Eigenvalues imbedded in the band spectrum for the periodic array of quantum dots

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Abstract

Solvable model of a periodic array of quantum dots in a uniform magnetic field is proposed having eigenvalues imbedded in the continuous spectrum. The continuous spectrum of the array may be absolutely continuous as well as singular continuous one. The model is based on the operator extension theory.

I. Introduction

The problem of eigenvalues imbedded in the continuous spectrum is of great interest both from mathematical and physical points of view. Mathematicians are studying it actively because there is no general theory allowing one to describe situation effectively, and to develop it they are trying to reveal general features by studying concrete problems [1]. An important reason for physicists is that the problem in question is closely related with the description of transport properties of many physical systems in fluid mechanics, acoustics, and quantum mechanics [2–4].

Among quantum transport problems great attention is attracted to unusual phenomena in such quantum nanostructures as arrays of quantum dots and antidots [5–9], espesially subjected to magnetic fields. The magnetic field may result in such an exotic effect as an appearence of a fractal structure of the energy spectrum predicted in [10–12]. On the other hand, it is known that magnetic field leads to the localization effects in periodic systems. This phenomena was discovered by Ando [13] in numerical experiments. Later the existence of the localizations was proved rigorously [14-17]. One can expect that there may be eigenvalues imbedded in the continuous spectrum in such systems. In our paper this effect is considered in the framework of the model based on the operator extension theory. The model was proposed and studied on the physical level of rigour in [18, 19]. The main goal of the present paper is to prove rigorously the existence of eigenvalues imbedded in the continuous spectrum of the model Hamiltonian for an appropriate choice of the model parameters. We stress that the continuous spectrum in question may be singular as well as absolutely continuous. As for the physical motivation, our model is based on physical assumptions like those in the Hubbard model (see [19] for details).

The paper is organized as follows. In the beginning we give a construction of the model. The essential point is that we use a variant of the extension theory model which is known as "zero-range potentials with internal structure" [20, 21]. The main result is contained in the third section where we study the band structure of the spectrum. In the particular case of the mono-atomic square lattice we reduce the spectral problem for the model Hamiltonian to the spectral problem for the well-known Harper operator [11, 12]. Note that proofs presented here are valid also for arbitrary lattice including multi-atomic one.

II. Model

We consider an array of quantum dots displaced in the nodes of a plane lattice Γ which is invariant with respect to translations by vectors of a Bravais lattice Λ. We assume that $\mathbf{0} \in \Gamma$ and fix some primitive vectors $\mathbf{a}_1$, $\mathbf{a}_2$ of Λ. The corresponding unit cell is marked as F_Λ:

$$F_\Lambda = \{t_1\mathbf{a}_1 + t_2\mathbf{a}_2 : 0 \leq t_1, t_2 < 1\}. \tag{1}$$

Let K be the set of all points of Γ belonging to F_Λ. Thus each point γ, $\gamma \in \Gamma$, has the unique representation in the form $\gamma = \lambda + \kappa$ where $\lambda \in \Lambda$, $\kappa \in \mathrm{K}$.

The Hamiltonian H_d of spinless charged particle in a single quantum dot will be chosen in the form $H_d = H_0 + V(r)$ where $V(r)$ is the confining potential of the dot and H_0 is free particle Hamiltonian with the presence of a magnetic field, i.e. the Schrödinger operator in $L^2(\mathbf{R}^2)$ defined by the differential expression

$$H_0 = \left(-i\frac{\partial}{\partial x} - \pi\xi y\right)^2 + \left(-i\frac{\partial}{\partial y} + \pi\xi x\right)^2 \tag{2}$$

(which is known also as the Landau operator). Here ξ is the flux density mesured in the magnetic flux quanta Φ_0, $\Phi_0 = (2\pi\hbar c)^{-1}e$, x and y are Cartesian coordinates on the plane. We choose the confining potential in a parabolic form: $V(r) = \omega_0^2 r^2/4$. Below we need the description of the spectrum and eigenfunctions of the operator H_d. To describe them let us introduce the notations: $\omega_c = 4\pi|\xi|$ (ω_c is known as the cyclotron frequency), $\Omega = \sqrt{\omega_c^2 + 4\omega_0^2}$, and $\omega_{1,2} = (\Omega \pm \omega_c)/2$ (the hybrid frequencies). The spectrum of H_d is pure point and consists of the eigenvalues known as the "Fock–Darwin" levels [23]:

$$E_{mn} = \frac{\omega_c}{2}m + \frac{\Omega}{2}(2n + |m| + 1), \qquad m \in \mathbf{Z}, \quad n \in \mathbf{N} \tag{3}$$

or in a more symmetric form

$$E_{n_1 n_2} = \omega_1(n_1 + 1/2) + \omega_2(n_2 + 1/2) \tag{4}$$

where

$$n_1 = n + \frac{|m| + m}{2}, \quad n_2 = n + \frac{|m| - m}{2}.$$

Using the polar coordinates $(\rho,\, \phi)$, we have the following representation of the corresponding eigenfunctions Ψ_{mn}

$$\Psi_{mn} = c_{mn}\, e^{im\phi}\, \rho^{|m|}\, \exp\left(-\rho^2/4l_0^2\right)\, L_n^{|m|}\left(\rho^2/2l_0^2\right). \tag{5}$$

Here $l_0 = (2/\Omega)^{1/2}$ is the so-called magnetic length, $L_n^k(x)$ are the generalized Laguerre polynomials, and c_{mn} are norming constants. The state space $\mathcal{H}$ of our model is the direct sum

$$\mathcal{H} = \sum_{\gamma\in\Gamma}{}^{\oplus} \mathcal{H}_\gamma \tag{6}$$

where each space $\mathcal{H}_\gamma$ is the state space $L^2(\mathbf{R}^2)$ of H_d. Assuming that the tunneling between dots is absent, we come to the following "background" Hamiltonian H^0_{arr}:

$$H^0_{arr} = \sum_{\gamma\in\Gamma}{}^{\oplus} H_\gamma \tag{7}$$

where each operator H_γ is equal to H_d. To take into account the charge carrier tunneling between the dots we use the "restriction – extension procedure" of the operator extension theory [20–22]. Let $\mathcal{D}_d$ be the set of all the functions f from the domain $D(H_d)$ each of which vanishes in a neighbourhood of zero. We denote by S_d the symmetric operator in $L^2(\mathbf{R}^2)$ which is the restriction of H_d to the domain $\mathcal{D}_d$. Let $S_\gamma = S_d$ for each $\gamma \in \Gamma$ and $S = \sum^{\oplus}_{\gamma\in\Gamma} S_\gamma$. We shall seek the "true" Hamiltonian H of the quantum dot array among self-adjoint extensions of S. Let $R(\zeta)$ be the resolvent of the desired operator: $R(\zeta) = (H - \zeta)^{-1}$, and let $R^0(\zeta) = (H^0_{arr} - \zeta)^{-1}$. Then R and R^0 are related by means of the Krein resolvent formula [24]:

$$R(\zeta) = R^0(\zeta) - \mathrm{B}(\zeta)[Q(\zeta) + A]^{-1}\mathrm{B}^*(\bar{\zeta}). \tag{8}$$

Here the operator valued holomorphic functions $\mathrm{B}(\zeta)$ and $Q(\zeta)$ are so-called Krein Γ- and $\mathcal{Q}$-functions respectively, and A is a self-adjoint operator in the deficiency space of S which parametrizes the self-adjoint extensions of S. In the considered case we can rewrite Eq.(8) in the following more detailed form. In virtue of (6)

$$R^0(\zeta) = \sum_{\gamma\in\Gamma}{}^{\oplus} R_\gamma(\zeta) \tag{9}$$

where each $R_\gamma(\zeta)$ is the resolvent of the operator H_d. Due to the same formula (6) we can get the representations:

$$\mathrm{B}(\zeta) = \sum_{\gamma\in\Gamma}{}^{\oplus}\mathrm{B}_\gamma(\zeta), \quad Q(\zeta) = \sum_{\gamma\in\Gamma}{}^{\oplus}Q_\gamma(\zeta) \tag{10}$$

where for each $\gamma\in\Gamma$ $\mathrm{B}_\gamma(\zeta)$ (respectively $Q_\gamma(\zeta)$) coincides with the Krein Γ-function $\mathrm{B}_d(\zeta)$ (respectively, with the $\mathcal{Q}$-function $Q_d(\zeta)$)) of the pair $(H_d,\, S_d)$. Identifying the deficiency subspace of S_γ with the standard one-dimensional space $\mathbf{C}$, we obtain the operator $\mathrm{B}_\gamma(\zeta)$ as an operator from $\mathbf{C}$ into $L^2(\mathbf{R}^2)$ which acts as multiplication by the function $G_d(x,0;\zeta)$ [25] where $G_d(x,y;\zeta)$ is the Green function of H_d, i.e. the integral kernel of its resolvent $R_d(\zeta)$. Thus the operator $\mathrm{B}_d(\zeta)\mathrm{B}^*_d(\bar\zeta)$ is an integral operator in $L^2(\mathbf{R}^2)$ with the kernel $B_d(x,y;\zeta) = G_d(x,0;\zeta)G_d(0,y;\zeta)$. The complex-valued function $Q_d(\zeta)$ can be obtained up to an additive constant by the formula [25]:

$$Q_d(\zeta) = \lim_{x\to 0}(G_d(x,0;\zeta) - G_d(x,0;\zeta_0)) \tag{11}$$

where ζ_0 is some fixed regular point of the operator H_d. To get G_d one can use the Laplace transform of time-dependent Green function $K_d(x,y;t)$ (i.e. the integral kernel of the propagator of H_d). This kernel has been obtained in [26]:

$$K_d(x,y;t) = \frac{\Omega}{8\pi i\sin\frac{\Omega t}{2}}\times$$

$$\exp\left\{\frac{i\Omega}{4\sin\frac{\Omega t}{2}}\left[(x_2y_1 - x_1y_2)\sin\frac{\omega_c t}{2} - (x_1y_1 + x_2y_2)\cos\frac{\omega_c t}{2} + \frac{1}{2}(x^2+y^2)\cos\frac{\Omega t}{2}\right]\right\}. \tag{12}$$

In particular, the time-dependent Green function K_0 for the Landau operator has the form:

$$K_0(x, y; t) = \frac{\omega_c}{8\pi i \sin\frac{\omega_c t}{2}} \times$$

$$\exp\left\{\frac{i\omega_c}{4\sin\frac{\omega_c t}{2}}\left[(x_1 y_2 - x_2 y_1)\sin\frac{\omega_c t}{2} - (x_1 y_1 + x_2 y_2)\cos\frac{\omega_c t}{2} + \frac{1}{2}(x^2 + y^2)\cos\frac{\omega_c t}{2}\right]\right\}. \tag{13}$$

If Im $\zeta > 0$, then $G_d(x, y; \zeta)$ is the Laplace transform of $K_d(x, y, ; t)$, namely

$$G_d(x, y, ; \zeta) = -i\int_0^\infty K_d(x, y; t)\, e^{it\zeta}\, dt\,; \tag{14}$$

the relation between G_0 and K_0 is analogous to Eq.(14). In the case of $\omega_0 = 0$ (i.e. in the case of the Landau operator H_0) the Krein $\mathcal{Q}$-function $Q_0(\zeta)$ is known and has the form [15]:

$$Q_0(\zeta) = -\frac{1}{4\pi}[\psi(1/2 - \zeta/\omega_c) + \log(\omega_c/4\pi) + 2C_{Euler}] \tag{15}$$

where ψ is the logarithmic derivative of the Euler Γ-function and C_{Euler} is the Euler constant. Comparing (12), (13), and (14) one can see that (15) and (11) result in

$$Q_d(\zeta) = -\frac{1}{4\pi}[\psi(1/2 - \zeta/\Omega) + \log(\Omega/4\pi) + 2C_{Euler}]. \tag{16}$$

It is convenient to rewrite Eq.(8) representing the state space $\mathcal{H}$ in the form of tensor product $\mathcal{H} = L^2(\mathbf{R}^2) \otimes l^2(\Gamma)$. Then, obviously, $R^0(\zeta) = R_d(\zeta) \otimes I$ and $\mathrm{B}(\zeta)[Q(\zeta) + A]^{-1}\mathrm{B}^*(\bar{\zeta})$ has the form $B_d(\zeta) \otimes T(\zeta)$ where $T(\zeta)$ is an operator in $l^2(\Gamma)$ having a diagonal matrix $T_{\gamma\gamma'}(\zeta) = Q(\zeta)\delta_{\gamma\gamma'}$. The operator A must be invariant with respect to a unitary representation of the magnetic translation group [27] in the space $l^2(\Lambda)$ (see [15]) and hence the matrix $A_{\gamma\gamma'}$ of the operator A is fully determined according to the formula

$$A_{\gamma+\lambda, \kappa'+\lambda} = \exp[\pi i \xi(\lambda \wedge (\gamma - \kappa'))] A_{\gamma\kappa'}$$

by the elements $A_{\gamma\kappa}$ ($\kappa \in \mathrm{K}$) only [18].

III. Structure of the spectrum

We start from a simple proposition.

Proposition 1. *The Fock-Darwin levels E_{mn} with $m \neq 0$ are eigenvalues of H.*

This proposition is a consequence of the following statement:

Proposition 2. *Let $\tilde{H}$ be any self-adjoint extension of S_d. The Fock-Darwin levels E_{mn} with $m \neq 0$ are eigenvalues of $\tilde{H}$ and the corresponding eigenvectors are equal to Ψ_{mn}.*

Proof. The statement expresses a well-known fact that a point perturbation leaves fixed the states with non-zero angular momentum. However, as far as we known, there are no proofs of this assertion in the sufficiently general case. To prove the proposition we use the Krein resolvent formula (8) once again. According to it, the Green function $\tilde{G}(x, y; \zeta)$ of $\tilde{H}$ has the following form

$$\tilde{G}(x, y; \zeta) = G_d(x, y; \zeta) - [Q_d(\zeta) + \alpha]^{-1} G_d(x, 0; \zeta) G_d(0, y; \zeta) \tag{17}$$

where α is a real number. With the help of (17) we can obtain the following explicit description of $\tilde{H}$. Let ζ_0, Im $\zeta_0 \neq 0$, be fixed. Then the domain $\tilde{\mathcal{D}}$ of $\tilde{H}$ consists of all the functions $f \in L^2(\mathbf{R}^2)$ having the form

$$f(x) = g(x) - [Q_d(\zeta) + \alpha]^{-1} g(0) G_d(x, 0; \zeta) \tag{18}$$

where g is an arbitrary function from $D(H_d)$ (it is easy to show that $D(H_d) \subset C(\mathbf{R}^2)$). For given f such a function g is uniquely defined, and

$$\tilde{H}(f) = \zeta_0 f + (H_d - \zeta_0) g. \tag{19}$$

The formula (18) and (19) imply in view of (5) the statement of the proposition.

Before to study the continuous part of the spectrum of H we note that the spectral properties of a periodic operator with a uniform magnetic field depend on whether geometric parameters of the system are commensurable with the so-called magnetic length. More precisely, let η be the number of magnetic flux quanta through the unit cell F_Λ, i.e. $\eta = \xi |F_\Lambda|$ where $|F_\Lambda|$ is the area of F_Λ. Then the spectral properties of H drastically depend on whether η is a rational number or not [10–12, 29]. To avoid some technical difficulties connected with this circumstance we restrict ourselves to the most interesting case being simultaneously a relatively simple one. Namely, we shall consider only the case when $\Lambda = \Gamma$ and the operator A in (8) has the form $A = A' + A''$ where A' is a scalar operator, i.e. an operator with a diagonal matrix $A'_{\lambda\mu} = a'\delta_{\lambda\mu}$, and A'' is a nearest-neighbour hopping operator. Thus matrix of A'' has the form

$$A''_{\lambda 0} = a''[\delta(\lambda_1, 0)(\delta(\lambda_2, 1) + \delta(\lambda_2, -1)) + \delta(\lambda_2, 0)(\delta(\lambda_1, 1) + \delta(\lambda_1, -1))]. \qquad (20)$$

Here a' and a'', $a'' \neq 0$, are some real non-zero constants (coupling constants), and we write $\delta(k, l)$ instead of δ_{kl} for convenience. In this case we are able to describe the continuous spectrum of H completely [18, 19], some mathematical details will appear in [28].

To do it we denote by $\chi(x)$ the inverse function to $\psi(x)$, $x \in \mathbf{R}$; χ is a multivalued real-analytic function defined for all x in $\mathbf{R}$ having continuous single-valued branches $\chi_n(x)$ $(n = 0, 1, \ldots)$ with values in the intervals $(0, +\infty)$, $(-1, 0)$, $(-2, -1)$, ... We denote by τ_n the functions

$$\tau_n(x) = \Omega[1/2 - \chi_n(4\pi a'' x + 4\pi a' - \log(\Omega/4\pi) - 2C_{Euler})]. \qquad (21)$$

Finally, let us consider the Harper operator, i.e. the direct integral with respect to ν, $\nu \in [0,\, 2\pi)$, of difference operators $h(\eta; \nu)$ which are defined in the space $l^2(\mathbf{Z})$ by the formula:

$$(h(\eta)\phi)_m = \phi_{m+1} + \phi_{m-1} + 2\cos(2\pi m\eta + \nu)\phi_m.$$

Here $\phi = (\phi_m)_{m\in\mathbf{Z}} \in l^2(\mathbf{Z})$.

The following proposition take place [19, 28]:

Proposition 3. *The continuous spectrum of H consists of the magneto-Bloch bands B_n. The band B_n is the set of all values of the function $E = \tau_n(x)$ on the* *spectrum of the Harper operator $h(\eta)$; B_n lies below the level E_{0n} and (if $n \geq 1$) above the level $E_{0,n-1}$.*

Corollary. *Let an eigenvalue E_{mn} be distinct from every E_{0k}, $k \in \mathbf{N}$. Then for every $a'' \neq 0$ there exists a coupling constan a' such that E_{mn} is imbedded in the continuous spectrum of H.*

Proof. Since $\omega_c < \Omega$, the eigenvalue E_{mn} lies between some eigenvalues of the form $E_{0,n-1}$ and E_{0n} where $n \geq 1$, i.e. between the levels $\Omega(n-1)/2$ and $\Omega n/2$. On the other hand, the image of the function χ_n is the whole interval $(-n, -n+1)$. Fix x in the spectrum of $h(\eta)$, then according to (21) $\tau_n(x)$ run from $\Omega(n-1)/2$ to $\Omega n/2$ as a' run from $-\infty$ to $+\infty$; this proves the collorary.

Remark. We stress that the spectrum of $h(\eta)$ is singular if η is an irrational number satisfying some arithmetic conditions [29, 30]. For such η the continuous spectrum of H is singular as well. Hence we have an example of operators having eigenvalues imbedded into the singular continuous spectrum.

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