INVERSE SPECTRAL ANALYSIS WITH PARTIAL INFORMATION ON THE POTENTIAL, II. THE CASE OF DISCRETE SPECTRUM

Fritz Gesztesy[1] and Barry Simon[2]

March 3, 1997

Abstract. We discuss results where the discrete spectrum (or partial information on the discrete spectrum) and partial information on the potential q of a one-dimensional Schrödinger operator $H = -\frac{d^2}{dx^2} + q$ determine the potential completely. Included are theorems for finite intervals and for the whole line. In particular, we pose and solve a new type of inverse spectral problem involving fractions of the eigenvalues of H on a finite interval and knowledge of q over a corresponding fraction of the interval. The methods employed rest on Weyl m-function techniques and densities of zeros of a class of entire functions.

§1. Introduction

In 1978, Hochstadt-Lieberman [14] proved the following remarkable theorem:

Theorem 1.1. *Let $h_0 \in \mathbb{R}$, $h_1 \in \mathbb{R} \cup \{\infty\}$ and assume $q_1, q_2 \in L^1((0,1))$ to be real-valued. Consider the Schrödinger operators H_1, H_2 in $L^2((0,1))$ given by*

$$H_j = -\frac{d^2}{dx^2} + q_j, \quad j = 1, 2,$$

with the boundary conditions

$$u'(0) + h_0 u(0) = 0, \tag{1.1a}$$

$$u'(1) + h_1 u(1) = 0. \tag{1.1b}$$

Let $\sigma(H_j) = \{\lambda_{j,n}\}$ be the (necessarily simple) spectra of H_j, $j = 1, 2$. Suppose that $q_1 = q_2$ (a.e.) on $[0, \frac{1}{2}]$ and that $\lambda_{1,n} = \lambda_{2,n}$ for all n. Then $q_1 = q_2$ (a.e.) on $[0, 1]$.

Here, in obvious notation, $h_1 = \infty$ in (1.1b) singles out the Dirichlet boundary condition $u(1) = 0$.

---

[1] Department of Mathematics, University of Missouri, Columbia, MO 65211, USA. E-mail: fritz@
math.missouri.edu

[2] Division of Physics, Mathematics, and Astronomy, California Institute of Technology, Pasadena, CA 91125, USA. E-mail: bsimon@caltech.edu

This material is based upon work supported by the National Science Foundation under Grant Nos. DMS-9623121 and DMS-9401491.

Typeset by $\mathcal{AMS}$-TeX

For each $\varepsilon > 0$, there are simple examples where $q_1 = q_2$ on $[0, \frac{1}{2} - \varepsilon]$ and $\sigma(H_1) = \sigma(H_2)$ but $q_1 \neq q_2$. (Choose $h_0 = -h_1$, $q_1(x) = 0$ for $x \in (0, \frac{1}{2} - \varepsilon] \cup [\frac{1}{2}, 1]$ and nonzero on $(\frac{1}{2} - \varepsilon, \frac{1}{2})$, and $q_2(x) = q_1(1-x)$. See also Theorem I$'$ in the appendix of [35].)

Later refinements of Theorem 1.1 in [11, 35] (see also the summary in [33]) showed that the boundary condition for H_1 and H_2 at $x = 1$ need not be assumed a priori to be the same, and that if q is continuous, then one only needs $\lambda_{1,n} = \lambda_{2,m(n)}$ for all values of n but one. ([35] claims the result does not require continuity of q, but we will see in Section 3 that this assertion is false.) The same boundary condition for H_1 and H_2 at $x = 0$, however, is crucial for Theorem 1.1 to hold (see [11, 31]).

Moreover, analogs of Theorem 1.1 for certain Schrödinger operators are considered in [17] and the interval $[0, \frac{1}{2}]$ replaced by different subsets of $[0, 1]$ was studied in [16] (see also [29], Ch. 4). Reconstruction techniques for $q(x)$ in this context are discussed in [32].

Our purpose in this paper is to provide a new approach to Theorem 1.1 that we feel is more transparent and, moreover, capable of vast generalizations. To state our generalizations, we will introduce a shorthand notation to paraphrase Theorem 1.1 by saying "q on $[0, \frac{1}{2}]$ and the eigenvalues of H uniquely determine q." This is just a shorthand notation for saying $q_1 = q_2$ if the obvious conditions hold.

Unless explicitly stated otherwise, all potentials q, q_1, and q_2 will be real-valued and in $L^1((0,1))$ for the remainder of this paper. Moreover, to avoid too many case distinctions in the proofs below, we shall assume $h_0, h_1 \in \mathbb{R}$ in (1.1) throughout the main body of this paper. In particular, for $h_0, h_1 \in \mathbb{R}$ we index the corresponding eigenvalues λ_n of H by $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The case of Dirichlet boundary conditions, where $h_0 = \infty$ and/or $h_1 = \infty$ will be dealt with in Appendix A.

Here are some of the generalizations we will prove for Schrödinger operators on $[0, 1]$:

Theorem 1.2. *Let $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ with boundary conditions* (1.1) *and $h_0, h_1 \in \mathbb{R}$. Suppose q is $C^{2k}((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon))$ for some $k = 0, 1, \dots$ and for some $\varepsilon > 0$. Then q on $[0, \frac{1}{2}]$, h_0, and all the eigenvalues of H except for $(k+1)$ uniquely determine h_1 and q on all of $[0, 1]$.*

Remarks. 1. The case $k = 0$ in Theorem 1.2 is due to Hald [11].

2. In the non-shorthand form of this theorem, we mean that both q_1 and q_2 are C^{2k} near $x = \frac{1}{2}$.

3. One need not know which eigenvalues are missing. Since the eigenvalues asymptotically satisfy

$$\lambda_n = (\pi n)^2 + 2(h_1 - h_0) + \int_0^1 dx\, q(x) + o(1) \qquad \text{as } n \to \infty, \tag{1.2}$$

given a set of candidates for the spectrum, one can tell how many are missing.

4. For the sake of completeness we mention the precise definition of H in $L^2((0,1))$ for real-valued $q \in L^1((0,1))$ and boundary condition parameters $h_0, h_1 \in$

$\mathbb{R} \cup \{\infty\}$ in (1.1):

$$\begin{aligned} H &= -\frac{d^2}{dx^2} + q, \\ D(H) &= \{g \in L^2((0,1)) \mid g, g' \in AC([0,1]);\ (-g'' + qg) \in L^2((0,1)); \\ &\qquad g'(0) + h_0 g(0) = 0,\ g'(1) + h_1 g(1) = 0\}, \end{aligned} \tag{1.3}$$

where $AC([0,1])$ denotes the set of absolutely continuous functions on $[0,1]$ and $h_{x_0} = \infty$ represents the Dirichlet boundary condition $g(x_0) = 0$ for $x_0 \in \{0,1\}$ in (1.3).

In Section 3, we discuss examples which show that Theorem 1.2 is optimal in the sense that if q is only assumed to be C^{2k-1} near $x = \frac{1}{2}$ for some $k \geq 1$, then it is not uniquely determined by $q \restriction [0, \frac{1}{2}]$ and all the eigenvalues but $(k+1)$.

Theorem 1.2 works because the condition that q is C^{2k} near $x = \frac{1}{2}$ gives us partial information about q on $[\frac{1}{2}, 1]$; namely, we know $q(\frac{1}{2}), q'(\frac{1}{2}), \dots, q^{(2k)}(\frac{1}{2})$ computed on $[\frac{1}{2}, 1]$ since we can compute them on $[0, \frac{1}{2}]$. This suggests that knowing q on more than $[0, \frac{1}{2}]$ should let one dispense with a finite density of eigenvalues. That this is indeed the case is the content of the following theorem:

Theorem 1.3. *Let $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ with boundary conditions* (1.1) *and $h_0, h_1 \in \mathbb{R}$. Then q on $[0, \frac{1}{2} + \frac{\alpha}{2}]$ for some $\alpha \in (0,1)$, h_0, and a subset $S \subseteq \sigma(H)$ of all the eigenvalues $\sigma(H)$ of H satisfying*

$$\#\{\lambda \in S \mid \lambda \leq \lambda_0\} \geq (1-\alpha)\#\{\lambda \in \sigma(H) \mid \lambda \leq \lambda_0\} + \tfrac{\alpha}{2} \tag{1.4}$$

for all sufficiently large $\lambda_0 \in \mathbb{R}$, uniquely determine h_1 and q on all of $[0,1]$.

Remarks. 1. As a typical example, knowing slightly more than half the eigenvalues and knowing q on $[0, \frac{3}{4}]$ determines q uniquely on all of $[0,1]$. To the best of our knowledge, Theorem 1.3 solves a new type of inverse spectral problem. In particular, we are not aware of any inverse spectral result involving fractions of the set of eigenvalues as in (1.4).

2. As in the case $\alpha = 0$, we have an extension of the same type as Theorem 1.2. Explicitly, if q is assumed to be C^{2k} near $x = \frac{1}{2} + \frac{\alpha}{2}$, we only need

$$\#\{\lambda \in S \mid \lambda \leq \lambda_0\} \geq (1-\alpha)\#\{\lambda \in \sigma(H) \mid \lambda \leq \lambda_0\} + \tfrac{\alpha}{2} - (k+1) \tag{1.5}$$

instead of (1.4).

We can also derive results about problems on all of $\mathbb{R}$. In Section 5, we will prove

Theorem 1.4. *Suppose that $q \in L^1_{\text{loc}}(\mathbb{R})$ obeys*

(i) *$q(x) \geq C|x|^{2+\varepsilon} - D$ for some $C, \varepsilon, D > 0$, and that*
(ii) *$q(-x) \geq q(x) \quad x \geq 0$.*

Then q on $[0, \infty)$ and the spectrum of $H = -\frac{d^2}{dx^2} + q$ in $L^2(\mathbb{R})$ uniquely determine q on all of $\mathbb{R}$.

In Section 5, we will also present further conjectures and explain how condition (i) is related to the class of entire functions of type less than one.

All these results are related to two other papers we have written. In [10], we consider, among other topics, analogs of Theorems 1.1 and 1.3 for finite tridiagonal (Jacobi) matrices extending a result in [13]. The approach there is very similar to the current one except that the somewhat subtle theorems on zeros of entire functions in this paper are replaced by the elementary fact that a polynomial of degree at most N with $N+1$ zeros must be identically zero. In [9], we consider results related to Theorem 1.4 in that for Schrödinger operators on $(-\infty,\infty)$, "spectral" information plus the potential on one of the half-lines determine the potential on all of $(-\infty,\infty)$. In that paper, we consider situations where there are scattering states for some set of energies and the "spectral" data are given by a reflection coefficient on a set of positive Lebesgue measure in the a.c. spectrum of H. The approach is not as close to this paper as is [10], but m-function techniques (see also [8]) are critical in all three papers.

Hochstadt-Lieberman [14] use the details of the inverse spectral theory in their proof. In a sense, we only use the main uniqueness theorem of that theory due to Marchenko [26], which we now describe. For $q \in L^1((a,b))$ real-valued, $-\infty < a < b < \infty$, consider

$$-u'' + qu = zu \tag{1.6}$$

with the boundary condition

$$u'(b) + h_b u(b) = 0 \tag{1.7}$$

at $x = b$. Let $u_+(z,x)$ denote the solution of this equation, normalized, say, by $u_+(z,b) = 1$. The m_+-function is then defined by

$$m_+(z,a) = \frac{u'_+(z,a)}{u_+(z,a)}\,. \tag{1.8}$$

Similarly, given a boundary condition at $x = a$,

$$u'(a) + h_a u(a) = 0, \tag{1.9}$$

we define the solution $u_-(z,x)$ of (1.6) normalized by $u_-(z,a) = 1$ and then define

$$m_-(z,b) = -\frac{u'_-(z,b)}{u_-(z,b)}\,. \tag{1.10}$$

The differing signs in (1.8) and (1.10) are picked so that both m_+ and m_- are Herglotz functions, that is, $m_\pm : \mathbb{C}_+ \to \mathbb{C}_+$ are analytic (in our present context where $-\infty < a < b < \infty$, $m_\pm$ are even meromorphic on $\mathbb{C}$), $\mathbb{C}_+$ the open complex upper half-plane. In particular,

$$\operatorname{Im}(z) > 0 \Longrightarrow \operatorname{Im}(m_-(z,b)) > 0, \quad \operatorname{Im}(m_+(z,a)) > 0. \tag{1.11}$$

Marchenko's [26] fundamental uniqueness theorem of inverse spectral theory then reads as follows:

Theorem 1.5. *$m_+(z,a)$ uniquely determines h_b as well as q (a.e.) on $[a,b]$.*

If $q \in L^1_{\text{loc}}([a,\infty))$ is real-valued (with $|a| < \infty$) and $-\frac{d^2}{dx^2} + q$ is in the limit point case at infinity, one can still define a unique $m_+(z,a)$ function but now for $\operatorname{Im}(z) \neq 0$ rather than all $z \in \mathbb{C}$. For such z, there is a unique function $u_+(z,\,\cdot\,)$ which is L^2 at infinity (unique up to an overall scale factor which drops out of $m_+(z,a)$ defined by (1.8)). Again, one has the following uniqueness result independently proved by Borg [3] and Marchenko [26]

Theorem 1.6. *$m_+(z,a)$ uniquely determines q (a.e.) on $[a,\infty)$.*

It is useful to have $m_-(z,b)$ because of the following basic fact:

Theorem 1.7. *Let $H = -\frac{d^2}{dx^2} + q$ be a Schrödinger operator in $L^2((a,b))$ with boundary conditions (1.7), (1.9) and let $G(z,x,y)$ be the integral kernel of $(H-z)^{-1}$. Suppose $c \in (a,b)$ and let $m_+(z,c)$ be the corresponding m_+-function for $[c,b]$ and $m_-(z,c)$ the m_--function for $[a,c]$. Then*

$$G(z,c,c) = -\frac{1}{m_+(z,c) + m_-(z,c)}\,. \tag{1.12}$$

Theorems 1.5 and 1.6 are deep facts; Theorem 1.7 is an elementary calculation from the explicit formula for the integral kernel of $(H-z)^{-1}$,

$$G(z,x,y) = \frac{u_-(z,\min(x,y))u_+(z,\max(x,y))}{W(u_-(z),u_+(z))(x)}\,,$$

where $W(\,\cdot\,,\,\cdot\,)$ is the Wronskian defined by

$$W(f,g)(x) = f'(x)g(x) - f(x)g'(x).$$

An analog of Theorem 1.7 holds in case $[a,b]$ is replaced by $(-\infty,\infty)$.

We can now describe the strategy of our proofs of Theorems 1.1–1.4. $G(z,c,c)$ has poles at the eigenvalues of H (this is not quite true; see below), so by (1.12), at eigenvalues λ_n of H:

$$m_+(\lambda_n,c) = -m_-(\lambda_n,c). \tag{1.13}$$

If we know q on a left partial interval $[a,c]$ and we know some eigenvalue λ_n, then we know $m_-(z,c)$ exactly; so by (1.13), we know the value of $m_+(\lambda_n,c)$ at the point λ_n. In Appendix B we discuss when knowing the values of $f(\lambda_n)$ of an analytic function of the type of the m-functions uniquely determines $f(z)$. If $m_+(z,c)$ is determined, then by Theorem 1.5, q is determined on $[a,b]$ and so is h_b.

So the logic of the argument for a theorem like Theorem 1.1 is the following:

(i) q on $[0,\frac{1}{2}]$ and h_0 determine $m_-(z,\frac{1}{2})$ by direct spectral theory.
(ii) The λ_n and (1.13) determine $m_+(\lambda_n,\frac{1}{2})$, and then by suitable theorems in complex analysis, $m_+(z,\frac{1}{2})$ is uniquely determined for all z.
(iii) $m_+(z,\frac{1}{2})$ uniquely determines q (a.e.) on $[\frac{1}{2},1]$ and h_1 by inverse spectral theory.

It is clear from this approach why h_0 is required and h_1 is free in the context of Theorem 1.1 (see [31] for examples where h_1 and $q \restriction [0, \frac{1}{2}]$ do not determine q); without h_0 we cannot compute $m_-(z, \frac{1}{2})$ and so start the process.

As indicated before (1.13), $G(z, c, c)$ may not have a pole at an eigenvalue λ_n of H. It will if $u_n(c) \neq 0$, but if $u_n(c) = 0$, then $G(z, c, c) = 0$ rather than ∞. Here u_n denotes the eigenfunction of H associated with the (necessarily simple) eigenvalue λ_n. Nevertheless, (1.13) holds at points where $u_n(c) = 0$ since then $u_-(c) = u_+(c) = 0$, and so both sides of (1.13) are infinite. (In spite of (1.13), $m_+ + m_-$ is also infinite at $z = \lambda_n$ and so $G(\lambda_n, c, c) = 0$.) We summarize this discussion in the following

Theorem 1.8. *For any $c \in (a, b)$, (1.13) holds at any eigenvalue λ_n of $H_{[a,b]}$ (with the possibility of both sides of (1.13) being infinite).*

An alternative way of proving (1.13) is that λ_n is an eigenvalue if and only if the Wronskian of u_+ and u_- is zero, which is precisely (1.13).

Here is a sketch of the contents of this paper. In Section 2 we present our proofs of Theorems 1.1 and 1.2. In Section 3 we discuss an example that delimits Theorem 1.2 and shows that Theorem 1.2 is optimal with respect to smoothness conditions on q. In Section 4 we prove Theorem 1.3, and in Section 5 we prove Theorem 1.4. Appendix A is devoted to the case of Dirichlet boundary conditions, and Appendix B presents some facts on entire functions that are necessary to prove our principal results.

§2. Theorems for a Half Interval

In this section, we will prove the original Hochstadt-Lieberman theorem (Theorem 1.1) and our extension of it (Theorem 1.2) for $h_0, h_1 \in \mathbb{R}$. Consider a problem on $[0, 1]$ with boundary condition (1.2) at $x = 1$. Let $u_+(z, x)$ be defined by $-u_+'' + qu_+ = zu_+$ and

$$u_+'(z, 1) = -h_1, \quad u_+(z, 1) = 1. \tag{2.1}$$

Then u_+ is known to have the following properties:

(1) For each $x \in [0, 1]$, $u_+(z, x), u_+'(z, x)$ are entire functions of z. (This follows from the fact that $u_+(z, 1) = 1$ and $u_+'(z, 1) = -h_1$ are independent of z, see, e.g., [4], Theorem I.8.4, Problem I.7, and p. 226.)

(2)
$$u_+(z, x) = \cos\big(\sqrt{z}\,(1-x)\big) + O\bigg(\frac{e^{\operatorname{Im}(\sqrt{z})\,(1-x)}}{\sqrt{z}}\bigg), \tag{2.2}$$
$$u_+'(z, x) = \sqrt{z}\,\sin\big(\sqrt{z}\,(1-x)\big) + O\big(e^{\operatorname{Im}(\sqrt{z})\,(1-x)}\big) \tag{2.3}$$
as $|z| \to \infty$ for all $x \in [0, 1]$, where $\sqrt{z}$ is the square root branch with $\operatorname{Im}(\sqrt{z}) \geq 0$ (see, e.g., [27], Sect. 1.4).

(3) The zeros of $u_+(\,\cdot\,, x)$ and $u_+'(\,\cdot\,, x)$ are all real for any $x \in [0, 1]$ and they all lie in some λ-interval $[c, \infty)$ (this is because these zeros are eigenvalues of self-adjoint boundary value problems for Schrödinger operators in $L^2((0, 1))$ bounded from below).

The final pair of preliminary results we need concerns the high-energy asymptotics of the m_+-function,

$$m_+(z,x) = \frac{u_+'(z,x)}{u_+(z,x)}\,.$$

(4) It is known [1, 6] that under the general hypothesis $q \in L^1((0,1))$,

$$m_+(z,x)^{-1} = -i(\sqrt{z}\,)^{-1}(1+o(z^{-1/2})) \tag{2.4}$$

uniformly in $x \in [0,1-\delta]$, $\delta > 0$ as $|z| \to \infty$ in any sector $\varepsilon < \operatorname{Arg}(z) < 2\pi - \varepsilon$, $\varepsilon > 0$.

(5) If q is C^{2k} near $x_0 \in (0,1)$, $k = 0,1,2,\dots$, then $m_+(z,x_0)$ and $m_+(z,x_0)^{-1}$ are known to have asymptotic expansions of the form [5],

$$m_+(z,x_0) = i(\sqrt{z}\,)\bigg(\sum_{\ell=0}^{2k+2} C_\ell(x_0) z^{-\ell/2} + o(z^{-k-1})\bigg), \qquad C_0(x_0) = 1, \tag{2.5}$$

$$m_+(z,x_0)^{-1} = -i(\sqrt{z}\,)^{-1}\bigg(\sum_{\ell=0}^{2k+2} D_\ell(x_0) z^{-\ell/2} + o(z^{-k-1})\bigg), \quad D_0(x_0) = 1, \tag{2.6}$$

as $|z| \to \infty$ in any sector $\varepsilon < \operatorname{Arg}(z) < 2\pi - \varepsilon$, $\varepsilon > 0$, where $C_\ell(x_0)$ and $D_\ell(x_0)$ are universal functions of $q(x_0),\dots,q^{(\ell-2)}(x_0)$. In fact, $C_\ell(x)$ and $D_\ell(x_0)$ have a well-known connection to the conserved densities of the KdV hierarchy [7] and they can be computed recursively as follows. Consider the Riccati-type equations for $m_+(z,x)$ and $m_+(z,x)^{-1}$,

$$m_+'(z,x) + m_+(z,x)^2 = q(x) - z, \tag{2.7}$$

$$[m_+(z,x)^{-1}]' + m_+(z,x)^{-2}[q(x)-z] = 1. \tag{2.8}$$

Inserting the asymptotic expansions (2.5) and (2.6) into (2.7) and (2.8) then yields the recursion relations

$$\begin{aligned}
&C_0(x) = 1, \quad C_1(x) = 0, \quad C_2(x) = -\tfrac12\, q(x),\\
&C_j(x) = \tfrac{i}{2}\, C_{j-1}'(x) - \tfrac12 \sum_{\ell=1}^{j-1} C_\ell(x) C_{j-\ell}(x), \quad j \geq 3,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&D_0(x) = 1, \quad D_1(x) = 0, \quad D_2(x) = \tfrac12\, q(x),\\
&D_j(x) = \tfrac{i}{2}\, D_{j-1}'(x) + \tfrac12\, q(x) \sum_{\ell=0}^{j-2} D_\ell(x) D_{j-\ell-2}(x) - \tfrac12 \sum_{\ell=1}^{j-1} D_\ell(x) D_{j-\ell}(x), \quad j \geq 3.
\end{aligned} \tag{2.10}$$

With these preliminaries out of the way, let q be given (a.e.) on $[0, \frac{1}{2}]$ and let q_1, q_2 be two candidates for q extended to all of $[0,1]$. Let $\sigma(H_1) = \{\lambda_{1,n}\}_{n=0}^\infty$ be the set of all the eigenvalues of $H_1 = -\frac{d^2}{dx^2} + q_1$. Define for $j = 1, 2$,

$$P_j(z) = u_{j,+}(z, \tfrac{1}{2}), \tag{2.11}$$

$$Q_j(z) = u'_{j,+}(z, \tfrac{1}{2}), \tag{2.12}$$

$$f_j(z) = \frac{P_j(z)}{Q_j(z)} = m_{j,+}(z, \tfrac{1}{2})^{-1}, \tag{2.13}$$

$$g(z) = u'_{1,+}(z, 0) + h_0 u_{1,+}(z, 0), \tag{2.14}$$

so that $\{\lambda_{1,n}\}_{n=0}^\infty$ are precisely the zeros of $g(z)$. (Note in this context that $u_{1,+}(z,x)$ satisfies (1.1b) at $x = 1$ for all $z \in \mathbb{C}$. Thus, if and only if $g(\hat\lambda) = 0$, $u_{1,+}(\hat\lambda, x)$ also satisfies (1.1a) at $x = 0$ and hence $\hat\lambda \in \sigma(H_1)$.) Here $u_{j,\pm}$ are the corresponding solutions of $-u'' + q_j u = zu$ used in (1.8) and (1.10). By adding a sufficiently large constant to q_1 and q_2, we can suppose all the zeros of P_j, Q_j, and g are in $[1, \infty)$.

By (1)–(5) above, we infer:

(a) P_j, Q_j, and g are all of the form (see, e.g., [19], Ch. I; [28], Sect. II.48)

$$c \prod_{n=0}^\infty \left(1 - \frac{z}{x_n}\right) \tag{2.15}$$

for suitable $\{x_n\}_{n=0}^\infty \subseteq [1, \infty)$ (which a priori could differ for the five functions).

(b) P_j, Q_j, and g are all bounded by $C_1 \exp(C_2 |z|^{1/2})$ for some $C_1, C_2 > 0$.

(c) $|f_1(iy) - f_2(iy)| = o(|y|^{-1})$ as y (real) $\to \pm\infty$.

(d) If $q_j \in C^{2k}$ near $x = \frac{1}{2}$, then as y (real) $\to \pm\infty$,

$$|f_1(iy) - f_2(iy)| = o(|y|^{-(2k+3)/2}).$$

(e) $|Q_j(iy)| = \frac{1}{2}|y|^{1/2} \,|\exp(\frac{1}{2}\mathrm{Im}\,(\sqrt{i})\,|y|^{1/2})|(1 + o(1))$ as y (real) $\to \infty$.

(f) $|g(iy)| = \frac{1}{2}|y|^{1/2}\,|\exp(\mathrm{Im}\,(\sqrt{i})\,|y|^{1/2})|\,(1 + o(1))$ as y (real) $\to \infty$.

(g) For n sufficiently large, $\inf_{\theta \in [0,2\pi]} |g((\pi(n+\frac{1}{2}))^2 e^{i\theta})| \geq \pi n + O(1)$.

Part (d) holds by (2.6) and (2.10) because $q_1^{(\ell)}(\frac{1}{2}) = q_2^{(\ell)}(\frac{1}{2})$ for all $0 \leq \ell \leq 2k$ by the regularity of q near $x = \frac{1}{2}$ and hence the terms $D_0(x_0), \dots, D_{2k+2}(x_0)$ in (2.6) in connection with q_1 and q_2 will cancel when inserted into $[f_1(iy) - f_2(iy)]$. (g) follows from (2.3) since by (2.15), the infimum is taken at $\theta = 0$.

Proof of Theorem 1.1 (*for* $h_0, h_1 \in \mathbb{R}$). Define

$$F(z) = \frac{[P_1(z)Q_2(z) - P_2(z)Q_1(z)]}{g(z)}. \tag{2.16}$$

By Theorem 1.8, (1.13) holds at the points $\lambda_{1,n}$. Hence $\frac{Q_1(z)}{P_1(z)} = \frac{Q_2(z)}{P_2(z)}$ at $z = \lambda_{1,n}$. Moreover, at points where both sides are infinite, one infers $P_1 = P_2 = 0$. Thus, the cross ratio $P_1 Q_2 - P_2 Q_1$ vanishes at each point where g vanishes, and since g necessarily has simple zeros (H_1 has simple spectrum), F is an entire function.

In addition, by (b) and (g), $F(z)$ satisfies

$$|F(z)| \leq C_1 \exp(C_2 |z|^{1/2}) \tag{2.17}$$

since (2.17) first holds when $|z| = (\pi(n + \frac{1}{2}))^2$ for n sufficiently large (by (f)) and then by the maximum modulus principle for all z. By Proposition B.6 (a Phragmén-Lindelöf argument) and (2.17), if we show that $|F(iy)| \to 0$ as $y \to \infty$ (y real), then $F \equiv 0$.

But

$$F(z) = \frac{Q_1(z) Q_2(z)}{g(z)} \, [f_1(z) - f_2(z)], \tag{2.18}$$

so by (c), (e), and (f),

$$|F(iy)| = \tfrac{1}{2} \, |y|^{1/2} \, [o(|y|^{-1})](1 + o(1)) = o(|y|^{-1/2}) \quad \text{as } y \text{ (real)} \to \infty, \tag{2.19}$$

goes to zero as required.

Once $F \equiv 0$, we can multiply by $\frac{g(z)}{Q_1(z) Q_2(z)}$ (which has isolated zeros and poles) to conclude that $f_1 = f_2$, and so by Theorem 1.5, $q_1 = q_2$ (a.e.). $\square$

Remark. There is a (patchable) gap in the paper of Hochstadt-Lieberman [14]. They consider an entire function $\psi(z) = \frac{H(z)}{\omega(z)}$ where they show $|H(z)| \leq M \exp(\operatorname{Im}(\sqrt{|z|}))$ and $\omega(z) = C\sqrt{z}\, \sin(\sqrt{z}\,) + O(e^{\operatorname{Im}(\sqrt{z})})$ and then claim $|\psi(z)| = O(\frac{1}{\sqrt{|z|}})$ without comment. Because of the zeros of $\sin(\,\cdot\,)$, this is not evident and one needs a Phragmén-Lindelöf-type argument to complete their proof.

Proof of Theorem 1.2. Let $\{\lambda_\ell\}_{\ell=1}^{k+1}$ be the $k+1$ eigenvalues of $-\frac{d^2}{dx^2} + q_1$ in $L^2((0,1))$ which a priori are not assumed to be the same for the two potentials. Now define $\tilde{F}(z)$ by

$$\tilde{F}(z) = \frac{[P_1(z) Q_2(z) - P_2(z) Q_1(z)]}{g(z)} \prod_{\ell=1}^{k+1} (z - \lambda_\ell) \tag{2.20}$$

instead of (2.16).

(2.17) still holds, and as in (2.18) and (2.19), one now infers from (d), (e), and (f),

$$|\tilde{F}(iy)| = O(|y|^{k+1}) |y|^{1/2} \, [o(|y|^{-(2k+3)/2})](1 + o(1)) = o(1) \quad \text{as } y \text{ (real)} \to \infty.$$

Thus $m_+(z, \frac{1}{2})$ determines h_1 (cf. Theorem 1.5) and $q_1 = q_2$ (a.e.) follows as in the previous proof. $\square$

§3. An Example

Our goal in this section is to construct, for each k, a function q on $[0,1]$ with the following properties:

(1) q is C^∞ on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$; q is C^{2k-1} on $[0,1]$.

(2) $\frac{d^{2k}q}{dx^{2k}}$ is discontinuous at $x = \frac{1}{2}$.
(3) $q = 0$ on $[0, \frac{1}{2}]$.
(4) For a suitable boundary condition parameter $h_1 \in \mathbb{R}$, the eigenvalues of $-\frac{d^2}{dx^2}+q$ in $L^2((0,1))$ with $u'(0) = 0$, $u'(1)+h_1u(1) = 0$ boundary conditions agree with those for $-\frac{d^2}{dx^2}$ in $L^2((0,1))$ with $u'(0) = u'(1) = 0$ boundary conditions with precisely $(k+1)$ exceptions.

For $k = 0$, (2) means that q is discontinuous at $x = \frac{1}{2}$.

This example shows that in Theorem 1.2, one cannot weaken the continuity requirement on q. In particular, it provides a counterexample to the claim in Suzuki [35] that his Theorem I in his Appendix only requires $q \in L^1((0,1))$. Continuity of q at $x = \frac{1}{2}$ is critical for his result to hold.

Our results depend on the following well-known fact:

Proposition 3.1. *Suppose that $x_0 < y_0 < x_1 < y_1 < \cdots$ are given so that for n sufficiently large,*

$$x_n = [(2n)\pi]^2, \quad y_n = [(2n+1)\pi]^2. \tag{3.1}$$

Then, there exists (a unique) h_1 and a C^∞ function q on $[\frac{1}{2}, 1]$ so that

$$-\frac{d^2}{dx^2} + q \quad \text{in } L^2((\tfrac{1}{2}, 1)); \qquad u'(\tfrac{1}{2}) = 0,\ u'(1) + h_1u(1) = 0 \tag{3.2}$$

has eigenvalues $\{x_n\}_{n=0}^\infty$ and

$$-\frac{d^2}{dx^2} + q \quad \text{in } L^2((\tfrac{1}{2}, 1)); \qquad u(\tfrac{1}{2}) = 0,\ u'(1) + h_1u(1) = 0 \tag{3.3}$$

has eigenvalues $\{y_n\}_{n=0}^\infty$. (By (1.2) and (A.5c), $h_1 = -\frac{1}{2}\int_{1/2}^1 dx q(x)$.)

This is just a special case of the construction of Levitan and Gasymov [23]. Historically, this classical two-spectra inverse problem goes back to Borg's seminal paper [2]. Subsequently, Levinson [20] found considerable simplifications of Borg's uniqueness arguments, and Krein [18] developed his own solution of these inverse spectral problems. This circle of ideas was further developed in [12], [21], [22], Ch. 3, [23], [24], Sect. 6.11 and continues to generate interest (see, e.g., [15], [25], [34]).

We also need the elementary:

Lemma 3.2. *Suppose that h_0, h_1, and q are given, and for some $\hat\lambda$, there exists an $h_{1/2}$ (with the Dirichlet boundary condition $h_{1/2} = \infty$ at $x = \frac{1}{2}$ allowed; $u'(\frac{1}{2}) + h_{1/2}u(\frac{1}{2}) = 0$ is then interpreted as $u(\frac{1}{2}) = 0$) so that $\hat\lambda$ is an eigenvalue of both*

$$-\frac{d^2}{dx^2} + q \quad \text{in } L^2((0, \tfrac{1}{2})); \qquad u'(0) + h_0u(0) = 0,\ u'(\tfrac{1}{2}) + h_{1/2}u(\tfrac{1}{2}) = 0$$

and

$$-\frac{d^2}{dx^2} + q \quad \text{in } L^2((\tfrac{1}{2}, 1)); \qquad u'(\tfrac{1}{2}) + h_{1/2}u(\tfrac{1}{2}) = 0,\ u'(1) + h_1u(1) = 0.$$

Then $\hat{\lambda}$ is also an eigenvalue of

$$-\frac{d^2}{dx^2}+q \quad in\ L^2((0,1)); \qquad u'(0)+h_0u(0)=0,\ u'(1)+h_1u(1)=0.$$

Proof. One can match the solutions in the two halves so that they and their first derivatives become absolutely continuous near $x=\frac{1}{2}$. $\square$

Let $x_n^{(0)}=[(2n)\pi]^2$ and $y_n^{(0)}=[(2n+1)\pi]^2$, $n=0,1,\dots$ be the eigenvalues that lead to $q=0$ on $[\frac{1}{2},1]$ (and $h_1^{(0)}=0$) in Proposition 3.1. To construct our example, we will take $x_n=x_n^{(0)}$ for all $n=0,1,\dots$ and $y_n=y_n^{(0)}$ for $n=0,1\dots$ with $k+1$ exceptions, say, $y_{n_0}\neq y_{n_0}^{(0)}, y_{n_1}\neq y_{n_1}^{(0)},\dots,y_{n_k}\neq y_{n_k}^{(0)}$. Make the choice so that for $\ell=1,2,\dots,k$,

$$\sum_{j=0}^{k} y_{n_j}^{\ell}=\sum_{j=0}^{k}[y_{n_j}^{(0)}]^{\ell}. \tag{3.4}$$

Choices satisfying (3.4) can certainly be made. For example, we can take $y_{n_0}=y_{n_0}^{(0)}+\varepsilon$ with ε small, and solve the k equations, (3.4), for $y_{n_1},\dots,y_{n_k}$ using the fact that the Jacobian determinant that needs to be non-zero to apply the inverse function theorem is essentially just a $k\times k$ Vandermonde determinant, $\det(a)$, with $a_{j\ell}=\ell(y_{n_j}^{(0)})^{\ell-1}$.

Let $m_{N,+}^{(0)}(z,\frac{1}{2})=-\frac{1}{\sqrt{z}}\cot(\frac{1}{2}\sqrt{z})$, the Neumann m_+-function on $[\frac{1}{2},1]$ for $q=0$ (and $h_1^{(0)}=0$). Let $m_{N,+}(z)$ be the corresponding Neumann m_+-function on $[\frac{1}{2},1]$ for the q constructed in Proposition 3.1 (whose poles and zeros are given by the eigenvalues x_n and y_n, $n=0,1\dots$ of (3.2) and (3.3)). We claim that

$$m_{N,+}(z,\tfrac{1}{2})=\prod_{j=0}^{k}\left(\frac{z-y_{n_j}}{z-y_{n_j}^{(0)}}\right)m_{N,+}^{(0)}(z,\tfrac{1}{2}). \tag{3.5}$$

Indeed, the two sides have the same zeros and poles and both are ratios of functions of order $\frac{1}{2}$; thus they are constant multiples of each other. Since both sides behave asymptotically like $\sim -|z|^{-1/2}$ as $z\to-\infty$, the constant multiple must be 1.

Because of (3.4),

$$\ln\left(\prod_{j=0}^{k}\left(\frac{z-y_{n_j}}{z-y_{n_j}^{(0)}}\right)\right)=\sum_{j=1}^{k}\left[\ln\left(1-\frac{y_{n_j}}{z}\right)-\ln\left(1-\frac{y_{n_j}^{(0)}}{z}\right)\right]=O(z^{-k-1}). \tag{3.6}$$

Since $m_{N,+}^{(0)}(z,\frac{1}{2})=-\frac{i}{\sqrt{z}}(1+o(z^{-K}))$ for all K consistent with $q=0$ in (2.10), (3.5) and (3.6) imply that

$$m_{N,+}(z,\tfrac{1}{2})=-\frac{i}{\sqrt{z}}\,(1+O(z^{-k-1})).$$

Thus in (2.6), $D_\ell(\frac{1}{2})=0$ for $\ell=1,2,\dots,2k+1$. But by (2.10) and induction, this implies that

$$q^{(m)}(\tfrac{1}{2})=0 \qquad \text{for } m=0,1,2,\dots,2k-1. \tag{3.7}$$

Next, let $q(x)$ be defined a.e. on $[0,1]$ by

$$\begin{aligned} q(x) &= 0, && 0 \leq x < \tfrac{1}{2} \\ &= \text{constructed } q \restriction (\tfrac{1}{2}, 1), && \tfrac{1}{2} < x \leq 1. \end{aligned}$$

By (3.7), q is C^{2k-1} at $x = \frac{1}{2}$. By Lemma 3.2, $-\frac{d^2}{dx^2} + q$ in $L^2((0,1))$, with $u'(0) = 0$, $u'(1) + h_1 u(1) = 0$ boundary conditions, has $\{x_n^{(0)}\}_{n=0}^\infty \cup \{y_n^{(0)}\}_{n=0, n \neq n_0, n_1, \dots, n_k}^\infty$ as eigenvalues, so at most $(k+1)$ eigenvalues differ from the free Neumann case (i.e., $q = 0$ on $[0,1]$, $h_0 = h_1 = 0$). If fewer than $k+1$ eigenvalues differed, then by Theorem 1.2, $q \equiv 0$ and $h_1 = 0$. Since the y_{n_i} are not $y_{n_i}^{(0)}$, this cannot be true. Thus, exactly $k+1$ eigenvalues differ. If $q^{(2k)}(\frac{1}{2})$ were zero, then by Theorem 1.2, again $q \equiv 0$ and $h_1 = 0$, so q is not C^{2k}.

There remains an interesting open question: Can one replace information on the missing eigenvalue by knowledge of the boundary condition h_1?

§4. The Case of Partially Known Spectra

Our goal in this section is to prove Theorem 1.3. Define

$$g_S(z) = \prod_{\lambda_n \in S} \left(1 - \frac{z}{\lambda_n}\right), \quad g_{\sigma(H)}(z) = \prod_{n=0}^\infty \left(1 - \frac{z}{\lambda_n}\right), \quad S \subseteq \sigma(H) = \{\lambda_n\}_{n=0}^\infty.$$

By the hypothesis (1.4) on S and $\sigma(H)$ in Theorem 1.3 and the method of proof of Theorem B.4 (see the critical equality (B.16)), we infer

$$\ln(|g_S(iy)|) \geq (1-\alpha) \ln(|g_{\sigma(H)}(iy)|) + \tfrac{\alpha}{4} \ln(1+y^2) + C_0. \tag{4.1}$$

Since $\sigma(H)$ is a complete set of eigenvalues for a self-adjoint problem on $[0,1]$, we know that asymptotically

$$|g_{\sigma(H)}(iy)| \sim \tfrac{1}{2} \, |y|^{1/2} \left| e^{\operatorname{Im}(\sqrt{i}) \, |y|^{1/2}} \right| \quad \text{as } y \text{ (real)} \to \infty.$$

Thus by (4.1), there exists a constant $C > 0$ such that

$$|g_S(iy)| \geq C |y|^{1/2} \left| e^{\operatorname{Im}(\sqrt{i})(1-\alpha)|y|^{1/2}} \right| \tag{4.2}$$

for $|y|$ sufficiently large.

Let $P_j(z) = u_{j,+}(z, \frac{1}{2} + \frac{\alpha}{2})$, $Q_j(z) = u'_{j,+}(z, \frac{1}{2} + \frac{\alpha}{2})$, $j = 1, 2$ for the two candidate potentials. Then, since $1 - (\frac{1}{2} + \frac{\alpha}{2}) = \frac{1}{2}(1-\alpha)$, we use (2.3) to infer asymptotically

$$|Q_j(iy)| \sim \tfrac{1}{2} \, |y|^{1/2} \left| e^{\operatorname{Im}(\sqrt{i}) \, (1/2)(1-\alpha)|y|^{1/2}} \right| \quad \text{as } y \text{ (real)} \to \infty. \tag{4.3}$$

With (4.2), (4.3), the arguments in Section 2 extend to prove Theorem 1.3. $\square$

§5. Theorems for the Whole Real Line

Our main goal in this section is to prove Theorem 1.4. So we suppose that $q_1(x), q_2(x)$ are two potentials on $\mathbb{R}$ satisfying

$$q_j(x) \geq C|x|^{2+\varepsilon} + 1, \quad j = 1, 2 \tag{5.1}$$

for some $C, \varepsilon > 0$. (The condition in Theorem 1.4 has $-D$ in place of 1. Just add $D+1$ to initial q_j's to get (5.1) if need be.) Thus for any $z \in \mathbb{C}$, there exist solutions $u_{j,\pm}(x,z)$ of $-u''(x) + q_j(x)u(x) = zu(x)$ which are L^2 near $\pm\infty$ and

$$m_{j,\pm}(z) := \pm \frac{u'_{j,\pm}(0,z)}{u_{j,\pm}(0,z)}, \quad j = 1, 2$$

are meromorphic functions of z. Let $\{\lambda_m\}_{m=1}^{\infty}$ be the eigenvalues of $-\frac{d^2}{dx^2} + q_1(x)$ on $(-\infty, \infty)$ and denote by $\{\mu_{j,m}\}_{m=1}^{\infty}$ the eigenvalues of $-\frac{d^2}{dx^2} + q_j(x)$ on $(-\infty, 0]$ with Dirichlet (i.e., $u(0) = 0$) boundary conditions at $x = 0$. We claim:

Proposition 5.1. (i) *Let $\rho_0 = 1 - \frac{\varepsilon}{(4+2\varepsilon)}$. Then for all $\rho > \rho_0$, $j = 1, 2$,*

$$\sum_{m=1}^{\infty} \lambda_m^{-\rho} < \infty, \tag{5.2}$$

$$\sum_{m=1}^{\infty} (\mu_{j,m})^{-\rho} < \infty. \tag{5.3}$$

(ii)

$$\mu_{j,m} \geq \lambda_{2m}\,, \quad m \in \mathbb{N}. \tag{5.4}$$

Proof. Let α_m be the m^{th} eigenvalue of $-\frac{d^2}{dx^2} + Cx^{2+\varepsilon}$ on $(-\infty, \infty)$. The large m asymptotics of α_m is given by a classical phase space argument (see, e.g., Theorem XIII.81 in [32], or Section 7.1 in [36]), that is, for an explicit constant $K \in (0, \infty)$, $\lim_{m\to\infty} \alpha_m / m^{1/\rho_0} = K$. Thus, $\sum_{m\in\mathbb{N}} \alpha_m^{-\rho} < \infty$ if $\rho > \rho_0$, and so (5.2) holds since $\lambda_m \geq \alpha_m$. Let $\beta_{j,m}$ be the m^{th} eigenvalue of $-\frac{d^2}{dx^2} + q_j(-|x|)$ on $(-\infty, \infty)$. By the hypothesis $q_j(-|x|) \geq q_j(x)$, we infer that $\beta_{j,m} \geq \lambda_m$. But by Dirichlet-Neumann alternation, $\mu_{j,m} = \beta_{j,2m}$ for $m = 1, 2, \ldots$ proving (5.4). (5.3) then follows from (5.4). $\square$

Define

$$Q_j(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z}{\mu_{j,m}}\right),$$

$$g(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z}{\lambda_m}\right).$$

Proposition 5.2. *For all $y \in \mathbb{R}$,*

$$\frac{|Q_1(iy)Q_2(iy)|}{|g(iy)|} \le 1. \tag{5.5}$$

Proof. $|1 - \frac{iy}{w}| = (1 + \frac{y^2}{w^2})^{1/2}$ for y, w real, is monotone decreasing in w, so by (5.4),

$$\left|1 - \frac{iy}{\mu_{j,m}}\right| \le \left|1 - \frac{iy}{\lambda_{2m}}\right| \le \left|1 - \frac{iy}{\lambda_{2m-1}}\right|.$$

It follows that

$$|Q_j(iy)| \le \prod_{m=1}^{\infty} |1 - \frac{iy}{\lambda_{2m}}| \le \prod_{m=1}^{\infty} |1 - \frac{iy}{\lambda_{2m-1}}|,$$

so (5.5) holds. $\square$

Now let

$$P_j(z) = m_{j,-}(z)Q_j(z), \quad j = 1, 2,$$

which are entire functions, and

$$f_j(z) = m_{j,-}(z), \quad j = 1, 2.$$

Define

$$F(z) = \frac{P_1(z)Q_2(z) - Q_1(z)P_1(z)}{g(z)} = \frac{Q_1(z)Q_2(z)}{g(z)}\,(f_1(z) - f_2(z)). \tag{5.6}$$

Proof of Theorem 1.4. At $z = \lambda_k$, the eigenfunctions on the left half-line for both q_1 and q_2 must match to the common eigenfunctions on the right, so $P_1(\lambda_k)Q_2(\lambda_k) - P_2(\lambda_k)Q_1(\lambda_k) = 0$, that is, $F(z)$ is an entire function.

By (5.2), $g(z)$ is a function of m-type as defined in Appendix B. Thus by Proposition B.5, there exists a sequence $R_k \to \infty$ so that $\sup\{\frac{1}{|g(z)|} \mid |z| = R_k\} \le C_1 \exp(+C_2 R_k^{\rho})$ for some $\rho < 1$. By (5.3) and a similar estimate for the Dirichlet eigenvalues, P_j, Q_j are functions of m-type. It follows by Proposition B.6 that if

$$\lim_{|y|\to\infty} |F(iy)| = 0, \tag{5.7}$$

then $F \equiv 0$. If we prove that, then $m_{1,-}(z) = m_{2,-}(z)$, and thus $q_1 = q_2$ a.e. on $(-\infty, 0]$ and hence on $\mathbb{R}$. Thus, we need only prove (5.7).

By (2.4), which holds for half-line m-functions [1, 6], $|f_1(iy) - f_2(iy)| = o(1)$. Thus, Proposition 5.2 and (5.6) show that (5.7) holds. $\square$

Several questions remain open. We do not believe that hypothesis (i) is needed in Theorem 1.4:

Conjecture 5.3. *Theorem* 1.4 *remains true if* (i) *is replaced by* $\lim_{|x|\to\infty} q(x) = \infty$.

This will require dealing with entire functions of type larger than 1.
We also believe:

Conjecture 5.4. *Suppose* $\lim_{x\to\infty} \frac{q(-x)}{q(x)} = \infty$ *and* $\lim_{|x|\to\infty} q(x) = \infty$. *Then* q *near* $+\infty$ *and a finite density subset of eigenvalues for* $-\frac{d^2}{dx^2} + q(x)$ *uniquely determine* q *on* $\mathbb{R}$.

Acknowledgments. We would like to thank R. del Rio for discussions and pertinent hints to the literature. F.G. is indebted to A.S. Kechris and C.W. Peck for a kind invitation to Caltech during the summer of 1996 where some of this work was done. The extraordinary hospitality and support by the Department of Mathematics at Caltech are gratefully acknowledged. B.S. would like to thank M. Ben-Artzi and the Hebrew University during a visit where some of this work was done.

Appendix A: Dirichlet Boundary Conditions

In this appendix we provide some details in the remaining cases, which involve Dirichlet boundary conditions at $x = 0$ and/or $x = 1$. We need to distinguish three cases (cf. (1.1)):

(I) H has a Dirichlet boundary condition at $x = 0$ and $x = 1$, that is,

$$u(0) = 0, \tag{A.1a}$$
$$u(1) = 0. \tag{A.1b}$$

(II) H has a non-Dirichlet boundary condition at $x = 0$ and a Dirichlet boundary condition at $x = 1$, that is,

$$u'(0) + h_0 u(0) = 0, \quad h_0 \in \mathbb{R}, \tag{A.2a}$$
$$u(1) = 0. \tag{A.2b}$$

(III) H has a Dirichlet boundary condition at $x = 0$ and a non-Dirichlet boundary condition at $x = 1$, that is,

$$u(0) = 0, \tag{A.3a}$$
$$u'(1) + h_1 u(1) = 0, \quad h_1 \in \mathbb{R}. \tag{A.3b}$$

Since later on, q is supposed to be known on $[0, \frac{1}{2} + \varepsilon]$ for some $0 \le \varepsilon < \frac{1}{2}$, cases II and III represent inequivalent situations and need to be treated separately in connection with Theorems A.1–A.3. Depending on the case at hand, we index the corresponding eigenvalues λ_n of H by

$$\{\lambda_n\}_{n=1}^{\infty} \quad \text{in case I} \quad \text{and} \quad \{\lambda_n\}_{n=0}^{\infty} \quad \text{in cases II and III.} \tag{A.4}$$

The asymptotic expansion (1.2) then becomes as $n \to \infty$,

$$\lambda_n = (\pi n)^2 + \int_0^1 dx\, q(x) + o(1) \qquad \text{in case I,} \tag{A.5a}$$

$$\lambda_n = (\pi(n+\tfrac{1}{2}))^2 - 2h_0 + \int_0^1 dx\, q(x) + o(1) \qquad \text{in case II,} \tag{A.5b}$$

$$\lambda_n = (\pi(n+\tfrac{1}{2}))^2 + 2h_1 + \int_0^1 dx\, q(x) + o(1) \qquad \text{in case III.} \tag{A.5c}$$

Let $u_+(z,x)$ be defined by $-u_+'' + qu_+ = zu_+$ subject to the boundary conditions and normalizations

$$u_+'(z,1) = 1, \qquad u_+(z,1) = 0 \qquad \text{in cases I and II,} \tag{A.6a}$$

$$u_+'(z,1) = -h_1, \quad u_+(z,1) = 1 \qquad \text{in case III.} \tag{A.6b}$$

Next we note that items (1) and (3)–(5) of Section 2 remain valid in the present cases I–III, whereas item (2) becomes

$$u_+(z,x) = -\frac{\sin\big(\sqrt{z}\,(1-x)\big)}{\sqrt{z}} + O\bigg(\frac{e^{\operatorname{Im}(\sqrt{z})\,(1-x)}}{z}\bigg), \tag{A.7}$$

$$u_+'(z,x) = \cos\big(\sqrt{z}\,(1-x)\big) + O\bigg(\frac{e^{\operatorname{Im}(\sqrt{z})\,(1-x)}}{\sqrt{z}}\bigg) \tag{A.8}$$

in cases I and II, and

$$u_+(z,x) = \cos\big(\sqrt{z}\,(1-x)\big) + O\bigg(\frac{e^{\operatorname{Im}(\sqrt{z})\,(1-x)}}{\sqrt{z}}\bigg), \tag{A.9}$$

$$u_+'(z,x) = \sqrt{z}\,\sin\big(\sqrt{z}\,(1-x)\big) + O\big(e^{\operatorname{Im}(\sqrt{z})\,(1-x)}\big) \tag{A.10}$$

in case III, as $|z| \to \infty$ for all $x \in [0,1]$.

Introducing P_j, Q_j, f_j, $j=1,2$ as in (2.11)–(2.13), and g by

$$g(z) = u_{1,+}(z,0) \qquad \text{in cases I and III,} \tag{A.11a}$$

$$g(z) = u_{1,+}'(z,0) + h_0 u_{1,+}(z,0) \qquad \text{in case II,} \tag{A.11b}$$

one infers again that $\{\lambda_{1,n}\}_{n=1}^\infty$ in case I and $\{\lambda_{1,n}\}_{n=0}^\infty$ in cases II and III are precisely the zeros of $g(z)$. The corresponding properties (a)–(g) listed in Section 2, suitably adapted to the present cases I–III, then read as follows:

(a′) Item (a) remains valid except that n in (2.15) runs through $\mathbb{N}$ in case I and through $\mathbb{N}_0$ in cases II and III.

(b′–d′) Items (b), (c), and (d) remain valid.

(e′) As y (real) $\to \infty$,

$$|Q_j(iy)| = \tfrac{1}{2}\,|\exp(\tfrac{1}{2}\operatorname{Im}(\sqrt{i})\,|y|^{1/2})|(1+o(1)) \qquad \text{in cases I and II,}$$

$$|Q_j(iy)| = \tfrac{1}{2}\,|y|^{1/2}\,|\exp(\tfrac{1}{2}\operatorname{Im}(\sqrt{i})\,|y|^{1/2})|(1+o(1)) \qquad \text{in case III.}$$

(f′) As y (real) $\to \infty$,

$$|g(iy)| = \tfrac{1}{2}\,|y|^{-1/2}\,|\exp(\operatorname{Im}\,(\sqrt{i})\,|y|^{1/2})|\,(1+o(1)) \quad \text{in case I,}$$
$$|g(iy)| = \tfrac{1}{2}\,|\exp(\operatorname{Im}\,(\sqrt{i})\,|y|^{1/2})|\,(1+o(1)) \quad \text{in cases II and III.}$$

(g′) For n sufficiently large, one obtains,

$$\inf_{\theta\in[0,2\pi]} |g((\pi(n+\tfrac{1}{2}))^2 e^{i\theta})| \geq \tfrac{1}{\pi n} + O(\tfrac{1}{n^2}) \quad \text{in case I,}$$
$$\inf_{\theta\in[0,2\pi]} |g((\pi(n+1))^2 e^{i\theta})| \geq 1 + O(\tfrac{1}{n}) \quad \text{in cases II and III.}$$

Introducing $F(z)$ as in (2.16) and (2.18), one verifies, using (b′) and (g′), that (2.17) remains valid. Items (c′), (e′), and (f′), however, yield the following modification of (2.19) as y (real) $\to \infty$,

$$|F(iy)| = \tfrac{1}{2}\,|y|^{1/2}\,[o(|y|^{-1})](1+o(1)) = o(|y|^{-1/2}) \quad \text{in case I,} \tag{A.12a}$$
$$|F(iy)| = \tfrac{1}{2}[o(|y|^{-1})](1+o(1)) = o(|y|^{-1}) \quad \text{in case II,} \tag{A.12b}$$
$$|F(iy)| = \tfrac{1}{2}\,|y|[o(|y|^{-1})](1+o(1)) = o(1) \quad \text{in case III.} \tag{A.12c}$$

Following the arguments in (2.16)–(2.19) step-by-step and taking into account the eigenvalue asymptotics in (A.5), the remaining Dirichlet cases in Theorem 1.1 then read as follows.

Theorem A.1. *Let $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ with boundary conditions* (A.1), (A.2), *or* (A.3) *according to cases* I, II, *or* III. *Then q on $[0,\frac{1}{2}]$, together with the knowledge of $h_0 = \infty$ or $h_0 \in \mathbb{R}$, and all the eigenvalues of H, uniquely determine h_1 (i.e., $h_1 = \infty$ in cases* I *and* II *and $h_1 \in \mathbb{R}$ in case* III) *and q (a.e.) on all of $[0,1]$.*

Remark. Case I for $q \in L^2((0,1))$ appears to be due to Pöschel and Trubowitz [29], Ch. 4. Much to our surprise, the extension of case I to $q \in L^1((0,1))$ in Theorem A.1 seems to be new. Case II is originally due to Hochstadt and Lieberman [14] as recorded in Theorem 1.1. To the best of our knowledge, case III is a new result.

The analog of Theorem 1.2 is now obtained as follows. Replace the definition of $\tilde{F}$ in (2.20) by

$$\tilde{F}(z) = \frac{[P_1(z)Q_2(z) - P_2(z)Q_1(z)]}{g(z)} \prod_{\ell=1}^{k+1}(z-\lambda_\ell) \quad \text{in cases I and II,} \tag{A.13a}$$
$$\tilde{F}(z) = \frac{[P_1(z)Q_2(z) - P_2(z)Q_1(z)]}{g(z)} \prod_{\ell=1}^{k}(z-\lambda_\ell) \quad \text{in case III.} \tag{A.13b}$$

Here $\{\lambda_\ell\}_{\ell=1}^{k+1}$ in cases I and II, and $\{\lambda_\ell\}_{\ell=1}^{k}$ in case III represent eigenvalues of H_1 and H_2 which are not a priori assumed to be equal.

The asymptotic behavior of $|\tilde{F}(iy)|$ as y (real) $\to \infty$,

$$|\tilde{F}(iy)| = o(1) \quad \text{in case I,} \tag{A.14a}$$
$$|\tilde{F}(iy)| = o(|y|^{-1/2}) \quad \text{in cases II and III,} \tag{A.14b}$$

then yields the following new result.

Theorem A.2. *Let $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ with boundary conditions* (A.1), (A.2), *or* (A.3) *according to cases* I, II, *or* III. *Suppose q is $C^{2k}((\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon))$, $k = 0, 1, \dots$ for some $\varepsilon > 0$. Then q on $[0, \frac{1}{2}]$, together with the knowledge of $h_0 = \infty$ or $h_0 \in \mathbb{R}$, and all the eigenvalues of H except for $(k+1)$ in cases* I *and* II *and k in case* III, *uniquely determine h_1 (i.e., $h_1 = \infty$ in cases* I *and* II *and $h_1 \in \mathbb{R}$ in case* III) *and q on all of $[0,1]$.*

Finally, we consider the analog of Theorem 1.3 in the Dirichlet context.

Theorem A.3. *Let $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ with boundary conditions* (A.1), (A.2), *or* (A.3). *Then q on $[0, \frac{1}{2} + \frac{\alpha}{2}]$ for some $\alpha \in (0,1)$, $h_0 = \infty$ or $h_0 \in \mathbb{R}$, and a subset $S \subseteq \sigma(H)$ of all the eigenvalues $\sigma(H)$ of H satisfying*

$$\#\{\lambda \in S \mid \lambda \leq \lambda_0\} \geq (1-\alpha)\#\{\lambda \in \sigma(H) \mid \lambda \leq \lambda_0\} + \tfrac{\alpha}{2} \tag{A.15}$$

for all sufficiently large $\lambda_0 \in \mathbb{R}$, uniquely determine h_1 (i.e., $h_1 = \infty$ in cases I *and* II *and $h_1 \in \mathbb{R}$ in case* III) *and q on all of $[0,1]$.*

Proof. Following the arguments employed in Section 4, we introduce again

$$\begin{aligned}
g_S(z) &= \prod_{\lambda_n \in S} \left(1 - \frac{z}{\lambda_n}\right), \quad && S \subseteq \sigma(H), \\
g_{\sigma(H)}(z) &= \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) && \text{in case I}, \\
g_{\sigma(H)}(z) &= \prod_{n=0}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) && \text{in cases II and III},
\end{aligned}$$

where

$$\sigma(H) = \{\lambda_n\}_{n=1}^{\infty} \quad \text{in case I} \quad \text{and} \quad \sigma(H) = \{\lambda_n\}_{n=0}^{\infty} \quad \text{in cases II and III.}$$

Then (A.15) and the method of proof of Theorem B.4 yield

$$\ln(|g_S(iy)|) \geq (1-\alpha)\ln(|g_{\sigma(H)}(iy)|) + \tfrac{\alpha}{4}\ln(1+y^2) + C_0. \tag{A.16}$$

Since asymptotically (cf. (f′)) for $|y|$ large enough,

$$\begin{aligned}
|g_{\sigma(H)}(iy)| &\sim \tfrac{1}{2}\, |y|^{-1/2} \big|e^{\operatorname{Im}(\sqrt{i})\,|y|^{1/2}}\big| && \text{in case I}, \\
|g_{\sigma(H)}(iy)| &\sim \tfrac{1}{2}\, \big|e^{\operatorname{Im}(\sqrt{i})\,|y|^{1/2}}\big| && \text{in cases II and III},
\end{aligned}$$

one infers from (A.16) that for some $C > 0$

$$|g_S(iy)| \geq C|y|^{-1/2} \big|e^{\operatorname{Im}(\sqrt{i})(1-\alpha)|y|^{1/2}}\big| \quad \text{in case I}, \tag{A.17a}$$

$$|g_S(iy)| \geq C \big|e^{\operatorname{Im}(\sqrt{i})(1-\alpha)|y|^{1/2}}\big| \quad \text{in cases II and III} \tag{A.17b}$$

for $|y|$ sufficiently large.

Introducing again $P_j(z) = u_{j,+}(z, \frac{1}{2}+\frac{\alpha}{2})$, $Q_j(z) = u'_{j,+}(z, \frac{1}{2}+\frac{\alpha}{2})$, $j = 1, 2$ for the two candidate potentials, noticing $1 - (\frac{1}{2} + \frac{\alpha}{2}) = \frac{1}{2}(1-\alpha)$, we infer asymptotically as y (real) $\to \infty$,

$$|Q_j(iy)| \sim \tfrac{1}{2}\left|e^{\operatorname{Im}(\sqrt{i})\,(1/2)(1-\alpha)|y|^{1/2}}\right| \quad \text{in cases I and II,} \tag{A.18a}$$

$$|Q_j(iy)| \sim \tfrac{1}{2}|y|^{1/2}\left|e^{\operatorname{Im}(\sqrt{i})\,(1/2)(1-\alpha)|y|^{1/2}}\right| \quad \text{in case III.} \tag{A.18b}$$

Given (A.17) and (A.18), one can now finish the proof of Theorem A.3 in the same way as that of Theorem 1.3 in Section 4. □

Remark. As in the case $\alpha = 0$, we have an extension of the same type as Theorem A.2. Explicitly, if q is assumed to be C^{2k} near $x = \frac{1}{2} + \frac{\alpha}{2}$, we only need

$$\#\{\lambda \in S \mid \lambda \leq \lambda_0\} \geq (1-\alpha)\#\{\lambda \in \sigma(H) \mid \lambda \leq \lambda_0\} + \tfrac{\alpha}{2} - N(k) \tag{A.19}$$

instead of (A.15), where $N(k) = k+1$ in cases I and II and $N(k) = k$ in case III.

Appendix B: Zeros of Entire Functions

In discussing extensions of Hochstadt's discrete (finite matrix) version [13] of the Hochstadt-Lieberman theorem in [10], we made use of the following simple lemma which is an elementary consequence of the fact that any polynomial of degree d with $d+1$ zeros must be the zero polynomial:

Lemma B.1. *Suppose $f_1 = \frac{P_1}{Q_1}$ and $f_2 = \frac{P_2}{Q_2}$ are two rational fractions where the polynomials satisfy $\deg(P_1) = \deg(P_2)$ and $\deg(Q_1) = \deg(Q_2)$. Suppose that $d = \deg(P_1) + \deg(Q_1)$ and that $f_1(z_n) = f_2(z_n)$ for $d+1$ distinct points $\{z_n\}_{n=1}^{d+1} \in \mathbb{C}$. Then $f_1 = f_2$.*

Our main goal in this appendix is to prove an analogous theorem for a class of entire functions. The theorem is sharp in the sense that it includes Lemma B.1 (at least the case of Lemma B.1 where the zeros of the entire functions involved and the z_n are all positive).

We will be interested here in entire functions of the form

$$f(z) = C \prod_{n=0}^{\infty} \left(1 - \frac{z}{x_n}\right), \tag{B.1}$$

where $0 < x_0 < x_1 < \cdots$ is a suitable sequence of positive numbers which are the zeros of f and C is some complex constant.

Given a sequence $\{x_n\}_{n=0}^{\infty}$ of positive reals, define

$$N(t) = \#\{n \in \mathbb{N} \cup \{0\} \mid x_n < t\}. \tag{B.2}$$

Recall the following basic theorem (see, e.g., [19], Ch. I; [28], Sects. II.48 and II.49):

Theorem B.2. *Fix $0 < \rho_0 < 1$. Then:*

(i) *If $\{x_n\}_{n=0}^\infty$ is a sequence of positive reals with*

$$\sum_{n=0}^{\infty} x_n^{-\rho} < \infty \qquad \text{for all } \rho > \rho_0 \tag{B.3}$$

then the product in (B.1) *defines an entire function f with*

$$|f(z)| \le C_1 \exp(C_2 |z|^\rho) \qquad \text{for all } \rho > \rho_0. \tag{B.4}$$

(ii) *Conversely, if f is an entire function satisfying* (B.4) *with all its (complex) zeros on $(0, \infty)$, then its zeros $\{x_n\}_{n=0}^\infty$ satisfy* (B.3)*, and f has the canonical product expansion* (B.1).

Moreover, (B.3) *holds if and only if*

$$N(t) \le C|t|^\rho \qquad \text{for all } \rho > \rho_0. \tag{B.5}$$

Given this theorem, we single out:

Definition B.3. A function f is called of m-type if and only if f is an entire function satisfying (B.4) (of type $0 < \rho < 1$ in the usual definition) with all the zeros of f on $(0, \infty)$.

Our choice of "m-type" in Definition B.3 comes from the fact that in many cases we discuss in this paper, the m-function is a ratio of functions of m-type. By Theorem B.2, f in Definition B.3 has the form (B.1) and $N(t)$, which we will denote as $N_f(t)$, satisfies (B.5). We are heading toward a proof of

Theorem B.4. *Let f_1, f_2, g be three functions of m-type so that:*

(i) *$f_1(z) = f_2(z)$ at any point z with $g(z) = 0$.*

(ii) *For all sufficiently large t,*

$$\max(N_{f_1}(t), N_{f_2}(t)) \le N_g(t) - 1.$$

Then, $f_1 = f_2$.

Proposition B.5. *Let f be a function of m-type. Then there exists a $0 < \rho < 1$ and a sequence $\{R_k\}_{k=1}^\infty$, $R_k \to \infty$ as $k \to \infty$, so that*

$$\inf\{|f(z)| \mid |z| = R_k\} \ge C_1 \exp(-C_2 R_k^\rho).$$

Proof. By hypothesis, for some $0 < \rho' < 1$,

$$N_f(t) \le C t^{\rho'}. \tag{B.6}$$

This implies

$$x_n \ge \left(\frac{n}{C}\right)^{1/\rho'}. \tag{B.7}$$

If for all $n \geq n_0$,

$$|x_n - x_{n-1}| \leq 2,$$

then

$$x_n \leq x_{n_0} + 2(n - n_0),$$

which contradicts (B.7). Thus for an infinite sequence $\{n(k)\}_{k=1}^{\infty}$, $n(k) \to \infty$ as $k \to \infty$, we necessarily must have

$$x_{n(k)} - x_{n(k)-1} \geq 2. \tag{B.8}$$

We will pick

$$R_k = \tfrac{1}{2}\left(x_{n(k)} + x_{n(k)-1}\right). \tag{B.9}$$

For any $\alpha > 0$, $|1-\alpha e^{i\theta}|^2 = 1+\alpha^2 - 2\alpha\cos(\theta)$ takes its minimum value at $\theta = 0$, so

$$\inf\{|f(z)| \mid z = R_k\} = |f(R_k)|. \tag{B.10}$$

By Theorem B.2, f has the form (B.1). We will write

$$\ln(|f(R_k)|) = A_1 + A_2, \tag{B.11}$$

where

$$A_1 = \sum_{n|x_n \geq 2x_{n(k)}} \ln\left(\left|1 - \frac{R_k}{x_n}\right|\right), \quad A_2 = \sum_{n|x_n < 2x_{n(k)}} \ln\left(\left|1 - \frac{R_k}{x_n}\right|\right).$$

We estimate A_1 by writing the sum as a Stieltjes integral, integrating by parts, and using (B.6):

$$\begin{aligned} A_1 &= \int_{2x_{n(k)}}^{\infty} \ln\left(1 - \frac{R_k}{t}\right) dN_f(t) \\ &= -\int_{2x_{n(k)}}^{\infty} \left(\frac{1}{1 - \frac{R_k}{t}}\right) \frac{R_k}{t^2}\left[N_f(t) - N_f(2x_{n(k)})\right] dt \\ &\geq -\int_{2R_k}^{\infty} \frac{2R_k}{t^2}\, Ct^{\rho'}\, dt = -CR_k^{\rho'}, \end{aligned} \tag{B.12}$$

where we have used C to represent a positive constant that varies from formula to formula.

For A_2, we write,

$$\ln\left(\left|1 - \frac{R_k}{x_n}\right|\right) = \ln(|x_n - R_k|) - \ln(|x_n|) \geq -\ln(|x_n|) \tag{B.13}$$

$$\geq -\ln(4R_k), \tag{B.14}$$

where (B.13) follows from (B.8) and (B.9), and (B.14) follows from

$$|x_n| < 2x_{n(k)} \leq 2(2R_k).$$

Thus by (B.6),

$$A_2 \geq -N_f(2x_{n(k)})\ln(4R_k) \geq -CR_k^{\rho'}\ln(4R_k) \geq -CR_k^{\rho''} \tag{B.15}$$

for some $1 > \rho'' > \rho'$ and suitable positive constants C. (B.10), (B.11), (B.12), and (B.15) prove the proposition. $\square$

Proposition B.6. *Let F be an entire function that satisfies*

(i) $\sup_{|z|=R_k} |F(z)| \leq C_1 \exp(C_2 R_k^\rho)$ *for some* $0 \leq \rho < 1$, $C_1, C_2 > 0$, *and some sequence* $R_k \to \infty$ *as* $k \to \infty$.

(ii) $\lim_{|x|\to\infty; x\in\mathbb{R}} |F(ix)| = 0$.

Then $F \equiv 0$.

Proof. A standard Phragmén-Lindelöf estimate separately applied to $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(z) < 0$ (i.e., to an angle of opening π; see, e.g., [28], Sect. II.34) shows that F is bounded. Liouville's theorem implies that F is constant, and then the fact that $|F(ix)| \to 0$ as $x \to \infty$ (x real), shows that $F = 0$. $\square$

Proof of Theorem B.4. By hypothesis (i), $\frac{f_1(z)-f_2(z)}{g(z)} := Q(z)$ is an entire function. By (B.4) (applied to f_1 and f_2) and Proposition B.5 (applied to g), there is a sequence $R_k \to \infty$ as $k \to \infty$ so that $Q(z)$ satisfies condition (i) of Proposition B.6. Thus, it suffices to prove $\lim_{x\to\pm\infty} \frac{f_1(ix)}{g(ix)} = \lim_{x\to\pm\infty} \frac{f_2(ix)}{g(ix)} = 0$. We will prove the f_1 result for definiteness. In fact, our proof will show that $\frac{f_1(ix)}{g(ix)} = O(x^{-1})$ as $|x| \to \infty$. Without loss of generality, we will assume that f_1, g satisfy (B.1) with $C = 1$, that is, that $f_1(0) = g(0) = 1$. We will also suppose that $N_{f_1}(t) = N_g(t) = 0$ if $t \leq 1$, which can be arranged by appropriate scaling.

Notice first that

$$\begin{aligned} \ln(|f_1(ix)|) &= \sum_{n=0}^{\infty} \tfrac{1}{2} \ln\biggl(1 + \frac{x^2}{x_n^2}\biggr) \\ &= \tfrac{1}{2} \int_0^\infty \ln\biggl(1 + \frac{x^2}{t^2}\biggr)\, dN_{f_1}(t) \\ &= \tfrac{1}{2} \int_0^\infty \frac{1}{1 + \frac{x^2}{t^2}} \frac{2x^2}{t^3} N_{f_1}(t)\, dt \\ &= \int_0^\infty \frac{x^2}{t^3 + tx^2} N_{f_1}(t)\, dt. \end{aligned} \tag{B.16}$$

The boundary term at $t = 0$ in the integration by parts step vanishes since $N_{f_1}(0) = 0$ and the one at $t = \infty$ vanishes by the estimate (B.5) and the fact that for x fixed, $\ln(1 + \frac{x^2}{t^2}) = O(t^{-2})$ as $t \to \infty$.

By hypothesis (ii) of the theorem, there are $t_0 \geq 1$ and $C \geq 0$ such that

$$N_{f_1}(t) \leq N_g(t) - 1, \qquad t \geq t_0 \tag{B.17a}$$

$$\leq N_g(t) + C, \qquad t \leq t_0. \tag{B.17b}$$

Hence, by (B.16),

$$\begin{aligned} \ln\biggl(\biggl|\frac{f_1(ix)}{g(ix)}\biggr|\biggr) &\leq (C+1) \int_1^{t_0} \frac{x^2}{t^3 + tx^2}\, dt - \int_1^\infty \frac{x^2}{t^3 + tx^2}\, dt \\ &\leq (C+1)\ln(t_0) - \tfrac{1}{2}\ln(1 + x^2), \end{aligned}$$

since $\frac{x^2}{t^3+tx^2} = -\frac{d}{dt}[\frac{1}{2}\ln(1+\frac{x^2}{t^2})]$. Thus, as claimed,

$$\frac{f_1(ix)}{g(ix)} = O(x^{-1})$$

as $|x| \to \infty$. $\square$.

One can replace (B.17a) by the following pair of conditions for $t \geq t_0 \geq 1$, t_0 sufficiently large:

$$N_{f_1}(t) \leq N_g(t) + D \quad \text{for some } D > -1, \tag{B.18}$$

$$\overline{\lim_{t\to\infty}}\, t^{-(D+1)^{-1}} \left|\{s \in [t_0, t] \mid N_{f_1}(s) > N_g(s) - 1\}\right| = 0, \tag{B.19}$$

where $|\cdot|$ abbreviates Lebesgue measure. Indeed, denoting by

$$\gamma(t) = |\{s \in [t_0, t] \mid N_{f_1}(s) > N_g(s) - 1\}|$$

the Lebesgue measure of the set in (B.19), the method of proof in Theorem B.4, together with (B.18) and (B.19), imply

$$\begin{aligned}
\ln\left(\left|\frac{f_1(ix)}{g(ix)}\right|\right) &= \int_0^\infty \frac{x^2}{t^3+tx^2}\,[N_{f_1}(t) - N_g(t)]\,dt \\
&\leq C\int_1^{t_0} \frac{x^2}{t^3+tx^2}\,dt + \int_{t_0}^\infty \frac{x^2}{t^3+tx^2}\,[N_{f_1}(t) - N_g(t)]\,dt \\
&= C\int_1^{t_0} \frac{x^2}{t^3+tx^2}\,dt + \int_{\{t\in[t_0,\infty) \mid N_{f_1}(t)\leq N_g(t)-1\}} \frac{x^2}{t^3+tx^2}\,[N_{f_1}(t) - N_g(t)]\,dt \\
&\quad + \int_{\{t\in[t_0,\infty) \mid N_{f_1}(t)> N_g(t)-1\}} \frac{x^2}{t^3+tx^2}\,[N_{f_1}(t) - N_g(t)]\,dt \\
&\leq C\int_1^{t_0} \frac{x^2}{t^3+tx^2}\,dt - \int_{t_0}^\infty \frac{x^2}{t^3+tx^2}\,dt \\
&\quad + (D+1)\int_{\{t\in[t_0,\infty) \mid N_{f_1}(t)> N_g(t)-1\}} \frac{x^2}{t^3+tx^2}\,dt \\
&\leq (C+1)\ln(t_0) - \tfrac{1}{2}\ln(1+x^2) \\
&\quad + (D+1)\int_x^\infty \frac{x^2}{t^3+tx^2}\,dt + (D+1)\int_{\{t\in[1,x] \mid N_{f_1}(t)> N_g(t)-1\}} \frac{x^2}{t^3+tx^2}\,dt \\
&\leq (C+1)\ln(t_0) - \tfrac{1}{2}\ln(1+x^2) + \tfrac{1}{2}(D+1)\ln(2) \\
&\quad + (D+1)\int_1^{\gamma(x)+1} \frac{x^2}{t^3+tx^2}\,dt \\
&\leq (C+1)\ln(t_0) - \tfrac{1}{2}\ln(1+x^2) + \tfrac{1}{2}(D+1)\ln(2) + (D+1)\int_1^{\gamma(x)+1}\frac{dt}{t} \\
&\leq -\tfrac{1}{2}\ln(1+x^2) + \tfrac{1}{2}(D+1)[\ln(2) + \ln((1+\gamma(x))^2)] + (C+1)\ln(t_0).
\end{aligned} \tag{B.20}$$

In particular, (B.19) is precisely the result needed in (B.20) to ensure that the limit $|x| \to \infty$ of $|\frac{f_1(ix)}{g(ix)}|$ is zero. In (B.20) we used the obvious inequality $\frac{x^2}{[t^3+tx^2]} < (\frac{1}{t})$ for $t > 0$ and the fact that

$$\int_\Omega f(t)\,dt \leq \int_1^{|\Omega|+1} f(t)\,dt$$

whenever $\Omega \subseteq [1,\infty)$ has finite Lebesgue measure, $|\Omega| < \infty$, and f is monotone decreasing on $[1,\infty)$.

An interesting case is $D = 0$ in (B.18)–(B.20).

References

1. F.V. Atkinson, *On the location of the Weyl Circles,* Proc. Roy. Soc. Edinburgh **88A** (1981), 345–356.

2. G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe,* Acta Math. **78** (1946), 1–96.

3. G. Borg, *Uniqueness theorems in the spectral theory of* $y'' + (\lambda - q(x))y = 0$, Proc. 11th Scandinavian Congress of Mathematicians, Johan Grundt Tanums Forlag, Oslo, 1952, 276–287.

4. E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Krieger, Malabar, 1985.

5. A.A. Danielyan and B.M. Levitan, *On the asymptotic behavior of the Weyl-Titchmarsh m-function,* Math. USSR Izv. **36** (1991), 487–496.

6. W.N. Everitt, *On a property of the m-coefficient of a second-order linear differential equation,* J. London Math. Soc. **4** (1972), 443–457.

7. F. Gesztesy, H. Holden, B. Simon, and Z.. Zhao, *Higher order trace relations for Schrö-dinger operators,* Revs. Math. Phys. **7** (1995), 893–922.

8. F. Gesztesy and B. Simon, *Uniqueness theorems in inverse spectral theory for one-dimensional Schrödinger operators,* Trans. Amer. Math. Soc. **348** (1996), 349–373.

9. F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential, I. The case of an a.c. component in the spectrum*, Helv. Phys. Acta, to appear.

10. F. Gesztesy and B. Simon, *m-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices*, preprint, 1996.

11. O.H. Hald, *Inverse eigenvalue problem for the mantle,* Geophys. J. R. Astr. Soc. **62** (1980), 41–48.

12. H. Hochstadt, *The inverse Sturm-Liouville problem,* Commun. Pure Appl. Math. **26** (1973), 715–729.

13. H. Hochstadt, *On the construction of a Jacobi matrix from mixed given data,* Lin. Algebra Appl. **28** (1979), 113–115.

14. H. Hochstadt and B. Lieberman, *An inverse Sturm-Liouville problem with mixed given data,* SIAM J. Appl. Math. **34** (1978), 676–680.

15. K. Iwasaki, *Inverse problem for Sturm-Liouville and Hill equations,* Ann. Mat. Pura Appl. Ser. 4 **149** (1987), 185–206.

16. S. Jayawardena, *The reconstruction of symmetric, anti-symmetric and partially known potentials from spectral data*, preprint.

17. A.B. Khaled, *Problème inverse de Sturm-Liouville associé à un opérateur différentiel singulier,* C. R. Acad. Sc. Paris **299** (1984), 221–224.

18. M.G. Krein, *On the solution of the inverse Sturm-Liouville problem*, Dokl. Akad. Nauk SSSR **76** (1951), 21–24 (Russian).

19. B. Ja. Levin, *Distribution of Zeros of Entire Functions* (rev. ed.), Amer. Math. Soc., Providence, RI, 1980.

20. N. Levinson, *The inverse Sturm-Liouville problem,* Mat. Tidskr. **B** (1949), 25–30.

21. B. Levitan, *On the determination of a Sturm-Liouville equation by two spectra,* Amer. Math. Soc. Transl. **68** (1968), 1–20.

22. B. Levitan, *Inverse Sturm-Liouville Problems*, VNU Science Press, Utrecht, 1987.

23. B.M. Levitan and M.G. Gasymov, *Determination of a differential equation by two of its spectra*, Russ. Math. Surv. **19:2** (1964), 1–63.

24. B.M. Levitan and I.S. Sargsjan, *Sturm-Liouville and Dirac Operators*, Kluwer, Dordrecht, 1991.

25. M.M. Malamud, *Similarity of Volterra operators and related questions of the theory of differential equations of fractional order,* Trans. Moscow Math. Soc. **55** (1994), 57–122.

26. V.A. Marchenko, *Some questions in the theory of one-dimensional linear differential operators of the second order, I*, Trudy Moskov. Mat. Obšč. **1** (1952), 327–420 (Russian); English transl. in Amer. Math. Soc. Transl. (2) **101** (1973), 1–104.

27. V. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

28. A.I. Markushevich, *Theory of Functions of a Complex Variable* (2nd ed.), Chelsea, New York, 1985.

29. J. Pöschel and E. Trubowitz, *Inverse Scattering Theory*, Academic Press, Boston, 1987.

32. M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV, Analysis of Operators*, Academic Press, New York, 1978.

31. R. del Rio Castillo, *On boundary conditions of an inverse Sturm-Liouville problem,* SIAM J. Appl. Math **50** (1990), 1745–1751.

32. W. Rundell and P.E. Sacks, *Reconstruction techniques for classical inverse Sturm-Liouville problems,* Math. Comp. **58** (1992), 161–183.

33. T. Suzuki, *Deformation formulas and their applications to spectral and evolutional inverse problems,* Lecture Notes in Num. Appl. Anal. **5** (1982), 289–311.

34. T. Suzuki, *Gel'fand-Levitan's theory, deformation formulas and inverse problems,* J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **32** (1985), 223–271.

35. T. Suzuki, *Inverse problems for heat equations on compact intervals and on circles, I,* J. Math. Soc. Japan **38** (1986), 39–65.

36. E.C. Titchmarsh, *Eigenfunction Expansions, Part I* (2nd ed.), Clarendon Press, Oxford, 1962.