Splitting in large dimension and infrared estimates II - Moment inequalities

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Abstract

This is the continuation of notes written for the NATO-ASI conference in Il Ciocco (Sept. 96) consisting in the analysis of the links between estimating the splitting between the two first eigenvalues for the Schrödinger operator H and the proof of infrared estimates for quantities attached to Gaussian type measures. These notes were mainly reporting on the "old" contributions of Dyson, Fröhlich, Glimm, Jaffe, Lieb, Simon, Spencer (in the seventies) in connection with more recent contributions of Pastur, Khoruzhenko, Barbulyak, Kondratev which treat in general more sophisticated models. Here we concentrate on the simplest model related to field theory and extend the results of Barbulyak-Kondratev by mixing ideas coming from Pastur-Khozurenko related to the use of Bogolyubov's inequality with classical inequalities due to Ginibre, Lebowitz, Sokal.... or in the case when the temperature T is zero by applying rather elementary estimates on Schrödinger operators, in order to find lower bounds for second order moments attached to the measure $\phi \mapsto \text{Tr } \phi \exp -\beta H / \text{Tr } \exp -\beta H$ with $\beta = \frac{1}{T}$. This question was "left to the reader" in lectures given by J. Fröhlich in 1976 [Fr], but we think that it is worthwhile to do this "home work" carefully.

1 Introduction

Our aim is to understand in the large dimension limit $L \to +\infty$ the splitting between the two lowest eigenvalues of the following Schrödinger operator

$$H_\Lambda^{per} := -h^2\Delta + \sum_{j\in\Lambda} v(x_j) + \frac{\mathcal{J}}{2}\sum_{j\sim k}|x_j - x_k|^2 \tag{1.1}$$

where

- Λ is a square periodic lattice $\Lambda := \Lambda(L) = (\mathbb{Z}/L\mathbb{Z})^d$ identified (up to translation) with $\{0, \cdots, L-1\}^d$ in $\mathbb{Z}^d$,
- $j \sim k$ means j nearest neighbor of k in Λ (seen as a periodic lattice),
- v is a one particle even potential on $\mathbb{R}^n$ defining as typical cases either a single well or a symmetric double well,
- $\mathcal{J}$ is the intensity of the interaction.

As well known, the first eigenvalue of the Schrödinger operator is simple, so the splitting is always strictly positive. Its behavior with respect to L as $L \to +\infty$ depends actually heavily on the nature of v and on the size on $\mathcal{J}$ which is always assumed positive.
The parameter $\mathcal{J}$ satisfies the condition

$$\mathcal{J} > 0\ . \tag{1.2}$$

The case $\mathcal{J} = 0$, which could be considered for comparison, corresponds to the case without interaction and the analysis in this case is immediately reduced to the semi-classical analysis of the so-called one-particle Hamiltonian on $L^2(\mathbb{R}^n)$

$$H_0 := -h^2\Delta_y + v(y)\ . \tag{1.3}$$

This model can also be written as

$$H_\Lambda^{per} = -h^2\Delta + \sum_{\ell\in\Lambda}\tilde{v}(x_\ell) - \mathcal{J}\sum_{i\sim j} x_i\cdot x_j\ , \tag{1.4}$$

with

$$\tilde{v}(y) = v(y) + \mathcal{J}\,d\,y^2\ ,\ \forall y \in \mathbb{R}^n\ . \tag{1.5}$$

This sometimes leads (See [BaKo1]) to an other notion of one-particle operator

$$\tilde{H}_0 := -h^2 \Delta_y + \tilde{v}(y) \, . \tag{1.6}$$

This article will in particularly discuss about the relevance of these various one-particle Hamiltonians in the problem of the phase transition.
For a one dimensional lattice (d=1), this model was analyzed in [He2], by extension of the study of another model analyzed in collaboration with J. Sjöstrand [HeSj3]. It was proved that

$$\lambda_2(L;h) - \lambda_1(L;h) \leq C \exp -(\frac{\epsilon L}{h}) \, , \tag{1.7}$$

for C, ϵ independent of L and for $h < h_0$, h_0 independent of L.
But we met an additional and unfortunate condition

$$L \leq C h^{-N_0} \, , \tag{1.8}$$

without to know if it is a technical or deep condition.
The main object of the NATO-ASI notes was to analyze another approach based on the so called infrared estimates which was developed by many authors in the late 70's Fröhlich-Simon-Spencer [FSS], Dyson-Lieb-Simon [DyLiSi], and Glimm-Jaffe [GlJa] and more recently by Pastur-Khozurenko [PaKh] and Barbulyak-Kondratev [BaKo1]. In particular how this infrared approach gives also information about the splitting for our initial questions. It gave actually a rather complete answer for all the lattices of dimension strictly greater than 1. In this case, the result does not seem to be related to tunneling properties. In the case when the lattice is of dimension 1, we obtained only a partial result under the condition

$$|\Lambda| \leq \exp \frac{\varepsilon}{h} \, , \tag{1.9}$$

for some suitable $\varepsilon > 0$. All these results were obtained relatively easy extensions of these contributions. Because, particularly in the paper by [BaKo1], semi-classical analysis is involved, we tried to be more precise that in the original paper, using our more precise knowledge of the tunneling [HeSj1]. On the other hand we got progressively the impression that some assumptions were actually irrelevant in the problem and our main goal is to present here rather optimal results, although limited by the bounds of this approach

(See [Fr] for discussions).
Let us now recall the main results obtained through the infrared estimates. Following [BaKo1], we now consider the operator $\exp -\beta H_\Lambda^{per}$ and the associated

$$\langle|\frac{1}{|\Lambda|}\sum_{k\in\Lambda} x_k|^2\rangle_{\beta,\Lambda} := \frac{\mathrm{Tr}\ (|\frac{1}{|\Lambda|}\sum_{k\in\Lambda} x_k|^2 \exp(-\beta H_\Lambda^{per}))}{\mathrm{Tr}\ (\exp(-\beta H_\Lambda^{per}))}\ . \tag{1.10}$$

Physically the strictly positive parameter β corresponds to the inverse of the temperature.
We now introduce the so called parameter of long-range order

$$P(\beta) = \lim_{|\Lambda|\to\infty} P_\Lambda(\beta)\ , \tag{1.11}$$

with

$$P_\Lambda(\beta) = \left\langle |\frac{1}{|\Lambda|}\sum_{k\in\Lambda} x_k|^2 \right\rangle_{\beta,\Lambda}\ . \tag{1.12}$$

The presence of the long range order, i.e. the strict positivity of $P(\beta)$, will serve as a test for phase transition (cf [DyLiSi] and [PaKh]).
When the limit of $\langle\cdot\rangle_{\beta,\Lambda}$ as $|\Lambda| \to +\infty$ exists, we shall denote it by $\langle\cdots\rangle_\beta$. Let

$$E(p) = \sum_{i=1}^{d} (1 - \cos p^{(i)})\ , \tag{1.13}$$

where $\Lambda^\star$ is the dual lattice

$$\Lambda^\star = \{p = (p^{(1)}, \cdots, p^{(d)})\ |\ p^{(i)} = 2\pi k^{(i)}/m\ ,\ 0 \le k^{(i)} \le m-1\ ;\ 1 \le i \le d\}\ . \tag{1.14}$$

The method of infrared estimates gives [1] the following lower bound

Theorem 1.1 . *If $d \ge 3$ and under the following assumption on v,*

$$v(x) \ge ax^2 + b\ ,\quad \text{with } a > 0\ , \tag{1.15}$$

we have

$$P(\beta) \ge \langle x_k^2\rangle_\beta - \frac{n}{2}\cdot\frac{1}{(2\pi)^d}\int_{]-\pi,\pi[^d} (\frac{h^2}{\mathcal{J}E(p)})^{\frac{1}{2}} \coth\left[\left(h^2\beta^2\mathcal{J}E(p)\right)^{\frac{1}{2}}\right]\ dp\ . \tag{1.16}$$

[1] Cf [DyLiSi], (52), p. 368 (cf also Theorem 3.2 and Theorem 5.1 in this article).

This lower bound is deduced by a limiting argument (thermodynamic limit) from the actually more useful inequality (for our questions relative to the splitting), which is relative to the finite lattice case, and which is given by the following theorem essentially due to [DyLiSi].

Theorem 1.2 . *Under the assumption (1.15) on v, we have, for any $k \in \Lambda$, the following universal estimates*

$$P_\Lambda(\beta) \geq \langle x_k^2 \rangle_{\beta,\Lambda} - \frac{n}{2} \cdot \frac{1}{|\Lambda|} \sum_{p \in \Lambda^\star \setminus \{0\}} (\frac{h^2}{\mathcal{J}E(p)})^{\frac{1}{2}} \coth\left[\left(\beta^2 h^2 \mathcal{J}E(p)\right)^{\frac{1}{2}}\right] . \quad (1.17)$$

Pastur and Khozurenko [PaKh] and more recently Barbulyak and Kondrat'ev [BaKo1] try (in particular) to analyze how one can control the quantity $\langle x_k^2 \rangle_{\beta,\Lambda}$ from below in order to deduce existence for long range order. The second authors use semi-classical analysis for the one particle Hamiltonian operator $\tilde{H}_0$ and standard inequalities (GKS) reducing the problem to this case. The first authors use for a rather specific model the Bogolyoubov inequality. Their result is no more semi-classical and explicit conditions for long range order are given.
Extending preliminary results announced in [He5], we push these results in various directions:

- very weak assumptions in the limit case $\beta = +\infty$, without using (GKS),
- stronger but still rather weak assumptions for the case β large, but using standard moment inequalities.

We get for example the following theorem

Theorem 1.3 . *Let us assume $d \geq 2$. If the potential v satisfies (1.15),*

$$v(x) = v(-x) \; , \quad (1.18)$$

$$v(0) > \inf v \; , \quad (1.19)$$

then there exists h_0 such that we have, for $0 < h < h_0$,

$$\lim_{|\Lambda| \to +\infty} (\lambda_2^\Lambda(h) - \lambda_1^\Lambda(h)) = 0 \; , \quad (1.20)$$

where λ_1^Λ and λ_2^Λ are the two lowest eigenvalues of H_Λ^{per}.

The direct application of the result by Barbulyak-Kondrat'ev would have given, in the limit $\beta \to +\infty$, the conditions $d \geq 3$, $n = 1$ and the conditions on $\tilde{v}$ that

$$\tilde{v}(y) \geq ay^2 + b \text{ , with , } a > \mathcal{J}d \text{ ,} \tag{1.21}$$

and that:

$$\begin{array}{l}\text{For some } q_0 > 0 \text{ the function } \tilde{v} \text{ attains its strong global minimum} \\ \quad \text{at the points } \pm q_0 \text{ .}\end{array} \tag{1.22}$$

On the other hand the conditions given by the authors are much stronger (but lead to a stronger result (treatment of the case β large) which will be analyzed later).

In any case, what seems still open is the complete analysis of the tunneling with control of the dimension and we recall from [Fr] the important point that the Peierls approch will give additional information particularly for $d = 1$ which can not be analyzed by this method. In reference to these enlighting lectures, what we are doing here is essentially what was left to the reader in these notes p. 237. We recently received a preprint of Albeverio, Kondratiev and Rebenko [AlKoRe] based on the Peierls argument.

This article is organized as follows.

- In Section 2, we recall the relations between the splitting and some notions related to the absence of long range order. We recall also the general strategy of the infrared estimates.
- In Section 3, we recall a family of estimates extending in various directions the FKG estimates: GKS, FKG, GHS, Lebowitz and Gaussian domination estimates.
  These estimates will play an important role in the treatment of the case of β large but are not used in the treatment of $\beta = +\infty$.
- Section 4 is devoted to the analysis of lower bounds for the second order moments in the case $\beta = +\infty$. This is a combination of standard semi-classical analysis with control with respect to the dimension (harmonic approximation) in combination with a priori estimates. Although this would be very interesting to analyze, no study of the tunneling is involved and only a rather weak localization of the first eigenfunction.

- In Section 5, we extend the results proposed by Barbulyak-Kondratev by following the same strategy but instead of reducing, through Ginibre type inequalities GKS, to a one particle problem, we take advantage of a reduction to a p-particle problem.
- Section 6 is mainly reporting on the partially alternative strategy followed by Pastur-Khozurenko and based on the use of the Bogolyubov inequality. This approach is not semi-classical and give more explicit estimates for specific models.
- Section 7 proposes extensions of the model treated by Pastur-Khozurenko by implementation of the results recalled in Section 3. This leads to less explicit results but permits for example to analyze the dependence on $\mathcal{J}$ of the transition of phase.
- The last section presents some open questions.

2 Connection with the Splitting.

We recall here rather standard relations connecting the splitting between the two first eigenvalues of the Schrödinger operator with various quantities introduced in the preceding section. They are quite general and true for any Schrödinger operator

$$H_V := -h^2\Delta + V \;,$$

where V is C^∞ semibounded, tending to ∞ as $|x| \to +\infty$ and satisfying

$$V(-x) = V(x) \;.$$

We have the following lemma

Lemma 2.1

$$\int |\frac{1}{|\Lambda|}\sum_i x_i|^2 \phi_1^\Lambda(x^\Lambda)^2 dx^\Lambda \leq \frac{n}{|\Lambda|}(\lambda_2^\Lambda - \lambda_1^\Lambda)^{-1}. \tag{2.1}$$

This is indeed just the minimax principle[2] saying that

$$(\lambda_2^\Lambda - \lambda_1^\Lambda) = \inf_{\substack{f \in C_0^\infty \\ \int f(x^\Lambda)\phi_1^\Lambda(x^\Lambda)^2 dx^\Lambda = 0}} \frac{\langle |\nabla f|^2\rangle}{\langle |f|^2\rangle}\;, \tag{2.2}$$

[2] One can extend C_0^∞ to slowly increasing C^∞ functions in our case.

where $\langle\cdot\rangle$ is computed for the measure $\phi_1^\Lambda(x^\Lambda)^2 dx^\Lambda$:

$$\langle\cdot\rangle = \langle\cdot\rangle_{\infty,\Lambda} \ , \tag{2.3}$$

(remembering that it corresponds to the case $\beta = +\infty$). Here dx^Λ is the Lebesgue measure on $(I\!R^n)^{|\Lambda|}$.
This is then applied to the function

$$f^{(j)}(x^\Lambda) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} x_i^{(j)} \ , \tag{2.4}$$

for $j = 1, \cdots, n$. The condition of symmetry on V (which is a consequence of the symmetry of v for our specific example) gives the orthogonality of $f^{(j)}\Phi_1^\Lambda$ to Φ_1^Λ.
In the case when $x_i \in I\!R^n$ with $n > 1$, we have to introduce the functions $f^{(j)}(x^\Lambda) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} x_i^{(j)}$ (for $j = 1, \cdots, n$).
This means that if we can prove by other means the property that

$$[\int |\frac{1}{|\Lambda|}\sum_i x_i|^2 \phi_1^\Lambda(x^\Lambda)^2 dx^\Lambda] \geq \rho > 0 \ , \tag{2.5}$$

with ρ independent of Λ, then we get that

$$\lim_{|\Lambda|\to+\infty} |\lambda_2^\Lambda - \lambda_1^\Lambda| = 0 \ . \tag{2.6}$$

A lower bound for $P_\Lambda(+\infty)$ gives an upper bound for the splitting but Theorem 1.2 gives the starting point for finding this lower bound and this will then permit to prove Theorem 1.3. We shall mainly follow the proof of Barbulyak-Kondrat'ev but with a small change.
In [BaKo1], the authors take indeed first the limit $|\Lambda| \to +\infty$ and then the limit $\beta \to +\infty$. We shall proceed for the application to the splitting in the inverse order.

Proof of Theorem 1.3, modulo a lower bound of $\langle x_k^2\rangle_{\beta,\Lambda}$:

We assume in this part that we have proved the existence of a constant $\delta(\beta, h) > 0$ such that for any Λ (with sometimes a restriction $L \geq L_0$)

$$\langle x_k^2\rangle_{\beta,\Lambda} \geq \delta(\beta, h) \tag{2.7}$$

We shall sometimes get in the next sections a control of $\delta(\beta, h)$, with β and h, of the type:

- In the case $\beta = +\infty$,
  (As1) there exists $\delta_0 > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$

$$\delta(\infty, h) \geq \delta_0 \ .$$

- In the case β large,

  - (As2) there exists $\delta_0 > 0$, $h_0 > 0$ such that for $h \in]0, h_0]$ there exists $\beta_0(h)$ such that, for all $\beta \geq \beta_0(h)$,

$$\delta(\beta, h) \geq \delta_0 \ ,$$

  or

  - (As3) there exists $\delta_0 > 0$, $h_0 > 0$ and β_0 such that for $h \in]0, h_0]$ and $\beta \geq \beta_0$,

$$\delta(\beta, h) \geq \delta_0 \ .$$

What is common to all these estimates is that they are independent of Λ with possibly the restriction L large enough. This last restriction is usually not important because we shall always take the limit $L \to +\infty$ at the end.
We start from (1.17)

$$P_\Lambda(\beta) \geq \langle x_k^2 \rangle_{\beta,\Lambda} - \frac{n\,h}{2} \cdot \frac{1}{|\Lambda|} \sum_{p \in \Lambda^\star \setminus \{0\}} (\frac{1}{\mathcal{J} E(p)})^{\frac{1}{2}} \coth \left[(h^2 \beta^2 \mathcal{J} E(p))^{\frac{1}{2}} \right] \ . \tag{2.8}$$

This leads, under assumption (2.7) to

$$P(\beta) \geq \delta(\beta, h) - \frac{nh}{2\mathcal{J}^{\frac{1}{2}}} \cdot \frac{1}{(2\pi)^d} \int_{]-\pi,\pi[^d} E(p)^{-\frac{1}{2}} \coth \left[h^2 \beta^2 \mathcal{J} E(p))^{\frac{1}{2}} \right] dp \ . \tag{2.9}$$

The discussion of the case β large is then related with the analysis of $\delta(\beta, h)$ for β large and its comparison with the quantity $\frac{nh}{2\mathcal{J}^{\frac{1}{2}}} \cdot \frac{1}{(2\pi)^d} \int_{]-\pi,\pi[^d} E(p)^{-\frac{1}{2}} \coth \left[h^2 \beta^2 \mathcal{J} E(p))^{\frac{1}{2}} \right] dp$. The condition $d \geq 3$ appears naturally in this case to keep the integral appearing in the r.h.s. of (2.9) convergent.
Taking the limit $\beta \to +\infty$ in (2.8), we first obtain

$$\lim_{\beta \to +\infty} P_\Lambda(\beta) \geq \langle x_k^2 \rangle_{+\infty,\Lambda} - \frac{n\,h}{2} \cdot \frac{1}{|\Lambda|} \sum_{p \in \Lambda^\star \setminus \{0\}} (\frac{1}{\mathcal{J} E(p)})^{\frac{1}{2}} \ . \tag{2.10}$$

Now, as $|\Lambda| \to +\infty$ and if $d \geq 2$, the right hand side is estimated from below, under the assumption (As1) by

$$P(+\infty) \geq \delta_0 - \frac{nh}{2}\mathcal{J}^{-\frac{1}{2}} I_d \;, \tag{2.11}$$

where we recall that

$$I_d = \frac{1}{(2\pi)^d} \int_{]-\pi,\pi[^d} E(p)^{-\frac{1}{2}} dp \;. \tag{2.12}$$

We now observe that I_d is finite for $d \geq 2$. Then $P(\infty)$ becomes strictly positive when $h \leq h_0$ and

$$\delta_0 - \frac{nh}{2}\mathcal{J}^{-\frac{1}{2}} I_d > 0 \;. \tag{2.13}$$

In some cases h_0 will be explicitly computed (e.g in the cases considered by [PaKh]) and we can in this case discuss the influence of $\mathcal{J}$ for fixed h, in particular (2.13) is satisfied for $\mathcal{J}$ large enough. In other cases (e.g. in the cases considered by [BaKo1]) h_0 is not explicited by the (semi-classical) analysis and we can only say that (2.13) is satisfied for h small enough.
In particular we obtain the proof of Theorem 1.3 modulo the verification of (As1).

The case: $d = 1$.

Another interesting point is to analyze the case $d = 1$. The sum

$$\frac{1}{|\Lambda|} \sum_{p \in \Lambda^\star \setminus \{0\}} \left(\frac{1}{\mathcal{J} E(p)}\right)^{\frac{1}{2}}$$

is divergent as $|\Lambda| \to +\infty$, but this divergence is controlled in $\ln(|\Lambda|)$. This leads to the statement that the splitting remains in $\mathcal{O}(\frac{1}{|\Lambda|})$ for $h \leq h_0$ and $|\Lambda| \leq \exp \frac{T}{h}$, where T is explicitely computable.
This condition is much weaker than the condition (1.8) given in [HeSj3].
As $h \to 0$, the estimate still work, under assumption (As1), if

$$\ln |\Lambda| \leq C \frac{\delta_0}{h} [2\mathcal{J}]^{\frac{1}{2}} \;, \tag{2.14}$$

where C is a universal constant. We note also that the phenomenon is effectively related to $\mathcal{J} \neq 0$.

To summarize, we prove in this case that

Theorem 2.2 . *Let $d = 1$ and v satisfying (1.15), (1.18) and (1.19), then there exists a constant C such that*

$$\lambda_2^\Lambda - \lambda_1^\Lambda \leq \frac{C}{|\Lambda|} \frac{1}{(1 - Ch \log |\Lambda|)_+} \,. \tag{2.15}$$

As mentioned in the introduction and as was communicated to us by J. Fröhlich, this condition on $|\Lambda|$ is due to the method (when $n = 1$). An approach using the Peierls trick could be more effective in this case [Fr].

3 Ginibre or Lebowitz type inequalities

Ginibre type inequalities[3] make it possible to estimate $\langle x_k^2 \rangle_{\beta,\Lambda}$ from below by the average $\langle x_k^2 \rangle_{\beta,\hat{H}}$ taken over the measure corresponding to the "formal" decoupled Hamiltonian

$$\hat{H} = -h^2 \sum_{k \in \mathbb{Z}^d} \frac{\partial^2}{\partial x_k^2} + \sum_{k \in \mathbb{Z}^d} \tilde{v}(x_k) \equiv \sum_{k \in \mathbb{Z}^d} \tilde{H}_k \,, \tag{3.1}$$

with separate variables.
Less formally, we introduce

$$\hat{H}^\Lambda = \sum_{k \in \Lambda} \tilde{H}_k \tag{3.2}$$

and we get ([BaKo1]) the

Proposition 3.1 . *Under the assumptions (1.15), (1.18) and (1.21)-(1.22) on the potential $\tilde{v}$ and $\mathcal{J}$, the following inequality is true, for any $k \in \Lambda$,*

$$\langle x_k^2 \rangle_{\beta,\Lambda} \geq \langle x_k^2 \rangle_{\beta,\Lambda,\hat{H}^\Lambda} \,. \tag{3.3}$$

[3] A good reference is [Si1] or [Si2]. These inequalities correspond to extensions of the so called FKG inequalities [FKG]. A more recent review can be found in [La].

Let us describe briefly the version of the Ginibre inequalities which is needed here. This is related to the control of the sign of the correlations attached to the more general measure:

$$Z^{-1} \exp \sum_{ij} J_{ij} x_i \, x_j \prod_{j=1}^{m} d\nu_j(x_j) \ . \tag{3.4}$$

We look at the partial derivative of $\langle (x_k)^2 \rangle$ with respect to J_{ij}. Here $\langle \cdot \rangle$ is taken with respect to the measure (3.4).
This gives

$$\partial_{J_{ij}} \langle (x_k)^2 \rangle = \langle x_i \, x_j \, (x_k)^2 \rangle - \langle (x_k)^2 \rangle \cdot \langle x_i \, x_j \, \rangle \ .$$

The right hand side appears as the pair correlation of the two functions $f = x_i \, x_j$ and $g = x_k^2$.
We shall deduce from the Ginibre's inequalities that this expression is positive.

$$\partial_{J_{ij}} \langle (x_k)^2 \rangle \geq 0 \ . \tag{3.5}$$

Let us briefly recall some elements of this theory (cf [Si2], p.119-124), for a nice exposition). We recall the

Theorem 3.2 . *Let $\mathcal{F}_1$ be the set of functions on $I\!R$ which are nonnegative and monotone on $[0, +\infty)$ and either even or odd. Let $\mathcal{F}_n$ be the functions on $I\!R^m$ of the form $f_1(x_1) \cdots f_n(x_n)$ with $f_i \in \mathcal{F}_1$. Let $d\mu$ be a probability measure of the form (3.4) where $J_{ij} \geq 0$ and each $d\nu_j$ has the form $\exp(f_j(x)) d\lambda_j(x)$ with $f_j \in \mathcal{F}_1$ and $d\lambda_j$ even. Then*

$$\begin{array}{ll} (GKS1) & \langle f \rangle \geq 0 \\ (GKS2) & \langle f \, g \rangle \geq \langle f \rangle \cdot \langle g \rangle \ , \end{array} \tag{3.6}$$

for all $f, g \in \mathcal{F}_{m+1}$.

The reader can find a proof of the theorem in [Si2].

The measure in consideration is attached to the restriction to the diagonal of the distribution kernel of $\exp -\beta H_\Lambda^{per}$, where $\mathcal{J}$ becomes a parameter in $[0, \mathcal{J}_0]$ and has not exactly the structure which is introduced above. But using the Trotter product formula which describes the kernel of $\exp -\beta H_\Lambda^{per}$ as the limit in a weak sense of $(\exp -\frac{\beta V(x)}{N} \exp \frac{\beta \Delta}{N})^N$, we observe that the

measure is a limit of measures satisfying the assumptions of the theorem for $\mathcal{J} \geq 0$.
We have just to control (Cf [Si2] p. 120 and p.122) the limit procedure in order to get the result.

Application:
The Hamiltonian $\hat{H}^\Lambda$ describes a system of non interacting particles. Consequently, we get immediately, for $k \in \Lambda$,

$$\langle x_k^2 \rangle_{\beta,\Lambda;\hat{H}} = \langle x_k^2 \rangle_{\beta,\tilde{H}_k} = \langle x_0^2 \rangle_{\beta,\tilde{H}_0} , \tag{3.7}$$

for all $k \in \mathbb{Z}^d$.
Here we recall that $\tilde{H}_k$ is the one-particle Hamiltonian at $k \in \mathbb{Z}^d$.
We consequently have obtained

Proposition 3.3 . *Under the previous assumptions on $\tilde{v}$ and $\mathcal{J}$ (in particular $n = 1$), we have*

$$\langle x_k^2 \rangle_{\beta,\Lambda} \geq \langle x_0^2 \rangle_{\beta,H_0} . \tag{3.8}$$

In particular we obtain the condition (As2) mentioned in the previous section by observing that it is sufficient for treating the case β large enough, to analyze the case $\beta = +\infty$ for $\langle x_0^2 \rangle_{\beta,H_0}$ and to use weak estimates for one-particle Hamiltonians. This will be generalized in one of the next sections.

Remark 3.4 . *When $n > 1$, we can no more apply this technique directly. The case when we have a rotational symmetry can probably be treated by taking polar coordinates. The results in this case is mentioned in [GlJa]. Pastur and Khoruzenko proceed differently in the case of the model $v(x) = (1 - |x|^2)^2$. We shall discuss this point later.*

We now present two other inequalities which may be important in order to compare moments relative to various hamiltonians. We discover these inequalities in the nice paper by A. Sokal [Sok] who refers to [EMN] and [EN]. The name of Lebowitz is in reference with the particular case of spin systems.

Theorem 3.5 *(GS and Lebowitz inequalities). Let $d\mu(x)$ a probability measure on $\mathbb{R}^m$ of the type*

$$d\mu(x) = Z^{-1} \exp \left[\frac{1}{2} \sum_{i,j} J_{ij} x_i x_j + \sum_i h_i x_i \right] \prod_{i=1}^m d\nu_i(x_i) , \tag{3.9}$$

with

$$J_{ij} = J_{ji} \geq 0 \ , \ h_i \geq 0 \ , \tag{3.10}$$

where $d\nu_i$ *is for each* i *a measure on* $I\!R$ *of the type*

$$d\nu_i(y) = \exp -\tilde{v}_i(y) dy \ , \tag{3.11}$$

where $\tilde{v}_i$ *is an even differentiable function such that*

$$\tilde{v}' \text{ is convex on }]0, +\infty[\ . \tag{3.12}$$

Then we have

$$(GS) \quad \langle x_i x_j x_k \rangle \leq \langle x_i x_j \rangle \langle x_k \rangle + + \langle x_i x_k \rangle \langle x_j \rangle + \langle x_j x_k \rangle \langle x_i \rangle - 2 \langle x_i \rangle \langle x_j \rangle \langle x_k \rangle \ , \tag{3.13}$$

and, when $h = (h_1, \cdots, h_m) = 0$,

$$(Leb) \quad \langle x_i x_j x_k x_\ell \rangle \leq \langle x_i x_j \rangle \langle x_k x_\ell \rangle + \langle x_i x_\ell \rangle \langle x_j x_k \rangle + \langle x_i x_k \rangle \langle x_j x_\ell \rangle \tag{3.14}$$

We shall also use the following extension mentioned by Sokal who refers to [BFS], [Ne], [Sy] and [Br].

Theorem 3.6 *(Gaussian domination inequalities). Under the assumptions of validity of Theorem 3.5, we have, with* $h = (h_1, \cdots, h_m) = 0$, *for any multindex* $\alpha = (\alpha_1, \cdots, \alpha_p)$ *with* p *even and* $\alpha_j \in \{1, \cdots, m\}$ *and with* $x^\alpha = \prod_{j=1}^p x_{\alpha_j}$,

$$\langle x^\alpha \rangle \leq \sum_{pairings} \prod \langle x_k x_\ell \rangle \tag{3.15}$$

where the sum is over all the pairs giving a decomposition of α.

4 Semiclassical analysis of the many particle problem, the case $\beta = +\infty$

We have seen in Section 3 that we can estimate from below, using the Ginibre's inequalities, the quantity $\langle x_j^2 \rangle_{\beta,\Lambda}$ (for $j \in \Lambda$), by the corresponding quantity $\langle y^2 \rangle_{\beta, H_{\tilde{v}}}$, attached to one particle potential $\tilde{v}$. We can then first analyze the limit case $\beta = +\infty$ by semiclassical analysis and proved that for

β large enough an analogous estimate from below remains true.
This was the approach developed by Barbulyak-Kondratev [BaKo1] that we further developed in [He5].
This result appears to be only semi-satisfactory because it does not seem so natural to impose that $\tilde{v}$ creates a double well and one is more likely waiting for the weaker assumption that v creates a double well. Of course this is not important when $\mathcal{J}$ is sufficiently small but it appears more important as $\mathcal{J}$ is large.
Our alternative idea is to treat the case $\beta = +\infty$ by analyzing directly the quantity $\langle x_j^2 \rangle_{\infty,\Lambda}$, without to reduce to a one-particle Hamiltonian. If successfull, the advantage is that we do not need any assumption related to the application of statistical mechanics like Ginibre' inequalities.
This is of course not sufficient for having the result for large β independently of Λ but is sufficient for the analysis of the splitting. The proposition is the following

Proposition 4.1 . *Let us assume that v is C^∞, positive, satisfying*

$$\lim \inf_{|x| \to +\infty} v(x) > 0, \tag{4.1}$$

such that there exists $q_0 > 0$, with

$$v(x) > 0 \;,\; \forall x \; s.t. \; |x| < q_0 \;, \tag{4.2}$$

and the property that there exists $y_0 \in I\!R^n$ such that $|y_0| = q_0$ and

$$v(y_0) = 0 \;. \tag{4.3}$$

Then, for any $\epsilon > 0$, there exists $h_0(\epsilon) > 0$ such that, for $h \in]0, h_0(\epsilon)]$, $\forall j \in \Lambda$, we have,

$$\langle x_j^2 \rangle_{\infty,\Lambda} \geq (q_0 - \epsilon)^2 \;. \tag{4.4}$$

This contains the double well case. The even character of v is not assumed in this proposition but note that this assumption is quite important in other parts of the problem in order for example to have $\langle x_j \rangle_{\infty,\Lambda} = 0$.
The argument is actually the conjonction of the existence of a minimum of v outside the origin and of the property that $v > 0$ in a neighborhood of 0.
Let us explain for simplification the proof in the case $n = 1$ and when v is

even.
Let us denote by Φ_1 the first normalized eigenfunction of H_Λ^{per}. We have to estimate from below $\int x_j^2 \Phi_1^2 \; dx$. It is trivial that, for any ϵ such that $q_0 > \epsilon > 0$, we have

$$\int x_j^2 \Phi_1^2 dx \geq (q_0 - \epsilon)^2 \int_{x \in I\!R^{|\Lambda|}\,;\, |x_j| \geq q_0 - \epsilon} \Phi_1^2 \; dx \; . \tag{4.5}$$

We rewrite this inequality in the form

$$\int x_j^2 \Phi_1^2 dx \geq (q_0 - \epsilon)^2 - (q_0 - \epsilon)^2 \int_{x \in I\!R^{|\Lambda|}\,;\, |x_j| \leq q_0 - \epsilon} \Phi_1^2 dx \; . \tag{4.6}$$

We are then reduced to the control of $\int_{x \in I\!R^{|\Lambda|}\,;\, |x_j| \leq q_0 - \epsilon} \Phi_1^2 dx$. For this estimate, we come back to the equation satisfied by Φ_1 and get, using also the invariance by translation of H_Λ^{per},

$$\int v(x_j) \Phi_1^2 dx \leq \lambda_1^\Lambda / |\Lambda| \; . \tag{4.7}$$

Let $m_\epsilon = \inf_{y \in [-q_0 + \epsilon, q_0 - \epsilon]} v(y)$. This quantity is strictly positive by assumption. We get in particular

$$\int_{|x_j| \leq q_0 - \epsilon} \Phi_1^2 dx \leq \frac{1}{m_\epsilon} \cdot \frac{\lambda_1^\Lambda}{|\Lambda|} \; . \tag{4.8}$$

We have to control $\frac{\lambda_1^\Lambda}{|\Lambda|}$ independently of $|\Lambda|$. This point is the consequence of the rather standard proof of the existence of the thermodynamic limit (see for example [He2]), but we need more, that is the control with dependence on h. We can add to v the potential $(y^2 - q_0^2)^2$ in order to get a potential with two non degenerate wells at $\pm q_0$. Using the minimax principle and the harmonic approximation, it is proved in [He2] the following lemma that we state here in a very weak form.

Lemma 4.2 . *Let us assume that v has non degenerate minima q_i with $v(q_i) = 0$. Then there exists a constant $C >$ and h_0 such that for $h \in]0, h_0]$ we have*

$$\frac{\lambda_1^\Lambda}{|\Lambda|} \leq Ch \; . \tag{4.9}$$

Proof of Theorem 1.3
As explained in Section 2, this is an immediate consequence of the infrared estimates once we have proven the existence of $\delta_0 > 0$ such that $\langle x_j^2\rangle_{\beta,\Lambda} \geq \delta_0$. We can first assume that $\inf v = 0$. Under the assumption that $v(0) > 0$ and (1.15), it is clear that there exists y_0 such that $v(y_0) = 0$ and such that the conditions of Proposition 4.1 are satisfied.

Remark 4.3 . *The proof gives also a "weak" control of the convergence of the type, for any $\epsilon > 0$, there exists $h_0(\epsilon)$ such that, for $h \in]0, h_0(\epsilon)]$*

$$\lambda_2^\Lambda(h) - \lambda_1^\Lambda(h) \leq n\frac{1+\epsilon}{q_0^2}\cdot\frac{1}{|\Lambda|}\,.$$

Remark 4.4 . *Let us emphasize that the assumptions are much weaker that the assumptions given by Barbulyak-Kondratev but the proof of these authors is still valid for β large. This is not the case of our proof. We shall see in the next section how we can analyze this case.*
Let us also emphasize that we have not used any result of Section 3.

Remark 4.5 . *Similarly to the case treated in Section 2, we can have, in the case $d = 1$, some estimate like (2.15) under the weaker assumptions of Theorem 1.3 (1.15), (1.18) and (1.19).*

5 The case β finite, large enough

We want to treat the case when β is finite by improving the ideas of Barbulyak-Kondratev. Our theorem will be the following

Theorem 5.1 . *Let us assume that v is a C^∞ potential on $I\!R$ with two non-degenerate minima at $\pm q_0$. Then, for all $\varepsilon > 0$, there exists $h(\varepsilon) > 0$ such that, for all $h \in]0, h(\varepsilon)]$, there exists $\beta_1(h)$ such that, for $|\Lambda| = L^d$, large enough, for any $k \in \Lambda$,*

$$\langle x_k^2\rangle_{\beta,\Lambda} \geq (1-\varepsilon)q_0^2\,, \tag{5.1}$$

This proof is modelled on the proof sketched by Barbulyak-Kondratev [BaKo1]. These authors reduce, through the Ginibre inequalities, to the study of a one particle problem attached to the potential $\tilde{v}$. The condition $n = 1$ appears only in the application of these inequalities.

For simplicity, we explain the case when $d = 1$. In the general case, this will correspond to the idea that Λ may be decomposed in smaller boxes.
Our unique idea is to replace the reduction to a one-particle problem by a reduction to a p-particle problem with p large enough but then fixed in the thermodynamic limit.
Let us divide L by p. The number p will be chosen large enough later. We start from

$$L = pL_0 + \ell \ .$$

This leads to a natural decomposition of $\{1, \cdots, L\}$ in $L_0 + 1$ intervals I_k $(k = 1, \cdots, L_0 + 1)$ with $I_k = \{(k-1)p, \cdots, kp - 1\}$ for $k = 1, \cdots, L_0$. Following [BaKo1], we destroy the terms $\mathcal{J} x_j x_k$ when j and k do not belong to the same interval. We recall that these interaction terms appear only for j and k nearest neighbors in $\mathbb{Z}/L\mathbb{Z}$.
Using the Ginibre inequalities, we obtain

$$\langle x_k^2 \rangle_{\beta,\Lambda} = \frac{1}{L} \sum_{k=1}^{L} \langle x_k^2 \rangle_{\beta,\Lambda} \geq (1 - \frac{\ell}{L}) \left(\frac{1}{p} \sum_{q=1}^{p} \langle x_q^2 \rangle_{\beta, H^{(p)}} \right) \ . \tag{5.2}$$

Here the last mean value is associated with the operator $\exp -\beta H^{(p)}$ and $H^{(p)}$ is the p-particle operator

$$H^{(p)} = \sum_{j=1}^{p} v(x_j) + \frac{\mathcal{J}}{2} \sum_{j=1}^{p-1} |x_j - x_{j+1}|^2 + \frac{\mathcal{J}}{2} (x_1^2 + x_p^2) \ . \tag{5.3}$$

When $p = 1$, we recover the one-particle hamiltonian $H^{(1)} = \tilde{H}_0$ introduced by [BaKo1]. So, what we have written above is just the extension of what was written in the case $p = 1$ by these authors.
In the case when $p = 1$, the inequalities obtained from the semi-classical analysis of $H^{(1)}$ were rather weak, particularly when $\mathcal{J}$ was large, and lead to (1.21)-(1.22). Our aim is to eliminate this condition by playing with p large enough.

We now implement the ideas appearing in Section 4 but for $H^{(p)}$. The new point is that we have destroyed the invariance by circular permutation of the model and we have to measure the effect of this modification.

The first problem is that we have now for the first eigenvalue $\lambda_1^{(p)}(h)$ of $H^{(p)}$ a weaker estimate than for the first eigenvalue of $H^{\Lambda}(h)$.

Lemma 5.2 . *There exists C and h_0 such that, for any $h \in]0, h_0]$ and any p,*

$$\lambda_1^{(p)}(h) \leq Cph + C \; . \tag{5.4}$$

This gives in particular that $\lambda_1^{(p)}(h)/p$ will be small for large p and small h. Following our previous analysis, we now observe that

$$\sum_{j=1}^{p} \int_{I\!R^p} v(x_j) \left(\varphi_1^{(p)}(x)\right)^2 \; dx^{(p)} \leq \lambda_1^{(p)}(h) \; . \tag{5.5}$$

The difference with the previous argument is that we can no more study independently the different terms of the sum in the left hand side of (4.6). This does not appear to be a big problem.
As for (4.6), we can write

$$\sum_j \int x_j^2 \left(\varphi_1^{(p)}(x)\right)^2 \; dx^{(p)} \geq p(q_0 - \epsilon)^2 - q_0^2 \sum_j \int_{|x_j| \leq q_0 - \epsilon} \left(\varphi_1^{(p)}(x)\right)^2 \; dx^{(p)} \; . \tag{5.6}$$

We then obtain

$$\begin{aligned} &\textstyle\sum_j \int x_j^2 \left(\varphi_1^{(p)}(x)\right)^2 \; dx^{(p)} \\ &\quad \geq p(q_0 - \epsilon)^2 - \frac{q_0^2}{m_\epsilon} \cdot \textstyle\sum_j \int_{|x_j| \leq q_0 - \epsilon} v(x_j) \left(\varphi_1^{(p)}(x)\right)^2 \; dx^{(p)} \; , \end{aligned} \tag{5.7}$$

hence

$$\begin{aligned} \textstyle\sum_j \int x_j^2 \, |\varphi_1^{(p)}(x)|^2 \, dx^{(p)} \; &\geq p(q_0 - \epsilon)^2 - \frac{q_0^2}{m_\epsilon} \cdot \textstyle\sum_j \int v(x_j) \, |\varphi_1^{(p)}(x)|^2 \, dx^{(p)} \; , \\ &\geq p(q_0 - \epsilon)^2 - \frac{q_0^2}{m_\epsilon} \cdot \lambda_1^{(p)}(h) \; . \end{aligned} \tag{5.8}$$

Finally, using Lemma 5.2, we prove,

$$\sum_j \int x_j^2 \, |\varphi_1^{(p)}(x)|^2 \, dx^{(p)} \geq p(q_0 - \epsilon)^2 - \tilde{C}(ph + 1) \; . \tag{5.9}$$

This is equivalent, dividing by p, to

$$\frac{1}{p} \left(\sum_j \int x_j^2 \, |\varphi_1^{(p)}(x)|^2 \, dx^{(p)} \right) \geq (q_0 - \epsilon)^2 - \tilde{C}h - \tilde{C}\frac{1}{p} \; . \tag{5.10}$$

So we have proved the following lemma.

Lemma 5.3 . *For any $\epsilon > 0$, there exists h_ϵ and p_ϵ such that, for $h \in]0, h_\epsilon]$ and $p \geq p_\epsilon$,*

$$\frac{1}{p}\left(\sum_{j=1}^{p}\int x_j^2\,|\varphi_1^{(p)}|^2\,dx^{(p)}\right) \geq (q_0-\epsilon)^2\ . \tag{5.11}$$

Using (5.2), this gives Theorem 5.1. The argument for comparing the situation $\beta = +\infty$ and the situation β large is now possible, once p is fixed.

6 About the Bogolyubov's inequality

Let us first recall some of the results obtained by Pastur and Khozurenko. We shall then analyze the technique which is involved in order to prepare the improvements of the next section. These authors discussed the case when

$$v(x) = -\frac{a}{2}|x|^2 + \frac{b}{4}|x|^4 \tag{6.1}$$

with $a > 0$, $b > 0$.
They prove the following theorem,

Theorem 6.1 . *If $d \geq 3$ and if*

$$\mathcal{J} > h^2(n+2)^2 b^2 I_d^2/4a^2\ , \tag{6.2}$$

then there exists a temperature β_0^{-1} such that, for $\beta > \beta_0$, the corresponding $P(\beta)$ is strictly positive.

The control of $\langle x_k^2\rangle_{\beta,\Lambda}$ is obtained by using the Bogolyubov's inequality. We recall that this inequality (See [Ru], Lemma 5.5.1) says the following

Lemma 6.2 . *Under suitable assumptions*[4] *on the various domains of the possibly not bounded operator C and of the selfadjoint operator H,*

$$\langle[\,[C^\star, H], C]\rangle_{\beta,\Lambda} \geq 0\ . \tag{6.3}$$

[4]This is discussed in detail in [BoMa]

In our context, this inequality is applied with

$$C = \frac{1}{i}\sum_{j\in\Lambda}\partial_{x_j}\ . \tag{6.4}$$

This leads, by considering the limit $\beta \to +\infty$ before to take the thermodynamic limit, to the following result for the splitting between the two first eigenvalues as $|\Lambda|$ tends to ∞.

Theorem 6.3 . *If $d \geq 2$, then, for v defined by (6.1) and $h_0 = \frac{2a\mathcal{J}^{\frac{1}{2}}}{(n+2)bI_d}$, we have, for $h \in]0, h_0[$,*

$$\lim_{|\Lambda|\to+\infty} |\lambda_2^\Lambda - \lambda_1^\Lambda| = 0\ . \tag{6.5}$$

For this theorem, we need only the Bogolyubov inequality in the limiting case $\beta = +\infty$. In this case and when $H = -\Delta + V$ is a Schrödinger operator, the inequality (6.3) becomes simply

$$\int ([\,[C^\star, H], C]\Phi)(x)\Phi(x)dx \geq 0\ , \tag{6.6}$$

where Φ is the first normalized positive eigenfunction of H. This is easily and directly obtained by the minimax principle. In particular, if C is given by (6.4), we get

Lemma 6.4 .

$$\sum_{jk}\int(\partial^2 V/\partial x_j\partial x_k)(x)\Phi(x)^2 dx \geq 0\ . \tag{6.7}$$

When $V(x)$ has the form (we take for simplification $n = 1$)

$$V(x) = \sum_j v(x_j) + J\sum_{i\sim j}|x_i - x_j|^2\ ,$$

we simply get

$$\sum_j \int v''(x_j)\Phi(x)^2 dx \geq 0\ . \tag{6.8}$$

By invariance of Φ in this case, we get,

Proposition 6.5 . *For any j,*

$$\int v''(x_j)\Phi(x)^2 dx \geq 0 \; . \tag{6.9}$$

More generally, for any β, we have the property

$$\langle v''(x_j)\rangle_{\beta,\Lambda} \geq 0 \; . \tag{6.10}$$

In the more specific case when $v(x) = \frac{b}{4}x^4 - \frac{a}{2}x^2$ we obtain

$$\int x_j^2 \Phi(x)^2 dx \geq \frac{a}{3b} \; . \tag{6.11}$$

The proof through the infrared estimates is then easy using the arguments presented in Section 2.

7 Extensions of Pastur-Khozurenko: universal lower bounds

7.1 Introduction

If we compare (6.11) with the semi-classical lower bound obtained by Barbulyak-Kondratev, we note that when $n = 1$ and $a > 2\mathcal{J}d$, the minimum of $\tilde{v}$ is for $q = q_0$ with $q_0^2 = \frac{a-2\mathcal{J}d}{b}$. This leads asymptotically (as $h \to 0$) to (6.11) with $\frac{a}{3b}$ replaced by $\frac{a-2\mathcal{J}d}{b} - \varepsilon$ with $\varepsilon > 0$. The improvement we gave in Section 4 leads asymptotically (as $h \to 0$) to a lower bound by $\frac{a}{b} - \varepsilon$ with $\varepsilon > 0$, which seems optimal in the semi-classical regime.
The main goal of this section is to extend the results proposed by Pastur-Khozurenko.

Theorem 7.1 . *Let us assume that $n = 1$ and that v satisfies (1.15), (1.18) and the condition*

$$v''(x) \leq -\gamma_0 + \gamma_1 x^2 \; , \tag{7.1}$$

for some real γ_0, γ_1 with $\gamma_0 > 0$. Then, for any $h > 0$,

$$\frac{\gamma_0}{\gamma_1} \leq \langle x_j^2\rangle_{\beta,\Lambda} \; . \tag{7.2}$$

In many cases, we can take $\gamma_0 = -v''(0)$. The point 0 corresponds for example to the top between the two wells.
We note also that this approach does not make use of the Ginibre inequalities. This leaves possible extensions for the case $n > 1$ in the case when $\mathrm{Hess}\, v(0)$ has a strictly negative eigenvalue which seems generic for a double well problem and also true in the radial case considered by [PaKh].

We observe also that the left hand side in (7.2) is independent of h.

7.2 Proof of the theorem

The Bogolyubov inequality was giving

$$\langle v''(x_j)\rangle_{\beta,\Lambda} \geq 0 \; . \tag{7.3}$$

The theorem follows immediately from (7.1).

Because the left hand side of (7.2) is independent of β and Λ, the arguments developed in Section 2 work in the same way: long range order for h small enough or $\mathcal{J}$ large enough. Moreover, the control of "h small enough" is explicit.
We recall that in the semi-classical approach developed in the spirit of Barbulyak-Kondratev "h small enough" is implicit.

As already mentioned, the condition (7.2) can be verified when

$$v''(0) < 0 \; , \tag{7.4}$$

which is a condition of concavity at 0, and when the condition at ∞

$$v''(x) \leq Cx^2 \; , \; \forall x > C \; , \tag{7.5}$$

is satisfied. The first one seems natural in the context. The second one is apparently more technical, at least in the semi-classical context, as we shall see in the next subsection for an example.

7.3 The case $n = 1$, $v(x) = bx^{2k} - ax^2$

In this case we have

$$v''(x) = 2k(2k-1)bx^{2k-2} - 2a \; , \tag{7.6}$$

with $a > 0$. We then obtain the inequality, for any $j \in \Lambda$,

$$\frac{a}{k(2k-1)b} \leq \langle x_j^{(2k-2)} \rangle_{\beta,\Lambda} \; , \tag{7.7}$$

as a consequence of the Bogolyubov inequality.
For $k = 2$, this was enough for having a universal lower bound for $\langle x_j^2 \rangle_{\beta,\Lambda}$ which was decisive for the use of the infrared estimate.
The problem is then to treat the case $k > 2$. The Gaussian domination inequalities give a rather satisfactory answer. They give indeed the inequality

$$\langle x_j^{(2k-2)} \rangle_{\beta,\Lambda} \leq C(k) \langle x_j^2 \rangle^{k-1} \; , \tag{7.8}$$

where $C(k)$ is a universal constant counting the number of pairings. Newman [Ne] gives,

$$C(k) = \frac{(2k-2)!}{2^{k-1}(k-1)!} \; . \tag{7.9}$$

Consequently, a lower bound for $\langle x_j^2 \rangle_{\beta,\Lambda}$ (which has to be uniform with respect to Λ, β and j) will be a consequence of an upper bound for $\langle x_j^{2k} \rangle_{\beta,\Lambda}$. We note here that everything is actually independent of j.
The first idea could be to analyze first the case when $\beta = +\infty$. This leads to some results which are not basically better than the results obtained in Section 4. We just mention that these results are obtained by combination of (6.9) and of the inequality

$$\langle v(x_j) \rangle_{\infty,\Lambda} \leq \frac{\lambda_1^\Lambda}{|\Lambda|} \; . \tag{7.10}$$

In the case β large, the arguments of the preceding example work if we have a control of $\langle x_j^{2k} \rangle_{\beta,\Lambda}$ by a constant (using in this case an Hölder inequality) or if we can more directly have a control of $\langle x_j^{2k-2} \rangle_{\beta,\Lambda}$ by a power of $\langle x_j^2 \rangle_{\beta,\Lambda}$. In the next section, we shall exploit this idea in combination with the Ginibre's inequalities.

7.4 Using Lebowitz inequality or gaussian domination inequalities.

As already said, the general princip of the Lebowitz inequalities is to control any correlation by the pair correlations under rather weak assumptions. Our strategy will be to modify the initial potential v in order to get a new potential v_1 satisfying the assumptions of Theorem 3.6. This will lead to the following improvement of Theorem 7.1

Theorem 7.2 . *Let us assume that $n = 1$, that v satisfies (1.15), (1.18) and the condition*

$$v''(0) < 0 \,. \tag{7.11}$$

Let us also assume the existence of ℓ and C such that

$$|v''(x)| \leq Cx^\ell, \forall x > C \,. \tag{7.12}$$

Then, there exists $\rho > 0$ such that, for any $h > 0$, $\beta > 0$ and Λ, we have

$$\rho \leq \langle x_j^2 \rangle_{\beta,\Lambda} \,. \tag{7.13}$$

Outside this unfortunate condition $n = 1$ due to the fact that we use Ginibre's inequality, this theorem is rather satisfactory. The computation of ρ is not very explicit but can be done on examples.

Proof of Theorem 7.2
The main new step is the following lemma

Lemma 7.3 . *Under the assumption of Theorem 7.2, there exists $k \in \mathbb{N}$, $\alpha_0 > 0$ and $\alpha_1 > 0$ such that*

$$v_1(x) = v(x) + \alpha_0 x^4 + \alpha_1 x^{2k} \tag{7.14}$$

satisfies:

$$v_1'(x) \text{ is convex on } \mathbb{R}^+ \,. \tag{7.15}$$

We observe that v_1 satisfies (1.15), (1.18) and for suitable $\gamma_0 > 0$ and $\gamma_1 > 0$ the estimate

$$v_1''(x) \leq -\gamma_0 + \gamma_1 x^{2k} \tag{7.16}$$

The key remark is that the Ginibre's inequalities give the estimate

$$\langle x_j^2 \rangle_{\beta,\Lambda,v} \geq \langle x_j^2 \rangle_{\beta,\Lambda,v_1} . \tag{7.17}$$

Here $\langle x_j^2 \rangle_{\beta,\Lambda,v}$(resp. $\langle x_j^2 \rangle_{\beta,\Lambda,v_1}$) means that we consider H_Λ^{per} with the one particle potential v (resp v_1).

We are now reduced to the problem of finding a lower bound for $\langle x_j^2 \rangle_{\beta,\Lambda,v_1}$. With the properties of convexity of v_1' and the estimate (7.17), this case is easily treated by combination of the Bogolyubov inequality and of the gaussian domination inequality.

8 Conclusion

In this article, we have given explicit conditions on the potential v under which a good lower bound can be found for the second order moment.
When $\beta = +\infty$, the results are not far from the optimal one and no restriction in the dimension n is involved. Nethertheless, restrictions on d appear when applying these inequalities in the infrared estimate strategy for proving transition of phase.
When we want to extend this study to the case β large, we meet immediately the condition $n = 1$ (which could probably be extended to the case $n > 1$ but with invariance by $SO(n)$) and this is due to the fact that we use these monotonicity arguments related to various extensions of FKG inequalities.
The importance of the two restrictions on d and n is sometimes technical, sometimes perhaps deeper and has surely to be investigated more carefully.
The analysis of the splitting given here is rather rough due to our weak control of the localization of the first eigenfunction in the thermodynamic limit. This has also to be analyzed in the future in order to see the effects of the tunneling between the wells in the spirit of our previous works with J. Sjöstrand [HeSj2], [HeSj3].

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References

[AlKoRe] Albeverio, S., Kondratiev, A.Yu and Rebenko, A.L. (1996) Peierls argument and long-range order behaviour of quantum lattice systems with unbounded spins, Preprint.

[BaKo1] Barbulyak, V.S. and Kondrat'ev, Y. (1992) The quasiclassical limit for the Schrödinger operator and phase transitions in Quantum statistical physics, *Functional Analysis and applications*, **26**(2), 61-64.

[BaKo2] Barbulyak, V.S. and Kondrat'ev, Y. (1990) Methods of functional analysis in Problems of mathematical Physics (in Russian), Institut Mat. AN USSR, Kiev, 30-41.

[BoMa] Bouziane, M. and Martin, Ph. A. (1976) Bogolyubov inequality for unbounded operators and the Bose gas, *J. Math. Phys.* **17**(10), 1848-1851.

[BraLi] Brascamp, H.J. and Lieb, E. (1976) On extensions of the Brunn-Minkovski and Prékopa-Leindler Theorems, including inequalities for Logconcave functions, and with an application to diffusion equation, *Journal of Functional Analysis* **22**, 366-389.

[Br] Bricmont, J. (1977) The gaussian inequalities for multicomponent rotators, *J. Stat. Phys* **17**, 289-300.

[BFS] Brydges, D., Fröhlich, J., and Spencer, T. (1978) The random walk representation of classical spin systems and correlation inequalities, *Commun. Math. Phys.***83**, 123-150 .

[DyLiSi] Dyson, F.J., Lieb,E.H. and Simon, B. (1978) Phase transitions in quantum spins systems with isotropic and nonisotropic interactions, *J. Statistic. Phys.* **18**(4), 335-383.

[EN] Ellis, R.S. and Newman, C.M.(1978) Necessary and sufficient conditions for the GHS inequality with applications to analysis and probability *Transactions of the AMS* **237**, 83-99.

[EMN] Ellis, R.S., Monroe, J.L. and Newman, C.M.(1976) The GHS and other correlations ineqqualities for a class of even ferromagnets, *Commun. math. Phys.* **46**, 167-182.

[FKG] Fortuin, C., Kastelyn, P. and Ginibre, J. (1971) Correlation inequalities on some partially ordered sets, *Comm. in Math. Physics* **22**, 89-103.

[Fr] Fröhlich, J. (1976) Phase transitions, Goldstone bosons and topological superselection rules, *Acta Phys. Austriaca* Suppl. XV, 133-269.

[FILS] Fröhlich, J., Israel, R., Lieb, E.H. and Simon, B. (1978) Phase transition and reflection positivity, I General theory and long range lattice models, *Comm. in Math. Physics* **62** (1), 1-34.

[FSS] Fröhlich, J., Simon, B. and Spencer, T. (1976) Infrared bounds, phase transitions, and continuous symmetry breaking, *Comm. Math. Phys.* **50**, 79-85.

[GlJa] Glimm, J. and Jaffe, A. (1987) *Quantum physics (a functional integral point of view),* Springer Verlag, Second edition.

[GJS] Glimm, J., Jaffe, A. and Spencer, T. (1975) Phase transitions for ϕ_2^4 quantum fields, *Comm. Math. Phys.* **45**, 203-216.

[GlKo] Globa, S.A. and Kondrat'ev, Yu. G. (1987) Application of functional analysis in Problems of mathematical Physics (in Russian), Institut Mat. AN USSR, Kiev, 4-16.

[He1] Helffer, B. (1988) *Introduction to the semiclassical analysis for the Schrödinger operator and applications,* Springer lecture Notes in Math., n^0 1336.

[He2] Helffer, B. (1996) Recent results and open problems on Schrödinger operators, Laplace integrals and transfer operators in large dimension, Advances in PDE, in Schrödinger operators, Markov semigroups, Wavelet analysis, operator algebras, edited by M. Demuth, E. Schrohe, B.W. Schultze and J. Sjöstrand, Akademie Verlag, 11-162.

[He3] Helffer, B. (1995) *Semiclassical analysis for Schrödinger operators, Laplace integrals and transfer operators in large dimension: an introduction,* Paris 11- Edition.

[He4] Helffer, B. (1996) On Laplace integrals and transfer operators in large dimension: examples in the non convex case, *Letters in Math. Physics* 38, 297-312 (1996).

[He5] Helffer, B. (1996) Splitting in large dimension and infrared estimates, To appear 1997, in Proceedings of the NATO-ASI conference, Kluwer.

[HeSj1] Helffer, B. and Sjöstrand, J. (1984) Multiple wells in the semiclassical limit, *Comm. in PDE* **9** (4), 337-408.

[HeSj2] Helffer, B. and Sjöstrand, J. (1992) Semiclassical expansions of the thermodynamic limit for a Schrödinger equation, *Astérisque* **210**, 135-181.

[HeSj3] Helffer, B. and Sjöstrand, J. (1992) Semiclassical expansions of the thermodynamic limit for a Schrödinger equation II, *Helvetica Physica Acta* **65**, 748-765 and Erratum **67** (1994), 1-3.

[HeSj4] Helffer, B. and Sjöstrand, J. (1994) On the correlation for Kac like models in the convex case, *J. Statist. Physics*, **74** (1-2), 349-369.

[La] Laroche, E. (1993) Inégalités de corrélation sur $\{-1,+1\}^n$ et dans $I\!R^n$, *Ann. Inst. Henri Poincaré* **29**(4), 531-567.

[Leb1] Lebowitz, J. (1972) Bounds on the correlations and analyticity properties of ferromagnetic Ising spin systems, *Commun. Math. Phys.* **28**, 313-322.

[Leb2] Lebowitz, J. (1974) GHS and other inequalities, *Commun. Math. Phys.* **35**, 87-92.

[LebPre] Lebowitz, J., Pressutti, E. (1976) Statistical Mechanics of systems of unbounded spins, Comm. in Math. Phys. **50**, 195-218.

[Ne] Newman, C.M. (1975) Gaussian correlation inequalities for ferromagnets, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **33**, 75-93.

[PaKh] Pastur, L.A. and Khoruzhenko, B. A. (1987) Phase transitions in quantum models of rotators and ferroelectrics, *Teor. Mat. Fiz.*, **73** (1), 111-124 .

[Re] Rebenko, A.L. (1996) Peierls argument and long range behavior of quantum lattice systems with unbounded spins. Lecture in Ascona (June 1996).

[Ru] Ruelle, D. (1969) *Statistical mechanics,* Math. Physics monograph series, W.A.Benjamin, Inc..

[Si1] Simon, B. (1974) *The* $P(\phi)_2$ *Euclidean quantum field theory,* Princeton Series in Physics.

[Si2] Simon, B. (1979) *Functional integration and quantum physics.* Pure and Applied mathematics, n^0 86, Academic press, New york.

[Sj1] Sjöstrand, J. (1993) Potential wells in high dimension II, more about the one well case, *Ann. Inst. H. Poincaré*, Section Physique théorique **58**(1), 1-41.

[Sj2] Sjöstrand, J. (1994) Evolution equations in a large number of variables, Mathematische Nachrichten **166**, 17-53.

[Sok] Sokal, A.D. (1982) Mean-field bounds and correlation inequalities, *Journal of statistical Physics*, **28** (3), 431-439.

[Sy] Sylvester, G.S. (1975) Representations and inequalities for Ising model Ursell functions, *Comm. Math. Phys.* **42**, 209-220.