EVOLUTION SEMIGROUPS AND ROBUST STABILITY OF EVOLUTION OPERATORS ON BANACH SPACES

Y. LATUSHKIN, S. MONTGOMERY-SMITH, AND T. RANDOLPH

ABSTRACT. Using the theory of evolution semigroups, this paper investigates the stability radius for a wide class of linear nonautonomous systems in Banach spaces. A spectral-mapping theorem for evolution semigroups acting on vector-valued functions on $[0, \infty)$ is proven first. This allows the stability radius to be expressed in terms of the spectrum of the generator of the evolution semigroup. Consequences include explicit formulas for bounds on the stability radius for Banach space systems; time-varying and constant perturbations are considered.

0. INTRODUCTION

The concept of the stability radius was introduced for finite-dimensional autonomous systems by D. Hinrichsen and A. J. Pritchard in [7] as a "distance from instability," that is, as the size of the largest operator, Δ, for which the perturbation, $x' = (A + \Delta)x$, of an exponentially stable system

$$x'(t) = Ax(t), \quad t \geq 0, \tag{0.1}$$

remains stable. More recently, these authors have developed a very general theory on stability of control systems and have studied the property of robustness of stability for infinite-dimensional nonautonomous systems with unbounded time-varying linear control operators (see [6] and the references therein). The in-depth works of B. Jacob [8] (see also [9]) and F. Wirth [30] build on this study and consider more general classes of perturbations, including nonlinear perturbations.

Our work also addresses infinite-dimensional nonautonomous systems

$$x'(t) = A(t)x(t), \qquad t \geq 0, \tag{0.2}$$

1991 *Mathematics Subject Classification.* 93D09, 47D99, 34G10, 34C35.

Key words and phrases. Exponential stability, stability radius, evolution semigroups, spectral mapping theorems.

The first author was supported by the NSF grant DMS-9400518; the second author was supported by the NSF grant DMS-9201357.

where $A(t)$ is a (possibly unbounded) linear operator on a Banach space X, but does so by using recent developments involving so-called evolution semigroups and exploiting their role in determining the asymptotic behavior of solutions for systems such as (0.2) (on $\mathbb{R}$, see [10, 11, 12, 13, 21, 22, 24, 25]). We describe, in particular, a new way of viewing the stability radius and obtain new bounds for it. For autonomous equations we generalize a known formula for the stability radius in Hilbert space by obtaining upper and lower bounds for it in the setting of Banach spaces.

Assume, for a moment, that (0.2), for $t \in \mathbb{R}$, is well-posed in the sense that there exists an evolution (solving) family of operators $\mathcal{U} = \{U(t,s)\}_{t \geq s}$ which gives a differentiable solution. This means that $x(\cdot) : t \mapsto U(t,s)x(s)$, $t \geq s$ in $\mathbb{R}$, is differentiable for any given initial conditions $x(s) = x_s \in \mathcal{D}(A(s))$ from the domain $\mathcal{D}(A(s))$ of the operator $A(s)$, $x(t) \in \mathcal{D}(A(t))$, and (0.2) holds. For a given evolution family $\mathcal{U}$ an "evolution semigroup" $\{T^t\}_{t \geq 0}$ is defined on a "super-space", F, of functions $f : \mathbb{R} \to X$ by the rule

$$(T^t f)(\tau) = U(\tau, \tau - t) f(\tau - t), \quad \tau \in \mathbb{R},\ t \geq 0. \tag{0.3}$$

For appropriate spaces F (e.g., $F = L_p(\mathbb{R}, X)$, $1 \leq p < \infty$, or $C_0(\mathbb{R}, X)$, the space of continuous functions vanishing at infinity), the semigroup $\{T^t\}_{t \geq 0}$ is strongly continuous on F with a generator, Γ.

A significant feature of the evolution semigroup is that it can be used to overcome the fact that the asymptotic behavior of solutions to (0.1) or (0.2) is not, in general, determined by the spectral properties of the operators A or $A(t)$. For example, there exist strongly continuous semigroups $\{e^{tA}\}_{t \geq 0}$ on infinite-dimensional Banach spaces that *are not* uniformly exponentially stable while $\mathrm{Re}\lambda \leq s < 0$ for all $\lambda \in \sigma(A)$, the spectrum of A; see, e.g., [15]. For nonautonomous equations the situation is worse. Indeed, even for finite-dimensional X, the spectra of the operators $A(t)$ do not determine the asymptotic behavior of solutions to (0.2). However, in all of these cases *it is determined* by the spectrum of T^t or Γ; we refer to [10, 11, 12, 13, 22, 25] which study the exponential dichotomy of solutions to (0.2) on $\mathbb{R}$.

In this paper we consider (0.2) and prove that the property of uniform exponential stability is completely determined by the spectrum of an evolution semigroup or its generator defined for "super-spaces" F_+ of X-valued functions on the half-line, $\mathbb{R}_+ = [0, \infty)$. More specifically, we extend a result of J. van Neerven [27] (concerning autonomous equations) in order to show that exponential stability of solutions to (0.2) on $\mathbb{R}_+$ can be characterized by the boundedness of a particular convolution operator on F_+; this operator is, in fact, the inverse of the generator of the evolution semigroup. This result is then used to address questions concerning stability, including the robustness of stability and the stability radius. In a sense, a nonautonomous problem is reduced to an autonomous one by considering the evolution semigroup. So for certain questions concerning stability, complications arising from the dependence on time are removed.

The class of problems being addressed is very large since we allow the operators $A(t)$ to be unbounded. Moreover, since the asymptotic properties are completely determined by the evolution family, we do not require solutions to be differentiable. Instead, we begin with an exponentially stable evolution family, $\mathcal{U}$, and study perturbations of the evolution semigroup. This allows us to consider properties of a perturbed equation, $x'(t) = (A(t) + B(t))x(t)$, even if there is no classical solution; that is, even when solutions exist only in a "mild" sense. We proceed by verifying that, if $\mathcal{U}$ corresponds to solutions of (0.2) (possibly in a mild sense), and if Γ denotes the generator of the induced evolution semigroup, then, under certain assumptions, the corresponding perturbation, $\Gamma + \mathcal{B}$, of Γ also generates an evolution semigroup. The stability of the perturbed problem is then determined by the spectrum of this semigroup or its generator. Robustness of the stability is an immediate consequence.

In order to prove that the exponential stability of solutions to (0.2) is determined by the spectrum of the generator of the evolution semigroup, we first prove a spectral mapping theorem for such a semigroup defined on functions $f\colon \mathbb{R}_+ \to X$ by the rule

$$(T_+^t f)(s) = \begin{cases} U(s, s-t)f(s-t), & s-t \geq 0; \\ 0, & \text{otherwise}. \end{cases} \tag{0.4}$$

Formally, this defines a semigroup of operators, $\{T_+^t\}_{t\geq 0}$; on appropriate Banach spaces of functions, F_+, such a semigroup is strongly continuous (one can take, for example, $F_+ = L_p(\mathbb{R}_+, X)$ or $F_+ = \{f \in C_0(\mathbb{R}_+, X) : f(0) = 0\}$). Letting Γ_+ denote the generator for $\{T_+^t\}_{t\geq 0}$, we begin in Section 1 by showing that

$$e^{t\sigma(\Gamma_+)} = \sigma(T_+^t) \setminus \{0\}, \quad t > 0. \tag{0.5}$$

Spectral mapping theorems of this type have been proven for evolution semigroups (0.3) on spaces of vector-valued functions on $\mathbb{R}$ [11, 12, 21]. The analogous theorem for functions on $\mathbb{R}_+$ is natural and of particular interest in the setting of the present paper. For autonomous equations (0.1) the solutions are given by a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ generated by A. In this case $U(t,s) = e^{(t-s)A}$. The question of whether the spectral mapping theorem holds for T_+^t was posed in [27] and later proven for the autonomous case in [28]. We stress that the spectral mapping theorem (0.5) holds for the evolution semigroup $\{T_+^t\}_{t\geq 0}$ even if $\sigma(e^{tA})\setminus\{0\}$ *is not* equal to $e^{t\sigma(A)}$. For the nonautonomous case the spectral mapping theorem (0.5) has recently been proven by R. Schnaubelt in [26]. A different proof is given here which is quite brief; it uses a simple change of variables and exploits the previous result for functions on $\mathbb{R}$.

The spectral mapping theorem (0.5) allows one to prove that the exponential stability of an evolution family $\mathcal{U}$ is equivalent to the fact that Γ_+ is invertible. This observation begins Section 2 and leads to a natural extension of a theorem by J. van Neerven [27] which characterizes the property of uniform exponential stability of a semigroup, $\{e^{tA}\}_{t\geq 0}$, in terms of a convolution operator: $f \mapsto \int_0^\cdot e^{tA} f(\cdot - t)\,dt$, on F_+. In our consideration of nonautonomous equations, we define the analogue, $\hat{G}$, of this operator by the rule

$$(0.6)\qquad (\hat{G}f)(s) = \int_0^s U(s, s-\tau) f(s-\tau)\,d\tau, \quad f\in F_+, \quad s\geq 0.$$

It turns out that this operator equals $-\Gamma_+^{-1}$ whenever either one is a bounded operator on F_+ and, hence, it can be used to characterize the property of exponential stability. Our arguments in Section 2 are based entirely on the insights given by J. van Neerven in [27] for the autonomous case.

In Section 3 the generator Γ_+ of the evolutionary semigroup $\{T_+^t\}_{t\geq 0}$ again plays an important role in the study of the stability radius. Assume, for now, that $\mathcal{U}$ is an evolution family corresponding to a uniformly exponentially stable equation (0.2). If $\Delta(\cdot)\in C_b(\mathbb{R}_+, B_s(X))$ (the set of strongly continuous $B(X)$-valued functions bounded on $\mathbb{R}_+$), denote by $\tilde{\Delta}$ the multiplication operator, $\tilde{\Delta}f(t) = \Delta(t)f(t)$, for $f\in F_+$. Then stability of a perturbed equation

$$(0.7)\qquad x'(t) = (A(t) + \Delta(t))x(t), \quad t\geq 0,$$

is determined by whether or not $\Gamma_+ + \tilde{\Delta}$ is invertible. Therefore, the stability radius, $r_{stab}(\mathcal{U})$, of $\mathcal{U}$ with respect to a perturbation in $C_b(\mathbb{R}_+, B_s(X))$ may be viewed as the largest $\|\Delta(\cdot)\|_\infty$ such that $\Gamma_+ + \tilde{\Delta}$ is invertible. When viewed in this context, a straightforward algebraic argument can be used to show that $r_{stab}(\mathcal{U})$ satisfies

$$\frac{1}{\|\Gamma_+^{-1}\|} \leq r_{stab}(\mathcal{U}) \leq \frac{1}{r(\Gamma_+^{-1})};$$

here $r(\cdot)$ denotes the spectral radius.

More generally, we consider "structured" perturbations of the form

$$x'(t) = (A(t) + D(t)\Delta(t)E(t))x(t), \quad t \in \mathbb{R}_+, \tag{0.8}$$

where, for Banach spaces U and Y, the operator-valued functions

$$D(\cdot) \in C_b(\mathbb{R}_+, B_s(U,X)) \quad \text{and} \quad E(\cdot) \in C_b(\mathbb{R}_+, B_s(X,Y))$$

are fixed. In the context of control systems, the operators D and E are given "scaling operators" describing the structure of the perturbation, and $\Delta(\cdot) \in C_b(\mathbb{R}_+, B_s(Y,U))$ represents an unknown disturbance. Let $\mathcal{D}$ denote the "multiplication" operator $\mathcal{D} : L_p(\mathbb{R}_+, U) \to L_p(\mathbb{R}_+, X)$, $\mathcal{D}u(t) = D(t)u(t)$. Let $\tilde{\Delta}$ and $\mathcal{E}$ denote analogously defined multiplication operators.

In studying the stability of (0.8), we again view the stability radius, $r_{stab}(\mathcal{U}, D, E)$, of $\mathcal{U}$ with respect to a perturbation in $C_b(\mathbb{R}_+, B_s(Y,U))$, as the largest $\|\Delta(\cdot)\|_\infty$ such that $\Gamma_+ + \mathcal{D}\tilde{\Delta}\mathcal{E}$ is invertible. It is well-known [2, 6] that the stability radius $r_{stab}(\mathcal{U}, D, E)$ can be characterized in terms of the so-called "input-output" operator $\mathbb{L}_0 : L_p(\mathbb{R}_+, U) \to L_p(\mathbb{R}_+, Y)$,

$$(\mathbb{L}_0 u)(t) = E(t) \int_0^t U(t,\tau)D(\tau)u(\tau)\, d\tau, \quad u \in L_p(\mathbb{R}_+, U).$$

We observe that if $\mathcal{U}$ is exponentially stable, then Γ_+ is invertible and $\Gamma_+^{-1} = -\hat{G}$. This implies that the "input-output" operator $\mathbb{L}_0$ can be expressed via the generator Γ_+ of the evolution semigroup as follows:

$$\mathbb{L}_0 = \mathcal{E}\hat{G}\mathcal{D} = -\mathcal{E}\Gamma_+^{-1}\mathcal{D}.$$

This allows for a new, relatively simple proof of the following lower bound on the stability radius (cf. [5, 6]) for the Banach space setting:

$$\frac{1}{\|\mathbb{L}_0\|} \le r_{stab}(\mathcal{U}, D, E).$$

In the case where A, D, and E are constant and $\mathcal{U}$ is replaced by the semigroup $\{e^{tA}\}_{t \ge 0}$ the following upper and lower bounds can be obtained for the Banach space setting:

$$\frac{1}{\|\mathbb{L}_0\|} \le r_{stab}(\{e^{tA}\}, D, E) \le \frac{1}{\sup_{s\in\mathbb{R}} \|E(A-is)^{-1}D\|} \tag{0.9}$$

(see also [6, Theorem 3.5] concerning equality of these three quantities when X, U and Y are Hilbert spaces).

Our proof of this inequality is quite elementary. It uses a recent result of [10] which extends a spectral mapping theorem of L. Gearhart (see, e.g., [15]) for semigroups on Hilbert spaces to a theorem for Banach spaces. Using the spectral mapping theorem, we give a simple proof of the fact that all three quantities in (0.9) are equal for the Hilbert space setting. Also, we give an example where at least one inequality in (0.9) is strict for the Banach space setting. We obtain, in addition, in the Banach space setting, the following formula for the lower bound of $rc_{stab}(\{e^{tA}\}, D, E)$, the *constant* stability radius related to the time-independent perturbations $\Delta(t) \equiv \Delta$:

$$rc_{stab}(\{e^{tA}\}, D, E) \geq \left(\sup_{\xi\in[0,1]} \sup_{\{u_k\}} \frac{\| \sum_k E(A - i\xi - ik)^{-1} D u_k e^{ik(\cdot)} \|_{L_p([0,2\pi],Y)}}{\| \sum_k u_k e^{ik(\cdot)} \|_{L_p([0,2\pi],U)}} \right)^{-1}.$$

Here the second sup is taken over all finite sequences $\{u_k\}_{k=-N}^{N}$ in U. In the development which leads to the bounds mentioned above we introduce the new idea of a "pointwise" stability radius,

$$rc^{\lambda}_{stab}(e^{t_0 A}, D, E) := \sup\{r > 0 : \|\Delta\|_{B(Y,U)} \leq r \Rightarrow \lambda \in \rho(e^{t_0(A+D\Delta E)})\},$$

for a point $\lambda \in \rho(e^{t_0 A})$, $t_0 > 0$ and obtain upper and lower bounds for it. In addition, for a *hyperbolic* strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ we study the constant *dichotomy radius* which is the size of largest Δ such that the differential equation $x' = (A + D\Delta E)x$ has exponential dichotomy.

For infinite-dimensional nonautonomous systems (0.8), the stability radius as defined in [6] allows for unbounded perturbations of the form $D(t)\Delta(t)E(t)$. We remark that the structure of three Banach spaces, $\underline{X} \subseteq X \subseteq \overline{X}$ as described by [6], would allow one to consider unbounded perturbation operators $D(t)$ and $E(t)$ under the assumption that $E(\cdot) \in C_b(\mathbb{R}_+, B_s(\underline{X}, Y))$ and $D(\cdot) \in C_b(\mathbb{R}_+, B_s(U, \overline{X}))$. Many of the arguments follow much as in the bounded case after making appropriate assumptions (and applying theorems of J. Voigt [29] and F. Räbiger et. al [23]) to ensure that the unbounded perturbation, $\Gamma_+ + \mathcal{D}\tilde{\Delta}\mathcal{E}$, of Γ_+ generates an evolution semigroup. This, however, will not be pursued in the present paper.

The following notation will be used throughout. If A is a linear operator on a Banach space, X, then $\mathcal{D}(A)$ denotes the domain of A, $\rho(A)$ denotes the resolvent set and $\sigma(A)$ denotes the spectrum relative to $B(X)$, the algebra of bounded linear operators on X. For $\lambda \in \rho(A)$, $R(\lambda, A)$ denotes the resolvent operator $(\lambda - A)^{-1} \in B(X)$. If A is a bounded linear operator, $r(A)$ denotes the spectral radius. If A is the generator of a strongly continuous semigroup, $\{e^{tA}\}_{t\geq 0}$, on X, define the growth bound of the semigroup by $\omega_0(\{e^{tA}\}) := \inf\{\omega \in \mathbb{R} : \|e^{tA}\| \leq Me^{t\omega}$ for some $M \geq 0$, all $t \geq 0\}$, and the spectral bound of A by $s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$. Also, we denote: $C_0(\mathbb{R}_+, X)$ —

the space of continuous functions vanishing at infinity; $C_{00}(\mathbb{R}_+, X)$ — its subspace of functions with $f(0) = 0$; C_c^1 — differentiable functions with compact support.

1. Evolution semigroups; spectral mapping theorem

An *evolution family* (see, e.g., [3]) is a family of bounded linear operators $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ on X with the following properties:

1. $U(t,s) = U(t,r)U(r,s)$ and $U(t,t) = I$ for all $t \geq r \geq s$;
2. for each $x \in X$ the function $(t,s) \mapsto U(t,s)x$ is continuous for $t \geq s$;
3. $\|U(t,s)\| \leq Ce^{\gamma(t-s)}$, $t \geq s$, for some constants $C > 1$ and γ.

An evolution family is said to be *exponentially stable* if there exist constants $M > 1$, $\beta > 0$ such that $\|U(t,s)\| \leq Me^{-\beta(t-s)}$ for all $t \geq s$.

An evolution family $\mathcal{U}$ is said to *solve* the nonautonomous Cauchy problem (0.2) with initial condition $x(s) = x_s \in \mathcal{D}(A(s))$ $(t \geq s)$ if $x(\cdot) = U(\cdot,s)x_s$ is the unique classical solution of (0.2) for every $x_s \in \mathcal{D}(A(s))$; that is, $x(\cdot)$ is differentiable, $x(t) \in \mathcal{D}(A(t))$ for $t \geq s$, and (0.2) holds. Since we assume that $\mathcal{U}$ is a *strongly* continuous family, the operators $A(t)$ here might be unbounded. In particular, for the autonomous case, if A is a generator of a strongly continuous semigroup, $\{e^{tA}\}_{t\geq 0}$, on X and $A(t) \equiv A$, then $U(t,s) = e^{(t-s)A}$.

Now let $\{e^{tA_0}\}_{t\geq 0}$ be a strongly continuous semigroup generated by A_0, let $A_1(t) \in B(X)$ for $t \geq 0$, and define $A(t) = A_0 + A_1(t)$. Then even if $t \mapsto A_1(t)$ is continuous, equation (0.2) may not have a differentiable solution for all initial conditions $x(0) = x \in \mathcal{D}(A) = \mathcal{D}(A_0)$ (see, e.g.,[18]). Consequently, it is important to be able to consider solutions which may exist only in a "mild sense".

Let $\mathcal{U}$ be an evolution family of operators corresponding to the solution of (0.2), and consider the nonautonomous inhomogeneous equation

$$x'(t) = A(t)x(t) + f(t), \qquad t \geq 0, \tag{1.1}$$

where f is a locally integrable X-valued function on $\mathbb{R}_+$. Recall, that $x(\cdot)$ is a *mild* solution of (1.1) with initial value $x(s) = x \in \mathcal{D}(A(s))$ if

$$x(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau)\,d\tau, \quad t \geq s.$$

Now let $\mathcal{U}$ be an evolution family. We define the *evolution semigroup* $\{T_+^t\}_\tau \geq 0$ *induced by* $\mathcal{U}$ by the rule given in (0.4). This defines a strongly continuous semigroup on $F_+ = L_p(\mathbb{R}_+, X)$, $1 \leq p < \infty$, and $F_+ = C_{00}(\mathbb{R}_+, X)$. In either case, the evolution semigroup on F_+ will be denoted by $\{T_+^t\}_t \geq 0$, and its generator will be denoted by Γ_+.

In order to study the asymptotic *stability* of solutions to (0.2) on $\mathbb{R}_+$, we now prove the spectral mapping theorem for the operators T_+^t and Γ_+. Of particular interest is the consequence that the growth bound $\omega_0(\{T_+^t\})$ equals the spectral bound $s(\Gamma_+)$.

We begin by observing that the spectrum $\sigma(\Gamma_+)$ is invariant under translations along $i\mathbb{R}$ and the spectrum $\sigma(T_+^t)$ is invariant under rotations about origin. This spectral symmetry is a consequence of the fact that for $\xi \in \mathbb{R}$,

$$T_+^t e^{i\xi\cdot} f = e^{i\xi\cdot} e^{-i\xi t} T_+^t f, \quad \text{and} \quad \Gamma_+ e^{i\xi\cdot} = e^{\xi\cdot}(\Gamma_+ - i\xi). \tag{1.2}$$

To prove the relationship $e^{t\sigma(\Gamma_+)} = \sigma(T_+^t)\setminus\{0\}, \quad t > 0$, we employ a "change-of-variables" technique as used in [11, Theorem 3.1]. That is, we exploit the known spectral relationships between a strongly continuous semigroup and the evolution semigroup it induces for functions on $\mathbb{R}$. Specifically, let $\{e^{tA}\}_{t\geq 0}$ denote any strongly continuous semigroup generated by A on a Banach space E. Let $F = C_0(\mathbb{R}, E)$ or $L_p(\mathbb{R}, E)$. Let $\{S^t\}_{t\geq 0}$ be the induced evolution semigroup

$$(S^t f)(\tau) = e^{tA} f(\tau - t), \quad f \in F,$$

and denote the generator of $\{S^t\}_{t\geq 0}$ by B. Then Theorem 2.5 in [11] shows that the following are equivalent:

i) $\sigma(e^{tA}) \cap \mathbb{T} = \emptyset$ on E;
ii) $\sigma(S^t) \cap \mathbb{T} = \emptyset$ on F;
iii) $0 \in \rho(B)$ on F.

Theorem 1.1. *Let F_+ denote $C_{00}(\mathbb{R}_+, X)$ or $L_p(\mathbb{R}_+, X)$. The spectrum $\sigma(\Gamma_+)$ is invariant under translations along $i\mathbb{R}$, the spectrum $\sigma(T_+^t)$ is invariant under rotations about origin, and*

$$e^{t\sigma(\Gamma_+)} = \sigma(T_+^t) \setminus \{0\}, \quad t > 0. \tag{1.3}$$

Proof. The arguments for the two cases $F_+ = C_{00}(\mathbb{R}_+, X)$ and $F_+ = L_p(\mathbb{R}_+, X)$ are similar, so only the first one is considered here. The inclusion $e^{t\sigma(\Gamma_+)} \subseteq \sigma(T_+^t) \setminus \{0\}$ follows from the standard spectral inclusion for strongly continuous semigroups [15]. In view of the spectral symmetry implied by (1.2), it suffices to show that $\sigma(T_+^t) \cap \mathbb{T} = \emptyset$ whenever $0 \in \rho(\Gamma_+)$.

To this end, we replace the Banach space E by $C_{00}(\mathbb{R}_+, X)$ in the discussion preceding the theorem and consider the evolution semigroup $\{\mathcal{S}^t\}_{t\geq 0}$ on $C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$ induced by T_+^t:

$$(\mathcal{S}^t F)(s) = T_+^t(F(s - t, \cdot)), \quad s \in \mathbb{R},\ t \geq 0. \tag{1.4}$$

Denoting the generator of $\{\mathcal{S}^t\}_{t\geq 0}$ by $\mathcal{B}$, we note that the equivalence of (i)–(iii), above, holds in this setting with T_+^t, $\mathcal{S}^t$ and $\mathcal{B}$ replacing e^{tA}, S^t and B, respectively.

Next, consider the semigroup $\{\mathcal{T}^t\}_{t\geq 0}$ on $C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$ defined by

$$(\mathcal{T}^t F)(s) = T_+^t F(s, \cdot).$$

The generator of $\{\mathcal{T}^t\}_{t\geq 0}$ will be denoted by $\mathcal{G}$. Note that $\mathcal{T}^t$ is the operator of multiplication by T_+^t. Similarly, the generator, $\mathcal{G}$, of $\{\mathcal{T}^t\}_{t\geq 0}$ satisfies

$$(\mathcal{G}F)(s) = \Gamma_+ F(s, \cdot), \qquad \text{where } F(s, \cdot) \in \mathcal{D}(\Gamma_+) \text{ for } s \in \mathbb{R}.$$

In particular, if $0 \in \rho(\Gamma_+)$, then $\Gamma_+^{-1}(g(s)) = (\mathcal{G}^{-1}g)(s)$, for $g \in C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$, and so $0 \in \rho(\mathcal{G})$.

Now define an isometry $\mathcal{J}$ on $C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$ by $(\mathcal{J}F)(s, \tau) = F(s + \tau, \tau)$, for $s \in \mathbb{R}$, $\tau \in \mathbb{R}_+$. This gives the relationship

$$(\mathcal{T}^t \mathcal{J} F)(s, \tau) = (\mathcal{J}\mathcal{S}^t F)(s, \tau), \quad s \in \mathbb{R},\ \tau \in \mathbb{R}_+, \tag{1.5}$$

and so $\mathcal{G}\mathcal{J}F = \mathcal{J}\mathcal{B}F$ for $F \in \mathcal{D}(\mathcal{B})$, and $\mathcal{J}^{-1}\mathcal{G}F = \mathcal{B}\mathcal{J}^{-1}F$ for $F \in \mathcal{D}(\mathcal{G})$. Consequently, $\sigma(\mathcal{G}) = \sigma(\mathcal{B})$ on $C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$. We conclude that if $0 \in \rho(\Gamma_+)$, then $0 \in \rho(\mathcal{G}) = \rho(\mathcal{B})$. The equivalence of (i)–(iii) above (with T_+^t, $\mathcal{S}^t$, and $\mathcal{B}$ on $C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$) implies that $\sigma(T_+^t) \cap \mathbb{T} = \emptyset$. □

Evolution semigroups induced by an evolution family as in (0.3) have been studied by several authors including R. Schnaubelt [26] who has characterized such semigroups in terms of their generators on general Banach function spaces of X-valued functions (e.g., $L_p(\mathbb{R}, X)$ and $C_0(\mathbb{R}, X)$). Of particular interest to us is the following theorem of R. Schnaubelt [26] (see also Räbiger et al. [23]) which shows exactly when a strongly continuous semigroup on a Banach function space arises from a strongly continuous evolution family on X. We state a version of the result that will be used in Section 3; a more general version is proven in [26].

Theorem 1.2. *Let $\{T_+^t\}_{t\geq 0}$ be a strongly continuous semigroup on $F_+ = L_p(\mathbb{R}_+, X)$ generated by Γ_+. The following are equivalent:*

- *i) $\{T_+^t\}_{t\geq 0}$ is an evolution semigroup (induced by an evolution family $\mathcal{U}$);*
- *ii) there exists a core, $\mathcal{C}$, of Γ_+ such that for all $\varphi \in C_c^1(\mathbb{R}_+)$, and $f \in \mathcal{C}$, it follows that $\varphi f \in \mathcal{D}(\Gamma_+)$ and $\Gamma_+(\varphi f) = -\varphi' f + \varphi \Gamma_+ f$. Moreover, there exists $\lambda \in \rho(\Gamma_+)$ such that $R(\lambda, \Gamma_+) : F_+ \to C_{00}(\mathbb{R}_+, X)$ is continuous with dense range.*

2. Exponential Stability

To make use of Theorem 1.1 in this section, the following connection between the spectral properties of the evolution semigroup $\{T_+^t\}_{t\geq 0}$ on F_+ and the evolution family $\mathcal{U}$ on X is needed, $F_+ = L_p(\mathbb{R}_+, X)$, $1 \leq p < \infty$, or $F_+ = C_{00}(\mathbb{R}_+, X)$. In the case of the whole line, $\mathbb{R}$, the exponential dichotomy of an evolution family $\mathcal{U}$ has been characterized by the property that the spectrum of the corresponding evolution semigroup operators does not intersect the unit circle (see the references in Section 1). For the property of exponential stability on $\mathbb{R}_+$, the argument is straightforward:

Theorem 2.1. *The evolution family $\mathcal{U}$ is exponentially stable if and only if the growth bound, $\omega_0(\{T_+^t\})$, of the the evolution semigroup on F_+, is less than zero.*

Proof. Let $F_+ = C_{00}(\mathbb{R}_+, X)$. If $\mathcal{U}$ is exponentially stable, then there exist $M > 1$, $\beta > 0$ such that $\|U(t,s)\|_{B(X)} \leq Me^{-\beta(t-s)}$, $t \geq s$. For $\tau \geq 0$ and $f \in C_{00}(\mathbb{R}_+, X)$,

$$\begin{aligned}\|T_+^\tau f\|_{C_{00}(\mathbb{R}_+,X)} = \sup_{t>0}\|T_+^\tau f(t)\|_X &= \sup_{t>\tau}\|U(t,t-\tau)f(t-\tau)\|_X \\ &\leq \sup_{t>\tau}\|U(t,t-\tau)\|_{B(X)}\|f(t-\tau)\|_X \\ &\leq Me^{-\beta\tau}\|f\|_{C_{00}(\mathbb{R}_+,X)}.\end{aligned}$$

Conversely, assume there exist $M > 1$, $\alpha > 0$ such that $\|T_+^t\| \leq Me^{-\alpha t}$, $t \geq 0$. Let $x \in X$, $\|x\| = 1$. For fixed $t > s > 0$, choose $f \in C_{00}(\mathbb{R}_+, X)$ such that $\|f\|_{C_{00}(\mathbb{R}_+,X)} = 1$ and $f(s) = x$. Then,

$$\begin{aligned}\|U(t,s)x\|_X = \|U(t,s)f(s)\|_X &= \|T_+^{(t-s)}f(t)\|_X \\ &\leq \sup_{\tau>0}\|T_+^{(t-s)}f(\tau)\|_X \\ &= \|T_+^{(t-s)}f\|_{C_{00}(\mathbb{R}_+,X)} \\ &\leq Me^{-\alpha(t-s)}.\end{aligned}$$

A similar argument works for $F_+ = L_p(\mathbb{R}_+, X)$. $\square$

J. van Neerven has characterized the exponential stability of solutions to (0.1) in terms of a convolution operator, $\hat{G}$, induced by $\{e^{tA}\}_{t\geq 0}$. In this autonomous setting, $T_+^t f(s) = e^{tA}f(s-t)$, for $s \geq t$, and 0 otherwise; the convolution operator takes the form

$$(\hat{G}f)(s) := \int_0^s e^{\tau A}f(s-\tau)\,d\tau = \int_0^\infty (T_+^\tau f)(s)\,d\tau, \quad s \geq 0, \tag{2.1}$$

for $f \in F_+$ (see [27], Theorem 1.3, and Corollary 2.2). Specifically, if $\{e^{tA}\}_{t\geq 0}$ is a strongly continuous semigroup on X, and $1 \leq p < \infty$, then the following are equivalent:

i) $\omega_0(\{e^{tA}\}) < 0$;

ii) $\hat{G}f \in L_p(\mathbb{R}_+, X)$ for all $f \in L_p(\mathbb{R}_+, X)$;

iii) $\hat{G}f \in C_0(\mathbb{R}_+, X)$ for all $f \in C_0(\mathbb{R}_+, X)$;

In this section this result will be extended so that it may be used to describe exponential stability of the nonautonomous equation (0.2). We define the convolution operator $\hat{G}$ in a natural way as follows. If $\mathcal{U}$ is an evolution family and $\{T_+^t\}_{t\geq 0}$ is the evolution semigroup defined in (0.4), define $\hat{G}$ on F_+ by:

$$\begin{aligned} \hat{G}f(s) &:= \int_0^\infty (T_+^\tau f)(s)\,d\tau = \int_0^s U(s, s-\tau)f(s-\tau)\,d\tau \\ &= \int_0^s U(s,\tau)f(\tau)\,d\tau, \qquad s \geq 0. \end{aligned} \tag{2.2}$$

By standard semigroups properties, $\hat{G} = -\Gamma_+^{-1}$ provided the semigroup $\{T_+^t\}_{t\geq 0}$ or the evolution family $\mathcal{U}$ is uniformly exponentially stable.

With this setup in mind, the following theorem can be proven using the technique of [27, Theorem 1.3]. The argument here is in fact shorter since, in view of Theorem 1.1, we may consider the resolvent, $(\lambda - \Gamma_+)^{-1}$, for only *real* λ.

Theorem 2.2. *The following are equivalent for an evolution family of operators $\mathcal{U}$ on X.*

i) $\mathcal{U}$ is exponentially stable;

ii) $\hat{G}$ is a bounded operator on $L_p(\mathbb{R}_+, X)$;

iii) $\hat{G}$ is a bounded operator on $C_0(\mathbb{R}_+, X)$.

Before proceeding with the proof, note that (ii) is equivalent to the statement that $\hat{G}f \in L_p(\mathbb{R}_+, X)$ for each $f \in L_p(\mathbb{R}_+, X)$. Indeed, an application of the Closed-Graph Theorem shows (as in [27]) that if the latter holds, then there exists a $M > 0$ such that $\|\hat{G}f\|_{L_p(\mathbb{R}_+,X)} \leq M\|f\|_{L_p(\mathbb{R}_+,X)}$, for all $f \in L_p(\mathbb{R}_+, X)$. Similarly, for any $\lambda > 0$, the rescaled semigroup $\{e^{-\lambda t}T_+^t\}_{t\geq 0}$ induces a convolution operator $\hat{G}_\lambda$, on $L_p(\mathbb{R}_+, X)$,

$$\hat{G}_\lambda f(s) = \int_0^\infty (e^{-\lambda t}T_+^\tau f)(s)\,d\tau = \int_0^s e^{-\lambda\tau}U(s, s-\tau)f(s-\tau)\,d\tau,$$

with the property that there exists a $M_\lambda > 0$ such that $\|\hat{G}_\lambda f\|_{L_p(\mathbb{R}_+,X)} \leq M_\lambda\|f\|_{L_p(\mathbb{R}_+,X)}$, for all $f \in L_p(\mathbb{R}_+, X)$.

As for the proof of $(ii) \Rightarrow (i)$, the idea is to prove the existence of the operator $(\lambda - \Gamma_+)^{-1}$ for any $\lambda \geq 0$. This shows that $s(\Gamma_+) < 0$ and hence by Theorem 1.1, $\omega_0(\{T_+^t\}) < 0$; Theorem 2.1 then shows that $\mathcal{U}$ is exponentially stable.

Proof. Beginning with the proof of $(ii) \Rightarrow (i)$, we make the following definitions: For $T > 0$ and $f \in L_p(\mathbb{R}_+, X)$, set

$$F_{T,f}(\lambda) = \int_0^T e^{-\lambda\tau} T_+^\tau f \, d\tau, \qquad \lambda \in \mathbb{R}_+.$$

This defines a continuous $L_p(\mathbb{R}_+, X)$-valued function on $\mathbb{R}_+$. For $T_0 > 0$, and $g \in L_p(\mathbb{R}_+, X)$, let $\pi_{T_0}(g)$ denote the restriction, $g|_{[0,T_0]}$. Then for $T \geq T_0$, set

$$F_{T,T_0,f} = \pi_{T_0}(F_{T,f}).$$

For each $\lambda \in \mathbb{R}_+$, the map $f \mapsto F_{T,T_0,f}(\lambda)$ is a bounded operator from $L_p(\mathbb{R}_+, X)$ to $L_p([0, T_0], X)$.

Now consider the growth bound $\omega_0(\{T_+^t\})$. A standard property of strongly continuous semigroups is that for $\lambda > \omega_0(\{T_+^t\})$, the resolvent $(\lambda - \Gamma_+)^{-1}$ exists and can be expressed as

$$(\lambda - \Gamma_+)^{-1} f = \int_0^\infty e^{-\lambda\tau} T_+^\tau f \, d\tau, \qquad f \in L_p(\mathbb{R}_+, X).$$

We now show that this holds for all $\lambda \geq 0$. To this end, fix $\omega > \omega_0(\{T_+^t\})$. Let $T \geq T_0 \geq 0$, and consider the operators $f \mapsto F_{T,T_0,f}(\lambda)$ for $0 \leq \lambda \leq \omega$. Then for $f \in C_{00}(\mathbb{R}_+, X) \cap L_p(\mathbb{R}_+, X)$ and $0 \leq s \leq T_0$,

$$\begin{aligned}
[F_{T,f}(\lambda)](s) &= \left(\int_0^T e^{-\lambda\tau} T_+^\tau f \, d\tau \right)(s) \\
&= \int_0^s e^{-\lambda\tau} U(s, s-\tau) f(s-\tau) \, d\tau \\
&= (\hat{G}_\lambda f)(s).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|F_{T,T_0,f}(\lambda)\|_{L_p([0,T_0],X)} &= \left\| \pi_{T_0} \left(\int_0^{(\cdot)} e^{-\lambda\tau} T_+^\tau f \, d\tau \right) \right\|_{L_p(\mathbb{R}_+,X)} \\
&\leq \|\pi_{T_0}\| \, \|\hat{G}_\lambda f\|_{L_p(\mathbb{R}_+,X)} \\
&\leq M_\lambda \|f\|_{L_p(\mathbb{R}_+,X)}
\end{aligned}$$

for some $M_\lambda > 0$ (see the remark preceding the proof). On the interval $[0, \omega]$, the function $F_{T,T_0,f}(\lambda)$ is a uniformly continuous function of λ, so there exists $K > 0$ such that $\|F_{T,T_0,f}(\lambda)\|_{L_p([0,T_0],X)} \leq K\|f\|_{L_p(\mathbb{R}_+,X)}$

for all $\lambda \in [0,\omega]$. Since $C_{00}(\mathbb{R}_+,X)\cap L_p(\mathbb{R}_+,X)$ is dense in $L_p(\mathbb{R}_+,X)$, this holds for all $f \in L_p(\mathbb{R}_+,X)$.

As a consequence, the limit

$$F_{\infty,T_0,f}(\lambda) := \lim_{T\to\infty} F_{T,T_0,f}(\lambda) = \pi_{T_0}\left(\int_0^\infty e^{-\lambda\tau} T_+^\tau f \, d\tau\right)$$

exists for all $\lambda \in [0,\omega]$.

Finally, the limit

$$F_{\infty,\infty,f}(\lambda) := \lim_{T_0\to\infty} F_{\infty,T_0,f}(\lambda) = \int_0^\infty e^{-\lambda t} T_+^\tau f \, d\tau$$

exists for all $\lambda \geq 0$ since for each T_0, $L_p([0,T_0],X)$ is a closed subspace of $L_p(\mathbb{R}_+,X)$. Consequently, if $\lambda \geq 0$, then $\lambda \in \rho(\Gamma_+)$ and $(\lambda-\Gamma_+)^{-1} = F_{\infty,\infty,f}(\lambda)$ [15, p. 6]. This proves $(ii) \Rightarrow (i)$.

The above argument can be modified, as in [27], and used to prove $(iii) \Rightarrow (i)$. First note that if, as above, $\hat{G}_\lambda$ denotes the convolution operator on $C_0(\mathbb{R}_+,X)$ induced by $\{e^{-\lambda t}T_+^t\}_{t\geq 0}$ $(\lambda > 0)$, then there exists $M_\lambda > 0$ such that $\|\hat{G}_\lambda f\|_{C_0(\mathbb{R}_+,X)} \leq M_\lambda \|f\|_{C_0(\mathbb{R}_+,X)}$ for $f \in C_0(\mathbb{R}_+,X)$ (this follows from the uniform boundedness principle). Next, applying the truncation, π_{T_0}, to a function in $C_{00}(\mathbb{R}_+,X)$ may not result in a function in $C_{00}(\mathbb{R}_+,X)$, so it is necessary to replace the operator π_{T_0} by a "continuous" truncation: for $T_0 \geq 1$, redefine π_{T_0} by the rule $\pi_{T_0}f(t) = \alpha_{T_0}(t)f(t)$, for $0 \leq t \leq T_0$, where

$$\alpha_{T_0} = \begin{cases} 1, & 0 \leq t \leq T_0 - 1 \\ T_0 - t, & T_0 - 1 \leq t \leq T_0 \\ 0, & t \geq T_0. \end{cases}$$

Then π_{T_0} is an operator from $C_{00}(\mathbb{R}_+,X)$ to $C_{00}([0,T_0],X)$, the subspace of functions vanishing on $[T_0,\infty)$; it satisfies $\|\pi_{T_0}f\|_{C_{00}([0,T_0],X)} \leq \|f\|_{C_{00}(\mathbb{R}_+,X)}$, for $f \in C_{00}(\mathbb{R}_+,X)$. The previous argument can now be used to prove $(iii) \Rightarrow (i)$.

Finally, assume that (i) holds. Then there exist $M > 0$, $\beta > 0$ such that $\|U(t,s)\| \leq Me^{-\beta(t-s)}$. Therefore,

$$\|\hat{G}f(s)\| = \|\int_0^s U(s,s-\tau)f(s-\tau)\, d\tau\| \leq \int_0^s Me^{-\beta\tau}\|f(s-\tau)\|\, d\tau.$$

This coincides with the inequality that occurs in the autonomous case; that is, the above inequality holds if $\|e^{tA}\| \leq Me^{-\beta t}$ for some $M,\beta > 0$. Therefore, the proofs of $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ will follow exactly as in [27]. □

This theorem makes explicit, in the case of the half line $\mathbb{R}_+$, the relationship between the stability of an evolution family $\mathcal{U}$ and the generator, Γ_+, of the corresponding evolution semigroup (0.4). Indeed, as shown above, stability is equivalent to the boundedness of $\hat{G}$, in which case $\hat{G} = -\Gamma_+^{-1}$. Combining Theorems 2.1 and 2.2 yields:

Corollary 2.3. *Let $\mathcal{U}$ be an exponentially bounded evolution family and let Γ_+ denote the generator of the induced evolution semigroup on $F_+ = L_p(\mathbb{R}_+, X)$, $1 \le p < \infty$ or $F_+ = C_{00}(\mathbb{R}_+, X)$. The following are equivalent:*

i) $\mathcal{U}$ is exponentially stable;
ii) Γ_+ is invertible;
iii) $s(\Gamma_+) < 0$.

3. Perturbations: the stability radius

In this section we consider perturbations of (0.2) of the form

$$x'(t) = (A(t) + B(t))x(t), \quad t \in \mathbb{R}_+, \tag{3.1}$$

with initial condition $x(s) = x \in \mathcal{D}(A(s))$, $s \le t$. As mentioned in the introduction, we do not restrict attention to differentiable solutions, and so we do not assume that either (0.2) or (3.1) are well-posed. Instead we begin with an evolution family $\mathcal{U}$ and, under appropriate hypotheses, prove the existence of a perturbed evolution *semigroup* from which the stability properties of the corresponding perturbed evolution family, $\mathcal{U}_1$, are determined.

If X and Y are Banach spaces, let $B_s(X, Y)$ denote the set of bounded operators from X to Y endowed with the strong operator topology, and let $C_b(\mathbb{R}_+, B_s(X, Y))$ denote the set of strongly continuous functions, $B(\cdot)\colon \mathbb{R}_+ \to B(X, Y)$, which are bounded on $\mathbb{R}_+$; the norm is defined by $\|B(\cdot)\|_\infty = \sup_{t\ge 0} \|B(t)\|_{B(X,Y)}$. Given a function $B(\cdot) \in C_b(\mathbb{R}_+, B_s(X, Y))$, a calligraphic letter will be used to denote the multiplication operator $\mathcal{B}\colon L_p(\mathbb{R}_+, X) \to L_p(\mathbb{R}_+, Y)$ defined by $\mathcal{B}f(t) = B(t)f(t)$. The map $B(\cdot) \mapsto \mathcal{B}$ is an isometric embedding from

$$C_b(\mathbb{R}_+, B_s(X, Y)) \quad \text{into} \quad B(L_p(\mathbb{R}_+, X), L_p(\mathbb{R}_+, Y)).$$

We begin in Subsection 3.1 with some observations concerning bounded perturbations of (0.2) of the form (3.1) where $B(\cdot) \in C_b(\mathbb{R}_+, B_s(X))$. For such perturbations, (3.1) has a mild solution in the sense that there exists a unique evolution family $\mathcal{U}_1 = \{U_1(t, s)\}_{t\ge s}$ satisfying

$$U_1(t, s)x = U(t, s)x + \int_s^t U(t, \tau)B(\tau)U_1(\tau, s)x \, d\tau. \tag{3.2}$$

for all $x \in X$. A byproduct of our approach is immediate information concerning stability and its robustness for such solutions, and a straightforward algebraic argument which identifies bounds for the stability radius.

Assume, for a moment, that $\mathcal{U}$ is an exponentially stable evolution family corresponding to solutions of (0.2). Let X, U and Y denote Banach spaces. Let $\Delta(\cdot) \in C_b(\mathbb{R}_+, B(Y,U))$ denote a "disturbance" operator. Let

$$E(\cdot) \in C_b(\mathbb{R}_+, B(X,Y)) \quad \text{and} \quad D(\cdot) \in C_b(\mathbb{R}_+, B(U,X))$$

denote given operators describing the "structure" of the perturbation of (0.2). Then (3.1) takes the form

$$x'(t) = [A(t) + D(t)\Delta(t)E(t)]x(t), \quad t \geq 0. \tag{3.3}$$

The stability radius measures the size of the largest disturbance, $\Delta(\cdot)$, such that solutions (possibly, in the "mild sense") to equation (3.3) remain stable. Our next goal is to obtain upper and lower bounds for this quantity. For the nonautonomous situation some general facts are obtained in Subsection 3.1. More specific results for autonomous problems are considered in Subsection 3.2.

We will drop the subscript "+" when referring to the space $F = L_p(\mathbb{R}_+, X)$ and when referring to the evolution semigroup (0.4) or its generator. Hence, if $\mathcal{U}$ is an evolution family on X, the evolution semigroup (0.4) and its generator on $F = L_p(\mathbb{R}_+, X)$ will be denoted simply by $\{T^t\}_{t\geq 0}$ and Γ.

3.1. **Nonautonomous case.**

Let $\mathcal{U}$ be an evolution family on X and let Γ be the generator of the corresponding evolution semigroup. If $B(\cdot) \in C_b(\mathbb{R}_+, B_s(X))$, then the multiplication operator $\mathcal{B}$ is a bounded operator on F and so the operator $\Gamma_1 = \Gamma + \mathcal{B}$ generates a strongly continuous semigroup, $\{T_1^t\}_{t\geq 0}$ on F (see, e.g., [17]). Moreover, the following simple fact holds.

Proposition 3.1. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family, and $B(\cdot) \in C_b(\mathbb{R}_+, B_s(X))$. Then there exists a unique evolution family $\mathcal{U}_1 = \{U_1(t,s)\}_{t\geq s}$ which solves the integral equation* (3.2). *Moreover, $\mathcal{U}_1$ is exponentially stable if and only if $\Gamma + \mathcal{B}$ is invertible.*

Proof. As already observed, $\Gamma_1 := \Gamma + \mathcal{B}$ generates a strongly continuous semigroup, $\{T_1^t\}_{t\geq 0}$ on F. To see that this is, in fact, an *evolution semigroup*, recall Theorem 1.2 and note that for $\lambda \in \rho(\Gamma) \cap \rho(\Gamma_1)$,

$$\mathrm{Range}(R(\lambda,\Gamma)) = D(\Gamma) = D(\Gamma + \mathcal{B}) = \mathrm{Range}(R(\lambda,\Gamma_1))$$

is dense in $C_0(\mathbb{R}_+, X)$. Also, if $\mathcal{C}$ is a core for Γ, then it is a core for Γ_1, and so for $\varphi \in C_c^1(\mathbb{R})$, $f \in \mathcal{C}$,

$$\Gamma_1(\varphi f) = \Gamma(\varphi f) + \mathcal{B}(\varphi f) = -\varphi' f + \varphi \Gamma f + \varphi \mathcal{B} f = -\varphi' f + \varphi(\Gamma_A + \mathcal{B}) f.$$

Consequently, $\{T_1^t\}_{t\geq 0}$ corresponds to an evolutionary family, $\mathcal{U}_1 = \{U_1(t,s)\}_{t\geq s}$, and $x(t) = U_1(t,s)x(s)$ is seen to define a mild solution to (3.1). Indeed,

$$T_1^t f = T_0^t f + \int_0^t T_0^{(t-\tau)} \mathcal{B} T_1^\tau f \, d\tau, \tag{3.4}$$

holds for all $f \in F$. In particular, for $x \in X$, and any $\varphi \in C_c^1(\mathbb{R})$, setting $f = \varphi \otimes x$ in (3.4) and using a change of variables leads to

$$\varphi(s) U_1(t,s)x = \varphi(s) U(t,s)x + \varphi(s) \int_s^t U(t,\tau) B(\tau) U_1(\tau,s)x \, d\tau.$$

Therefore, (3.2) holds for all $x \in X$.

Finally, Theorem 2.1 shows that $\mathcal{U}_1$ is exponentially stable if and only if Γ_1 is invertible. □

The existence of mild solutions under bounded perturbations of this type is well known (see, e.g., [2]), but an immediate consequence of the approach given here is the property of robustness for the stability of $\mathcal{U}$. Indeed, by continuity properties of the spectrum of an operator Γ, there exists $\epsilon > 0$ such that Γ_1 is invertible whenever $\|\Gamma_1 - \Gamma\| < \epsilon$; that is, $\mathcal{U}_1$ is exponentially stable whenever $\|B(\cdot)\|_\infty < \epsilon$. Moreover, this approach provides insight into the concept of the stability radius of a controlled system.

Let $\mathcal{U}$ be an evolution family on X. Fix $D(\cdot) \in C_b(\mathbb{R}_+, B_s(U,X))$, and $E(\cdot) \in C_b(\mathbb{R}_+, B_s(X,Y))$. For any $\Delta(\cdot) \in C_b(\mathbb{R}_+, B_s(Y,U))$, we consider the perturbation $B(t) = D(t)\Delta(t)E(t)$.

The *stability radius* of an exponentially stable evolution family $\mathcal{U}$ with respect to the perturbation structure $D(\cdot)$ and $E(\cdot)$ is defined as

$$r_{stab}(\mathcal{U}, D, E) = \sup\{r > 0 : \Delta(\cdot) \in C_b(\mathbb{R}_+, B_s(Y,U)), \;\; \|\Delta(\cdot)\|_\infty \leq r \text{ implies } \mathcal{U}_1 \text{ is exponentially stable}\}.$$

Here, $\mathcal{U}_1 = \{U_1(t,s)\}_{t\geq s}$ is the perturbed evolution family as in Proposition 3.1 with $B(t) = D(t)\Delta(t)E(t)$. In view of this proposition, we have

$$r_{stab}(\mathcal{U}, D, E) = \sup\{r > 0 : \Delta(\cdot) \in C_b(\mathbb{R}_+, B_s(Y,U)), \;\; \|\Delta(\cdot)\|_\infty \leq r \text{ implies } \Gamma + \mathcal{D}\tilde{\Delta}\mathcal{E} \text{ is invertible}\}.$$

We will also need to consider the *constant stability radius*, denoted by $rc_{stab}(\{e^{tA}\}, D, E)$, which is defined in the same way as $r_{stab}(\{e^{tA}\}, D, E)$ except that Δ does not depend on t.

For an exponentially stable evolution family $\mathcal{U}$ consider the "input-output" operator $\mathbb{L}_0 : L_p(\mathbb{R}_+, U) \to L_p(\mathbb{R}_+, Y)$ defined as:

$$(\mathbb{L}_0 u)(t) = E(t) \int_0^t U(t,\tau) D(\tau) u(\tau)\, d\tau, \quad u \in L_p(\mathbb{R}_+, U).$$

As shown by the next theorem, the norm of this operator provides a lower bound for $r_{stab}(\mathcal{U}, D, E)$. (The inequality (3.5), below, is proven in [6, Theorem 3.2] using a completely different approach.)

Theorem 3.2. *Let $\mathcal{U}$ be an exponentially stable evolution family. Let $D(\cdot) \in C_b(\mathbb{R}_+, B_s(U, X))$ and $E(\cdot) \in C_b(\mathbb{R}_+, B_s(X, Y))$. Then*

$$\mathbb{L}_0 = \mathcal{E}\hat{G}\mathcal{D} = -\mathcal{E}\Gamma^{-1}\mathcal{D}$$

and

$$\frac{1}{\|\mathbb{L}_0\|} \le r_{stab}(\mathcal{U}, D, E). \tag{3.5}$$

In the "unstructured" case, where $U = Y = X$ and $D = E = I$, one has

$$\mathbb{L}_0 = -\Gamma^{-1}, \quad \text{and} \quad \frac{1}{\|\Gamma^{-1}\|} \le r_{stab}(\mathcal{U}, I, I) \le \frac{1}{r(\Gamma^{-1})},$$

where $r(\cdot)$ denotes the spectral radius.

Proof. Since $\mathcal{U}$ is exponentially stable, Γ is invertible and $\Gamma^{-1} = \hat{G}$. The required formula for $\mathbb{L}_0$ follows from (2.2).

Set $\mathcal{H} := \Gamma^{-1}\mathcal{D}\tilde{\Delta}$. To prove (3.5), let $\Delta(\cdot) \in C_b(\mathbb{R}_+, B_s(Y, U))$ and suppose that $\|\Delta(\cdot)\| < 1/\|\mathbb{L}_0\|$. Then $\|\mathbb{L}_0\tilde{\Delta}\| < 1$, and hence $I - \mathbb{L}_0\tilde{\Delta} = I + \mathcal{E}\Gamma^{-1}\mathcal{D}\tilde{\Delta}$ is invertible on $L_p(\mathbb{R}_+, Y)$. That is, $I + \mathcal{E}\mathcal{H}$ is invertible on $L_p(\mathbb{R}_+, Y)$, and hence $I + \mathcal{H}\mathcal{E}$ is invertible on $L_p(\mathbb{R}_+, X)$ (with inverse $(I - \mathcal{H}(I + \mathcal{E}\mathcal{H})^{-1}\mathcal{E})$). Now,

$$\Gamma + \mathcal{D}\tilde{\Delta}\mathcal{E} = \Gamma(I + \Gamma^{-1}\mathcal{D}\tilde{\Delta}\mathcal{E}) = \Gamma(I + \mathcal{H}\mathcal{E})$$

and so $\Gamma + \mathcal{D}\tilde{\Delta}\mathcal{E}$ is invertible. Since $\Gamma + \mathcal{D}\tilde{\Delta}\mathcal{E}$ generates the evolutionary semigroup corresponding to $U_1(\cdot,\cdot)$, it follows that $1/\|\mathbb{L}_0\| \le r_{stab}(\mathcal{U}, D, E)$.

For the last assertion, suppose that $r_{stab}(\mathcal{U}, I, I) > 1/r(\Gamma^{-1})$. Then there exists λ such that $|\lambda| = r(\Gamma^{-1})$ and $\lambda + \Gamma^{-1}$ is not invertible. But then setting $\tilde{\Delta} \equiv \frac{1}{\lambda}$ gives $\|\tilde{\Delta}\| = \frac{1}{|\lambda|} < r_{stab}(\mathcal{U}, I, I)$, and so $\Gamma + \tilde{\Delta} = \tilde{\Delta}(\lambda + \Gamma^{-1})\Gamma$ is invertible, a contradiction. ∎

The nonautonomous scalar equation given by Example 4.4 of [5] shows that, in general, the inequality in (3.5) is strict.

3.2. **Autonomous case.** The development in this subsection is motivated by considering the autonomous equation (0.1) and constant perturbation operators $D \in B(U, X)$ and $E \in B(X, Y)$:

$$x'(t) = (A + D\Delta(t)E)x(t), \quad t \geq 0.$$

The evolution family $\mathcal{U}$ is now replaced by a semigroup $\{e^{tA}\}_{t\geq 0}$ generated by A. The primary focus is on properties of the constant stability radius, in which case Δ is independent of t.

When U and Y are Hilbert spaces it is known [6, Theorem 3.5] that

$$\frac{1}{\|\mathbb{L}_0\|} = rc_{stab}(\{e^{tA}\}, D, E) = \frac{1}{\sup_{s\in\mathbb{R}} \|E(A - is)^{-1}D\|}. \tag{3.6}$$

In this section we obtain explicit formulas for upper and lower bounds on the constant stability radius when U and Y are Banach spaces. The development uses a generalization of the spectral mapping theorem of L. Gearhart. Gearhart's theorem (see [15, p. 95]) says: if A generates a C_0 semigroup $\{e^{tA}\}_{t\geq 0}$ on a Hilbert space, then $1 \in \rho(e^{2\pi A})$ *if and only if* $i\mathbb{Z} \subset \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|(A - ik)^{-1}\| < \infty$.

A generalization (see [11]) of this result for Banach spaces is stated next. Here, F_{per} denotes the Banach space $L_p([0, 2\pi], X)$, $1 \leq p < \infty$. If $\{e^{tA}\}_{t\geq 0}$ is a strongly continuous semigroup on X, $\{T^t_{per}\}_{t\geq 0}$ will denote the evolution semigroup defined on F_{per} by the rule $T^t_{per} f(s) = e^{tA} f([s-t](\mathrm{mod}\, 2\pi))$; its generator will be denoted by Γ_{per}. The symbol Λ will be used to denote the set of all finite sequences $\{v_k\}_{k=-N}^N$ in X, or $\{u_k\}_{k=-N}^N$ in U.

Theorem 3.3. *Let A generate a C_0 semigroup $\{e^{tA}\}_{t\geq 0}$ on X. Let D and E be as above, and $\Delta \in B(Y, U)$. Let $\{e^{t(A+D\Delta E)}\}_{t\geq 0}$ be the strongly continuous semigroup generated by $A + D\Delta E$. Then the following are equivalent:*

i) $1 \in \rho(e^{2\pi(A+D\Delta E)})$;

ii) $i\mathbb{Z} \subset \rho(A + D\Delta E)$ *and*

$$\sup_{\{v_k\}\in\Lambda} \frac{\|\sum_k (A - ik + D\Delta E)^{-1} v_k e^{ik(\cdot)}\|_{F_{per}}}{\|\sum_k v_k e^{ik(\cdot)}\|_{F_{per}}} < \infty;$$

iii) $i\mathbb{Z} \subset \rho(A + D\Delta E)$ *and*

$$\inf_{\{v_k\}\in\Lambda} \frac{\|\sum_k (A - ik + D\Delta E) v_k e^{ik(\cdot)}\|_{F_{per}}}{\|\sum_k v_k e^{ik(\cdot)}\|_{F_{per}}} > 0.$$

Further, if Γ_{per} denotes the generator of the evolution semigroup on F_{per}, as above, and if $1 \in \rho(e^{2\pi A})$, then Γ_{per} is invertible and

$$\|\mathcal{E}\Gamma_{per}^{-1}\mathcal{D}\| = \sup_{\{u_k\}\in\Lambda} \frac{\|\sum_k E(A-ik)^{-1}Du_k e^{ik(\cdot)}\|_{L_p(0,2\pi],Y)}}{\|\sum_k u_k e^{ik(\cdot)}\|_{L_p([0,2\pi],U)}} \tag{3.7}$$

where $\mathcal{E}\Gamma_{per}^{-1}\mathcal{D} \in B(L_p([0,2\pi],U), L_p([0,2\pi],Y))$.

Proof. The equivalence of *(i)–(iii)* follows as in Theorem 2.3 of [11]. For the last statement, let $\{u_k\}$ be a finite set in U and consider functions f and g of the form

$$f(s) = \sum_k (A-ik)^{-1}Du_k e^{iks}, \quad \text{and} \quad g(s) = \sum_k Du_k e^{iks}.$$

Then $f = \Gamma_{per}^{-1}g$. For,

$$\begin{aligned}(\Gamma_{per}f)(s) &= \frac{d}{dt}\bigg|_{t=0} e^{tA}f([s-t]\mathrm{mod}2\pi) \\ &= \sum_k [A(A-ik)^{-1}Du_k e^{iks} - ik(A-ik)^{-1}Du_k e^{iks}] = g(s).\end{aligned}$$

For functions of the form $h(s) = \sum_k u_k e^{iks}$, where $\{u_k\}_k$ is a finite set in U, we have $\mathcal{E}\Gamma^{-1}\mathcal{D}h = \sum_k E(A-ik)^{-1}Du_k e^{ik(\cdot)}$. Taking the supremum over all such functions gives:

$$\begin{aligned}\|\mathcal{E}\Gamma_{per}^{-1}\mathcal{D}\| &= \sup_h \frac{\|\mathcal{E}\Gamma_{per}^{-1}\mathcal{D}h\|}{\|h\|} \\ &= \sup_{\{u_k\}\in\Lambda} \frac{\|\sum_k E(A-ik)^{-1}Du_k e^{ik(\cdot)}\|_{L_p([0,2\pi],Y)}}{\|\sum_k u_k e^{ik(\cdot)}\|_{L_p([0,2\pi],U)}}.\end{aligned}$$

□

In view of these facts we introduce a "pointwise" variant of the constant stability radius: for $t_0 > 0$ and $\lambda \in \rho(e^{t_0 A})$, define

$$rc_{stab}^{\lambda}(e^{t_0 A}, D, E) := \sup\{r > 0 : \|\Delta\|_{B(Y,U)} \le r \Rightarrow \lambda \in \rho(e^{t_0(A+D\Delta E)})\}.$$

By rescaling, the study of this quantity can be reduced to the case of $\lambda = 1$ and $t_0 = 2\pi$. Indeed,

$$rc_{stab}^{\lambda}(e^{t_0 A}, D, E) = \frac{2\pi}{t_0} rc_{stab}^{\lambda}(e^{2\pi C}, D, E), \quad \text{where} \quad C = \frac{t_0}{2\pi}A.$$

Also, after writing $\lambda = |\lambda|e^{i\theta}$ $(\theta \in \mathbb{R})$, note that

$$rc_{stab}^{\lambda}(e^{2\pi A}, D, E) = rc_{stab}^{1}(e^{2\pi B}, D, E) \quad \text{for} \quad B = A - \frac{1}{2\pi}(\ln|\lambda| + i\theta).$$

Therefore,

$$rc^{\lambda}_{stab}(e^{t_0 A}, D, E) = \frac{2\pi}{t_0} rc^{1}_{stab}(e^{2\pi B}, D, E),$$

where

$$B = \frac{1}{2\pi}(t_0 A - \ln|\lambda| - i\theta)).$$

In the following theorem we estimate $rc^{1}_{stab}(e^{2\pi A}, D, E)$.

Theorem 3.4. *Let $\{e^{tA}\}_{t\geq 0}$ be a strongly continuous semigroup generated by A on X, and assume $1 \in \rho(e^{2\pi A})$. Let Γ_{per} denote the generator of the induced evolution semigroup on F_{per}. Let $D \in B(U, X)$, and $E \in B(X, Y)$. Then*

$$\frac{1}{\|\mathcal{E}\Gamma_{per}^{-1}\mathcal{D}\|} \leq rc^{1}_{stab}(e^{2\pi A}, D, E) \leq \frac{1}{\sup_{k\in\mathbb{Z}} \|E(A - ik)^{-1}D\|}. \tag{3.8}$$

If U and Y are a Hilbert spaces and $p = 2$, then equalities hold in (3.8).

Proof. The first inequality follows from an argument as in Theorem 3.2. For the second inequality, let $\epsilon > 0$, and choose $\bar{u} \in U$ with $\|\bar{u}\| = 1$ and $k_0 \in \mathbb{Z}$ such that

$$\|E(A - ik_0)^{-1}D\bar{u}\|_Y \geq \sup_{k\in\mathbb{Z}} \|E(A - ik)^{-1}D\| - \epsilon > 0.$$

Choose $y^* \in Y^*$ with $\|y^*\| \leq 1$ such that

$$\left\langle y^*, \frac{E(A - ik_0)^{-1}D\bar{u}}{\|E(A - ik_0)^{-1}D\bar{u}\|_Y} \right\rangle = 1.$$

Define $\Delta \in B(Y, U)$ by

$$\Delta y = -\frac{\langle y^*, y\rangle}{\|E(A - ik_0)^{-1}D\bar{u}\|_Y}\bar{u}, \quad y \in Y.$$

We note that

$$\Delta E(A - ik_0)^{-1}D\bar{u} = -\frac{\langle y^*, E(A - ik_0)^{-1}D\bar{u}\rangle}{\|E(A - ik_0)^{-1}D\bar{u}\|_Y}\bar{u} = -\bar{u}, \tag{3.9}$$

and

(3.10)

$$\|\Delta\| \leq \frac{1}{\|E(A - ik_0)^{-1}D\bar{u}\|_Y} \leq \frac{1}{\sup_{k\in\mathbb{Z}} \|E(A - ik_0)^{-1}D\bar{u}\|_Y - \epsilon}.$$

Now set $\bar{v} := (A - ik_0)^{-1}D\bar{u}$ in X. By (3.9), $\Delta E\bar{v} = -\bar{u}$, and so

$$(A - ik_0 + D\Delta E)\bar{v} = (A - ik_0)\bar{v} + D\Delta E\bar{v} = D\bar{u} - D\bar{u} = 0.$$

Therefore,

$$\inf_{\{v_k\}\in\Lambda} \frac{\|\sum_k (A-ik+D\Delta E)v_k e^{ik(\cdot)}\|_{F_{per}}}{\|\sum_k u_k e^{ik(\cdot)}\|_{F_{per}}} \le \frac{\|(A-ik_0+D\Delta E)\bar{v}e^{ik_0(\cdot)}\|_{F_{per}}}{\|\bar{v}e^{ik_0(\cdot)}\|_{F_{per}}} = 0.$$

By Theorem 3.3, $1 \notin \rho(e^{2\pi(A+D\Delta E)})$. This shows that

$$rc^1_{stab}(e^{2\pi A}, D, E) \le \|\Delta\|.$$

To finish the proof, suppose that

$$rc^1_{stab}(e^{2\pi A}, D, E) > \frac{1}{\sup_{k\in\mathbb{Z}} \|E(A-ik)^{-1}D\|}.$$

Then $\epsilon > 0$ can be chosen sufficiently small so that for

$$r := \frac{1}{\sup_{k\in\mathbb{Z}} \|E(A-ik_0)^{-1}D\bar{u}\|_Y - \epsilon},$$

one has

$$\frac{1}{\sup_{k\in\mathbb{Z}} \|E(A-ik_0)^{-1}D\bar{u}\|_Y} < r < rc^1_{stab}(e^{2\pi A}, D, E).$$

But then by (3.10), $\|\Delta\| \le r < rc^1_{stab}(e^{2\pi A}, D, E)$, which is a contradiction.

For the last statement of the theorem, note that Parseval's formula applied to (3.7) gives

$$\|\mathcal{E}\Gamma^{-1}_{per}\mathcal{D}\| = \sup_{\{u_k\}\in\Lambda} \frac{\left(\sum_k \|E(A-ik)^{-1}Du_k\|^2_Y\right)^{1/2}}{\left(\sum_k \|u_k\|^2_U\right)^{1/2}} \le \sup_{k\in\mathbb{Z}} \|E(A-ik)^{-1}D\|.$$

Therefore,

$$\frac{1}{\|\mathcal{E}\Gamma^{-1}_{per}\mathcal{D}\|} \ge \frac{1}{\sup_{k\in\mathbb{Z}} \|E(A-ik)^{-1}D\|}$$

and hence equalities hold in (3.8). □

Next we consider the following "hyperbolic" variant of the constant stability radius. Recall, that a strongly continuous semigroup $\{e^{tA}\}_{t\ge 0}$ on X is called *hyperbolic* if

$$\sigma(e^{tA}) \cap \mathbb{T} = \emptyset, \quad \text{where } \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\},$$

for some (and, hence, for all) $t > 0$ (see, e.g., [28]). The hyperbolic semigroups are those for which the differential equation $x' = Ax$ has exponential dichotomy (see, e.g., [1, 3]) with the dichotomy projection P being the Riesz projection for the operator, say, e^A that corresponds to the part of its spectrum contained in the open unit disc.

For a given hyperbolic semigroup $\{e^{tA}\}_{t\geq 0}$ and operators D, E we define the *constant dichotomy radius* as follows:

$$rc_{dich}(\{e^{tA}\}, D, E) := \sup\{r > 0 : \|\Delta\|_{B(Y,U)} \leq r \text{ implies } \sigma(e^{t(A+D\Delta E)}) \cap \mathbb{T} = \emptyset \text{ for all } t > 0\}.$$

The dichotomy radius measures the size of the smallest Δ for which the perturbed equation $x' = [A + D\Delta E]x$ looses the exponential dichotomy.

Now for any $\xi \in [0, 1]$, consider the rescaled semigroup generated by $A_\xi := A - i\xi$ consisting of operators $e^{tA_\xi} = e^{-i\xi t}e^{tA}$, $t \geq 0$. The pointwise stability radius can be related to the dichotomy radius as follows.

Lemma 3.5. *Let $\{e^{tA}\}_{t\geq 0}$ be a hyperbolic semigroup. Then*

$$rc_{dich}(\{e^{tA}\}, D, E) = \inf_{\xi\in[0,1]} rc^1_{stab}(e^{2\pi A_\xi}, D, E).$$

Proof. Denote the left-hand side by α and the right-hand side by β. First fix $r < \beta$. Let $\xi \in [0, 1]$. If $\|\Delta\| \leq r$, then $1 \in \rho(e^{2\pi(A_\xi+D\Delta E)})$ and so $e^{i\xi 2\pi} \in \rho(e^{2\pi(A+D\Delta E)})$ for all $\xi \in [0, 1]$. That is, $e^{is} \in \rho(e^{2\pi(A+D\Delta E)})$ for all $s \in \mathbb{R}$, and so $\sigma(e^{2\pi(A+D\Delta E)}) \cap \mathbb{T} = \emptyset$. This shows that $r \leq \alpha$, and so $\beta \leq \alpha$.

Now suppose $r < \alpha$. If $\|\Delta\| \leq r$, then $\sigma(\{e^{t(A+D\Delta E)}\}) \cap \mathbb{T} = \emptyset$, and so $e^{i\xi t} \in \rho(e^{t(A+D\Delta E)})$ for all $\xi \in [0, 1], t \in \mathbb{R}$. That is, $1 \in \rho(e^{t(A_\xi+D\Delta E)})$. This says $r \leq \beta$ and so $\alpha \leq \beta$. □

Under the additional assumption that the semigroup $\{e^{tA}\}_{t\geq 0}$ is exponentially stable (that is, hyperbolic with a trivial dichotomy projection $P = I$), Lemma 3.5 gives, in fact, a formula for the constant *stability* radius. Indeed, the following simple proposition holds.

Proposition 3.6. *Let $\{e^{tA}\}_{t\geq 0}$ be an exponentially stable semigroup. Then*

$$rc_{dich}(\{e^{tA}\}, D, E) = rc_{stab}(\{e^{tA}\}, D, E).$$

Proof. Denote the left-hand side by α and the right-hand side by β. Take $r < \beta$ and any Δ with $\|\Delta\| \leq r$. By definition of the constant stability radius, $\omega_0(\{e^{t(A+D\Delta E)}\}) < 0$. In particular, $\sigma(e^{t(A+D\Delta E)}) \cap \mathbb{T} = \emptyset$, and $r \leq \alpha$ shows that $\beta \leq \alpha$.

Suppose that $\beta < r < \alpha$ for some r. By the definition of the stability radius β, there exists a Δ with $\|\Delta\| \in (\beta, r)$ such that the semigroup $\{e^{t(A+D\Delta E)}\}_{t\geq 0}$ is not stable.

For any $\tau \in [0,1]$ one has $\|\tau\Delta\| \le r < \alpha$. By the definition of the dichotomy radius α it follows that the semigroup $\{e^{t(A+\tau D\Delta E)}\}_{t\ge 0}$ is hyperbolic for each $\tau \in [0,1]$. Now consider its dichotomy projection

$$P(\tau) = (2\pi i)^{-1} \int_{\mathbb{T}} \left(\lambda - e^{A+\tau D\Delta E}\right)^{-1} d\lambda,$$

which is the Riesz projection corresponding to the part of $\sigma(e^{A+\tau D\Delta E})$ located inside of the open unit disk. The function $\tau \mapsto P(\tau)$ is norm continuous. Indeed, since the bounded perturbation $\tau D\Delta E$ of the generator A is continuous in τ, the operators $e^{t(A+\tau D\Delta E)}$, $t \ge 0$ depend on τ continuously (see, e.g., [17, Cor. 3.1.3]); this implies (see, e.g., [3, Thm. I.2.2]) the continuity of $P(\cdot)$.

By assumption $\{e^{tA}\}_{t\ge 0}$ is exponentially stable, so $P(0) = I$. Also, $P(1) \ne I$ since the semigroup $\{e^{t(A+D\Delta E)}\}_{t\ge 0}$ with $\|\Delta\| \le r < \alpha$ is hyperbolic but not stable. Since either $\|I - P(\tau)\| = 0$ or $\|I - P(\tau)\| \ge 1$, this contradicts the continuity of $\|P(\cdot)\|$. □

As a consequence, the following explicit formulas for bounds on the constant stability radius can be obtained. For this, let $\xi \in [0,1]$ and set

$$S_\xi := \sup_{\{u_k\}\in\Lambda} \frac{\|\sum_k E(A - i\xi - ik)^{-1} D u_k e^{ik(\cdot)}\|_{L_p([0,2\pi],Y)}}{\|\sum_k u_k e^{ik(\cdot)}\|_{L_p([0,2\pi],U)}}.$$

We note that S_ξ is computed as in equation (3.7) with A replaced by $A_\xi = A - i\xi$.

Corollary 3.7. *Let $\{e^{tA}\}_{t\ge 0}$ be an exponentially stable semigroup generated by A. Then*

$$\frac{1}{\sup_{\xi\in[0,1]} S_\xi} \le rc_{stab}(\{e^{tA}\}, D, E) \le \frac{1}{\sup_{s\in\mathbb{R}} \|E(A - is)^{-1} D\|}.$$

Proof. Fix $\xi \in [0,1]$, and let $\Gamma_{per,\xi}$ denote the generator of evolution semigroup on $L_p([0,2\pi], X)$ induced by $\{e^{tA_\xi}\}_{t\ge 0}$. By Theorem 3.3, $\|\mathcal{E}\Gamma^{-1}_{per,\xi}\mathcal{D}\| = S_\xi$, and so by Theorem 3.4,

$$\frac{1}{S_\xi} \le rc^1_{stab}(e^{2\pi A_\xi}, D, E) \le \frac{1}{\sup_{k\in\mathbb{Z}} \|E(A_\xi - ik)^{-1} D\|}.$$

By Proposition 3.6, taking the infimum over $\xi \in [0,1]$ gives

$$\begin{aligned}\frac{1}{\sup_{\xi\in[0,1]} S_\xi} &\le \inf_{\xi\in[0,1]} rc^1_{stab}(e^{2\pi A_\xi}, D, E) = rc_{stab}(\{e^{tA}\}, D, E)\\ &\le \inf_{\xi\in[0,1]} \frac{1}{\sup_{k\in\mathbb{Z}} \|E(A_\xi - ik)^{-1}D\|}\\ &= \frac{1}{\sup_{s\in\mathbb{R}} \|E(A-is)^{-1}D\|}.\end{aligned}$$

□

Remark 3.8. The inequality (3.5) together with the observation that $r_{stab}(\{e^{tA}\}, D, E) \le rc_{stab}(\{e^{tA}\}, D, E)$ shows that

$$\frac{1}{\|\mathbb{L}_0\|} \le r_{stab}(\{e^{tA}\}, D, E) \le \frac{1}{\sup_{s\in\mathbb{R}} \|E(A-is)^{-1}D\|}.$$

Moreover,

$$\frac{1}{\|\mathbb{L}_0\|} = \frac{1}{\|\mathcal{E}\Gamma^{-1}\mathcal{D}\|} \le \frac{1}{\sup_{\xi\in[0,1]} \|\mathcal{E}\Gamma^{-1}_{per,\xi}\mathcal{D}\|} = \frac{1}{\sup_{\xi\in[0,1]} S_\xi}.$$

If equality holds in the last statement, then $r_{stab}(\{e^{tA}\}, D, E)$ can replace $rc_{stab}(\{e^{tA}\}, D, E)$ in Corollary 3.7.

In contrast to the Hilbert space setting, the following Banach space example shows that equality in (3.6) does not generally hold.

Example 3.9. If A generates a strongly continuous semigroup on a Banach space X, the *abscissa of uniform boundedness of the resolvent*, denoted by $s_0(A)$, is the infimum of all $\omega \in \mathbb{R}$ such that $\{\lambda : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$ and $\sup\{\|R(\lambda, A)\| : \mathrm{Re}\lambda > \omega\} < \infty$. If $s(A)$ and $\omega_0(A)$ denote the spectral bound and growth bound of the semigroup, respectively, then it is always the case that $s(A) \le s_0(A) \le \omega_0(A)$. An example due to W. Arendt (see [28], Example 1.4.5) exhibits a (positive) strongly continuous semigroup with the property that $s_0(A) < \omega_0(A) < 0$. Now, for α such that $0 \le \alpha \le -\omega_0(A)$, consider a rescaled semigroup generated by $A + \alpha$, and denote by $\Gamma_{A+\alpha}$ the generator of the induced evolution semigroup on $L_p(\mathbb{R}_+, X)$. The following relationships hold: for $0 \le \alpha < -\omega_0(A)$,

$$s_0(A+\alpha) = s_0(A) + \alpha < \omega_0(A) + \alpha = \omega_0(A+\alpha) < 0;$$

for $\alpha_0 := -\omega_0(A)$,

$$s_0(A+\alpha_0) < \omega_0(A+\alpha_0) = 0.$$

This says that $s_0(A+\alpha) < 0$ for all $\alpha \in [0, \alpha_0]$ and hence

$$M := \sup_{\alpha\in[0,\alpha_0]} \sup_{s\in\mathbb{R}} \|(A+\alpha-is)^{-1}\| < \infty.$$

Now note that $\omega_0(A+\alpha) < 0$ if and only if $\|\Gamma_{A+\alpha}^{-1}\| < \infty$, and $\|\Gamma_{A+\alpha}^{-1}\|$ is a continuous function of α on $[0, \alpha_0)$. Therefore, there exists $\alpha_1 \in [0, \alpha_0)$ such that $\|\Gamma_{A+\alpha_1}^{-1}\| > M$, and so at least one of the following inequalities is strict:

$$\frac{1}{\|\mathbb{L}_0\|} = \frac{1}{\|\Gamma_{A+\alpha_1}^{-1}\|} \leq rc_{stab}(\{e^{t(A+\alpha_1)}\}, I, I) \leq \frac{1}{\sup_{s\in\mathbb{R}} \|(A+\alpha_1 - is)^{-1}\|}.$$

Acknowledgments: The authors would like to thank Professor S. Clark, University of Missouri-Rolla, for several helpful discussions.

References

[1] A. Ben-Artzi and I. Gohberg, Dichotomy of systems and invertibility of linear ordinary differential operators, *Oper. Theory Adv. Appl.* **56** (1992), 90–119.

[2] R. Curtain and A. J. Pritchard, "Infinite Dimensional Linear system Theory," Lecture Notes in Control and Information Sciences, Vol. 8, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[3] J. Daleckij and M. Krein, "Stability of Differential Equations in Banach Space," Amer. Math. Soc., Providence, RI, 1974.

[4] R. Datko, "Uniform asymptotic stability of evolutionary processes in a Banach space," SIAM J. Math. Anal., **3** (1972), 428–445.

[5] D. Hinrichsen, A. Ilchmann and A. J. Pritchard, Robustness of stability of time-varying linear systems, *J. Differential Equations*, **82** (1989), 219–250.

[6] D. Hinrichsen and A. J. Pritchard, Robust stability of linear evolution operators on Banach spaces, *SIAM J. Control Optim.*, **32** no. 6, (1994), 1503–1541.

[7] D. Hinrichsen and A. J. Pritchard, Stability radius for structured perturbations and the algebraic Riccati equation, *Systems Control Lett.*, **8** (1986), 105–113.

[8] B. Jacob, Robust stability of infinite dimensional time-varying systems with respect to nonlinear perturbations, *Integral Equations Operator Theory*, (1995), to appear.

[9] B. Jacob, Time-Varying Infinite Dimensional State-Space Systems, Dissertation, Universität Bremen, 1995.

[10] Y. Latushkin and S. Montgomery-Smith, Lyapunov theorems for Banach spaces, *Bull. Amer. Math. Soc. (N.S.)*, **31** no. 1 (1994), 44–49.

[11] Y. Latushkin and S. Montgomery-Smith, Evolutionary semigroups and Lyapunov theorems in Banach spaces, *J. Funct. Anal.* **127** (1995), 173–197.

[12] Y. Latushkin and S. Montgomery-Smith and T. Randolph, Evolutionary semigroups and dichotomy of linear skew-product flows on locally compact spaces with Banach fibers, *J. Differential Equations*, **125** no. 1 (1996), 73–116.

[13] Y. Latushkin and T. Randolph, Dichotomy of differential equations on Banach spaces and an algebra of weighted composition operators, *Integral Equations Operator Theory*, **23** (1995), 472–500.

[14] Y. Latushkin and A. M. Stepin, Weighted composition operators and linear extensions of dynamical systems, *Russian Math. Surveys*, **46** no. 2 (1992), 95–165.

[15] R. Nagel (ed.) "One Parameters Semigroups of Positive Operators", Lecture Notes in Math., no. 1184, Springer-Verlag, Berlin, 1984.

[16] R. Nagel and A. Rhandi, A characterization of Lipschitz continuous evolution families on Banach spaces, *Integral Equations Operator Theory*, to appear.
[17] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, N.Y./Berlin, 1983.
[18] R. S. Phillips, Perturbation theory for semi-groups of linear operators, *Trans. Amer. Math. Soc.* **74** (1953), 199–221.
[19] A. J. Pritchard and S. Townley, Robustness of linear systems, *J. Differential Equations* **77** (1989), 254–286.
[20] J. Prüss, On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.* **284** no. 2 (1984), 847–857.
[21] F. Räbiger and R. Schnaubelt, The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions, *Semigroup Forum* **52** (1996), 225–239.
[22] F. Räbiger and R. Schnaubelt, A spectral characterization of exponentially dichotomic and hyperbolic evolution families, *Tubingen Berichte zur Funktionalanalysis*, **3**, 1993/94, 204–221.
[23] F. Räbiger, A. Rhandi, R. Schnaubelt, Perturbation and an abstract characterization of evolution semigroups, *J. Math. Anal. Appl.*, **198** (1996), 516–533.
[24] R. Rau, Hyperbolic evolution groups and dichotomic of evolution families, *J. Dynamics Diff. Eqns.* **6**, no. 2 (1994), 335–350.
[25] R. Rau, Hyperbolic evolutionary semigroups on vector-valued function spaces, *Semigroup Forum*, **48** (1994), 107–118.
[26] R. Schnaubelt, *Exponential Bounds and Hyperbolicity of Evolution Families*, Dissertation, Eberhard-Karls-Univeristät Tübingen, 1996.
[27] J. M. A. M. van Neerven, Characterization of exponential stability of a semigroup of operators in terms of its action by convolution on vector-valued function spaces over $\mathbb{R}_+$, *Tubingen Berichte zur Funktionalanalysis*, Heft 3, Jahrang 1993/94, (1994), 153–168.
[28] J. M. A. M. van Neerven, "The Asymptotic Behavior of a Semigroup of Linear Operators," Lecture Notes in Math., Springer-Verlag, Berlin, 1996.
[29] J. Voigt, On the perturbation theory for strongly continuous semigroups, Math. Ann., **229** (1977), 163–171.
[30] F. R. Wirth, Robust Stability of Discrete-Time Systems under Time-Varying Perturbations, Dissertation, Universität Bremen, 1995.

Department of Mathematics, University of Missouri, Columbia, MO 65211
E-mail address: `mathyl@mizzou1.missouri.edu`

Department of Mathematics, University of Missouri, Columbia, MO 65211
E-mail address: `stephen@mont.cs.missouri.edu`

Department of Mathematics, University of Missouri, Rolla, MO 65409
E-mail address: `randolph@umr.edu`