potential; hence the proof will be inductive. We shall suppose that $\hat{\varphi}_m \neq 0$, $h \leq -1$ and that, if $h+1 \leq h' \leq 0$,

$$|\sigma_{h'} - \lambda\hat{\varphi}_m| \leq A|\lambda|^2 e^{-m\frac{\xi}{2}} \,, \qquad |\nu_{h'}| \leq B_3|\lambda| \,, \tag{A2.15}$$

and we shall prove that it is possible to choose A and B_3 so that (A2.15) is true also for $h' = h$, together with the bound (3.50) on the effective potential. The proof of Lemma 2 will follow from the remark that (A2.15) is verified for $h = -1$, by (3.5), if $A \geq A_0$.

Let $\vartheta_{\mathcal{R}} \in \mathcal{T}^h_{\mathcal{R},n,q}$ and $q > 1$ (the case $q = 1$ is trivial, except for the ν_h vertex). Let us consider the collection $\mathbf{V}_1$ of *maximal resonances*, *i.e.* resonances which are not strictly contained in any other resonance. If V is such a resonance, ℓ^i_V and ℓ^o_V are its external lines, and $k'_{\ell^i_V} = k'_{\ell^o_V}$. Then (recall the definition in item (5) of §2.4)

$$\mathrm{Val}(\vartheta_{\mathcal{R}}) = \Big(\prod_{v\in\vartheta_{\mathcal{R}}} F_v\Big)\Big(\prod_{\ell\cap\mathbf{V}_1=\emptyset} \tilde{g}_\ell\Big) \prod_{V\in\mathbf{V}_1} \Big[\tilde{g}_{\ell^i_V}\, \tilde{g}_{\ell^o_V}\, \mathcal{R}\Xi^{h_V}_V(\mathbf{k}'_{\ell^o_V})\Big] \,, \tag{A2.16}$$

where $\tilde{g}_\ell$ is a shorthand for $\tilde{g}^{(h_\ell)}_{\omega^1_\ell,\omega^2_\ell}(\mathbf{k}'_\ell)$, if ℓ is an internal line of $\vartheta_{\mathcal{R}}$, $\tilde{g}_\ell = 1$ otherwise; $\prod_{\ell\cap\mathbf{V}_1=\emptyset}\tilde{g}_\ell = 1$, if $\vartheta_{\mathcal{R}}$ itself is a resonance (so that all lines intersect $\mathbf{V}_1$); $F_v = \gamma^{h_v}\nu_{h_v}$ if $n_v = 0$, $F_v = \lambda\hat{\varphi}_{n_v}$ otherwise. Finally the *resonance value* $\Xi^{h_V}_V(\mathbf{k}'_{\ell^o_V})$ is given by

$$\Xi^{h_V}_V(\mathbf{k}'_{\ell^o_V}) = \Big(\prod_{\ell\in V\,:\,\ell\cap\mathbf{V}_2=\emptyset} \tilde{g}_\ell\Big) \prod_{V'\in\mathbf{V}_2\cap V} \Big[\tilde{g}_{\ell^i_{V'}}\, \tilde{g}_{\ell^o_{V'}}\, \mathcal{R}\Xi^{h_{V'}}_{V'}(\mathbf{k}'_{\ell^o_{V'}})\Big] \,, \tag{A2.17}$$

where $\mathbf{V}_2$ is the collection of resonances which are strictly contained inside some resonance in $\mathbf{V}_1$, and which are maximal, and $\mathbf{V}_2 \cap V$ is the subset of resonances in $\mathbf{V}_2$ which are contained in V. Note that (A2.17) extends (3.47) to the case in which V contains other resonances.

We can write $\mathcal{R}\Xi^{h_V}_V(\mathbf{k}'_{\ell^o_V})$ as in (3.48), that is

$$\mathcal{R}\Xi^{h_V}_V(\mathbf{k}'_{\ell^o_V}) \equiv \Xi^{h_V}_V(\mathbf{k}'_{\ell^o_V}) - \Xi^h_V(\mathbf{0}) = \int_0^1 dt\Big[\frac{d}{dt}\Xi^h_V(t\mathbf{k}'_{\ell^o_V})\Big] \,. \tag{A2.18}$$

Note that $\Xi^h_V(t\mathbf{k}'_{\ell^o_V})$ can be written as in (A2.17), by substituting the momentum $\mathbf{k}'_\ell$ of any line with $t\mathbf{k}'_{\ell^o_V} + \mathbf{q}_\ell$, for suitable values of $\mathbf{q}_\ell$. Therefore the r.h.s. of (A2.18) can be

written as a sum of terms of the form (A2.17) with a derivative d/dt acting either
(1) on one of the propagators corresponding to a line outside $\mathbf{V}_2$, or
(2) on one of the $\mathcal{R}\Xi_{V'}^{h_{V'}}$.

In case (2), we write

$$\frac{d}{dt}\mathcal{R}\Xi_{V'}^{h_{V'}}(t\mathbf{k}'_{\ell_V^\circ}+\mathbf{q}_{\ell_{V'}^\circ}) = \frac{d}{dt}\Big[\Xi_{V'}^{h_{V'}}(t\mathbf{k}'_{\ell_V^\circ}+\mathbf{q}_{\ell_{V'}^\circ}) - \Xi_{V'}^{h_{V'}}(\mathbf{0})\Big] = \frac{d}{dt}\Xi_{V'}^{h_{V'}}(t\mathbf{k}'_{\ell_V^\circ}+\mathbf{q}_{\ell_{V'}^\circ}) \; , \qquad (A2.19)$$

so that, if the derivative corresponding to a resonance V acts on the value of some resonance $V' \subset V$, one can replace with $\mathbb{1}$ the $\mathcal{R}$ operator corresponding to V'.

We can now iterate this procedure, by applying to $\Xi_{V'}^{h_{V'}}(t\mathbf{k}'_{\ell_V^\circ}+\mathbf{q}_{\ell_{V'}^\circ})$ the equation (A2.17), with $\mathbf{V}_3$ (the family of resonances which are strictly contained inside some resonance belonging to $\mathbf{V}_2$ in place of $\mathbf{V}_2$), and so on. At the end the r.h.s. of (A2.18) can be written as a sum of $q_V - 1$ terms, if q_V denotes the number of vertices contained in V, which can be described in the following way.
(1) There is one term for each line $\bar\ell \in V$;
(2) if $\bar\ell \in T_0$, where T is a cluster contained in V (see item (1) in §2.4 and note that T can be equal to V), and $T = T^{(r)} \subset T^{(r-1)} \ldots \subset T^{(1)} = V$ is the chain of r clusters containing T and contained in V, then the graph value can be computed by replacing with $\mathbb{1}$ the $\mathcal{R}$ operator acting on $T^{(i)}$, $i = 1, \ldots, r$, even if $T^{(i)}$ is a resonance, because of the comments after (A2.19);
(3) the $\mathcal{R}$ operation acts on all other resonances contained in V;
(4) the derivative d/dt acts on the propagator of $\bar\ell$, whose momentum is of the form $t\mathbf{k}'_{\ell_V^\circ}+\mathbf{q}_\ell$.

A similar decomposition of the resonance value is now applied, for each term of the previous sum, to all resonant clusters, which are still affected by the $\mathcal{R}$ operation. This procedure is iterated, until no $\mathcal{R}$ operation is explicitly present; it is easy to see that we end with an expression of the form

$$\mathrm{Val}(\vartheta_{\mathcal{R}}) = \sum \int dt_1 \ldots dt_s \Big(\prod_{v\in\vartheta_{\mathcal{R}}} F_v \Big) \prod_{T\in\mathbf{T}} \Big[\prod_{\ell\in T_0} \Big(\frac{d}{dt_{i(\ell)}} \Big)^{d_\ell} \tilde g_\ell \Big] \; , \qquad (A2.20)$$

where the sum is over all possible choices of s, $\{d_\ell\}$ and $\{i(\ell)\}$, which satisfy the following conditions:
(1) d_ℓ is equal to 0 or 1;
(2) if $d_\ell = 0$, $i(\ell)$ is arbitrarily defined, otherwise $i(\ell) \in \{1, \ldots, s\}$ and $i(\ell) \neq i(\ell')$, if $\ell \neq \ell'$;
(3) the number of lines for which $d_\ell = 1$ is equal to the number of interpolating parameters s;
(4) for each derived line ℓ there is a chain of r clusters $T = T^{(r)} \subset T^{(r-1)} \ldots \subset T^{(1)} = V$, such that $\ell \in T_0$ and V is a resonance;
(5) no cluster can belong to more than one chain of clusters;
(6) each resonance belongs to one of the chains of clusters;
(7) the momentum of the derived line is of the form $t_{i(\ell)}\mathbf{k}' + \mathbf{q}_l$, with $|\mathbf{k}'| \le a_o\gamma^{-h_V^e}$, (in general $\mathbf{k}'$ is not $\mathbf{k}'_{\ell_V^0}$, but it can depend also on the interpolation parameters corresponding to resonances containing V, if any), where h_V^e is the external scale of V, that is the scale of the smaller cluster containing it.

The item (7) above implies that, for each derived line, by (3.44),

$$\Big| \frac{d}{dt_{i(\ell)}} \tilde g_\ell \Big| \le a_0 G_2 \gamma^{h_V^e - h_\ell} \gamma^{-h_\ell} \; . \qquad (A2.21)$$

Note that

$$h_V^e - h_\ell = \sum_{i=1}^{r} [h_{T^{(i)}}^e - h_{T^{(i)}}] \; ; \qquad (A2.22)$$

hence the "gain" $\gamma^{h_V^e - h_\ell}$ in the bound (A2.21), with respect to the bound of a non derived propagator, can be divided between the clusters of the chain associated to the derived line

ℓ, so that each cluster has a factor $\gamma^{h^e_{T^{(i)}}-h_{T^{(i)}}}\le 1$ associated with it; in particular we have a factor of this type associated with each resonance, for each term in the sum of (A2.20). Since the number of terms in this sum is bounded by 2^{q-1}, we can write, if we denote by $\mathbf{V}$ the family of resonant clusters,

$$\Big|\mathrm{Val}(\vartheta_{\mathcal{R}})\Big| \le 2^{q-1}\prod_{v\in\vartheta_{\mathcal{R}}}|F_v|\prod_{T\in\mathbf{T}}(G_3\gamma^{-h_T})^{L_T}\prod_{T\in\mathbf{V}}\gamma^{h^e_T-h_T}\ , \tag{A2.23}$$

where $G_3=\max\{G_1,a_0G_2\}$.

We shall now consider, as in the proof of Lemma 1, a minimal cluster $T\in\mathcal{T}_1(\vartheta_{\mathcal{R}})$ with $h_T\le 0$ and note that

$$\prod_{v\in T}|F_v|\le|\nu_{h_T}\gamma^{h_T}|^{M_T^{(\nu)}}(|\lambda|F_0)^{\tilde M_T^{(2)}}\prod_{v\in T}e^{-\xi|n_v|}\ , \tag{A2.24}$$

where $M_T^{(\nu)}$ is the number of resonant vertices contained in T and $\tilde M_T^{(2)}$ is the number of non resonant vertices, so that $M_T=M_T^{(2)}=\tilde M_T^{(2)}+M_T^{(\nu)}=L_T+1$. If we now recall that $n_T=\sum_{v\in T}n_v+\sum_{\ell\in T}(\omega^1_\ell-\omega^2_\ell)m/2$ and we use (A2.6) for non resonant vertices, we get

$$\prod_{v\in T}e^{-\xi|n_v|}\le\bar C_2^{\tilde M_T^{(2)}+L_T}\Big(\prod_{v\in T}e^{-\frac{3\xi}{4}|n_v|}\Big)e^{-2^{-3}\xi|n_T|}e^{-2^{-3}\xi\tilde M_T^{(2)}C_1\gamma^{-h_T/\tau}}\ . \tag{A2.25}$$

Hence, if $|\nu_h|\le B_3|\lambda|$, by (A2.24) and (A2.25) we have

$$\begin{aligned}\Big(\prod_{v\in\vartheta}|F_v|\Big)(G_3\gamma^{-h_T})^{L_T}&\le(|\lambda|C_2^4)^{M_T}\Big(\prod_{v\in T}e^{-\frac{3\xi}{4}|n_v|}\Big)\\ &e^{-2^{-3}\xi|n_T|}\gamma^{h_T}\Big[\gamma^{-h_T}e^{-2^{-3}\xi C_1\gamma^{-h_T/\tau}}\Big]^{\tilde M_T^{(2)}}\ ,\end{aligned} \tag{A2.26}$$

where $C_2=\max\{B_3,G_3,\bar C_2,F_0\}$.

Next we consider a cluster $T\in\mathcal{T}_2(\vartheta_{\mathcal{R}})$; by using (A2.6) for non resonant vertices and (A2.8) for non resonant clusters, whose number will be called $\tilde M_T^{(1)}$, we get

$$\begin{aligned}\Big(\prod_{\tilde T\subset T}e^{-2^{-3}\xi|n_{\tilde T}|}\Big)\Big(\prod_{v\in T_0}|F_v|\Big)\big(G_3\gamma^{-h_T}\big)^{L_T}&\le|\lambda|^{M_T^{(2)}}C_2^{4M_T}\\ \Big(\prod_{v\in T_0}e^{-\frac{3\xi}{4}|n_v|}\Big)e^{-2^{-4}\xi|n_T|}\gamma^{-h_TM_T^{(r)}}\gamma^{h_T}&\Big[\gamma^{-h_T}\,e^{-2^{-4}\xi C_1\gamma^{-h_T/\tau}}\Big]^{\tilde M_T}\ ,\end{aligned} \tag{A2.27}$$

where $M_T^{(r)}=M_T^{(1)}-\tilde M_T^{(1)}$ is the number of resonant clusters strictly contained in T and $\tilde M_T=\tilde M_T^{(1)}+\tilde M_T^{(2)}$.

We iterate the previous procedure, as in the proof of Lemma 1, and we get

$$\begin{aligned}|\mathrm{Val}(\vartheta_{\mathcal{R}})|&\le(2|\lambda|)^qC_2^{4M_{\mathbf{T}}}e^{-\frac{\xi}{2}|n|}\Big(\prod_{v\in\vartheta_{\mathcal{R}}}e^{-\frac{\xi}{4}|n_v|}\Big)\cdot\\ &\cdot\Big\{\prod_{T\in\mathbf{T}}\gamma^{-h_TM_T^{(r)}}\gamma^{h_T}\Big[\gamma^{-h_T}\,e^{-2^{-4}\xi C_1\gamma^{-h_T/\tau}}\Big]^{\tilde M_T}\Big\}\prod_{T\in\mathbf{V}}\gamma^{h^e_T-h_T}\ ,\end{aligned} \tag{A2.28}$$

where $\mathbf{V}$ is the family of all resonant clusters strictly contained in $\vartheta_{\mathcal{R}}$.

Note that

$$\Big[\prod_{T\in\mathbf{T}}\gamma^{-h_TM_T^{(r)}}\Big]\Big[\prod_{T\in\mathbf{V},T\ne\vartheta_{\mathcal{R}}}\gamma^{h^e_T-h_T}\Big]=\prod_{T\in\mathbf{V},T\ne\vartheta_{\mathcal{R}}}\gamma^{-h_T}\ ; \tag{A2.29}$$

hence (A2.28) can be written also as

$$\begin{aligned}\Big|\mathrm{Val}(\vartheta_{\mathcal{R}})\Big| \le \gamma^{h_{\vartheta_{\mathcal{R}}}}(2|\lambda|)^q C_2^{4M_{\mathbf{T}}} e^{-\frac{\xi}{2}|n|}\Big(\prod_{v\in\vartheta_{\mathcal{R}}} e^{-\frac{\xi}{4}|n_v|}\Big) \\ \Big\{\prod_{T\in\mathbf{T}}\Big[\gamma^{-h_T}\, e^{-2^{-4}\xi C_1\gamma^{-h_T/\tau}}\Big]^{\tilde M_T}\Big\}\ .\end{aligned} \tag{A2.30}$$

In order to bound $W^{(h)}_{\mathcal{R},n,q}(\mathbf{k})$ we have to perform the sum of (A2.30) over the n_v, ω^i_ℓ and h_T labels. The sum over n_v is trivial, as well as the sum over h_T, for the clusters with $\tilde M_T \ne 0$. The sum over h_T would give some bad factor, when $\tilde M_T = 0$, but it turns out that there is indeed no sum in this case. In fact, if all the clusters and vertices strictly contained in T are resonant, then T itself must be a resonance and all its internal lines have the same $\mathbf{k}'$ as the external ones, implying, by support properties of the f_h functions, that the frequency label of the external lines is equal to $h_T - 1$. Hence we can proceed as in the proof of Lemma 1 and we get

$$\Big|\mathcal{W}^{(h)}_{\mathcal{R},n,q}(\mathbf{k})\Big| \le e^{-\frac{\xi}{2}|n|} B_2^q \tag{A2.31}$$

for a suitable constant $B_2 > B_1$.

We still have to check that the bound (A2.15) is satisfied also by σ_h and ν_h. Note that, $\forall h < 0$,

$$\begin{aligned} s_h = \sigma_h - \sigma_{h+1} &= \sum_{q=2}^{\infty} \bar{\mathcal{W}}^{(h)}_{\mathcal{R},m,q}(-m\mathbf{p})\ , \\ \nu_h = \gamma\nu_{h+1} &+ \gamma^{-h}\sum_{q=2}^{\infty}\bar{\mathcal{W}}^{(h)}_{\mathcal{R},0,q}(-m\mathbf{p})\ , \end{aligned} \tag{A2.32}$$

where $\bar{\mathcal{W}}^{(h)}_{\mathcal{R},0,q}(-m\mathbf{p})$ and $\bar{\mathcal{W}}^{(h)}_{\mathcal{R},m,q}(-m\mathbf{p})$ admit an expansion in terms of graphs $\vartheta_{\mathcal{R}}$, differing from the corresponding expansion of $\mathcal{W}^{(h)}_{\mathcal{R},0,q}(-m\mathbf{p})$ and $\mathcal{W}^{(h)}_{\mathcal{R},m,q}(-m\mathbf{p})$ in the following respects:

(1) the $\mathcal{R}$ operation on the whole graph, which is necessarily a resonance, is substituted with the localization operation, hence in the previous analysis $\vartheta_{\mathcal{R}}$ must not be included in the set $\mathbf{V}$;

(2) the internal scale of $\vartheta_{\mathcal{R}}$ is equal to $h+1$, that is there is in the graph at least one line of frequency $h+1$;

(3) if all maximal clusters strictly contained in $\vartheta_{\mathcal{R}}$ are resonant, as well as the vertices belonging to $\vartheta_{\mathcal{R}0}$ (see item (3) in §2.4), that is if $\tilde M_{\vartheta_{\mathcal{R}}} = 0$, then $\mathrm{Val}(\vartheta_{\mathcal{R}}) = 0$.

Item (3) follows from the support properties of the propagators, the definition of resonance in §2.5 and from the observation that all lines $\ell \in \vartheta_{\mathcal{R}0}$ would have $\mathbf{k}'_\ell = 0$, if $\tilde M_{\vartheta_{\mathcal{R}}} = 0$, since $\mathbf{k}'_\ell = 0$ for the external lines.

Item (2) easily implies that the bound (A2.30) is valid also for the new graphs, possibly with a different value of C_2, even if there is no $\mathcal{R}$ operation on $\vartheta_{\mathcal{R}}$. Even more, items (2) and (3) together imply that bound (A2.31) can be improved and we can write, for any N and a suitable constant C_N, for $n = 0$ or $n = m$,

$$|\bar{\mathcal{W}}^{(h)}_{\mathcal{R},n,q}(-m\mathbf{p})| \le \gamma^{Nh}(|\lambda| C_N)^q\ . \tag{A2.33}$$

Note that item (2) alone implies the bound (A2.33) with $N = 1$, by (A2.30), which is sufficient for iterating the bound on ν_h.

Hence, we have, for λ small enough and $h < 0$,

$$|s_h| \le B_4\gamma^h|\lambda|^2 e^{-\frac{m}{2}\xi}\ , \tag{A2.34}$$

$$|\nu_h| \le \gamma|\nu_{h+1}| + B_4|\lambda|^2\ , \tag{A2.35}$$

where B_4 is a suitable constant.

The constant B_4 depends in principle on the constants A and B_3, appearing in the inductive hypothesis (A2.15), through the constant C_3 in (A2.30), defined after (A2.26). However, it is easy to prove that in fact it can be chosen independently of A and B_3, if λ is small enough. The independence of A follows from the remark that the constant C_2 of (A2.30) can be made independent of the constant A, if λ is chosen so small that (3.41) is satisfied. The independence of B_3 is a bit more involved. One has to observe that there is no graph contributing to $\bar{\mathcal{W}}^{(h)}_{\mathcal{R},n,2}(-m\mathbf{p})$, $n=0,m$, (that is no second order contribution to ν_h and s_h), containing resonant vertices. In fact one can construct graphs of this type, but their value is zero, since they contain necessarily a line with $\mathbf{k}'_\ell=0$, whose propagator vanishes by its support properties, (see item (2) after (A2.32)). It easily follows that it is possible to choose B_4 independent of B_3, if λ is small enough.

By iterating the bound (A2.35) and using the bound (3.5) on ν_0, we get, for $h\ge h^*$,

$$|\nu_h|\le\gamma^{-h}\left[|\nu_0|+B_4|\lambda|^2\sum_{j=h}^{-1}\gamma^j\right]\le\gamma^{-h^*}|\lambda|^2\left(A_0+B_4\sum_{r=1}^{\infty}\gamma^{-r}\right)\;. \tag{A2.36}$$

Moreover, the definition (3.40) of h^* implies that $\gamma^{-h^*}|\lambda|\le G_3/(2|\hat\varphi_m|)$.

Hence (A2.15) is satisfied also for $h'=h$, if we choose $A=A_0+B_4\sum_{r=0}^{\infty}\gamma^{-r}$, $B_3=AG_3/(2|\hat\varphi_m|)$. ■

Appendix 3. Proof of the bounds (4.20) and (4.21)

A3.1. *Proof of (4.20).* By using (3.27)÷(3.31), it is easy to prove that, for any $N\ge 0$, there is a constant G_N such that, if $0\ge h\ge h^*$, $N_0,N_1\ge 0$ and $N_0+N_1=N$,

$$\left|D_0^{N_0}D_1^{N_1}\tilde g^{(h)}_{\omega,\omega'}(\mathbf{k}')\right|\le G_N\gamma^{-hN}\frac{\max\{\gamma^h,|\sigma_h|\}}{\gamma^{2h}+\sigma_h^2}\;, \tag{A3.1}$$

where D_0 and D_1 denote the discrete derivative with respect to k_0 and k', respectively.

Hence, if $|x_0-y_0|\le\beta/2$ and $|x-y|\le L_i/2$, we have

$$\begin{aligned}&\left(\frac{\sqrt2}{\pi}\right)^N|x_0-y_0|^{N_0}|x-y|^{N_1}\left|g^{(h)}(\mathbf{x};\mathbf{y})\right|\le\\ &\le\left|\frac{\beta}{2\pi}[e^{-i\frac{2\pi}{\beta}(x_0-y_0)}-1]\right|^{N_0}\left|\frac{L_i}{2\pi}[e^{-i\frac{2\pi}{L_i}(x-y)}-1]\right|^{N_1}\left|g^{(h)}(\mathbf{x};\mathbf{y})\right|=\\ &=\left|\sum_{\omega,\omega'=\pm1}e^{-i(\omega x-\omega' y)p_F}\frac{1}{L_i\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L_i,\beta}}e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}D_0^{N_0}D_1^{N_1}\tilde g^{(h)}_{\omega,\omega'}(\mathbf{k}')\right|\le\\ &\le C_N\gamma^h(\max\{\gamma^h,L^{-1}\})\,\gamma^{-hN}\frac{\max\{\gamma^h,|\sigma_h|\}}{\gamma^{2h}+\sigma_h^2}\le C_N\gamma^{-hN}(\max\{\gamma^h,L^{-1}\})\;,\end{aligned} \tag{A3.2}$$

where C_N denotes a varying constant, depending only on N, and the factor $(\max\{\gamma^h,L^{-1}\})$ arises from the sum over $\mathbf{k}'$ (note that the sum over k_0 always gives a factor γ^h, since $h_\beta\le h^*\le h$). Therefore we have

$$\left|g^{(h)}(\mathbf{x};\mathbf{y})\right|\le\frac{C_N\max\{\gamma^h,L^{-1}\}}{1+\gamma^{hN}|\mathbf{x}-\mathbf{y}|^N}\;. \tag{A3.3}$$

and a similar bound is verified for $g^{(\le h^*)}(\mathbf{x};\mathbf{y})$, if λ is real. ■

A3.2. *Proof of (4.21).* Let $\vartheta_{\mathcal{R}}$ be one of the graphs contributing to the kernel $K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y})$, see (4.19) and let us consider the two vertices, v_1 and v_q, connected to the external lines (which are associated with the external field).

Suppose first that neither v_1 nor v_q, the *external vertices*, are contained in any cluster, different from $\vartheta_{\mathcal{R}}$ itself. In this case, we can bound $\mathrm{Val}(\vartheta_{\mathcal{R}})$ as in §A2.3, by taking into account that

(1) there is no factor associated to the external vertices;

(2) $h_{\vartheta_{\mathcal{R}}}=h+1$;

(3) there are at least two lines of scale $h+1$, the external propagators.

Hence we get a bound differing from (A2.30) only because the power of $|\lambda|$ is $q-2$ instead of q and each external propagator gives a contribution proportional to $\gamma^{-h_{\vartheta_{\mathcal{R}}}}$:

$$\begin{aligned}|\mathrm{Val}(\vartheta_{\mathcal{R}})| \le& \gamma^{-h_{\vartheta_{\mathcal{R}}}}(2|\lambda|)^{q-2}C_2^{4M_{\mathbf{T}}}e^{-\frac{\xi}{2}|n|}\Big(\prod_{v\in\vartheta}e^{-\frac{\xi}{4}|n_v|}\Big)\\ &\left\{\prod_{T\in\mathbf{T}}\left[\gamma^{-h_T}\,e^{-2^{-4}\xi C_1\gamma^{-h_T/\tau}}\right]^{\tilde M_T}\right\}\,,\end{aligned}\tag{A3.4}$$

where the same notation of §A2.3 is used, except for the definition of $\tilde M_T$, which differs from the previous one, since we do not consider the external vertices in the calculation of $\tilde M_T^{(2)}$; moreover we assigned a label $n_v=0$ to the external vertices.

Suppose now that v_1 is contained in some cluster strictly contained in $\vartheta_{\mathcal{R}}$ and that the scale of the external propagator emerging from v_1 is h_1. In this case, there is a chain of clusters $T^{(1)}\subset T^{(2)}\ldots\subset T^{(r)}=\vartheta_{\mathcal{R}}$, such that $v_1\in T^{(i)}$ and $h_{T^{(1)}}=h_1$; moreover $\mathcal{R}=\mathbb{1}$ on $T^{(i)}$, $i=1,\ldots,r$, even if $T^{(i)}$ is a resonance.

We proceed again as in §A2.3, by we have to take into account the lack of the factor $\gamma^{h^e_{T^{(i)}}-h_{T^{(i)}}}$, which was present before, when $T^{(i)}$ is a resonance. Since $\vartheta_{\mathcal{R}}$ is not a resonance (by definition) and $h^e_{T^{(i)}}=h_{T^{(i+1)}}$, we loose at most a factor $\gamma^{h_{\vartheta_{\mathcal{R}}}-h_{T^{(1)}}}=\gamma^{h+1-h_1}$. If we also consider the bound of the external propagator emerging from v_1, we see that the overall effect of the vertex v_1 in the bound of $\mathrm{Val}(\vartheta_{\mathcal{R}})$ is to add a factor γ^{-h-1} to the expression in the r.h.s. of (A2.30), that is the same effect that we should get, if the only cluster containing v_1 was $\vartheta_{\mathcal{R}}$.

A similar argument can be used for studying the effect of the vertex v_q. Hence we get the bound (A3.4) for all graphs contributing to $K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y})$.

We can now bound as in §A2.3 the sum over $\vartheta_{\mathcal{R}}$ in the r.h.s. of (4.19). Since the sum over $\mathbf{k}'$ gives a factor $\gamma^h\max\{\gamma^h,L^{-1}\}$, we get, for λ sufficiently small,

$$\left|K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\right|\le B_5\sum_{n=1}^{\infty}\sum_{q=3}^{\infty}B_2^q|\lambda|^{q-2}e^{-\frac{\xi}{2}|n|}\max\{\gamma^h,L^{-1}\}\le|\lambda|B_6\max\{\gamma^h,L^{-1}\}\,,\tag{A3.5}$$

for suitable constants B_5 and B_6.

In the same way, for any $N_1,N_2\ge 0$, $N=N_1+N_2>0$, we have:

$$\begin{aligned}&\left(\frac{\sqrt{2}}{\pi}\right)^N|x_0-y_0|^{N_0}|x-y|^{N_1}\left|K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\right|\le\\ &\le\left|\sum_{n=-[L/2]}^{[(L-1)/2]}\frac{1}{L_i\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L_i,\beta}}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})+2inpy}\left[D_{N_0}D_{N_1}\hat K^{(h)}_{\phi,\phi,n}(\mathbf{k})\right]\right|\,.\end{aligned}\tag{A3.6}$$

We can now proceed as in §A3.1, by using (A3.1); we get

$$|\mathbf{x}-\mathbf{y}|^N\left|K^{(h)}_{\phi,\phi}(\mathbf{x};\mathbf{y})\right|\le C_N\gamma^{-hN}\,|\lambda|\max\{\gamma^h,L^{-1}\}\,;\tag{A3.7}$$

a similar bound can be obtained for $K^{(<h^*)}_{\phi,\phi}(\mathbf{x};\mathbf{y})$. ■

A3.3. *Limit* $i,\beta\to\infty$. Let us consider one of the finite $L=L_i$ and β quantities appearing in the r.h.s. of (4.18); if we interpret it as a Riemann sum of the corresponding $L_i=\beta=\infty$ quantity, as defined in §4.4, it is easy to prove that, for any $h>h^*$, the difference between the two quantities can be bounded by a constant times $(1/L_i+1/\beta)$. In fact one gets essentially the same bounds as in the proof of (4.20) and (4.21), up to a factor $\gamma^{-h}(1/L_i+1/\beta)$, coming from the comparison of the integral and the corresponding Riemann sum; we shall not give the details, which are completely straightforward. It follows that

$$|S(\mathbf{x};\mathbf{y})-S^{L_i,\beta}(\mathbf{x};\mathbf{y})|\le C|h^*|(1/L_i+1/\beta)\xrightarrow[i,\beta\to\infty]{}0\ . \tag{A3.8}$$

Appendix 4.

A4.1. In this section we shall discuss how the analysis of §A2.3 has to be modified, if the support functions $\hat f_h$ do not depend on k_0. As we have discussed in §4.6, we must use now the bounds (4.46), instead of (2.14), (3.42) and (3.43). The main difference (see (4.47)) is that the bound (A2.21) has to be replaced by

$$\left|\frac{d}{dt_{i(\ell)}}\tilde g_\ell\right|\le a_1\frac{G_2}{2}\left[\frac{|-i\tau_Vk_0+\gamma^{h^e_V}|}{|-it_{i(\ell)}\tau_Vk_0+\gamma^{h_\ell}|^2}+\frac{\gamma^{h^e_V}}{\gamma^{h_\ell}|-it_{i(\ell)}\tau_Vk_0+\gamma^{h_\ell}|}\right]\ , \tag{A4.1}$$

with $a_1=\max\{a_0,1\}$ and $\tau_V=\prod_i t_i$, where the product is over all resonances strictly containing V.

In §4.6 we remarked that this bound is not good for k_0 large; however this problem can be cared by using, for each resonance, the decay of the one of the propagators external to it, if the set of such propagators is not empty. Of course, it is not possible to use one fixed propagator for two different resonances, hence we decide to use, for this "balance" of large k_0 behaviour, only the propagator on the right of each resonance. It follows that we can define the set of "bad" resonances as the the set $\tilde{\mathbf{V}}$ of the resonances V with $\ell^o_V=\ell_{q+1}$, that it the set of resonances, which have no internal line of the graph at the right. In the evaluation of $\mathrm{Val}(\vartheta_{\mathcal R})$ we shall not use for such resonances the interpolation formula (A2.18), but we shall simply bound $|\mathcal R\Xi^{h_V}_V(\mathbf{k}'_{\ell^o_V})|$ by $2\sup_{\mathbf{k}'_{\ell^o_V}}\{|\Xi^{h_V}_V(\mathbf{k}'_{\ell^o_V})|\}$. Note that this implies that, in (A4.1), τ_V must be substituted with the product over all interpolating parameters associated with the resonances strictly containing V, but not belonging to $\tilde{\mathbf{V}}$.

It is important to note that $\tilde{\mathbf{V}}$ can contain at most two resonances and that, if $|\tilde{\mathbf{V}}|=2$, one of them must coincide with the whole graph. This claim easily follows from the remark that, if a graph $\vartheta_{\mathcal R}$ with external scale h contains a resonance V with $\ell^o_V=\ell_{q+1}$, then the internal scale of $\vartheta_{\mathcal R}$ must be equal to $h+1$; hence no other cluster, except $\vartheta_{\mathcal R}$ itself, can contain V.

A4.2. In §A2.3 the interpolating formula (A2.18) was used to produce a "gain factor", that allowed to control the sum over the scale index of the cluster containing the resonance. The remark above implies that the bounds (A2.31), (A2.34), (A2.35) would not have been modified, if we did not exploit the gain factor associated with the resonances belonging to $\tilde{\mathbf{V}}$. We shall now prove that they survive also to the use of (A4.1) instead of (3.44).

Note that, instead of (A2.23), one has

$$\begin{aligned}\left|\mathrm{Val}(\vartheta_{\mathcal R})\right|\le 2^{q-1}\Big(\prod_{v\in\vartheta_{\mathcal R}}|F_v|\Big)\Big(\prod_{T\in\mathbf{T}}\frac{G_3^{L_T}}{|-it_T\tau_Tk_0+\gamma^{h_T}|^{L_T}}\Big)\\ \Big(\prod_{T\in\mathbf{V}\setminus\tilde{\mathbf{V}}}\frac{|-i\tau_Tk_0+\gamma^{h^e_T}|}{|-it_T\tau_Tk_0+\gamma^{h_T}|}\Big)\ ,\end{aligned}\tag{A4.2}$$

where t_T is the interpolation parameter corresponding to the resonance T, if $T \in \mathbf{V}$, while $t_T = 1$, if T is not a resonance. Note that, for $T \in \tilde{\mathbf{V}}$, $t_T = \tau_T = 1$.

If we recall that $L_T = \tilde{M}_T + M_T^{(r)} + M_T^{(\nu)} - 1$ (see §2.4 and §A2.3 for notations), we can write

$$\begin{aligned} &\Big(\prod_{T \in \mathbf{T}} \frac{1}{|-it_T \tau_T k_0 + \gamma^{h_T}|^{L_T}} \Big) \Big(\prod_{T \in \mathbf{T}} \gamma^{h_T M_T^{(\nu)}} \Big) \le \\ &\le \Big(\prod_{T \in \mathbf{T}} \frac{1}{|-it_T \tau_T k_0 + \gamma^{h_T}|^{\tilde{M}_T + M_T^{(r)} - 1}} \Big) , \end{aligned} \tag{A4.3}$$

and we can proceed as in the proof of (A2.30); the role of (A2.29) is taken by

$$\begin{aligned} &\Big(\prod_{T \in \mathbf{T}} \frac{1}{|-it_T \tau_T k_0 + \gamma^{h_T}|^{M_T^{(r)}}} \Big) \Big(\prod_{T \in \mathbf{V} \setminus \tilde{\mathbf{V}}} \frac{|-i\tau_T k_0 + \gamma^{h_T^e}|}{|-it_T \tau_T k_0 + \gamma^{h_T}|} \Big) = \\ &= \Big(\prod_{V \in \tilde{\mathbf{V}}} \frac{1}{|-ik_0 + \gamma^{h_V^e}|} \Big) \Big(\prod_{T \in \mathbf{V} \setminus \tilde{\mathbf{V}}} \frac{1}{|-it_T \tau_T k_0 + \gamma^{h_T}|} \Big) . \end{aligned} \tag{A4.4}$$

By the remarks above about $\tilde{\mathbf{V}}$, the r.h.s. of (A4.4) can be bounded by a constant times the r.h.s. of (A2.29). Hence we get again the bound (A2.30), with different values of the constants.

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